A

Outline of Functional Analysis

Introduction

Problems in PDE have provided a major impetus for the development of functional analysis. Here, we present some basic results, which are useful for the development of such subjects as distribution theory and Sobolev spaces, discussed in Chapters 3 and 4; the spectral theory of compact and of unbounded operators, applied to elliptic PDE in Chapter 5; the theory of Fredholm operators and their indices, needed for the study of the Atiyah-Singer index theorem in Chapter 10; and the theory of semigroups, of particular value in Chapter 9 on scattering theory, and also germane to studies of evolution equations in Chapters 3 and 6. Indeed, what is thought of as the subject of functional analysis naturally encompasses some of the development of these chapters as well as the material presented in this appendix. One particular case of this is the spectral theory of Chapter 8. In fact, it is there that we present a proof of the spectral theorem for general self-adjoint operators. One reason for choosing to do it this way is that my favorite approach to the spectral theorem uses Fourier analysis, which is not applied in this appendix, though some of the exercises make contact with it. Thus in this appendix the spectral theorem is proved only for compact operators, an extremely simple special case. On the other hand, it is hoped that by the time one gets through the Fourier analysis as developed in Chapter 3, the presentation of the general spectral theorem in Chapter 8 will appear to be very simple too.

1. Banach spaces

A *Banach space* is a complete, normed, linear space. A *norm* on a linear space V is a positive function $\|v\|$ having the properties

$$\begin{aligned} \|av\| &= |a| \cdot \|v\| \text{ for } v \in V,\ a \in \mathbb{C}\,(\text{or } \mathbb{R}), \\ \|v+w\| &\le \|v\| + \|w\|, \\ \|v\| &> 0 \text{ unless } v = 0. \end{aligned} \tag{1.1}$$

The second of these conditions is called the *triangle inequality*. Given a norm on V, there is a distance function $d(u,v) = \|u-v\|$, making V a metric space.

A metric space is a set X, with distance function $d : X \times X \to \mathbb{R}^+$, satisfying

$$\begin{aligned} d(u,v) &= d(v,u), \\ d(u,v) &\le d(u,w) + d(w,v), \\ d(u,v) &> 0 \text{ unless } u = v. \end{aligned} \tag{1.2}$$

A sequence (u_j) is Cauchy provided $d(v_n, v_m) \to 0$ as $m,n \to \infty$; completeness is the property that any Cauchy sequence converges. Further background on metric spaces is given in §1 of Appendix B.

We list some examples of Banach spaces. First, let X be any compact metric space, that is, a metric space with the property that any sequence (x_n) has a convergent subsequence. Then $C(X)$, the space of continuous functions on X, is a Banach space, with norm

$$\|u\|_{\sup} = \sup\{|u(x)| : x \in X\}. \tag{1.3}$$

Also, for any $\alpha \in [0,1]$, we set

$$\operatorname{Lip}^\alpha(X) = \{u \in C(X) : |u(x) - u(y)| \le C\, d(x,y)^\alpha \text{ for all } x, y \in X\}. \tag{1.4}$$

This is a Banach space, with norm

$$\|u\|_\alpha = \|u\|_{\sup} + \sup_{x,y \in X} \frac{|u(x)-u(y)|}{d(x,y)^\alpha}. \tag{1.5}$$

$\operatorname{Lip}^0(X) = C(X)$; the space $\operatorname{Lip}^1(X)$ is typically denoted $\operatorname{Lip}(X)$. For $\alpha \in (0,1)$, $\operatorname{Lip}^\alpha(X)$ is frequently denoted $C^\alpha(X)$. In all these cases, it is straightforward to verify the conditions (1.1) on the proposed norms and to establish completeness.

Related spaces arise when X is specialized to be a compact Riemannian manifold. We have $C^k(M)$, the space of functions whose derivatives of order $\le k$ are continuous on M. Norms on $C^k(M)$ can be constructed as follows. Pick $Z_1, \dots, Z_N$, smooth vector fields on M that span T_pM at each $p \in M$. Then we can set

$$\|u\|_{C^k} = \sum_{\ell \leq k} \|Z_{j_1} \cdots Z_{j_\ell} u\|_{\sup}. \tag{1.6}$$

If one replaces the sup norm on the right by the C^α-norm (1.5), for some $\alpha \in (0,1)$, one has a norm for the Banach space $C^{k,\alpha}(M)$.

More subtle examples of Banach spaces are the L^p-spaces, defined as follows. First take $p = 1$. Let (X, μ) be a measure space. We say a measurable function f belongs to $\mathcal{L}^1(X, \mu)$ provided

$$\int_X |f(x)| \, d\mu(x) < \infty. \tag{1.7}$$

Elements of $L^1(X, \mu)$ consist of equivalence classes of elements of $\mathcal{L}^1(X, \mu)$, where we say

$$f \sim \tilde{f} \Leftrightarrow f(x) = \tilde{f}(x), \quad \text{for } \mu\text{-almost every } x. \tag{1.8}$$

With a slight abuse of notation, we denote by f both a measurable function in $\mathcal{L}^1(X, \mu)$ and its equivalence class in $L^1(X, \mu)$. Also, we say that f, defined only almost everywhere on X, belongs to $L^1(X, \mu)$ if there exists $\tilde{f} \in \mathcal{L}^1(X, \mu)$ such that $\tilde{f} = f$ a.e. The norm $\|f\|_{L^1}$ is given by (1.7); it is easy to see that this norm has the properties (1.1).

The proof of completeness of $L^1(X, \mu)$ makes use of the following key convergence results in measure theory.

Monotone convergence theorem. *If $f_j \in \mathcal{L}^1(X, \mu)$, $0 \leq f_1(x) \leq f_2(x) \leq \cdots$, and $\|f_j\|_{L^1} \leq C < \infty$, then $\lim_{j \to \infty} f_j(x) = f(x)$, with $f \in L^1(X, \mu)$ and $\|f_j - f\|_{L^1} \to 0$ as $j \to \infty$.*

Dominated convergence theorem. *If $f_j \in \mathcal{L}^1(X, \mu), \lim f_j(x) = f(x)$, μ-a.e., and there is an $F \in \mathcal{L}^1(X, \mu)$ such that $|f_j(x)| \leq F(x)$ μ-a.e., for all j, then $f \in \mathcal{L}^1(X, \mu)$ and $\|f_j - f\|_{L^1} \to 0$.*

To show that $L^1(X, \mu)$ is complete, suppose (f_n) is Cauchy in L^1. Passing to a subsequence, we can assume $\|f_{n+1} - f_n\|_{L^1} \leq 2^{-n}$. Consider the infinite series

$$f_1(x) + \sum_{n=1}^{\infty} \big[f_{n+1}(x) - f_n(x)\big]. \tag{1.9}$$

Now the partial sums are dominated by

$$G_m(x) = \sum_{n=1}^{m} |f_{n+1}(x) - f_n(x)|,$$

and since $0 \leq G_1 \leq G_2 \leq \cdots$ and $\|G_m\|_{L^1} \leq \sum 2^{-n} \leq 1$, we deduce from the monotone convergence theorem that $G_m \nearrow G$ μ-a.e. and in L^1-norm.

Hence the infinite series (1.9) is convergent a.e., to a limit $f(x)$, and via the dominated convergence theorem we deduce that $f_n \to f$ in L^1-norm. This proves completeness.

Continuing with a description of L^p-spaces, we define $\mathcal{L}^\infty(X,\mu)$ to consist of bounded, measurable functions, $L^\infty(X,\mu)$ to consist of equivalence classes of such functions, via (1.8), and we define $\|f\|_{L^\infty}$ to be the smallest sup of $\tilde{f} \sim f$. It is easy to show that $L^\infty(X,\mu)$ is a Banach space.

For $p \in (1,\infty)$, we define $\mathcal{L}^p(X,\mu)$ to consist of measurable functions f such that

$$\Big[\int_X |f(x)|^p \, d\mu(x)\Big]^{1/p} \tag{1.10}$$

is finite. $L^p(X,\mu)$ consists of equivalence classes, via (1.8), and the L^p-norm $\|f\|_{L^p}$ is given by (1.10). This time it takes a little work to verify the triangle inequality. That this holds is the content of *Minkowski's inequality*:

$$\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}. \tag{1.11}$$

One neat way to establish this is by the following characterization of the L^p-norm. Suppose p and q are related by

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{1.12}$$

We claim that if $f \in L^p(X,\mu)$,

$$\|f\|_{L^p} = \sup\big\{\|fh\|_{L^1} : h \in L^q(X,\mu),\ \|h\|_{L^q} = 1\big\}. \tag{1.13}$$

We can apply (1.13) to $f+g$, which belongs to $L^p(X,\mu)$ if f and g do, since $|f+g|^p \le 2^p(|f|^p + |g|^p)$. Given this, (1.11) follows easily from the inequality $\|(f+g)h\|_{L^1} \le \|fh\|_{L^1} + \|gh\|_{L^1}$.

The identity (1.13) can be regarded as two inequalities. The "$\le$" part can be proven by choosing $h(x)$ to be an appropriate multiple $C|f(x)|^{p-1}$. We leave this as an exercise. The converse inequality, "$\ge$," is a consequence of Hölder's inequality:

$$\int |f(x)g(x)| \, d\mu(x) \le \|f\|_{L^p}\|g\|_{L^q}, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{1.14}$$

Hölder's inequality can be proved via the following inequality for positive numbers:

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \quad a,b>0, \tag{1.15}$$

assuming that $p \in (1,\infty)$ and (1.12) holds; (1.15) is equivalent to

$$x^{1/p}y^{1/q} \le \frac{x}{p} + \frac{y}{q}, \quad x,y>0. \tag{1.16}$$

Since both sides of this are homogeneous of degree 1 in (x, y), it suffices to prove it for $y = 1$, that is, to prove that $x^{1/p} \le x/p + 1/q$ for $x \in [0, \infty)$. Now $\varphi(x) = x^{1/p} - x/p$ can be maximized by elementary calculus; one finds a unique maximum at $x = 1$, with $\varphi(1) = 1 - 1/p = 1/q$. This establishes (1.16), hence (1.15). Applying this to the integrand in (1.14) gives

$$\int |f(x)g(x)|\, d\mu(x) \le \frac{1}{p}\, \|f\|_{L^p}^p + \frac{1}{q}\, \|g\|_{L^q}^q. \tag{1.17}$$

This looks weaker than (1.14), but now replace f by tf and g by $t^{-1}g$, so that the left side of (1.17) is dominated by

$$\frac{t^p}{p}\|f\|_{L^p}^p + \frac{1}{qt^q}\|g\|_{L^q}^q.$$

Minimizing over $t \in (0, \infty)$ then gives Hölder's inequality. Consequently, (1.10) defines a norm on $L^p(X, \mu)$. Completeness follows as in the $p = 1$ case discussed above.

We next give a discussion of one important method of manufacturing new Banach spaces from old. Namely, suppose V is a Banach space, W a closed linear subspace. Consider the linear space $L = V/W$, with norm

$$\|[v]\| = \inf \bigl\{ \|v - w\| : w \in W \bigr\}, \tag{1.18}$$

where $v \in V$, and $[v]$ denotes its class in V/W. It is easy to see that (1.18) defines a norm on V/W. We record a proof of the following.

Proposition 1.1. *If V is a Banach space and W is a closed linear subspace, then V/W, with norm (1.18), is a Banach space.*

It suffices to prove that V/W is complete. We use the following; compare the use of (1.9) in the proof of completeness of $L^1(X, \mu)$.

Lemma 1.2. *A normed linear space L is complete provided the hypothesis*

$$x_j \in L, \quad \sum_{j=1}^{\infty} \|x_j\| < \infty,$$

implies that $\sum_{j=1}^{\infty} x_j$ converges in L.

Proof. If (y_k) is Cauchy in L, take a subsequence so that $\|y_{k+1} - y_k\| \le 2^{-k}$, and consider $y_1 + \sum_{j=1}^{\infty}(y_{j+1} - y_j)$.

To prove Proposition 1.1 now, say $[v_j] \in V/W$, $\sum \|[v_j]\| < \infty$. Then pick $w_j \in W$ such that $\|v_j - w_j\| \le \|[v_j]\| + 2^{-j}$, to get $\sum_{j=1}^{\infty} \|v_j - w_j\| < \infty$. Hence $\sum(v_j - w_j)$ converges in V, to a limit v, and it follows that $\sum[v_j]$ converges to $[v]$ in V/W.

Note that if W is a proper closed, linear subspace of V, given $v \in V \setminus W$, we can pick $w_n \in W$ such that $\|v - w_n\| \to \operatorname{dist}(v, W)$. Normalizing $v - w_n$ produces $v_n \in V$ such that the following holds.

Lemma 1.3. *If W is a proper closed, linear subspace of a Banach space V, there exist $v_n \in V$ such that*

$$\|v_n\| = 1, \quad \mathit{dist}(v_n, W) \nearrow 1. \tag{1.19}$$

In Proposition 2.1 we will produce an important sharpening of this for Hilbert spaces. For now we remark on the following application.

Proposition 1.4. *If V is an infinite-dimensional Banach space, then the closed unit ball $B_1 \subset V$ is not compact.*

Proof. If V_j is an increasing sequence of spaces, of dimension j, by (1.19) we can obtain $v_j \in V_j$, $\|v_j\| = 1$, each pair a distance $\geq 1/2$; thus (v_j) has no convergent subsequence.

It is frequently useful to show that a certain linear subspace L of a Banach space V is *dense*. We give a few important cases of this here.

Proposition 1.5. *If μ is a Borel measure on a compact metric space X, then $C(X)$ is dense in $L^p(X, \mu)$ for each $p \in [1, \infty)$.*

Proof. First, let K be any compact subset of X. The functions

$$f_{K,n}(x) = \big[1 + n \operatorname{dist}(x, K)\big]^{-1} \in C(X) \tag{1.20}$$

are all ≤ 1 and decrease monotonically to the characteristic function χ_K equal to 1 on K, 0 on $X \setminus K$. The monotone convergence theorem gives $f_{K,n} \to \chi_K$ in $L^p(X, \mu)$ for $1 \leq p < \infty$. Now let $A \subset X$ be any measurable set. Any Borel measure on a compact metric space is regular, that is,

$$\mu(A) = \sup\{\mu(K) : K \subset A,\ K \text{ compact}\}. \tag{1.21}$$

Thus there exists an increasing sequence K_j of compact subsets of A such that $\mu(A \setminus \cup_j K_j) = 0$. Again, the monotone convergence theorem implies $\chi_{K_j} \to \chi_A$ in $L^p(X, \mu)$ for $1 \leq p < \infty$. Thus all *simple* functions on X are in the closure of $C(X)$ in $L^p(X, \mu)$ for $p \in [1, \infty)$. The construction of $L^p(X, \mu)$ directly shows that each $f \in L^p(X, \mu)$ is a norm limit of simple functions, so the result is proved.

This result is easily extended to give the following:

Corollary 1.6. *If X is a metric space that is locally compact and a countable union of compact X_j, and μ is a (locally finite) Borel measure on X,* *then the space* $C_{00}(X)$ *of compactly supported, continuous functions on* X *is dense in* $L^p(X,\mu)$ *for each* $p \in [1,\infty)$.

Further extensions, involving more general locally compact spaces, can be found in [Lo].

The following is known as the Weierstrass approximation theorem.

Theorem 1.7. *If* $I = [a,b]$ *is an interval in* $\mathbb{R}$, *the space* $\mathcal{P}$ *of polynomials in one variable is dense in* $C(I)$.

There are many proofs of this. One close to Weierstrass's original (and my favorite) goes as follows. Given $f \in C(I)$, extend it to be continuous and compactly supported on $\mathbb{R}$; convolve this with a highly peaked Gaussian; and approximate the result by power series. For a more detailed sketch, in the context of other useful applications of highly peaked Gaussians, see Exercises 14 and 15 in §3 of Chapter 3.

The following generalization is known as the *Stone-Weierstrass theorem.*

Theorem 1.8. *Let* X *be a compact Hausdorff space and* $\mathcal{A}$ *a subalgebra of* $C_{\mathbb{R}}(X)$, *the algebra of real-valued, continuous functions on* X. *Suppose that* $1 \in \mathcal{A}$ *and that* $\mathcal{A}$ *separates points of* X, *that is, for distinct* $p, q \in X$, *there exists* $h_{pq} \in \mathcal{A}$ *with* $h_{pq}(p) \neq h_{pq}(q)$. *Then the closure* $\overline{\mathcal{A}}$ *is equal to* $C_{\mathbb{R}}(X)$.

We sketch a proof of Theorem 1.8, making use of Theorem 1.7, which implies that if $f \in \overline{\mathcal{A}}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous, then $\varphi \circ f \in \overline{\mathcal{A}}$. Consequently, if $f_j \in \overline{\mathcal{A}}$, then $\sup(f_1, f_2) = (1/2)|f_1 - f_2| + (1/2)(f_1 + f_2) \in \overline{\mathcal{A}}$.

The hypothesis of separating points implies that, for distinct $p, q \in X$, there exists $f_{pq} \in \overline{\mathcal{A}}$, equal to 1 at p, 0 at q. Applying appropriate φ, we can arrange also that $0 \le f_{pq}(x) \le 1$ on X and that f_{pq} is 1 near p and 0 near q. Taking infima, we can obtain $f_{pU} \in \overline{\mathcal{A}}$, equal to 1 on a neighborhood of p and equal to 0 off a given neighborhood U of p. Applying sups to these, we obtain, for each compact $K \subset X$ and open $U \supset K$, a function $g_{KU} \in \overline{\mathcal{A}}$ such that g_{KU} is 1 on K, 0 off U, and $0 \le g_{KU}(x) \le 1$ on X. Once we have gotten this far, it is easy to approximate any continuous $u \ge 0$ on X by a sup of (positive constants times) such g_{KU}, and from there it is easy to prove the theorem.

Theorem 1.8 has a complex analogue. In that case, we add the assumption that $f \in \mathcal{A} \Rightarrow \overline{f} \in \mathcal{A}$ and conclude that $\overline{\mathcal{A}} = C(X)$. This is easily reduced to the real case.

Exercises

1. Let $\mathcal{L}$ be the subspace of $C(S^1)$ consisting of finite linear combinations of the exponentials $e^{in\theta}$, $n \in \mathbb{Z}$. Use the Stone-Weierstrass theorem to show that $\mathcal{L}$ is dense in $C(S^1)$.
2. Show that the space of finite linear combinations of the functions
$$E_\zeta(t) = e^{-\zeta t},$$
as ζ ranges over $(0, \infty)$, is dense in $C_0(\mathbb{R}^+)$, the space of continuous functions on $\mathbb{R}^+ = [0, \infty)$, vanishing at infinity. (*Hint*: Make a slight generalization of the Stone-Weierstrass theorem.)
3. Given $f \in L^1(\mathbb{R}^+)$, the Laplace transform
$$(\mathcal{L}f)(\zeta) = \int_0^\infty e^{-\zeta t} f(t)\, dt$$
is defined and holomorphic for $\operatorname{Re}\zeta > 0$. Suppose $(\mathcal{L}f)(\zeta)$ vanishes for ζ on some open subset of $(0, \infty)$. Show that $f = 0$, using Exercise 2. (*Hint*: First show that $(\mathcal{L}f)(\zeta)$ is identically zero.)
4. Let I be a compact interval, V a Banach space, and $f : I \to V$ a continuous function. Show that the Riemann integral $\int_I f(x)\, dx$ is well-defined. Formulate and establish the fundamental theorem of calculus for V-valued functions. Formulate and verify appropriate basic results on multidimensional integrals of V-valued functions.
5. Let $\Omega \subset \mathbb{C}$ be open, V a (complex) Banach space, and $f : \Omega \to V$. We say f is holomorphic if it is a C^1-map and, for each $z \in \Omega$, $Df(z)$ is $\mathbb{C}$-linear. Establish for such V-valued holomorphic functions the Cauchy integral theorem, the Cauchy integral formula, power-series expansions, and the Liouville theorem.

A Banach space V is said to be *uniformly convex* provided that for each $\varepsilon > 0$, these exists $\delta > 0$ such that, for $x, y \in V$,
$$\|x\|, \|y\| \le 1, \quad \left\| \frac{1}{2}(x+y) \right\| \ge 1 - \delta \Longrightarrow \|x - y\| < \varepsilon.$$

6. Show that $L^p(X, \mu)$ is uniformly convex provided $2 \le p < \infty$.
(*Hint* Prove and use the fact that, for $a, b \in \mathbb{C},\ p \in [2, \infty)$,
$$|a+b|^p + |a-b|^p \le 2^{p-1}(|a|^p + |b|^p),$$
so that
$$\|f+g\|_{L^p}^p + \|f-g\|_{L^p}^p \le 2^{p-1}(\|f\|_{L^p}^p + \|g\|_{L^p}^p).)$$

REMARK. $L^p(X, \mu)$ is also uniformly convex for $p \in (1, 2)$, but the proof is harder. See [Kot], pp. 358–359.

2. Hilbert spaces

A *Hilbert space* is a complete inner-product space. That is to say, first the space H is a linear space provided with an inner product, denoted (u, v), for u and v in H, satisfying the following defining conditions:

$$\begin{aligned}(au_1 + u_2, v) &= a(u_1, v) + (u_2, v), \\ (u, v) &= \overline{(v, u)}, \\ (u, u) &> 0 \text{ unless } u = 0.\end{aligned} \tag{2.1}$$

To such an inner product is assigned a norm, by

$$\|u\| = \sqrt{(u, u)}. \tag{2.2}$$

To establish that the triangle inequality holds for $\|u + v\|$, we can expand $\|u + v\|^2 = (u + v, u + v)$ and deduce that this is $\le \big[\|u\| + \|v\|\big]^2$, as a consequence of Cauchy's inequality:

$$|(u, v)| \le \|u\| \cdot \|v\|, \tag{2.3}$$

a result that can be proved as follows. The fact that $(u - v, u - v) \ge 0$ implies $2\,\mathrm{Re}\,(u, v) \le \|u\|^2 + \|v\|^2$; replacing u by $e^{i\theta}u$ with $e^{i\theta}$ chosen so that $e^{i\theta}(u, v)$ is real and positive, we get

$$|(u, v)| \le \frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2. \tag{2.4}$$

Now in (2.4) we can replace u by tu and v by $t^{-1}v$, to get $|(u, v)| \le (t/2)\|u\|^2 + (1/2t)\|v\|^2$; minimizing over t gives (2.3). This establishes Cauchy's inequality, so we can deduce the triangle inequality. Thus (2.2) defines a norm, as in §1, and the notion of completeness is as stated there.

Prime examples of Hilbert spaces are the spaces $L^2(X, \mu)$ for a measure space (X, μ), that is, the case of $L^p(X, \mu)$ discussed in §1 with $p = 2$. In this case, the inner product is

$$(u, v) = \int_X u(x)\overline{v(x)}\, d\mu(x). \tag{2.5}$$

The nice properties of Hilbert spaces arise from their similarity with familiar Euclidean space, so a great deal of geometrical intuition is available. For example, we say u and v are orthogonal, and write $u \perp v$, provided $(u, v) = 0$. Note that the Pythagorean theorem holds on a general Hilbert space:

$$u \perp v \Longrightarrow \|u + v\|^2 = \|u\|^2 + \|v\|^2. \tag{2.6}$$

This follows directly from expanding $(u + v, u + v)$.

Another useful identity is the following, called the "parallelogram law," valid for all $u, v \in H$:

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2. \tag{2.7}$$

This also follows directly by expanding $(u+v, u+v)+(u-v, u-v)$, observing some cancellations. One important application of this simple identity is to the following existence result.

Let K be any closed, convex subset of H. Convexity implies that $x, y \in K \Rightarrow (x+y)/2 \in K$. Given $x \in H$, we define the distance from x to K to be

$$d = \inf\{\|x-y\| : y \in K\}. \tag{2.8}$$

Proposition 2.1. *If $K \subset H$ is a closed, convex set, there is a unique $z \in K$ such that $d = \|x-z\|$.*

Proof. We can pick $y_n \in K$ such that $\|x-y_n\| \to d$. It will suffice to show that (y_n) must be a Cauchy sequence. Use (2.7) with $u = y_m - x$, $v = x - y_n$, to get

$$\|y_m - y_n\|^2 = 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\Big\|x - \frac{1}{2}(y_n + y_m)\Big\|^2.$$

Since K is convex, $(1/2)(y_n + y_m) \in K$, so $\|x - (1/2)(y_n + y_m)\| \geq d$. Therefore,

$$\limsup_{m,n\to\infty} \|y_n - y_m\|^2 \leq 2d^2 + 2d^2 - 4d^2 \leq 0,$$

which implies convergence.

In particular, this result applies when K is a closed, linear subspace of H. In this case, for $x \in H$, denote by $P_K x$ the point in K closest to x. We have

$$x = P_K x + (x - P_K x). \tag{2.9}$$

We claim that $x - P_K x$ belongs to the linear space $K^\perp$, called the *orthogonal complement* of K, defined by

$$K^\perp = \{u \in H : (u,v) = 0 \text{ for all } v \in K\}. \tag{2.10}$$

Indeed, take any $v \in K$. Then

$$\begin{aligned}\Delta(t) &= \|x - P_K x + tv\|^2 \\ &= \|x - P_K x\|^2 + 2t \,\mathrm{Re}\, (x - P_K x, v) + t^2\|v\|^2\end{aligned}$$

is minimal at $t = 0$, so $\Delta'(0) = 0$ (i.e., $\mathrm{Re}(x - P_K x, v) = 0$), for all $v \in K$. Replacing v by iv shows that $(x - P_K x, v)$ also has vanishing imaginary part for any $v \in K$, so our claim is established. The decomposition (2.9) gives

$$x = x_1 + x_2, \quad x_1 \in K,\ x_2 \in K^\perp, \tag{2.11}$$

with $x_1 = P_K x$, $x_2 = x - P_K x$. Clearly, such a decomposition is unique. It implies that H is an orthogonal direct sum of K and $K^\perp$; we write

$$H = K \oplus K^\perp. \tag{2.12}$$

From this it is clear that

$$(K^\perp)^\perp = K, \tag{2.13}$$

that

$$x - P_K x = P_{K^\perp} x, \tag{2.14}$$

and that P_K and $P_{K^\perp}$ are *linear* maps on H. We call P_K the *orthogonal projection* of H on K. Note that $P_K x$ is uniquely characterized by the condition

$$P_K x \in K,\ (P_K x, v) = (x, v), \quad \text{for all } v \in K. \tag{2.15}$$

We remark that if K is a linear subspace of H which is not closed, then $K^\perp$ coincides with $\overline{K}^\perp$, and (2.13) becomes $(K^\perp)^\perp = \overline{K}$.

Using the orthogonal projection discussed above, we can establish the following result.

Proposition 2.2. *If $\varphi : H \to \mathbb{C}$ is a continuous, linear map, there exists a unique $f \in H$ such that*

$$\varphi(u) = (u, f), \quad \text{for all } u \in H. \tag{2.16}$$

Proof. Consider $K = \ \mathrm{Ker}\ \varphi = \{u \in H : \varphi(u) = 0\}$, a closed, linear subspace of H. If $K = H$, then $\varphi = 0$ and we can take $f = 0$. Otherwise, $K^\perp \neq 0$; select a nonzero $x_0 \in K^\perp$ such that $\varphi(x_0) = 1$. We claim $K^\perp$ is one-dimensional in this case. Indeed, given any $y \in K^\perp$, $y - \varphi(y)x_0$ is annihilated by φ, so it belongs to K as well as to $K^\perp$, so it is zero. The result is now easily proved by setting $f = a x_0$ with $a \in \mathbb{C}$ chosen so that (2.16) works for $u = x_0$, namely $\overline{a}(x_0, x_0) = 1$.

We note that the correspondence $\varphi \mapsto f$ gives a conjugate linear isomorphism

$$H' \to H, \tag{2.17}$$

where H' denotes the space of all continuous linear maps $\varphi : H \to \mathbb{C}$.

We now discuss the existence of an orthonormal basis of a Hilbert space H. A set $\{e_\alpha : \alpha \in A\}$ is called an *orthonormal set* if each $\|e_\alpha\| = 1$ and $e_\alpha \perp e_\beta$ for $\alpha \neq \beta$. If $B \subset A$ is any finite set, it is easy to see via (2.15) that, for all $x \in H$,

$$P_V x = \sum_{\beta \in B} (x, e_\beta) e_\beta, \quad V = \ \mathrm{span}\ \{e_\beta : \beta \in B\}, \tag{2.18}$$

where P_V is the orthogonal projection on V discussed above. Note that

$$\sum_{\beta \in B} |(x, e_\beta)|^2 = \|P_V x\|^2 \leq \|x\|^2. \tag{2.19}$$

In particular, we have $(x, e_\alpha) \neq 0$ for at most countably many $\alpha \in A$, for any given x. (Sometimes, A can be an uncountable set.) By (2.19) we also deduce that, with $c_\alpha = (x, e_\alpha)$, $\sum_{\alpha\in A} |c_\alpha|^2 < \infty$, and $\sum_{\alpha\in A} c_\alpha e_\alpha$ is a convergent series in the norm topology of H. We can apply (2.15) again to show that

$$\sum_{\alpha\in A} (x, e_\alpha) e_\alpha = P_L x, \tag{2.20}$$

where P_L is the orthogonal projection on

$$L = \text{ closure of the linear span of } \{e_\alpha : \alpha \in A\}. \tag{2.21}$$

We call an orthonormal set $\{e_\alpha : \alpha \in A\}$ maximal if it is not contained in any larger orthonormal set. Such a maximal orthonormal set is a basis of H; the term "basis" is justified by the following result.

Proposition 2.3. *An orthonormal set $\{e_\alpha : \alpha \in A\}$ is maximal if and only if its linear span is dense in H, that is, if and only if L in (2.21) is all of H. In such a case, we have, for all $x \in H$,*

$$x = \sum_{\alpha\in A} c_\alpha e_\alpha, \quad c_\alpha = (x, e_\alpha). \tag{2.22}$$

The proof of the first assertion is obvious; the identity (2.22) then follows from (2.20).

The existence of a maximal orthonormal set in any Hilbert space can be inferred from Zorn's lemma; cf. [DS] and [RS]. This existence can be established on elementary logical principles in case H is separable (i.e., has a countable dense set $\{y_j : j = 1, 2, 3, \dots\}$). In this case, let V_n be the linear span of $\{y_j : j \le n\}$, throwing out any y_n for which V_n is not strictly larger than V_{n-1}. Then pick unit $e_1 \in V_1$, unit $e_2 \in V_2$, orthogonal to V_1, and so on, via the Gramm-Schmidt process, and consider the orthonormal set $\{e_j : j = 1, 2, 3, \dots\}$. The linear span of $\{e_j\}$ coincides with that of $\{y_j\}$, hence is dense in H.

As an example of an orthonormal basis, we mention

$$e^{in\theta}, \quad n \in \mathbb{Z}, \tag{2.23}$$

a basis of $L^2(S^1)$ with square norm $\|u\|^2 = (1/2\pi)\int_{S^1} |u(\theta)|^2\, d\theta$. See Chapter 3, §3, or the exercises for this section.

Exercises

1. Let $\mathcal{L}$ be the finite, linear span of the functions $e^{in\theta}$, $n \in \mathbb{Z}$, of (2.23). Use Exercise 1 of §1 to show that $\mathcal{L}$ is dense in $L^2(S^1)$ and hence that these exponentials form an orthonormal basis of $L^2(S^1)$.

2. Deduce that the Fourier coefficients

$$\mathcal{F}f(n) = \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta}\, d\theta \tag{2.24}$$

give a norm-preserving isomorphism

$$\mathcal{F} : L^2(S^1) \xrightarrow{\approx} \ell^2(\mathbb{Z}), \tag{2.25}$$

where $\ell^2(\mathbb{Z})$ is the set of sequences (c_n), indexed by $\mathbb{Z}$, such that $\sum |c_n|^2 < \infty$. Compare the approach to Fourier series in Chapter 3, §1.

In the next set of exercises, let μ and ν be two finite, positive measures on a space X, equipped with a σ-algebra $\mathcal{B}$. Let $\alpha = \mu + 2\nu$ and $\omega = 2\mu + \nu$.

3. On the Hilbert space $H = L^2(X, \alpha)$, consider the linear functional $\varphi : H \to \mathbb{C}$ given by $\varphi(f) = \int_X f(x)\, d\omega(x)$. Show that there exists $g \in L^2(X, \alpha)$ such that $1/2 \le g(x) \le 2$ and

$$\int\limits_X f(x)\, d\omega(x) = \int\limits_X f(x)g(x)\, d\alpha(x).$$

4. Suppose ν is absolutely continuous with respect to μ (i.e., $\mu(S) = 0 \Rightarrow \nu(S) = 0$). Show that $\{x \in X : g(x) = \frac{1}{2}\}$ has μ-measure zero, that

$$h(x) = \frac{2 - g(x)}{2g(x) - 1} \in L^1(X, \mu),$$

and that, for positive measurable F,

$$\int\limits_X F(x)\, d\nu(x) = \int\limits_X F(x)h(x)\, d\mu(x).$$

5. The conclusion of Exercise 4 is a special case of the Radon-Nikodym theorem, using an approach due to von Neumann. Deduce the more general case. Allow ν to be a signed measure. (You then need the Hahn decomposition of ν.) Cf. [T], Chapter 8.
6. Recall uniform convexity, defined in the exercise set for §1. Show that every Hilbert space is uniformly convex.

3. Fréchet spaces; locally convex spaces

Fréchet spaces form a class more general than Banach spaces. For this structure, we have a linear space V and a countable family of seminorms $p_j : V \to \mathbb{R}^+$, where a seminorm p_j satisfies part of (1.1), namely

$$p_j(av) = |a|p_j(v), \quad p_j(v + w) \le p_j(v) + p_j(w), \tag{3.1}$$

but not necessarily the last hypothesis of (1.1); that is, one is allowed to have $p_j(v) = 0$ but $v \neq 0$. However, we do assume that

$$v \neq 0 \Longrightarrow p_j(v) \neq 0, \quad \text{for some } p_j. \tag{3.2}$$

Then, if we set

$$d(u,v)=\sum_{j=0}^{\infty}2^{-j}\frac{p_j(u-v)}{1+p_j(u-v)}, \tag{3.3}$$

we have a distance function. That $d(u,v)$ satisfies the triangle inequality follows from the next lemma, with $\rho(a)=a/(1+a)$.

Lemma 3.1. *Let $\delta : X\times X\to\mathbb{R}^+$ satisfy*

$$\delta(x,z)\le\delta(x,y)+\delta(y,z), \tag{3.4}$$

for all $x,y,z\in X$. Let $\rho:\mathbb{R}^+\to\mathbb{R}^+$ satisfy

$$\rho(0)=0,\quad \rho'\ge 0,\quad \rho''\le 0,$$

so that $\rho(a+b)\le\rho(a)+\rho(b)$. Then $\delta_\rho(x,y)=\rho\big(\delta(x,y)\big)$ also satisfies (3.4).

Proof. We have

$$\rho\big(\delta(x,z)\big)\le\rho\big(\delta(x,y)+\delta(y,z)\big)\le\rho\big(\delta(x,y)\big)+\rho\big(\delta(y,z)\big).$$

Thus V, with seminorms as above, gets the structure of a metric space. If it is complete, we call V a Fréchet space. Note that one has convergence $u_n\to u$ in the metric (3.3) if and only if

$$p_j(u_n-u)\to 0 \text{ as } n\to\infty,\quad \text{for each } p_j. \tag{3.5}$$

A paradigm example of a Fréchet space is $C^\infty(M)$, the space of C^∞-functions on a compact Riemannian manifold M. Then one can take $p_k(u)=\|u\|_{C^k}$, defined by (1.6). These seminorms are actually norms, but one encounters real seminorms in the following situation. Suppose M is a noncompact, smooth manifold, a union of an increasing sequence $\overline{M}_k$ of compact manifolds with boundary. Then $C^\infty(M)$ is a Fréchet space with seminorms $p_k(u)=\|u\|_{C^k(M_k)}$. Also, for such M, and for $1\le p\le\infty$, $L^p_{\mathrm{loc}}(M)$ is a Fréchet space, with seminorms $p_k(u)=\|u\|_{L^p(\overline{M}_k)}$.

Another important Fréchet space is the Schwartz space of rapidly decreasing functions

$$\mathcal{S}(\mathbb{R}^n)=\{u\in C^\infty(\mathbb{R}^n):|D^\alpha u(x)|\le C_{N\alpha}\langle x\rangle^{-N}\text{ for all }\alpha,N\}, \tag{3.6}$$

with seminorms

$$p_k(u)=\sup_{x\in\mathbb{R}^n,|\alpha|\le k}\langle x\rangle^k|D^\alpha u(x)|. \tag{3.7}$$

This space is particularly useful for Fourier analysis; see Chapter 3.

A still more general class is the class of locally convex spaces. Such a space is a vector space V, equipped with a family of seminorms, satisfying (3.1)–(3.2). But now we drop the requirement that the family of seminorms be countable, that is, j ranges over some possibly uncountable set $\mathcal{J}$, rather than a countable set like $\mathbb{Z}^+$. Thus the construction (3.3) of a metric is not available. Such a space V has a natural topology, defined as follows. A neighborhood basis of a point $x \in V$ is given by

$$\mathcal{O}(x, \varepsilon, q) = \{y \in V : q(x - y) < \varepsilon\}, \quad \varepsilon > 0, \tag{3.8}$$

where q runs over finite sums of seminorms p_j. Then V is a topological vector space, that is, with respect to this topology, the vector operations are continuous. The term "locally convex" arises because the sets (3.8) are all convex.

Examples of such more general, locally convex structures will arise in the next section.

Exercises

1. Let E be a Fréchet space, with topology determined by seminorms p_j, arranged so that $p_1 \le p_2 \le \cdots$. Let F be a closed linear subspace. Form the quotient E/F. Show that E/F is a Fréchet space, with seminorms
$$q_j(x) = \inf \{p_j(y) : y \in E,\ \pi(y) = x\},$$
where $\pi : E \to E/F$ is the natural quotient map. (*Hint*: Extend the proof of Proposition 1.1. To begin, if $q_j(a) = 0$ for all j, pick $b_j \in E$ such that $\pi(b_j) = a$ and $p_j(b_j) \le 2^{-j}$; hence $p_j(b_k) \le 2^{-k}$, for $k \ge j$. Consider $b_1 + (b_2 - b_1) + (b_3 - b_2) + \cdots = b \in E$. Show that $\pi(b) = a$ and that $p_j(b) = 0$ for all j. Once this is done, proceed to establish completeness.)
2. If V is a Fréchet space, with topology given by seminorms $\{p_j\}$, a set $S \subset V$ is called *bounded* if each p_j is bounded on S. Show that every bounded subset of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is relatively compact. Show that no infinite-dimensional Banach space can have this property.
3. Let $T : V \to V$ be a continuous, linear map on a locally convex space. Suppose K is a compact, convex subset of V and $T(K) \subset K$. Show that T has a fixed point in K.
(*Hint*: Pick any $v_0 \in K$ and set
$$w_n = \frac{1}{n+1} \sum_{j=0}^{n} T^j v_0 \in K.$$
Show that any limit point of $\{w_n\}$ is a fixed point of T. Note that $Tw_n - w_n = (T^{n+1}v_0 - v_0)/(n + 1)$.)

4. Duality

Let V be a linear space such as discussed in §§1–3, for example, a Banach space, or more generally a Fréchet space, or even more generally a Hausdorff topological vector space. The dual of V, denoted V', consists of continuous, linear maps

$$\omega : V \longrightarrow \mathbb{C} \tag{4.1}$$

($\omega : V \to \mathbb{R}$ if V is a real vector space). Elements $\omega \in V'$ are called *linear functionals* on V. Sometimes one finds the following notation for the action of $\omega \in V'$ on $v \in V$:

$$\langle v, \omega \rangle = \omega(v). \tag{4.2}$$

If V is a Banach space, with norm $\| \quad \|$, the condition for the map (4.1) to be continuous is the following: The set of $v \in V$ such that $|\omega(v)| \le 1$ must be a neighborhood of $0 \in V$. Thus this set must contain a ball $B_R = \{v \in V : \|v\| \le R\}$, for some $R > 0$. With $C = 1/R$, it follows that ω must satisfy

$$|\omega(v)| \le C\|v\|, \tag{4.3}$$

for some $C < \infty$. The infimum of the C's for which this holds is defined to be $\|\omega\|$; equivalently,

$$\|\omega\| = \sup\{|\omega(v)| : \|v\| \le 1\}. \tag{4.4}$$

It is easy to verify that V', with this norm, is also a Banach space.

More generally, let ω be a continuous, linear functional on a Fréchet space V, equipped with a family $\{p_j : j \ge 0\}$ of seminorms and (complete) metric given by (3.3). For any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(u, 0) \le \delta$ implies $|\omega(u)| \le \varepsilon$. Take $\varepsilon = 1$ and the associated δ; pick N so large that $\sum_{N+1}^{\infty} 2^{-j} < \delta/2$. It follows that $\sum_1^N p_j(u) \le \delta/2$ implies $|\omega(u)| \le 1$. Consequently, we see that the continuity of $\omega : V \to \mathbb{C}$ is equivalent to the validity of an estimate of the form

$$|\omega(u)| \le C \sum_{j=1}^{N} p_j(u). \tag{4.5}$$

For general Fréchet spaces, there is no simple analogue of (4.4); V' is typically not a Fréchet space. We will give a further discussion of topologies on V' later in this section.

Next we consider identification of the duals of some specific Banach spaces mentioned before. First, if H is a Hilbert space, the inner product produces a conjugate linear isomorphism of H' with H, as noted in (2.17). We next identify the dual of $L^p(X, \mu)$.

Proposition 4.1. *Let (X, μ) be a σ-finite measure space. Let $1 \le p < \infty$. Then the dual space $L^p(X, \mu)'$, with norm given by (4.4), is naturally isomorphic to $L^q(X, \mu)$, with $1/p + 1/q = 1$.*

Note that Hölder's inequality and its refinement (1.13) show that there is a natural inclusion $\iota : L^q(X, \mu) \to L^p(X, \mu)'$, which is an isometry.

It remains to show that ι is surjective. We sketch a proof in the case when $\mu(X)$ is finite, from which the general case is easily deduced. If $\omega \in L^p(X,\mu)'$, define a set function ν on measurable sets $E \subset X$ by $\nu(E) = \langle \chi_E, \omega \rangle$, where χ_E is the characteristic function of E; ν is readily verified to be countably additive, as long as $p < \infty$. Furthermore, ν annihilates sets of μ-measure zero, so the Radon-Nikodym theorem implies

$$\int f \, d\nu = \int f w \, d\mu,$$

for some measurable function w. A variant of the proof of (1.13) gives $w \in L^q(X,\mu)$, with $\|w\|_{L^q} = \|\omega\|$.

Note that the countable additivity of ν fails for $p = \infty$; in fact, $L^\infty(X,\mu)'$ can be identified with the space of finitely additive set functions on the σ-algebra of μ-measurable sets that annihilate sets of μ-measure zero.

REMARK. In the argument above, you need the Radon-Nikodym theorem for signed measures. The result of Exercise 4, §2 does not suffice; see Exercise 5 of §2.

The following complement to Proposition 4.1 is one of the fundamental results of measure theory. For a proof, we refer to [Ru] and [Yo].

Proposition 4.2. *If X is a compact metric space, $C(X)'$ is isometrically isomorphic to the space $\mathcal{M}(X)$ of (complex) Borel measures on X, with the total variation norm.*

In fact, the generalization of this to the case where X is a compact Hausdorff space, not necessarily metrizable, is of interest. In that case, there is a distinction between the Borel σ-algebra, generated by all compact subsets of X, and the Baire σ-algebra, generated by the compact G_δ subsets of X. For $\mathcal{M}(X)$ here one takes the space of Baire measures to give $C(X)'$. It is then an important fact that each Baire measure has a unique extension to a regular Borel measure. For details, see [Hal].

If M is a smooth, compact manifold, the dual of the Fréchet space $C^\infty(M)$ is denoted $\mathcal{D}'(M)$ and is called the space of *distributions* on M. It is discussed in Chapter 3; also discussed there is the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions on $\mathbb{R}^n$, the dual of $\mathcal{S}(\mathbb{R}^n)$.

For a Banach space, since V' is a Banach space, one can construct its dual, V''. Note that the action (4.2) produces a natural linear map

$$\kappa : V \longrightarrow V'', \tag{4.6}$$

and it is obvious that $\|\kappa(v)\| \le \|v\|$. In fact, $\|\kappa(v)\| = \|v\|$, that is, κ is an isometry. In other words, for any $v \in V$, there exists $\omega \in V'$, $\|\omega\| = 1$, such that $\omega(v) = \|v\|$. This is a special case of the Hahn-Banach theorem, stated below in Proposition 4.3.

Sometimes κ in (4.6) is surjective, so it gives an isometric isomorphism of V with V''. In this case, we say V is *reflexive*. Clearly, any Hilbert space is reflexive. Also, in view of Proposition 4.1, we see that $L^p(X,\mu)$ is reflexive, provided $1 < p < \infty$. On the other hand, $L^1(X,\mu)$ is not reflexive; $L^\infty(X,\mu)'$ is strictly larger than $L^1(X,\mu)$, except for the trivial cases where $L^1(X,\mu)$ is finite-dimensional.

We now state the Hahn-Banach theorem, referred to above. It has a fairly general formulation, useful also for Fréchet spaces and more general, locally convex spaces.

Proposition 4.3. *Let V be a linear space (real or complex), W a linear subspace. Let p be a seminorm on V. Suppose ω is a linear functional on W satisfying $|\omega(v)| \le p(v)$, for $v \in W$. Then there exists an extension of ω to a linear functional Ω on V ($\Omega = \omega$ on W), such that $|\Omega(v)| \le p(v)$ for $v \in V$.*

Note that in case V is a Hilbert space and p the associated norm, this result follows readily from the orthogonal decomposition established in (2.9)–(2.10).

The key to the proof in general is to show that ω can be extended to V when V is spanned by W and one element $z \in V \setminus W$. So one looks for a constant c so that the prescription $\Omega(v + az) = \omega(v) + ac$ works; c is to be picked so that

$$|\omega(v) + ac| \le p(v + az), \quad \text{for } v \in W,\ a \in \mathbb{R} \text{ (or } \mathbb{C}). \tag{4.7}$$

First consider the case of a real vector space. Then (4.7) holds provided $\omega(v) + ac \le p(v + az)$, for all $v \in W$, $a \in \mathbb{R}$, or equivalently provided

$$\begin{aligned} c &\le a^{-1}\big[p(v + az) - \omega(v)\big], \\ -c &\le a^{-1}\big[p(v - az) - \omega(v)\big], \end{aligned} \tag{4.8}$$

for $v \in W$, $a > 0$. Such a constant will exist provided

$$\begin{aligned} &\sup_{v_1 \in W, a_1 > 0} a_1^{-1}\big[\omega(v_1) - p(v_1 - a_1 z)\big] \\ &\qquad \le \inf_{v_2 \in W, a_2 > 0} a_2^{-1}\big[p(v_2 + a_2 z) - \omega(v_2)\big]. \end{aligned} \tag{4.9}$$

Equivalently, for such v_j and a_j, one must have

$$\omega(a_2 v_1 + a_1 v_2) \le a_1 p(v_2 + a_2 z) + a_2 p(v_1 - a_1 z). \tag{4.10}$$

We know that the left side is dominated by

$$p(a_2 v_1 + a_1 v_2) = p(a_2 v_1 - a_2 a_1 z + a_1 a_2 z + a_1 v_2),$$

which is readily dominated by the right side of (4.10). Hence such a number c exists to make (4.7) work.

A Zorn's lemma argument will then work to show that ω can be extended to all of V in general (i.e., it has a "maximal" extension). In case V is a separable Fréchet space and p a continuous seminorm on V, an elementary inductive argument provides an extension from W to a space dense in V, and hence by continuity to V.

The complex case can be deduced from the real case as follows. Define $\gamma : W \to \mathbb{R}$ as $\gamma(v) = \operatorname{Re} \omega(v)$. Then $\omega(v) = \gamma(v) - i\gamma(iv)$. If $\Gamma : V \to \mathbb{R}$ is a desired real, linear extension of γ to V, then one can set $\Omega(v) = \Gamma(v) - i\Gamma(iv)$.

We now make note of some further topologies on the dual space V'. The first is called the *weak*-topology.* It is the topology of pointwise convergence and is specified by the family of seminorms

$$p_v(\omega) = |\omega(v)|, \tag{4.11}$$

as v varies over V. The following result, called *Alaoglu's theorem*, is useful.

Proposition 4.4. *If V is a Banach space, then the closed unit ball $B \subset V'$ is compact in the weak*-topology.*

This result is readily deduced from the following fundamental result in topology:

Tychonov's Theorem. *If $\{X_\alpha : \alpha \in A\}$ is any family of compact Hausdorff spaces, then the Cartesian product $\prod_\alpha X_\alpha$, with the product topology, is a compact Hausdorff space.*

Indeed, the space $B \subset V'$ above, with the weak*-topology, is homeomorphic to a closed subset of the Cartesian product $\prod\{X_v : v \in B_1\}$, where $B_1 \subset V$ is the unit ball in V, each X_v is a copy of the unit disk in $\mathbb{C}$, and $\kappa : B \to \prod X_v$ is given by $\kappa(\omega) = \{\omega(v) : v \in B_1\}$. For a proof of Tychonov's theorem, see [Dug] and [RS].

We remark that if V is separable, then B is a compact metric space. In fact, if $\{v_j : j \in \mathbb{Z}^+\}$ is a dense subset of $B_1 \subset V$, the weak*-topology on B is given by the metric

$$d(\omega, \sigma) = \sum_{j \geq 0} 2^{-j} |\langle v_j, \omega - \sigma\rangle|. \tag{4.12}$$

Conversely, on V there is the weak topology, the topology of pointwise convergence on V', with seminorms

$$p_\omega(v) = |\omega(v)|, \quad \omega \in V'. \tag{4.13}$$

When V is a reflexive Banach space, $V = V''$, then the weak topology of V coincides with its weak*-topology, as the dual of V'; thus Proposition 4.4 applies to the unit ball in V in this case.

More generally, we say two vector spaces V and W have a dual pairing if there is a bilinear form $\langle v, w\rangle$, defined for $v \in V$, $w \in W$, such that for each $v \neq 0, \langle v, w\rangle \neq 0$ for some $w \in W$, and for each $w \neq 0$, this form is nonzero for some $v \in V$. Then the seminorms $p_w(v) = |\langle v, w\rangle|$ on V define a Hausdorff topology called the $\sigma(V, W)$-topology, and symmetrically we have the $\sigma(W, V)$ topology on W. Thus the weak topology on V defined above is the $\sigma(V, V')$-topology, and the weak*-topology on V' is the $\sigma(V', V)$-topology.

We define another topology on the dual space V' of a locally convex space V, called the strong topology. This is the topology of uniform convergence on bounded subsets of V. A set $Y \subset V$ is bounded provided each seminorm p_j defining the topology of V is bounded on Y. The strong topology on V' is defined by the seminorms

$$p_Y(\omega) = \sup\{|\omega(y)| : y \in Y\}, \quad Y \subset V \text{ bounded}. \tag{4.14}$$

In case V is a Banach space, $Y \subset V$ is bounded if and only if it is contained in some ball of finite radius, and then each seminorm (4.14) is dominated by some multiple of the norm on V', given by (4.3). Thus in this case the strong topology and the norm topology on V' coincide. For more general Fréchet spaces, such as $V = C^\infty(M)$, the strong topology on V' does not make V' a normed space, or even a Fréchet space.

There are many interesting results in the subject of duality, concerning the topologies discussed above and other topologies, such as the Mackey topology, which we will not describe here. For further material, see [S].

We return to the setting of the Hahn-Banach theorem, Proposition 4.3, and produce some complementary results. First, instead of taking $p : V \to \mathbb{R}^+$ to be a seminorm, we can more generally take p to be a *gauge*, which is a map $p : V \to \mathbb{R}^+$ satisfying

$$p(av) = ap(v), \quad \forall\, a > 0, \quad p(v + w) \leq p(v) + p(w), \tag{4.15}$$

instead of (3.1). A simple variant of the proof of Proposition 4.3 gives the following.

Proposition 4.5. *Let V be a real linear space, W a linear subspace. Assume p is a gauge on V. If $\omega : W \to \mathbb{R}$ is a linear functional satisfying $\omega(v) \leq p(v)$, for $v \in W$, then there is an extension of ω to a linear functional Ω on V, such that $\Omega(v) \leq p(v)$ for all $v \in V$.*

Note that the conclusion gives $\Omega(-v) \leq p(-v)$, hence $|\Omega(v)| \leq \tilde{p}(v) = \max(p(v), p(-v))$, so Ω is continuous if $\tilde{p}(v)$ is dominated by a seminorm that helps define the topology of V.

Here is an example of a gauge. Let V be a locally convex space and $\mathcal{O}$ a convex neighborhood of $0 \in V$. Define $p_{\mathcal{O}} : V \to \mathbb{R}^+$ by

$$p_{\mathcal{O}}(v) = \inf\{a > 0 : a^{-1}v \in \mathcal{O}\}. \tag{4.16}$$

This is called the Minkowski gauge of $\mathcal{O}$. This object will take us from Proposition 4.5 to the following result, known as the separating hyperplane theorem.

Proposition 4.6. *Let V be a locally convex space (over $\mathbb{R}$), and let $K_1, K_2 \subset V$ be disjoint convex sets.*

(i) If K_1 is open, then K_1 and K_2 can be separated by a closed hyperplane.

(ii) If K_1 and K_2 are both open, they can be strictly separated by a closed hyperplane.

(iii) If K_1 is compact and K_2 is closed, they can be strictly separated by a closed hyperplane.

Here (i) means there exists a continuous linear functional $\Omega : V \to \mathbb{R}$ and a number $a \in \mathbb{R}$ such that $\Omega(v_1) \le a \le \Omega(v_2)$ for all $v_j \in K_j$, and (ii) means there exist such Ω and a with the property that $\Omega(v_1) < a < \Omega(v_2)$ for all $v_j \in K_j$. The separating hyperplane is given by $\{v \in V : \Omega(v) = a\}$.

Proof. For (i), pick $w \in K_2 - K_1 = \{v_2 - v_1 : v_j \in K_j\}$, and let $\mathcal{O} = K_1 - K_2 + w$. Then $\mathcal{O}$ is an open, convex neighborhood of 0, and $w \notin \mathcal{O}$. Let $p = p_{\mathcal{O}}$ be the associated Minkowski gauge, and define ω on $\mathrm{Span}(w)$ by $\omega(aw) = a$. Since $w \notin \mathcal{O}$, $p(w) \ge 1$, so $\omega(aw) \le p(aw)$ for all $a \ge 0$, hence for all $a \in \mathbb{R}$. By Proposition 4.5, ω can be extended to a continuous linear functional Ω on V such that $\Omega(v) \le p(v)$, for all $v \in V$. Hence $\Omega(v) \le 1$ for all $v \in \mathcal{O}$. Thus, for each $v_j \in K_j$,

$$\Omega(v_1) \le \Omega(v_2) + (1 - \omega(w)).$$

But $\omega(w) = 1$, so

$$\Omega(v_1) \le \Omega(v_1), \quad \forall\, v_j \in K_j. \tag{4.17}$$

This proves (i).

For (ii), take Ω as in (i). If K_j is open $\{\Omega(v) : v \in K_j\}$ is readily verified so be an open subset of $\mathbb{R}$. So we have two open subsets of $\mathbb{R}$, which by (4.17) share at most one point. They must hence be disjoint.

In case (iii), consider $C = K_2 - K_1$. Disjointness implies $0 \notin C$. Since K_1 is compact, C is closed. Thus there is an open, convex neighborhood U of 0, disjoint from C. Let $\widetilde{K}_1 = K_1 + (1/2)U$ and $\widetilde{K}_2 = K_2 - (1/2)U$. Then $\widetilde{K}_1$ and $\widetilde{K}_2$ are disjoint, open, convex sets, and (ii) applies. Any closed hyperplane that strictly separates $\widetilde{K}_1$ and $\widetilde{K}_2$ also strictly separates K_1 and K_2.

Proposition 4.6 has the following important topological consequence.

Proposition 4.7. *Let K be a closed, convex subset of the locally convex space V (over $\mathbb{R}$). Then K is weakly closed.*

Proof. Suppose $v_\alpha \in K$ and $v_\alpha \to v$ weakly, that is, $\Omega(v_\alpha) \to \Omega(v)$ for all $\Omega \in V'$. If $v \notin K$, this contradicts the conclusion of (iii) of Proposition 4.6 (with $K_1 = \{v\},\ K_2 = K$), so $v \in K$.

NOTE. If V is a linear space over $\mathbb{C}$ with a locally convex topology, its weak topology coincides with that produced by regarding V as a linear space over $\mathbb{R}$.

Proposition 4.7 interfaces as follows with Proposition 4.4.

Proposition 4.8. *Let V be a reflexive Banach space and $K \subset V$ a closed, bounded, convex set. Then K is compact in the weak topology.*

Proof. Proposition 4.4, with V and V' switched, implies that each closed ball $B_R \subset V$ is compact in the weak topology (which coincides with the weak* topology by reflexivity). The hypotheses imply $K \subset B_R$ for some R, and, by Proposition 4.7, K is a closed subset of B_R, in the weak topology.

Exercises

1. Suppose $\{u_j : j \in \mathbb{Z}^+\}$ is an orthonormal set in a Hilbert space H. Show that $u_j \to 0$ in the weak* topology as $j \to \infty$.
2. In the setting of Exercise 1, suppose $H = L^2(X, \mu)$, and the u_j also satisfy uniform bounds: $|u_j(x)| \le M$. Show that $u_j \to 0$ in the weak* topology of $L^\infty(X, \mu)$, as the dual to $L^1(X, \mu)$.
3. Deduce that if $f \in L^1(S^1)$, with Fourier coefficients $\hat{f}(n)$ given by (2.24), then $\hat{f}(n) \to 0$ as $n \to \infty$.
4. Prove the assertion made in the text that, when V is a separable Banach space, then the unit ball B in V', with the weak* topology, is metrizable. (*Hint*: To show that (4.12) defines a topology coinciding with the weak* topology, use the fact that if $\varphi : X \to Y$ is continuous and bijective, with X compact and Y Hausdorff, then φ is a homeomorphism.)
5. On a Hilbert space H, suppose $f_j \to f$ weakly. Show that if
$$\|f\| \ge \limsup_{j\to\infty} \|f_j\|, \tag{4.18}$$
then $f_j \to f$ in *norm*. (*Hint*: Expand $(f - f_j, f - f_j)$.)
6. Extend Exercise 5 as follows. Let V be a uniformly convex Banach space (cf. §1, Exercise 6). Suppose $f_j, f \in V$ and $f_j \to f$ weakly. Show that if (4.18)

holds, then $f_j \to f$ in norm. (*Hint.* Assume $\|f\| = 1$. Take $\omega \in V'$ such that $\|\omega\| = 1$ and $\langle f, \omega \rangle = 1$. Investigate implications of

$$\left\langle \frac{f + f_j}{2}, \omega \right\rangle \longrightarrow \langle f, \omega \rangle, \quad \text{as} \ \ j \to \infty,$$

in concert with (4.18).)

7. Suppose X is a closed, linear subspace of a reflexive Banach space V. Show that X is reflexive. (*Hint*: Use the Hahn-Banach theorem. First show that $X' \approx V'/X^{\perp}$, where $X^{\perp} = \{\omega \in V' : \omega(v) = 0, \forall\, v \in X\}$. Thus, a bounded linear functional on X' gives rise to a bounded linear functional on V', annihilating $X^{\perp}$.)
8. Let V be a $\mathbb{C}$-linear space, and let $\alpha : V \to \mathbb{R}$ be $\mathbb{R}$-linear. Define $\beta : V \to \mathbb{C}$ by $\beta(v) = \alpha(v) - i\alpha(iv)$. Show that β is $\mathbb{C}$-linear.
9. Suppose $V = H$ is a Hilbert space, $K \subset H$ a closed, convex subset, and $v \notin K$. As an alternative to Proposition 4.6, use Proposition 2.1 to produce a closed hyperplane strongly separating K and v. Apply this to Propositions 4.7 and 4.8, in case V is a Hilbert space.

5. Linear operators

If V and W are two Banach spaces, or more generally two locally convex spaces, we denote by $\mathcal{L}(V, W)$ the space of continuous, linear transformations from V to W. As in the derivation of (4.4), it is easy to see that, when V and W are Banach spaces, a linear map $T : V \to W$ is continuous if and only if there exists a constant $C < \infty$ such that

$$\|Tv\| \le C\|v\| \tag{5.1}$$

for all $v \in V$. Thus we call T a *bounded* operator. The infimum of all the C's for which this holds is defined to be $\|T\|$; equivalently,

$$\|T\| = \sup\{\|Tv\| : \|v\| \le 1\}. \tag{5.2}$$

It is clear that $\mathcal{L}(V, W)$ is a linear space. If V and W are Banach spaces and $T_j \in \mathcal{L}(V, W)$, then $\|T_1 + T_2\| \le \|T_1\| + \|T_2\|$; completeness is also easy to establish in this case, so $\mathcal{L}(V, W)$ is also a Banach space. If X is a third Banach space and $S \in \mathcal{L}(W, X)$, it is clear that $ST \in \mathcal{L}(V, X)$, and

$$\|ST\| \le \|S\| \cdot \|T\|. \tag{5.3}$$

The space $\mathcal{L}(V) = \mathcal{L}(V, V)$, with norm (5.2), is a Banach algebra for any Banach space V. Generally, a Banach algebra is defined to be a Banach space B with the structure of an algebra, so that, for any $S, T \in B$, the inequality (5.3) holds. Another example of a Banach algebra is the space $C(X)$, for compact X, with norm (1.3), the product being given by the pointwise product of functions.

If V and W are Banach spaces and $T \in \mathcal{L}(V,W)$, then the adjoint $T' \in \mathcal{L}(W',V')$ is uniquely defined to satisfy

$$\langle Tv, w\rangle = \langle v, T'w\rangle, \quad v \in V,\ w \in W'. \tag{5.4}$$

Using the Hahn-Banach theorem, it is easy to see that

$$\|T\| = \|T'\|, \tag{5.5}$$

both norms being the sup of the absolute value of (5.4) over $\|v\| = 1$, $\|w\| = 1$. When V and W are reflexive, it is clear that $T'' = T$. We remark that (5.4) also defines T' for general locally convex V and W.

In case V and W are Hilbert spaces and $T \in \mathcal{L}(V,W)$, then we also have an adjoint $T^* \in \mathcal{L}(W,V)$, given by

$$(Tv, w) = (v, T^*w), \quad v \in V,\ w \in W, \tag{5.6}$$

using the inner products on W and V, respectively. As in (5.5) we have $\|T\| = \|T^*\|$. Also it is clear that $T^{**} = T$.

When H is a Hilbert space, the Banach algebra $\mathcal{L}(H)$ is a C*-algebra. Generally, a C*-algebra B is a Banach algebra, equipped with a conjugate linear involution $T \mapsto T^*$, satisfying $\|T^*\| = \|T\|$ and

$$\|T^*T\| = \|T\|^2. \tag{5.7}$$

To see that (5.7) holds for $T \in \mathcal{L}(H)$, note that both sides are equal to the sup of the absolute value, over $\|v_1\| \le 1$, $\|v_2\| \le 1$, of

$$(T^*Tv_1, v_2) = (Tv_1, Tv_2), \tag{5.8}$$

such a supremum necessarily being obtained over the set of pairs satisfying $v_1 = v_2$. Note that $C(X)$, considered above, is also a C*-algebra. However, for a general Banach space V, $\mathcal{L}(V)$ will not have the structure of a C*-algebra.

We consider some simple examples of bounded linear operators. If (X,μ) is a measure space, $f \in L^\infty(X,\mu)$, then the multiplication operator M_f, defined by $M_f u = fu$, is bounded on $L^p(X,\mu)$ for each $p \in [1,\infty]$, with $\|M_f\| = \|f\|_{L^\infty}$. If X is a compact Hausdorff space and $f \in C(X)$, then $M_f \in \mathcal{L}(C(X))$, with $\|M_f\| = \|f\|_{\sup}$. In case X is a compact Riemannian manifold and P is a differential operator of order k on X, with smooth coefficients, then P does not give a bounded operator on $C(X)$, but one has $P \in \mathcal{L}(C^k(X), C(X))$, and more generally $P \in \mathcal{L}(C^{k+m}(X), C^m(X))$, for $m \ge 0$. For related results on Sobolev spaces, see Chapter 4.

Another class of examples, a little more elaborate than those just mentioned, is given by integral operators, of the form

$$Ku(x) = \int_X k(x,y)\ u(y)\ d\mu(y), \tag{5.9}$$

where (X,μ) is a measure space. We have the following result:

Proposition 5.1. *Suppose k is measurable on $X \times X$ and*

$$\int_X |k(x,y)|\, d\mu(x) \le C_1, \quad \int_X |k(x,y)|\, d\mu(y) \le C_2, \tag{5.10}$$

for all y and for all x, respectively. Then (5.9) defines K as a bounded operator on $L^p(X,\mu)$, for each $p \in [1,\infty]$, with

$$\|K\| \le C_1^{1/p}\, C_2^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{5.11}$$

Proof. For $p \in (1,\infty)$, we estimate

$$\Bigl| \int_X \int_X k(x,y) f(y) g(x)\, d\mu(x)\, d\mu(y) \Bigr| \tag{5.12}$$

via the estimate $ab \le a^p/p + b^q/q$ of (1.15), used to prove Hölder's inequality. Apply this to $|f(y)g(x)|$. Then (5.12) is dominated by

$$\frac{C_1}{p} \|f\|_{L^p}^p + \frac{C_2}{q} \|g\|_{L^q}^q \tag{5.13}$$

provided (5.10) holds. Replacing f, g by $tf, t^{-1}g$, we see that (5.12) is dominated by $(C_1 t^p/p)\|f\|_{L^p}^p + (C_2/qt^q)\|g\|_{L^q}^q$; minimizing over $t \in (0,\infty)$, via elementary calculus, we see that (5.12) is dominated by

$$C_1^{1/p} C_2^{1/q} \|f\|_{L^p} \|g\|_{L^q}, \tag{5.14}$$

proving the result. The exceptional cases $p = 1$ and $p = \infty$ are easily handled.

We call $k(x,y)$ the *integral kernel* of K. Note that K' is an integral operator, with kernel $k'(x,y) = k(y,x)$. In the case of the Hilbert space $L^2(X,\mu)$, K^* is an integral operator, with kernel $k^*(x,y) = \overline{k(y,x)}$.

Chapter 7 includes a study of a much more subtle class of operators called *singular integral operators*, or *pseudodifferential operators of order zero*; L^p-estimates for this class are made in Chapter 13.

We next consider some results about linear transformations on Banach spaces which use the following general result about complete metric spaces, known as the *Baire category theorem*.

Proposition 5.2. *Let X be a complete metric space, and X_j, $j \in \mathbb{Z}^+$, nowhere-dense subsets; that is, the closure $\overline{X}_j$ contains no nonempty open set. Then $\bigcup_j X_j \ne X$.*

Proof. The hypothesis on X_1 implies there is a closed ball $B_{r_1}(p_1) \subset X \setminus X_1$, for some $p_1 \in X$, $r_1 > 0$. Then the hypothesis on X_2 gives a ball

$B_{r_2}(p_2) \subset B_{r_1}(p_1) \setminus X_2$, $0 < r_2 \le r_1/2$. Continue, getting balls

$$B_{r_j}(p_j) \subset B_{r_{j-1}}(p_{j-1}) \setminus X_j, \quad 0 < r_j \le 2^{-j+1} r_1. \tag{5.15}$$

Then (p_j) is Cauchy; it must converge to a point $p \notin \cup_j X_j$, as p belongs to each $B_{r_j}(p_j)$.

Our first application is to a result called the *uniform boundedness principle.*

Proposition 5.3. *Let V, W be Banach spaces, $T_j \in \mathcal{L}(V, W)$, $j \in \mathbb{Z}^+$. Assume that for each $v \in V$, $\{T_j v\}$ is bounded in W. Then $\{\|T_j\|\}$ is bounded.*

Proof. Let $X = V$. Let $X_n = \{v \in X : \|T_j v\| \le n \text{ for all } j\}$. The hypothesis implies $\cup_n X_n = X$. Clearly, each X_n is closed. The Baire category theorem implies that some X_N has nonempty interior, so there exists $v_0, r > 0$ such that $\|v\| \le r \Rightarrow \|T_j(v_0 + v)\| \le N$, for all j. Hence

$$\|v\| \le r \Rightarrow \|T_j v\| \le N + \|T_j v_0\| \le R \quad \forall\, j, \tag{5.16}$$

using the boundedness of $\{T_j v_0\}$. This implies $\|T_j\| \le R/r$, completing the proof.

The next result is known as the *open mapping theorem.*

Proposition 5.4. *If V and W are Banach spaces and $T \in \mathcal{L}(V, W)$ is onto, then any neighborhood of 0 in V is mapped onto a neighborhood of 0 in W.*

Proof. Let B_1denote the unit ball in V, $X_n = T(nB_1) = nT(B_1)$. The hypothesis implies $\bigcup_{n \ge 1} X_n = W$. The Baire category theorem implies that some $\overline{X}_N$ has nonempty interior, hence contains a ball $B_r(w_0)$; symmetry under sign change implies $\overline{X}_N$ also contains $B_r(-w_0)$. Hence $\overline{X}_{2N} = 2\overline{X}_N$ contains $B_{2r}(0)$. By scaling, $\overline{X}_1$ contains a ball $B_\varepsilon(0)$. Our goal now is to show that X_1 itself contains a ball. This will follow if we can show that $\overline{X}_1 \subset X_2$.

So let $y \in \overline{X}_1 = \overline{T(B_1)}$. Thus there is an $x_1 \in B_1$ such that $y - Tx_1 \in B_{\varepsilon/2}(0) \subset \overline{X}_{1/2}$. For the same reason, there is an $x_2 \in B_{1/2}$ such that $(y - Tx_1) - Tx_2 \in B_{\varepsilon/4}(0) \subset \overline{X}_{1/4}$. Continue, getting $x_n \in B_{2^{1-n}}$ such that

$$y - \sum_{j=1}^{n} Tx_j \in B_{\varepsilon/2^n}(0).$$

Then $x = \sum_{j=1}^{\infty} x_j$ is in B_2 and $Tx = y$. This completes the proof.

Corollary 5.5. *If V and W are Banach spaces and $T : V \to W$ is continuous and bijective, then $T^{-1} : W \to V$ is continuous.*

In such a situation, we say that T is a topological isomorphism.

The third basic application of the Baire category theorem is called the *closed-graph theorem.* For a given linear map $T : V \to W$, its graph is defined to be

$$G_T = \{(v, Tv) \in V \oplus W : v \in V\}. \tag{5.17}$$

It is easy to see that, whenever V and W are topological vector spaces, then if T is continuous, G_T is closed. The following is a converse.

Proposition 5.6. *Let V and W be Banach spaces, $T : V \to W$ a linear map. If G_T is closed in $V \oplus W$, then T is continuous.*

Proof. The hypothesis implies that G_T is a Banach space, with norm $\|(v, Tv)\| = \|v\| + \|Tv\|$. Now the maps $J : G_T \to V$, $K : G_T \to W$, given by $J(v, Tv) = v$, $K(v, Tv) = Tv$, are clearly continuous, and J is bijective. Hence J^{-1} is continuous, and so $T = KJ^{-1}$ is also continuous.

Propositions 5.3–5.6 have extensions to Fréchet spaces, since they are also complete metric spaces. For example, let V be a Fréchet space in Proposition 5.3 (keep W a Banach space). In this case, the hypothesis that $\{T_j v\}$ is bounded in W for each $v \in V$ implies that there exists a neighborhood $\mathcal{O}$ of the origin in V, of the form (3.8), such that $v \in \mathcal{O} \Rightarrow \|T_j v\| \le 1$ for all j, that is, for some finite sum q of seminorms defining the Fréchet space structure of V,

$$\|T_j v\| \le K\ q(v), \quad \text{for all } j, \tag{5.18}$$

with K independent of j.

Propositions 5.4–5.6 extend directly to the case where V and W are Fréchet spaces, with only slight extra complications in the proofs.

We now give an important application of the open mapping theorem, to a result known as the *closed-range theorem.* If W is a Banach space and $L \subset W$ is a linear subspace, we denote by $L^\perp$ the subspace of W' consisting of linear functionals on W that annihilate L.

Proposition 5.7. *If V and W are Banach spaces and $T \in \mathcal{L}(V, W)$, then*

$$\text{Ker } T' = T(V)^\perp. \tag{5.19}$$

If, in addition, $T(V)$ is closed in W, then $T'(W')$ is closed in V' and

$$T'(W') = (\text{Ker } T)^\perp. \tag{5.20}$$

Proof. For the first identity, by $\langle Tv, w\rangle = \langle v, T'w\rangle$, it is obvious that $T(V)^\perp = \text{Ker } T'$. If $T(V)$ is closed, it follows from Corollary 5.5 that $\widetilde{T}$:

$V/\text{Ker}\ T \to T(V)$ is a topological isomorphism. Thus we have a topological isomorphism

$$\widetilde{T}' : T(V)' \xrightarrow{\approx} (V/\text{Ker}\ T)'. \tag{5.21}$$

Meanwhile, there is a natural isomorphism of Banach spaces

$$(V/\text{Ker}\ T)' \approx (\text{Ker}\ T)^{\perp}, \tag{5.22}$$

and, by the Hahn-Banach theorem, there is a natural surjection $W' \to T(V)'$. (See Exercise 4 below.) Composing these operators yields T'. Thus we have (5.20).

In the Hilbert space case, we have the same result for T^*.

Since one frequently looks at equations $Tu = v$, it is important to consider the notion of invertibility. An operator $T \in \mathcal{L}(V, W)$ is invertible if there is an $S \in \mathcal{L}(W, V)$ such that ST and TS are identity operators. One useful fact is that all operators close to the identity in $\mathcal{L}(V)$ are invertible.

Proposition 5.8. *Let V be a Banach space, $T \in \mathcal{L}(V)$, with $\|T\| < 1$. Then $I - T$ is invertible.*

Proof. The power series $\sum_{n=0}^{\infty} T^n$ converges to $(I - T)^{-1}$.

When V is a Banach space, we say $\zeta \in \mathbb{C}$ belongs to the resolvent set of an operator $T \in \mathcal{L}(V)$ (denoted $\rho(T)$) provided $\zeta I - T$ is invertible; then the resolvent of T is

$$R_\zeta = (\zeta I - T)^{-1}. \tag{5.23}$$

It easily follows from the method of proof of Proposition 5.8 that the resolvent set of any $T \in \mathcal{L}(V)$ is open in $\mathbb{C}$. Furthermore, R_ζ is a holomorphic function of $\zeta \in \rho(T)$. In fact, if $\zeta_0 \in \rho(T)$, then we can write $\zeta - T = (\zeta_0 - T)\big(I - (\zeta_0 - \zeta)R_{\zeta_0}\big)$, and hence, for ζ close to ζ_0,

$$R_\zeta = R_{\zeta_0} \sum_{n=0}^{\infty} R_{\zeta_0}^n (\zeta_0 - \zeta)^n.$$

It is also clear that ζ belongs to the resolvent set whenever $|\zeta| > \|T\|$, since

$$(\zeta - T)^{-1} = \zeta^{-1}(I - \zeta^{-1}T)^{-1}. \tag{5.24}$$

The complement of the resolvent set is called the *spectrum* of T. Thus, for any $T \in \mathcal{L}(V)$, the spectrum of T (denoted $\sigma(T)$) is a compact set in $\mathbb{C}$. By (5.24), $\|R_\zeta\| \to 0$ as $|\zeta| \to \infty$. Since R_ζ is holomorphic on $\rho(T)$, it follows by Liouville's theorem that, for any $T \in \mathcal{L}(V)$, $\rho(T)$ cannot be all of $\mathbb{C}$, so $\sigma(T)$ is nonempty.

Using the resolvent as a tool, we now discuss a holomorphic functional calculus for an operator $T \in \mathcal{L}(V)$, and applications to spectral theory.

Let Ω be a bounded region in $\mathbb{C}$, with smooth boundary, containing the spectrum $\sigma(T)$ in its interior. If f is holomorphic on a neighborhood of Ω, we set

$$f(T) = \frac{1}{2\pi i} \int_\gamma f(\zeta)\ (\zeta - T)^{-1}\ d\zeta, \tag{5.25}$$

where $\gamma = \partial\Omega$. Note that if T were a complex number in Ω, this would be Cauchy's formula. Here are a couple of very basic facts.

Lemma 5.9. *If $f(z) = 1$, then $f(T) = I$, and if $f(z) = z$, then $f(T) = T$.*

Proof. Deform γ to be a large circle and use (5.24), plus

$$(I - \zeta^{-1}T)^{-1} = I + \sum_{n=1}^{\infty} (\zeta^{-1}T)^n. \tag{5.26}$$

We next derive a multiplicative property of this functional calculus, making use of the following result, known as the *resolvent identity*.

Lemma 5.10. *If $z, \zeta \in \rho(T)$, then*

$$R_z - R_\zeta = (\zeta - z) R_z R_\zeta. \tag{5.27}$$

Proof. For any $\zeta \in \rho(T)$, R_ζ commutes with $\zeta - T$, hence with T, hence with any $z - T$. If, in addition, $z \in \rho(T)$, we have both $R_\zeta R_z (z - T) = R_\zeta$ and $R_z R_\zeta (z - T) = R_z (z - T) R_\zeta = R_\zeta$, hence

$$R_z R_\zeta = R_\zeta R_z. \tag{5.28}$$

Thus

$$\begin{aligned} R_z - R_\zeta &= (\zeta - T) R_\zeta R_z - (z - T) R_z R_\zeta \\ &= (\zeta - z) R_\zeta R_z, \end{aligned}$$

proving (5.27).

Now for our multiplicative property:

Proposition 5.11. *If f and g are holomorphic on a neighborhood of Ω, then*

$$f(T) g(T) = (fg)(T). \tag{5.29}$$

Proof. Let $\gamma = \partial\Omega$, as above, and let γ_1 be the boundary of a slightly larger region, on which f and g are holomorphic. Write

$$g(T) = \frac{1}{2\pi i} \int_{\gamma_1} g(z) (z - T)^{-1}\, dz,$$

and hence, using (5.25), write $f(T)g(T)$ as a double integral. The product $R_\zeta R_z$ of resolvents of T appears in the new integrand. Using the resolvent identity (5.27), we obtain

$$f(T)g(T) = -\frac{1}{4\pi^2} \int_{\gamma_1} \int_{\gamma} (\zeta - z)^{-1} f(\zeta) g(z) (R_z - R_\zeta)\, d\zeta\, dz. \tag{5.30}$$

The term involving R_z as a factor has $d\zeta$-integral equal to zero, by Cauchy's theorem. Doing the dz-integral for the other term, using Cauchy's identity

$$g(\zeta) = \frac{1}{2\pi i} \int_{\gamma_1} (z - \zeta)^{-1} g(z)\, dz,$$

we obtain from (5.30)

$$f(T)g(T) = \frac{1}{2\pi i} \int_{\gamma} f(\zeta) g(\zeta) R_\zeta\, d\zeta, \tag{5.31}$$

which gives (5.29).

One interesting situation that frequently arises is the following. Ω can have several connected components, $\Omega = \Omega_1 \cup \cdots \cup \Omega_M$, each Ω_j containing different pieces of $\sigma(T)$. Taking a function equal to 1 on Ω_j and 0 on the other components produces operators

$$P_j = \frac{1}{2\pi i} \int_{\gamma_j} (\zeta - T)^{-1}\, d\zeta, \quad \gamma_j = \partial \Omega_j. \tag{5.32}$$

By (5.29) we see that

$$P_j^2 = P_j, \quad P_j P_k = 0, \quad \text{for } j \neq k, \tag{5.33}$$

so $P_1, \dots, P_M$ are mutually disjoint projections. By Lemma 5.9, $P_1 + \cdots + P_M = I$. It follows easily that if T_j denotes the restriction of T to the range of P_j, then

$$\sigma(T_j) = \sigma(T) \cap \Omega_j. \tag{5.34}$$

Exercises

1. Extend the $p = 2$ case of Proposition 5.1 to the following result of Schur. Let (X, μ) and (Y, ν) be measure spaces, and let $k(x, y)$ be measurable on $(X \times Y, \mu \times \nu)$. Assume that there are measurable functions $p(x)$, $q(y)$, positive a.e. on X and Y, respectively, such that

$$\int_X |k(x,y)| p(x)\, d\mu(x) \le C_1 q(y), \quad \int_Y |k(x,y)| q(y)\, d\nu(y) \le C_2 p(x). \tag{5.35}$$

Show that $Ku(x) = \int_Y k(x,y)u(y)\,d\nu(y)$ defines a bounded operator

$$K : L^2(Y,\nu) \longrightarrow L^2(X,\mu), \quad \|K\|^2 \le C_1C_2.$$

Give an appropriate modification of the hypothesis (5.35) in order to obtain an operator bound $K : L^p(Y,\nu) \to L^p(X,\mu)$.

2. Show that $k(x,y)$ is the integral kernel of a bounded map $K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n_+)$ provided it has support in $\{x_1, y_1 \in [0,1]\}$ and satisfies the estimate

$$|k(x,y)| \le C\Big(|x'-y'|^2 + x_1^2 + y_1^2\Big)^{-n/2}, \quad x = (x_1, x'),\ y = (y_1, y'). \tag{5.36}$$

(*Hint*: Construct $p(x)$ and $q(y)$ so that (5.35) holds. Here, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 \ge 0\}$.)

3. Show that $k(x,y)$ is the integral kernel of a bounded map $K : L^p(\mathbb{R}^n_+) \to L^p(\mathbb{R}^n_+)$, for $1 \le p \le \infty$, provided it has support in $\{x_1, y_1 \in [0,1]\}$ and satisfies the estimates

$$|k(x,y)| \le Cx_1\Big(|x_1+y_1| + |x'-y'|\Big)^{-(n+1)}$$

and

$$|k(x,y)| \le Cy_1\Big(|x_1+y_1| + |x'-y'|\Big)^{-(n+1)}.$$

4. Let K be a closed, linear subspace of a Banach space V; consider the natural maps $j : K \hookrightarrow V$ and $\pi : V \to V/K$. Show that $j' : V' \to K'$ is surjective and that $\pi' : (V/K)' \to V'$ has range $K^\perp$.
5. Show that the set of invertible, bounded, linear maps on a Banach space V is open in $\mathcal{L}(V)$. (*Hint*: If T^{-1} exists, write $T + R = T(I + T^{-1}R)$.)
6. Let X be a compact metric space and $F : X \to X$ a continuous map. Define $T : C(X) \to C(X)$ by $Tu(x) = u(F(x))$. Show that $T' : \mathcal{M}(X) \to \mathcal{M}(X)$ is given by $(T'\mu)(E) = \mu(F^{-1}(E))$, for any Borel set $E \subset X$. Using Exercise 3 of §3, show that there is a *probability measure* μ on X such that $T'\mu = \mu$.

6. Compact operators

Throughout this section we will restrict attention to operators on Banach spaces. An operator $T \in \mathcal{L}(V,W)$ is said to be compact provided T takes any bounded subset of V to a relatively compact subset of W, that is, a set with compact closure. It suffices to assume that $T(B_1)$ is relatively compact in W, where B_1 is the closed unit ball in V. We denote the space of compact operators by $\mathcal{K}(V,W)$. The following proposition summarizes some elementary facts about $\mathcal{K}(V,W)$.

Proposition 6.1. *$\mathcal{K}(V,W)$ is a closed, linear subspace of $\mathcal{L}(V,W)$. Any T in $\mathcal{L}(V,W)$ with finite-dimensional range is compact. Furthermore, if $T \in \mathcal{K}(V,W)$, $S_1 \in \mathcal{L}(V_1,V)$, and $S_2 \in \mathcal{L}(W,W_2)$, then $S_2TS_1 \in \mathcal{K}(V_1,W_2)$.*

Most of these assertions are obvious. We show that if $T_j \in \mathcal{K}(V,W)$ is norm convergent to T, then T is compact. Given any sequence (x_n) in B_1, one can pick successive subsequences on which T_1x_n converges, then T_2x_n converges, and so on, and by a diagonal argument produce a single subsequence (which we'll still denote (x_n)) such that for each j, T_jx_n converges as $n \to \infty$. It is then easy to show that Tx_n converges, giving compactness of T.

A particular case of Proposition 6.1 is that $\mathcal{K}(V) = \mathcal{K}(V,V)$ is a closed, two-sided ideal of $\mathcal{L}(V)$.

The following gives a useful class of compact operators.

Proposition 6.2. *If X is a compact metric space, then the natural inclusion*

$$\iota : Lip(X) \longrightarrow C(X) \tag{6.1}$$

is compact.

Proof. It is easy to show that any compact metric space has a countable, dense subset; let $\{x_j : j = 1, 2, 3, \dots\}$ be dense in X. Say (f_n) is a bounded sequence in $\mathrm{Lip}(X)$. We want to prove that a subsequence converges in $C(X)$. Since bounded subsets of $\mathbb{C}$ are relatively compact, we can pick a subsequence of (f_n) converging at x_1; then we can pick a further subsequence of this subsequence, converging at x_2, and so forth. The standard diagonal argument then produces a subsequence (which we continue to denote (f_n)) converging at each x_j. We claim that (f_n) converges uniformly on X, as a consequence of the uniform estimate

$$|f_n(x) - f_n(y)| \le K\ d(x,y), \tag{6.2}$$

with K independent of n. Indeed, pick $\varepsilon > 0$. Then pick $\delta > 0$ such that $K\delta < \varepsilon/3$. Since X is compact, we can select from $\{x_j\}$ finitely many points, say $\{x_1, \dots, x_N\}$, such that any $x \in X$ is of distance $\le \delta$ from one of these. Then pick M so large that $f_n(x_j)$ is within $\varepsilon/3$ of its limit for $1 \le j \le N$, for all $n \ge M$. Now, for any $x \in X$, picking $\ell \in \{1, \dots, N\}$ such that $d(x, x_\ell) \le \delta$, we have, for $k \ge 0$, $n \ge M$,

$$\begin{aligned} |f_{n+k}(x) - f_n(x)| &\le |f_{n+k}(x) - f_{n+k}(x_\ell)| \\ &\quad + |f_{n+k}(x_\ell) - f_n(x_\ell)| + |f_n(x_\ell) - f_n(x)| \\ &\le K\delta + \frac{\varepsilon}{3} + K\delta < \varepsilon, \end{aligned} \tag{6.3}$$

proving the proposition.

The argument given above is easily modified to show that $\iota : \mathrm{Lip}^\alpha(X) \to C(X)$ is compact, for any $\alpha > 0$. Indeed, there is the following more general result. Let $\omega : X \times X \to [0, \infty)$ be any continuous function, vanishing on the diagonal $\Delta = \{(x,x) : x \in X\}$. Fix $K \in \mathbb{R}^+$. Let $\mathcal{F}$ be any subset of $C(X)$ satisfying

$$|u(x)| \le K, \quad |u(x) - u(y)| \le K\ \omega(x,y). \tag{6.4}$$

The latter condition is called *equicontinuity*. Ascoli's theorem states that such a set $\mathcal{F}$ is relatively compact in $C(X)$ whenever X is a compact Hausdorff space. The proof is a further extension of the argument given above.

We note another refinement of Proposition 6.2, namely that the inclusion $\iota : \mathrm{Lip}^{\alpha}(X) \to \mathrm{Lip}^{\beta}(X)$ is compact whenever $0 \le \beta < \alpha \le 1$, X a compact metric space. Compare results on inclusions of Sobolev spaces given in Chapter 4.

We next look at persistence of compactness upon taking adjoints.

Proposition 6.3. *If $T \in \mathcal{K}(V,W)$, then T' is also compact.*

Proof. Let (ω_n) be sequence in B_1', the closed unit ball in W'. Consider (ω_n) as a sequence of continuous functions on the compact space $X = \overline{T(B_1)}$, B_1 being the unit ball in V. Ascoli's theorem, indeed its special case, Proposition 6.2, applies; there exists a subsequence (ω_{n_k}) converging uniformly on X. Thus $(T'\omega_{n_k})$ is a sequence in V' converging uniformly on B_1, hence in the V'-norm. This completes the proof.

The following provides a useful improvement over the a priori statement that, for $T \in \mathcal{K}(V,W)$, the image $T(B_1)$ of the closed unit ball $B_1 \subset V$ is relatively compact in W.

Proposition 6.4. *Assume V is separable and reflexive. If $T : V \to W$ is compact, then the image of the closed unit ball $B_1 \subset V$ under T is compact.*

Proof. From Proposition 4.4 and the remark following its proof, B_1, with the weak*-topology (the $\sigma(V,V')$-topology, since $V = V''$), is a compact metric space, granted that V' is also separable, which we now demonstrate. Indeed, for any Banach space Y, it is a simple consequence of the Hahn-Banach theorem that Y is separable provided Y' is separable; if Y is reflexive, this implication can be reversed.

Consequently, given a sequence $v_n \in B_1$, possessing a subsequence $v_n^{(1)}$ such that $Tv_n^{(1)}$ converges in W, say to w, you can pass to a further subsequence $v_n^{(2)}$, which is weak*-convergent in V, with limit $v \in B_1$. It follows that $Tv_n^{(2)}$ is weakly convergent to Tv; for any $\omega \in W'$, $Tv_n^{(2)}(\omega) = v_n^{(2)}(T'\omega) \to v(T'\omega) = (Tv)(\omega)$. Hence $Tv = w$. This shows that $T(B_1)$ is closed in W, and hence completes the proof.

Remark: It is possible to drop the assumption that V is separable, via an argument replacing sequences by nets in order to construct the weak* limit point v.

We next derive some results on the spectral theory of a compact operator A on a Hilbert space H that is self-adjoint, so $A = A^*$. For simplicity, we will assume that H is separable, though that hypothesis can easily be dropped.

Proposition 6.5. *If $A \in \mathcal{L}(H)$ is compact and self-adjoint, then either $\|A\|$ or $-\|A\|$ is an eigenvalue of A, that is, there exists $u \neq 0$ in H such that*

$$Au = \lambda u, \tag{6.5}$$

with $\lambda = \pm\|A\|$.

Proof. By Proposition 6.4, we know that the image under A of the closed unit ball in H is compact, so the norm assumes a maximum on this image. Thus there exists $u \in H$ such that

$$\|u\| = 1, \quad \|Au\| = \|A\|. \tag{6.6}$$

Pick any unit $w \perp u$. Self-adjointness implies $\|Ax\|^2 = (A^2x, x)$, so we have, for all real s,

$$\big(A^2(u + sw), u + sw\big) \le \|A\|^2(1 + s^2), \tag{6.7}$$

equality holding at $s = 0$. Since the left side is equal to

$$\|A\|^2 + 2s \operatorname{Re}(A^2u, w) + s^2\|Aw\|^2,$$

this inequality for $s \to 0$ implies $\operatorname{Re}(A^2u, w) = 0$; replacing w by iw gives $(A^2u, w) = 0$ whenever $w \perp u$. Thus A^2u is parallel to u, that is, $A^2u = cu$ for some scalar c; (6.6) implies $c = \|A\|^2$. Now, assuming $A \neq 0$, set $v = \|A\|u + Au$. If $v = 0$, then u satisfies (6.5) with $\lambda = -\|A\|$. If $v \neq 0$, then v is an eigenvector of A with eigenvalue $\lambda = \|A\|$.

The space of $u \in H$ satisfying (6.5) is called the λ-*eigenspace* of A. Clearly, if A is compact and $\lambda \neq 0$, such a λ-eigenspace must be finite-dimensional. If $Au_j = \lambda_j u_j$, $A = A^*$, then

$$\lambda_1(u_1, u_2) = (Au_1, u_2) = (u_1, Au_2) = \overline{\lambda}_2(u_1, u_2). \tag{6.8}$$

With $\lambda_1 = \lambda_2$ and $u_1 = u_2$, this implies that each eigenvalue of $A = A^*$ is *real*. With $\lambda_1 \neq \lambda_2$, it then yields $(u_1, u_2) = 0$, so any distinct eigenspaces of $A = A^*$ are orthogonal. We also note that if $Au_1 = \lambda_1 u_1$ and $v \perp u_1$, then $(u_1, Av) = (Au_1, v) = \lambda_1(u_1, v) = 0$, so $A = A^*$ leaves invariant the orthogonal complement of any of its eigenspaces.

Now if A is compact and self-adjoint on H, we can apply Proposition 6.5, restrict A to the orthogonal complement of its $\pm\|A\|$-eigenspaces (where its norm must be strictly smaller, as a consequence of Proposition 6.5), apply the proposition again, to this restriction, and continue. In this fashion we arrive at the following result, known as the *spectral theorem* for compact, self-adjoint operators.

Proposition 6.6. *If $A \in \mathcal{L}(H)$ is a compact, self-adjoint operator on a Hilbert space H, then H has an orthonormal basis u_j of eigenvectors of A. With $Au_j = \lambda_j u_j$, (λ_j) is a sequence of real numbers with only 0 as an accumulation point.*

The spectral theorem has a more elaborate formulation for general self-adjoint operators. It is proved in Chapter 8.

We next give a result that will be useful in the study of spectral theory of compact operators that are not self-adjoint. It will also be useful in §7. Let V, W and Y be Banach spaces.

Proposition 6.7. *Let $T \in \mathcal{L}(V, W)$. Suppose $K \in \mathcal{K}(V, Y)$ and*

$$\|u\|_V \leq C\|Tu\|_W + C\|Ku\|_Y, \tag{6.9}$$

for all $u \in V$. Then T has closed range.

Proof. Let $Tu_n \to f$ in W. We need $v \in V$ with $Tv = f$. Let $L = \operatorname{Ker} T$. We divide the argument into two cases.

If $\operatorname{dist}(u_n, L) \leq a < \infty$, take $v_n = u_n \bmod L$, $\|v_n\| \leq 2a$; then $Tv_n = Tu_n \to f$. Passing to a subsequence, we have $Kv_n \to g$ in Y. Then (6.9), applied to $u = v_n - v_m$, implies that (v_n) is Cauchy, so $v_n \to v$ and $Tv = f$.

If $\operatorname{dist}(u_n, L) \to \infty$, we can assume that $\operatorname{dist}(u_n, L) \geq 2$ for all n. Pick $v_n = u_n \bmod L$ such that $\operatorname{dist}(u_n, L) \leq \|v_n\| \leq \operatorname{dist}(u_n, L) + 1$, and set $w_n = v_n/\|v_n\|$. Note that $\operatorname{dist}(w_n, L) \geq 1/2$. Since $\|w_n\| = 1$, we can take a subsequence and assume $Kw_n \to g$ in Y. Since $Tw_n \to 0$, (6.9) applied to $w_n - w_m$ implies (w_n) is Cauchy. Thus $w_n \to w$ in V, and we see that simultaneously $\operatorname{dist}(w, L) \geq 1/2$ and $Tw = 0$, a contradiction. Hence this latter case is impossible, and the proposition is proved.

Note that Proposition 6.7 applies to the case $V = W = Y$ and $T = \zeta I - K$, for $K \in \mathcal{K}(V)$ and ζ a nonzero scalar. Such an operator therefore has closed range. The next result is called the *Fredholm alternative*.

Proposition 6.8. *For $\zeta \neq 0$, $K \in \mathcal{K}(V)$, the operator $T = \zeta I - K$ is surjective if and only if it is injective.*

Proof. Assume T is injective. Then $T : V \to R(T)$ is bijective. By Proposition 6.7, $R(T)$ is a Banach space, so the open mapping theorem implies that $T : V \to R(T)$ is a topological isomorphism. If $R(T) = V_1$ is not all of V, then $V_2 = T(V_1)$, $V_3 = T(V_2)$, and so on, form a strictly decreasing family of closed subspaces. By Lemma 1.3, we can pick $v_v \in V_n$ with $\|v_n\| = 1$, $\operatorname{dist}(v_n, V_{n+1}) \geq 1/2$. Thus, for $n > m$,

$$\begin{aligned} Kv_m - Kv_n &= \zeta v_m + [-\zeta v_n - (Tv_m - Tv_n)] \\ &= \zeta v_m + w_{mn}, \end{aligned} \tag{6.10}$$

with $w_{mn} \in V_{m+1}$. Hence $\|Kv_n - Kv_m\| \geq |\zeta|/2$, contradicting compactness of K. Consequently, T is surjective if it is injective.

For the converse, we use Proposition 5.7. If T is surjective, (5.19) implies $T' = \zeta I - K'$ is injective on V'. Since K' is compact, the argument above implies T' is surjective, and hence, by (5.20), T is injective.

A substantial generalization of this last result will be contained in Proposition 7.4 and Corollary 7.5.

It follows that every $\zeta \neq 0$ in the spectrum of a compact K is an eigenvalue of K. We hence derive the following result on $\sigma(K)$.

Proposition 6.9. *If $K \in \mathcal{K}(V)$, the spectrum $\sigma(K)$ has only 0 as an accumulation point.*

Proof. Suppose we have linearly independent $v_n \in V$, $\|v_n\| = 1$, with $Kv_n = \lambda_n v_n$, $\lambda_n \to \lambda \neq 0$. Let V_n be the linear span of $\{v_1, \dots, v_n\}$. By Lemma 1.3, there exist $y_n \in V_n$, $\|y_n\| = 1$, such that $\operatorname{dist}(y_n, V_{n-1}) \geq 1/2$. With $T_\lambda = \lambda I - K$, we have, for $n > m$,

$$\begin{aligned} \lambda_n^{-1} K y_n - \lambda_m^{-1} K y_m &= y_n + \left[-y_m + \lambda_n^{-1} T_{\lambda_n} y_n + \lambda_m^{-1} T_{\lambda_m} y_m\right] \\ &= y_n + z_{nm}, \end{aligned} \tag{6.11}$$

where $z_{nm} \in V_{n-1}$ since $T_{\lambda_n} y_n \in V_{n-1}$. Hence $\|\lambda_n^{-1} K y_n - \lambda_m^{-1} K y_m\| \geq 1/2$, which contradicts compactness of K.

Note that if $\lambda_j \neq 0$ is such an isolated point in the spectrum $\sigma(K)$ of a compact operator K, and we take γ_j to be a small circle enclosing λ_j but no other points of $\sigma(K)$, then, as in (5.32), the operator

$$P_j = \frac{1}{2\pi i} \int_{\gamma_j} (\zeta - K)^{-1} \, d\zeta$$

is a projection onto a closed subspace V_j of V with the property that the restriction of K to V_j (equal to $P_j K P_j$) has spectrum consisting of the one point $\{\lambda_j\}$. Thus V_j must be finite-dimensional. $K|_{V_j}$ may perhaps not be scalar; it might have a Jordan normal form with λ_j down the diagonal and some ones directly above the diagonal.

Having established a number of general facts about compact operators, we take a look at an important class of compact operators on Hilbert spaces: the Hilbert-Schmidt operators, defined as follows. Let H be a separable

Hilbert space and $A \in \mathcal{L}(H)$. Let $\{u_j\}$ be an orthonormal basis of H. We say A is a *Hilbert-Schmidt operator*, or an HS operator for short, provided

$$\sum_j \|Au_j\|^2 < \infty, \tag{6.12}$$

or equivalently, if

$$\sum_{j,k} |a_{jk}|^2 < \infty, \quad a_{jk} = (Au_k, u_j). \tag{6.13}$$

The class of HS operators on H will be denoted $\mathrm{HS}(H)$. The first characterization makes it clear that if A is HS and B is bounded, then BA is HS. The second makes it clear that A^* is HS if A is; hence $AB = (B^*A^*)^*$ is HS if A is HS and B is bounded. Thus (6.12) is independent of the choice of orthonormal basis $\{u_j\}$. We also define the Hilbert-Schmidt norm of an HS operator:

$$\|A\|_{\mathrm{HS}}^2 = \sum_j \|Au_j\|^2 = \sum_{j,k} |a_{jk}|^2. \tag{6.14}$$

The first identity makes it clear that $\|BA\|_{\mathrm{HS}} \le \|B\| \cdot \|A\|_{\mathrm{HS}}$ if A is HS and B is bounded, and in particular

$$\|UA\|_{\mathrm{HS}} = \|A\|_{\mathrm{HS}}$$

when U is unitary. The second identity in (6.14) shows that

$$\|A^*\|_{\mathrm{HS}} = \|A\|_{\mathrm{HS}}.$$

Using $AU = (U^*A^*)^*$, we deduce that

$$\|AU\|_{\mathrm{HS}} = \|A\|_{\mathrm{HS}}$$

when U is unitary. Thus, for U unitary, $\|UAU^{-1}\|_{\mathrm{HS}} = \|A\|_{\mathrm{HS}}$, so the HS-norm in (6.14) is independent of the choice of orthonormal basis for H.

From (6.12) it follows that an HS operator A is a norm limit of finite-rank operators, hence compact. If $A = A^*$, and we choose an orthonormal basis of eigenvectors of A, with eigenvalues μ_j, then

$$\sum_j |\mu_j|^2 = \|A\|_{\mathrm{HS}}^2. \tag{6.15}$$

A compact, self-adjoint operator A is HS if and only if the left side of (6.15) is finite.

If $A : H_1 \to H_2$ is a bounded operator, we can say it is HS provided AV is HS for some unitary map $V : H_2 \to H_1$, with obvious adjustments when either H_1 or H_2 is finite-dimensional.

The following classical result might be called the Hilbert-Schmidt kernel theorem. In Chapter 4 it is used as an ingredient in the proof of the celebrated Schwartz kernel theorem.

Proposition 6.10. *If $T : L^2(X_1, \mu_1) \to L^2(X_2, \mu_2)$ is HS, then there exists a function $K \in L^2(X_1 \times X_2, \mu_1 \times \mu_2)$ such that*

$$(Tu, v)_{L^2} = \iint K(x_1, x_2) u(x_1) \overline{v(x_2)} \, d\mu_1(x_1) \, d\mu_2(x_2). \tag{6.16}$$

Proof. Pick orthonormal bases $\{f_j\}$ for $L^2(X_1)$ and $\{g_k\}$ for $L^2(X_2)$, and set

$$K(x_1, x_2) = \sum_{j,k} a_{jk} \overline{f_j(x_1)} g_k(x_2),$$

where $a_{jk} = (Tf_j, g_k)$. The hypothesis that T is HS is precisely what is necessary to guarantee that $K \in L^2(X_1 \times X_2)$, and then (6.16) is obvious. It is also clear that

$$\|T\|_{\text{HS}}^2 = \|K\|_{L^2}^2. \tag{6.17}$$

Also of interest is the converse, proved simply by reversing the argument:

Proposition 6.11. *If $K \in L^2(X_1 \times X_2, \mu_1 \times \mu_2)$, then (6.16) defines an HS operator T, satisfying (6.17).*

We note that the HS-square norm polarizes to a Hilbert space inner product on $\text{HS}(H)$:

$$(A, B)_{\text{HS}} = \sum_{j,k} a_{jk} \bar{b}_{jk} \tag{6.18}$$

if, parallel to (6.13), $b_{jk} = (Bu_k, u_j)$, given an orthonormal basis $\{u_j\}$. Since the norm uniquely determines the inner product, we have without further calculation the independence of $(A, B)_{\text{HS}}$ under change of orthonormal basis; more generally, $(A, B)_{\text{HS}} = (UAV, UBV)_{\text{HS}}$ for unitary U and V on H.

Note that $\sum_k a_{jk} \bar{b}_{\ell k} = c_{j\ell}$ form the matrix coefficients of $C = AB^*$, and (6.18) is the sum of the diagonal elements of C; we write

$$(A, B)_{\text{HS}} = \text{Tr } AB^*. \tag{6.19}$$

Generally, we say an operator $C \in \mathcal{L}(H)$ is *trace class* if it can be written as a product of two HS operators; call them A and B^*, and then $\text{Tr } C$ is defined to be given by (6.19). It is not clear at first glance that TR, the set of trace class operators, is a linear space, but this can be seen as follows. If $C_j = A_j B_j^*$, then

$$C_1 + C_2 = \begin{pmatrix} A_1 & A_2 \end{pmatrix} \begin{pmatrix} B_1^* \\ B_2^* \end{pmatrix}. \tag{6.20}$$

Note that a given $C \in \text{TR}$ may be written as a product of two HS operators in many different ways, but the computation of $\text{Tr } C$ is unaffected, since as we have already seen, the definition (6.19) leads to the computation

$$\operatorname{Tr} C = \sum_j c_{jj}, \quad c_{jj} = (Cu_j, u_j). \tag{6.21}$$

This formula shows that $\operatorname{Tr}:\mathrm{TR} \to \mathbb{C}$ is a linear map. Furthermore, by our previous remarks on $(\ ,\ )_{\mathrm{HS}}$, the trace formula (6.21) is independent of the choice of orthonormal basis of H.

There is an intrinsic characterization of trace class operators:

Proposition 6.12. *An operator $C \in \mathcal{L}(H)$ is trace class if and only if C is compact and the operator $(C^*C)^{1/2}$ has the property that its set of eigenvalues $\{\lambda_j\}$ is summable; $\sum \lambda_j < \infty$.*

Proof. Given C compact, let $\{u_j\}$ be an orthonormal basis of H consisting of eigenvectors of C^*C, which is compact and self-adjoint. Say $C^*Cu_j = \lambda_j^2 u_j$, $\lambda_j \geq 0$. Then the identity $(C^*C)^{1/2}u_j = \lambda_j u_j$ defines $(C^*C)^{1/2}$.

Note that, for all $v \in H$,

$$\|(C^*C)^{1/2}v\|^2 = (C^*Cv, v) = \|Cv\|^2. \tag{6.22}$$

Thus $Cv \mapsto (C^*C)^{1/2}v$ extends to an isometric isomorphism between the ranges of C and of $(C^*C)^{1/2}$, yielding in turn operators V and W of norm 1 such that

$$C = V(C^*C)^{1/2}, \quad (C^*C)^{1/2} = WC. \tag{6.23}$$

Now, if $\sum \lambda_j < \infty$, define $A \in \mathcal{L}(H)$ by $Au_j = \lambda_j^{1/2} u_j$. Hence A is Hilbert-Schmidt, and $C = VA \cdot A$, so C is trace class. Conversely, if $C = AB^*$ with $A, B \in \mathrm{HS}$, then $(C^*C)^{1/2} = WA \cdot B^*$ is a product of HS operators, hence of trace class. The computation (6.21), using the basis of eigenvectors of C^*C, then yields $\sum \lambda_j = \operatorname{Tr}(C^*C)^{1/2} < \infty$, and the proof is complete.

It is desirable to establish some results about TR as a linear space. Given $C \in \mathrm{TR}$, we define

$$\|C\|_{\mathrm{TR}} = \inf\,\{\|A\|_{\mathrm{HS}}\|B\|_{\mathrm{HS}} : C = AB^*\}. \tag{6.24}$$

This is a norm; in particular,

$$\|C_1 + C_2\|_{\mathrm{TR}} \leq \|C_1\|_{\mathrm{TR}} + \|C_2\|_{\mathrm{TR}}. \tag{6.25}$$

This can be seen by using (6.20), with A_2 replaced by tA_2 and B_2^* by $t^{-1}B_2^*$, and minimizing over $t \in (0, \infty)$ the quantity

$$\begin{aligned} &\|(A_1, tA_2)\|_{\mathrm{HS}}^2 \cdot \|(B_1, t^{-1}B_2)^t\|_{\mathrm{HS}}^2 \\ &\qquad = \big(\|A_1\|_{\mathrm{HS}}^2 + t^2\|A_2\|_{\mathrm{HS}}^2\big) \cdot \big(\|B_1\|_{\mathrm{HS}}^2 + t^{-2}\|B_2\|_{\mathrm{HS}}^2\big). \end{aligned}$$

Next, we note that (6.24) easily yields

$$\|C^*\|_{\mathrm{TR}} = \|C\|_{\mathrm{TR}} \tag{6.26}$$

and, for bounded S_j,

$$\|S_1 C S_2\|_{\mathrm{TR}} \le \|S_1\| \cdot \|C\|_{\mathrm{TR}} \cdot \|S_2\|, \tag{6.27}$$

with equality if S_1 and S_2 are unitary. Also, using (6.23), we have

$$\|C\|_{\mathrm{TR}} = \|(C^*C)^{1/2}\|_{\mathrm{TR}}. \tag{6.28}$$

Using (6.24) with C replaced by $D = (C^*C)^{1/2}$, the choice $A = B = D^{1/2}$ yields

$$\|(C^*C)^{1/2}\|_{\mathrm{TR}} \le \|(C^*C)^{1/4}\|_{\mathrm{HS}}^2 = \operatorname{Tr}(C^*C)^{1/2}. \tag{6.29}$$

On the other hand, we have, by (6.19) and Cauchy's inequality,

$$|\mathrm{Tr}(AB^*)| \le \|A\|_{\mathrm{HS}}\|B\|_{\mathrm{HS}}, \tag{6.30}$$

and hence, for $C \in \mathrm{TR}$,

$$|\mathrm{Tr}\ C| \le \|C\|_{\mathrm{TR}}. \tag{6.31}$$

If we apply this, with C replaced by $(C^*C)^{1/2}$, and compare with (6.28)–(6.29), we have

$$\|C\|_{\mathrm{TR}} = \mathrm{Tr}\ (C^*C)^{1/2}. \tag{6.32}$$

Either directly or as a simple consequence of this, we have

$$\|C\|_{\mathrm{TR}} \ge \|C\|_{\mathrm{HS}} \ge \|C\|. \tag{6.33}$$

We can now establish:

Proposition 6.13. *Given a Hilbert space H, the space TR of trace class operators on H is a Banach space, with norm (6.24).*

Proof. It suffices to prove completeness. Thus let (C_j) be Cauchy in TR. Passing to a subsequence, we can assume $\|C_{j+1} - C_j\|_{\mathrm{TR}} \le 8^{-j}$. Then write $C = \sum \widetilde{C}_j$, where $\widetilde{C}_1 = C_1$ and, for $j \ge 2$, $\widetilde{C}_j = C_j - C_{j-1}$. By (6.33), C is a bounded operator on H. Write

$$\widetilde{C}_j = \widetilde{A}_j \widetilde{B}_j^*, \quad \|\widetilde{A}\|_{\mathrm{HS}}, \|\widetilde{B}\|_{\mathrm{HS}} \le 2^{-j}.$$

Then we can form

$$A = \widetilde{A}_1 \oplus \widetilde{A}_2 \oplus \cdots, \quad B = \widetilde{B}_1 \oplus \widetilde{B}_2 \oplus \cdots \in \mathcal{L}(\mathcal{H}, H),$$

where $\mathcal{H} = H \oplus H \oplus \cdots$, check that A and B are Hilbert-Schmidt, and note that $C = AB^*$. Hence $C \in \mathrm{TR}$ and $C_j \to C$ in TR-norm.

The classes HS and TR are the most important cases of a continuum of ideals $\mathcal{I}_p \subset \mathcal{L}(H)$, $1 \le p < \infty$. One says $C \in \mathcal{K}(H)$ belongs to $\mathcal{I}_p$ if and only if $(C^*C)^{p/2}$ is trace class. Then $\mathrm{TR} = \mathcal{I}_1$ and $\mathrm{HS} = \mathcal{I}_2$. For more on this topic, see [Si].

We next discuss the trace of an integral operator. Let A and B be two HS operators on $L^2(X,\mu)$, with integral kernels $K_A, K_B \in L^2(X\times X, \mu\times\mu)$. Then $C = AB$ is given by

$$Cu(x) = \iint K_A(x,z)K_B(z,y)u(y)\ d\mu(y)\ d\mu(z), \tag{6.34}$$

and we have, by (6.17) and (6.19),

$$\operatorname{Tr} C = \iint K_A(x,z)K_B(z,x)\ d\mu(z)\ d\mu(x). \tag{6.35}$$

Now C has an integral kernel $K_C \in L^2(X\times X, \mu\times\mu)$:

$$K_C(x,y) = \int K_A(x,z)K_B(z,y)\ d\mu(z), \tag{6.36}$$

which strongly suggests the trace formula

$$\operatorname{Tr} C = \int K_C(x,x)\ d\mu(x). \tag{6.37}$$

The only sticky point is that the diagonal $\{(x,x) : x \in X\}$ may have measure 0 in $X\times X$, so one needs to define $K_C(x,y)$ carefully. The formula (6.35) implies, via Fubini's theorem, that

$$K_C(x,x) = \int K_A(x,z)K_B(z,x)\ d\mu(z)$$

exists for μ-almost every $x\in X$, and for this function, the identity (6.37) holds. In many cases of interest, X is a locally compact space and $K_C(x,y)$ is continuous, and then passing from (6.35) to (6.37) is straightforward.

We next give a treatment of the determinant of $I + A$, for trace class A. This is particularly useful for results on trace formulas and the scattering phase, in Chapter 9. Our treatment largely follows [Si]; another approach can be found in Chapter 11 of [DS].

With $\Lambda^j C$ the operator induced by C on $\Lambda^j H$, we define

$$\det(I+C) = 1 + \sum_{j\ge 1} \operatorname{Tr} \Lambda^j C. \tag{6.38}$$

It is not hard to show that if $C_j = \Lambda^j C$ and $D_j = (C_j^* C_j)^{1/2}$, then $D_j = \Lambda^j (C^*C)^{1/2}$, so

$$\|C_j\|_{\mathrm{TR}} = \operatorname{Tr} D_j = \sum_{i_1<\cdots<i_j} \mu_{i_1}\cdots\mu_{i_j}, \tag{6.39}$$

where μ_i, $i\ge 1$, are the positive eigenvalues of the compact, positive operator $(C^*C)^{1/2}$, counted with multiplicity. In particular,

$$\|C_j\|_{\mathrm{TR}} \le \frac{1}{j!}\|C\|_{\mathrm{TR}}^j, \tag{6.40}$$

so (6.38) is absolutely convergent for any $C \in \mathrm{TR}$. Note that in the finite-dimensional case, (6.38) is simply the well-known expansion of the characteristic polynomial. Replacing C by zC, $z \in \mathbb{C}$, we obtain an entire holomorphic function of z:

$$\det(I + zC) = 1 + \sum_{j\ge 1} z^j \ \mathrm{Tr}\ \Lambda^j C. \tag{6.41}$$

This replacement causes D_j to be replaced by $|z|^j D_j$, and (6.39) implies

$$|\det(I + zC)| \le \det(I + |z|D) = \prod_{i\ge 1}(1 + \mu_i|z|), \tag{6.42}$$

the latter identity following by diagonalization of the compact, self-adjoint operator D. Note that since $1 + r \le e^r$, for $r \ge 0$,

$$\prod_{i\ge \ell}(1 + \mu_i|z|) \le e^{\kappa_\ell |z|}, \quad \kappa_\ell = \sum_{i\ge \ell} \mu_i. \tag{6.43}$$

Taking $\ell = 1$, we have

$$|\det(I + zC)| \le e^{|z|\|C\|_{\mathrm{TR}}}. \tag{6.44}$$

Also,

$$|\det(I + zC)| \le \Bigl\{\prod_{i=1}^{\ell-1}(1 + \mu_i|z|)\Bigr\} e^{\kappa_\ell |z|}, \quad \forall\ \ell. \tag{6.45}$$

Hence, for any $C \in \mathrm{TR}$,

$$|\det(I + zC)| \le C_\varepsilon e^{\varepsilon |z|}, \quad \forall\ \varepsilon > 0. \tag{6.46}$$

We next establish the continuous dependence of the determinant.

Proposition 6.14. *We have a continuous map $F: TR \to \mathbb{C}$, given by*

$$F(A) = \det(I + A).$$

Proof. For fixed $C, D \in TR$, $g(z) = F(C + zD)$ is holomorphic, as one sees from (6.40) and (6.41). Now consider

$$h(z) = F\Bigl(\frac{1}{2}(A + B) + z(A - B)\Bigr). \tag{6.47}$$

Then

$$\begin{aligned} |F(A) - F(B)| = \Bigl|h\Bigl(\frac{1}{2}\Bigr) - h\Bigl(-\frac{1}{2}\Bigr)\Bigr| &\le \sup\Bigl\{|h'(t)| : -\frac{1}{2} \le t \le \frac{1}{2}\Bigr\} \\ &\le R^{-1} \sup_{|z|\le R+1/2} |h(z)|. \end{aligned} \tag{6.48}$$

In turn, we can estimate $|h(z)|$ using (6.45). If we take $R = \|A - B\|_{\mathrm{TR}}^{-1}$, we get

$$|F(A) - F(B)| \le \|A - B\|_{\mathrm{TR}} \exp\bigl\{\|A\|_{\mathrm{TR}} + \|B\|_{\mathrm{TR}} + 1\bigr\}, \tag{6.49}$$

which proves the proposition.

One use of Proposition 6.14 is as a tool to prove the following.

Proposition 6.15. *For any $A, B \in TR$,*

$$\det\big((I+A)(I+B)\big) = \det(I+A) \cdot \det(I+B). \tag{6.50}$$

Proof. By Proposition 6.14, it suffices to prove (6.50) when A and B are finite rank operators, in which case it is elementary.

The following is an important consequence of (6.50).

Proposition 6.16. *Given $A \in TR$, we have*

$$I + A \text{ invertible} \iff \det(I+A) \neq 0. \tag{6.51}$$

Proof. If $I + A$ is invertible, the inverse has the form

$$(I+A)^{-1} = I + B, \quad B = -A(I+A)^{-1} \in \mathrm{TR}. \tag{6.52}$$

Hence (6.50) implies $\det(I+A)\ \det(I+B) = 1$, so $\det(I+A) \neq 0$.

For the converse, assume $I+A$ is not invertible, so $-1 \in \operatorname{Spec}(A)$. Since A is compact, we can consider the associated spectral projection P of H onto the generalized (-1)-eigenspace of A. Since $(PA)(I-P)A = 0$, we have

$$\det(I+A) = \det(I+AP) \cdot \det(I+A(I-P)). \tag{6.53}$$

It is elementary that $\det(I+AP) = 0$, so the proposition is proved.

As another application of (6.50), we can use the identity

$$I + A + sB = (I+A)\big(I + s(I+A)^{-1}B\big) \tag{6.54}$$

to show that

$$\frac{d}{ds} \det\big(I + A(s)\big) = \det\big(I + A(s)\big) \cdot \operatorname{Tr}\big((I+A(s))^{-1}A'(s)\big), \tag{6.55}$$

when $A(s)$ is a differentiable function of s with values in TR.

Exercises

1. If A is a Hilbert-Schmidt operator, show that

$$\|A\| \le \|A\|_{\mathrm{HS}},$$

where the left side denotes the operator norm. (*Hint*: Pick unit u_1 such that $\|Au_1\| \ge \|A\| - \varepsilon$, and make that part of an orthonormal basis.)

2. Suppose $K \in L^2(X \times X, \mu \times \mu)$ satisfies $K(x, y) = \overline{K(y, x)}$. Show that
$$K(x, y) = \sum c_j \; u_j(x) \; \overline{u_j(y)}$$
with $\{u_j\}$ an orthonormal set in $L^2(X, \mu)$, $c_j \in \mathbb{R}$, and $\sum c_j^2 < \infty$.
(*Hint*: Apply the spectral theorem for compact, self-adjoint operators.)
3. Define $T : L^2(I) \to L^2(I)$, $I = [0, 1]$, by
$$Tf(x) = \int_0^x f(y) \; dy.$$
Show that T has range $\mathcal{R}(T) \subset \{u \in C(I) : u(0) = 0\}$. Show that T is compact, that T has no eigenvectors, and that $\sigma(T) = \{0\}$. Also, show that T is HS, but not trace class.
4. Let K be a closed bounded subset of a Banach space B. Suppose T_j are compact operators on B and $T_j x \to x$ for each $x \in B$. Show that K is compact if and only if $T_j \to I$ uniformly on K.
5. Prove the following result, also known as part of Ascoli's theorem. If X is a compact metric space, B_j are Banach spaces, and $K : B_1 \to B_2$ is a compact operator, then $\kappa f(x) = K(f(x))$ defines a compact map $\kappa : C^\alpha(X, B_1) \to C(X, B_2)$, for any $\alpha > 0$.
6. Let B be a bounded operator on a Hilbert space H, and let A be trace class. Show that
$$\operatorname{Tr}(AB) = \operatorname{Tr}(BA).$$
(*Hint*: Write $A = A_1 A_2$ with $A_j \in \text{HS}$.)
7. Given a Hilbert space H, define $\Lambda^j H$ as a Hilbert space and justify (6.39). Also, check the finite rank case of (6.50).
8. Assume $\{u_j : j \geq 1\}$ is an orthonormal basis of the Hilbert space H, and let P_n denote the orthogonal projection of H onto the span of $\{u_1, \dots, u_n\}$. Show that if $A \in \text{TR}$, then $P_n A P_n \to A$ in TR-norm. (This is used implicitly in the proof of Proposition 6.15.)

7. Fredholm operators

Again in this section we restrict attention to operators on Banach spaces. An operator $T \in \mathcal{L}(V, W)$ is said to be Fredholm provided

$$\text{Ker } T \text{ is finite-dimensional} \tag{7.1}$$

and

$$T(V) \text{ is closed in } W, \text{ of finite codimension,} \tag{7.2}$$

that is, $W/T(V)$ is finite-dimensional. We say T belongs to $\text{Fred}(V, W)$. We define the *index* of T to be

$$\text{Ind } T = \dim \text{Ker } T - \dim W/T(V), \tag{7.3}$$

the last term also denoted Codim $T(V)$. Note the isomorphism $\big(W/T(V)\big)'$ $\approx T(V)^\perp$. By (5.19), $T(V)^\perp =$ Ker T'. Consequently,

$$\text{Ind } T = \dim \text{Ker } T - \dim \text{Ker } T'. \tag{7.4}$$

Furthermore, using Proposition 5.7, and noting that

$$(\text{Ker } T)' \approx V'/(\text{Ker } T)^\perp = V'/T'(W'),$$

we deduce that if T is Fredholm, $T' \in \mathcal{L}(W', V')$ is also Fredholm, and

$$\text{Ind } T' = -\text{ Ind } T. \tag{7.5}$$

The following is a useful characterization of Fredholm operators.

Proposition 7.1. *Let $T \in \mathcal{L}(V, W)$. Then T is Fredholm if and only if there exist $S_j \in \mathcal{L}(W, V)$ such that*

$$S_1 T = I + K_1 \tag{7.6}$$

and

$$T S_2 = I + K_2, \tag{7.7}$$

with K_1 and K_2 compact.

Proof. The identity (7.6) implies Ker $T \subset \text{Ker}(I + K_1)$, which is finite-dimensional. Also, by Proposition 6.7, (7.6) implies T has closed range. On the other hand, (7.7) implies $T(V)$ contains the range of $I + K_2$, which has finite codimension in light of the spectral theory of K_2 derived in the last section. The converse result, that $T \in \text{Fred}(V, W)$ has such "Fredholm inverses" S_j, is easy.

Note that, by virtue of the identity

$$S_1(I + K_2) = S_1 T S_2 = (I + K_1) S_2, \tag{7.8}$$

we see that whenever (7.6) and (7.7) hold, S_1 and S_2 must differ by a compact operator. Thus we could take $S_1 = S_2$.

The following result is an immediate consequence of the characterization of the space $\text{Fred}(V, W)$ by (7.6)–(7.7).

Corollary 7.2. *If $T \in \text{Fred}(V, W)$ and $K : V \to W$ is compact, then $T + K \in \text{Fred}(V, W)$. If also $T_2 \in \text{Fred}(W, X)$, then $T_2 T \in \text{Fred}(V, X)$.*

Proposition 7.1 also makes it natural to consider the quotient space $\mathcal{Q}(V) = \mathcal{L}(V)/\mathcal{K}(V)$. Recall that $\mathcal{K}(V)$ is a closed, two-sided ideal of $\mathcal{L}(V)$. Thus the quotient is a Banach space, and in fact a Banach algebra. It is called the *Calkin algebra*. One has the natural algebra homomorphism $\pi : \mathcal{L}(V) \to \mathcal{Q}(V)$, and a consequence of Proposition 7.1 is that $T \in \mathcal{L}(V)$ is Fredholm if and only if $\pi(T)$ is invertible in $\mathcal{Q}(V)$. For general $T \in$

$\mathrm{Fred}(V,W)$, the operators S_1T and TS_2 in (7.6) and (7.7) project to the identity in $\mathcal{Q}(V)$ and $\mathcal{Q}(W)$, respectively. Now the argument made in §5 that the set of invertible elements of $\mathcal{L}(V)$ is open, via Proposition 5.8, applies equally well when $\mathcal{L}(V)$ is replaced by any Banach algebra with unit. Applying it to the Calkin algebra, we have the following:

Proposition 7.3. *$\mathrm{Fred}(V,W)$ is open in $\mathcal{L}(V,W)$.*

We now establish a fundamental result about the index of Fredholm operators.

Proposition 7.4. *The index map*

$$\mathrm{Ind} : \mathrm{Fred}(V,W) \longrightarrow \mathbb{Z} \tag{7.9}$$

defined by (7.3) is constant on each connected component of $\mathrm{Fred}(V,W)$.

Proof. Let $T \in \mathrm{Fred}(V,W)$. It suffices to show that if $S \in \mathcal{L}(V,W)$ and if $\|T-S\|$ is small enough, then $\mathrm{Ind}\ S = \mathrm{Ind}\ T$. We can pick a closed subspace $V_1 \subset V$, complementary to $\mathrm{Ker}\ T$ and a (finite-dimensional) $W_0 \subset W$, complementary to $T(V)$, so that

$$V = V_1 \oplus \ \mathrm{Ker}\ T, \quad W = T(V) \oplus W_0. \tag{7.10}$$

Given $S \in \mathcal{L}(V,W)$, define

$$\tau_S : V_1 \oplus W_0 \to W, \quad \tau_S(v,w) = Sv + w. \tag{7.11}$$

The map τ_T is an isomorphism of Banach spaces. Thus $\|T-S\|$ small implies τ_S is an isomorphism of $V_1 \oplus W_0$ onto W. We restrict attention to such S, lying in the same component of $\mathrm{Fred}(V,W)$ as T.

Note that $\tau_S(V_1)$ is closed in W, of codimension equal to $\dim W_0$; now $\tau_S(V_1) = S(V_1)$, so we have the semicontinuity property

$$\mathrm{Codim}\ S(V) \leq \ \mathrm{Codim}\ T(V). \tag{7.12}$$

We also see that $\mathrm{Ker}\ S \cap V_1 = 0$. Thus we can write

$$V = \ \mathrm{Ker}\ S \oplus Z \oplus V_1,$$

for a finite-dimensional $Z \subset V$. S is injective on $Z \oplus V_1$, taking it to $S(V) = S(Z) \oplus S(V_1)$, closed in W, of finite codimension. It follows that

$$\mathrm{Codim}\ S(V) = \ \mathrm{Codim}\ T(V) - \ \dim S(Z), \tag{7.13}$$

while

$$\dim \mathrm{Ker}\ S + \ \dim Z = \ \dim \mathrm{Ker}\ T. \tag{7.14}$$

Since $S(Z)$ and Z have the same dimension, this gives the desired identity, namely $\mathrm{Ind}\ S = \mathrm{Ind}\ T$.

Corollary 7.5. *If $T \in$ Fred(V,W) and $K \in \mathcal{K}(V,W)$, then $T+K$ and T have the same index.*

Proof. For $s \in [0,1]$, $T+sK \in \mathrm{Fred}(V,W)$.

The next result rounds out a useful collection of tools in the study of index theory.

Proposition 7.6. *If $T \in$ Fred(V,W) and $S \in$ Fred(W,X), then*

$$\textit{Ind}\, ST = \textit{Ind}\, S + \textit{Ind}\, T. \tag{7.15}$$

Proof. Consider the following family of operators in $\mathcal{L}(V \oplus W, W \oplus X)$:

$$\begin{pmatrix} I & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix}, \tag{7.16}$$

the middle factor belonging to $\mathcal{L}(W \oplus W)$. For each $t \in \mathbb{R}$, this is Fredholm. For $t = 0$, it is

$$\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix},$$

of index Ind $T+$ Ind S, while for $t = -\pi/2$, it is

$$\begin{pmatrix} 0 & -I \\ ST & 0 \end{pmatrix},$$

of index Ind ST. The identity of these two quantities now follows from Proposition 7.4.

Exercises

Exercises 1–4 may be compared to Exercises 3–7 in Chapter 4, §3. Let H denote the subspace of $L^2(S^1)$ that is the range of the projection P:

$$Pf(\theta) = \sum_{n=0}^{\infty} \hat{f}(n) e^{in\theta}.$$

Given $\varphi \in C(S^1)$, define the "Toeplitz operator" $T_\varphi : H \to H$ by $T_\varphi u = P(\varphi u)$. Clearly, $\|T_\varphi\| \le \|\varphi\|_{\sup}$.

1. By explicit calculation, for $\varphi(\theta) = E_k(\theta) = e^{ik\theta}$, show that
$$T_{E_k} T_{E_\ell} - T_{E_k E_\ell} \text{ is compact on } H.$$
2. Show that, for any $\varphi, \psi \in C(S^1)$, $T_\varphi T_\psi - T_{\varphi\psi}$ is compact on H. (*Hint*: Approximate φ and ψ by linear combinations of exponentials.)
3. Show that if $\varphi \in C(S^1)$ is nowhere vanishing, then $T_\varphi : H \to H$ is Fredholm. (*Hint*: Show that a Fredholm inverse is given by T_ψ, $\psi(\theta) = \varphi(\theta)^{-1}$.)

4. A nowhere-vanishing $\varphi \in C(S^1)$ is said to have degree $k \in \mathbb{Z}$ if φ is homotopic to $E_k(\theta) = e^{ik\theta}$, through continuous maps of S^1 to $\mathbb{C} \setminus 0$. Show that this implies

$$\text{Index } T_\varphi = \text{ Index } T_{E_k}.$$

Compute this index by explicitly describing Ker T_{E_k} and Ker $T^*_{E_k}$. Show that the calculation can be reduced to the case $k = 1$.

8. Unbounded operators

Here we consider unbounded linear operators on Banach spaces. Such an operator T between Banach spaces V and W will not be defined on all of V, though for simplicity we write $T : V \to W$. The domain of T, denoted $\mathcal{D}(T)$, will be some linear subspace of T. Generalizing (5.17), we consider the graph of T:

$$G_T = \{(v, Tv) \in V \oplus W : v \in \mathcal{D}(T)\}. \tag{8.1}$$

Then G_T is a linear subspace of $V \oplus W$; if G_T is closed in $V \oplus W$, we say T is a closed operator. By the closed-graph theorem, if T is closed and $\mathcal{D}(T) = V$, then T is bounded. If T is a linear operator, the closure of its graph $\overline{G}_T$ may or may not be the graph of an operator. If it is, we write $\overline{G}_T = G_{\overline{T}}$ and call $\overline{T}$ the closure of T.

For a linear operator $T : V \to W$ with dense domain $\mathcal{D}(T)$, we define the adjoint $T' : W' \to V'$ as follows. There is the identity

$$\langle Tv, w'\rangle = \langle v, T'w'\rangle, \tag{8.2}$$

for $v \in \mathcal{D}(T)$, $w' \in \mathcal{D}(T') \subset W'$. We define $\mathcal{D}(T')$ to be the set of $w' \in W'$ such that the map $v \mapsto \langle Tv, w'\rangle$ extends from $\mathcal{D}(T) \to \mathbb{C}$ to a continuous, linear functional $V \to \mathbb{C}$. For such w', the identity (8.2) uniquely determines $T'w' \in V'$.

It is useful to note the following relation between the graphs of T and T'. The graph G_T has annihilator $G_T^\perp \subset V' \oplus W'$ given by

$$G_T^\perp = \{(v', w') \in V' \oplus W' : \langle Tv, w'\rangle = -\langle v, v'\rangle \text{ for all } v \in \mathcal{D}(T)\}. \tag{8.3}$$

Comparing the definition of T', we see that, with

$$\mathcal{J} : V' \oplus W' \to W' \oplus V', \quad \mathcal{J}(v', w') = (w', -v'),$$

we have

$$G_{T'} = \mathcal{J}\, G_T^\perp. \tag{8.4}$$

We remark that $\mathcal{D}(T)$ is dense if and only if the right side of (8.4) is the graph of a (single-valued) transformation. Using $X^{\perp\perp} = \overline{X}$ for a linear subspace of a reflexive Banach space, we have the following.

Proposition 8.1. *A densely defined linear operator $T : V \to W$ between reflexive Banach spaces has a closure $\overline{T}$ if and only if T' is densely defined. T' is always closed, and $T'' = \overline{T}$.*

If H_0 and H_1 are Hilbert spaces and $T : H_0 \to H_1$, with dense domain $\mathcal{D}(T)$, we define the adjoint $T^* : H_1 \to H_0$ by replacing the dual pairings in (8.2) by the Hilbert space inner products. Parallel to (8.4), we have

$$G_{T^*} = \mathcal{J}\ G_T^{\perp}, \tag{8.5}$$

where $\mathcal{J} : H_0 \oplus H_1 \to H_1 \oplus H_0$, $\mathcal{J}(v, w) = (w, -v)$, and one takes Hilbert space orthogonal complements. Again, T has a closure if and only if T^* is densely defined, T^* is always closed, and $T^{**} = \overline{T}$. Note that, generally, the range $\mathcal{R}(T)$ of T satisfies

$$\mathcal{R}(T)^{\perp} = \text{ Ker } T^*. \tag{8.6}$$

A densely defined operator $T : H \to H$ on a Hilbert space is said to be symmetric provided T^* is an extension of T (i.e., $\mathcal{D}(T^*) \supset \mathcal{D}(T)$ and $T = T^*$ on $\mathcal{D}(T)$). An equivalent condition is that $\mathcal{D}(T)$ is dense and

$$(Tu, v) = (u, Tv), \quad \text{for } u, v \in \mathcal{D}(T). \tag{8.7}$$

If $T^* = T$ (so $\mathcal{D}(T^*) = \mathcal{D}(T)$), we say T is *self-adjoint*. In light of (8.5), T is self-adjoint if and only if $\mathcal{D}(T)$ is dense and

$$G_T^{\perp} = \mathcal{J}\ G_T. \tag{8.8}$$

Note that if T is symmetric and $\mathcal{D}(T) = H$, then T^* cannot be a proper extension of T, so we must have $T^* = T$; hence T is closed. By the closed graph theorem, T must be bounded in this case; this result is called the *Hellinger-Toeplitz theorem.*

For a bounded operator defined on all of H, being symmetric is equivalent to being self-adjoint; in the case of unbounded operators, self-adjointness is a stronger and much more useful property. We discuss some results on self-adjointness. In preparation for this, it will be useful to note that if $T : H_0 \to H_1$ has range $\mathcal{R}(T)$, and if T is injective on $\mathcal{D}(T)$, then $T^{-1} : H_1 \to H_0$ is defined, with domain $\mathcal{D}(T^{-1}) = \mathcal{R}(T)$, and we have

$$G_{T^{-1}} = \mathcal{J}\ G_{-T}. \tag{8.9}$$

Since generally $\mathcal{R}(T)^{\perp} = \text{ Ker } T^*$, the following is an immediate consequence.

Proposition 8.2. *If T is self-adjoint on H and injective, then T^{-1}, with dense domain $\mathcal{R}(T)$, is self-adjoint.*

From this easy result we obtain the following more substantial conclusion.

Proposition 8.3. *If $T : H \to H$ is symmetric and $\mathcal{R}(T) = H$, then T is self-adjoint.*

Proof. The identity (8.6) implies Ker $T = 0$ if $\mathcal{R}(T) = H$, so T^{-1} is defined. Writing $f, g \in H$ as $f = Tu, g = Tv$, and using

$$(T^{-1}f, g) = (T^{-1}Tu, Tv) = (u, Tv) = (Tu, v) = (f, T^{-1}g),$$

we see that T^{-1} is symmetric. Since $\mathcal{D}(T^{-1}) = H$, the Hellinger-Toeplitz theorem implies that T^{-1} is bounded and self-adjoint, so Proposition 8.2 applies to T^{-1}.

Whenever $T : H_0 \to H_1$ is a closed, densely defined operator between Hilbert spaces, the spaces G_T and $\mathcal{J}G_{T^*}$ provide an orthogonal decomposition of $H_0 \oplus H_1$; that is,

$$H_0 \oplus H_1 = \{(v, Tv) + (-T^*u, u) : v \in \mathcal{D}(T), u \in \mathcal{D}(T^*)\}, \tag{8.10}$$

where the terms in the sum are mutually orthogonal. Using this observation, we will be able to prove the following important result, due to J. von Neumann.

Proposition 8.4. *If $T : H_0 \to H_1$ is closed and densely defined, then T^*T is self-adjoint, and $I + T^*T$ has a bounded inverse.*

Proof. Pick $f \in H_0$. Applying the decomposition (8.10) to $(f, 0) \in H_0 \oplus H_1$, we obtain unique $v \in \mathcal{D}(T)$, $u \in \mathcal{D}(T^*)$, such that

$$f = v - T^*u, \quad u = -Tv. \tag{8.11}$$

Hence

$$v \in \mathcal{D}(T^*T) \ \text{ and } \ (I + T^*T)v = f. \tag{8.12}$$

Consequently, $I + T^*T : \mathcal{D}(T^*T) \to H_0$ is bijective, with inverse $(I + T^*T)^{-1} : H_0 \to H_0$ having range $\mathcal{D}(T^*T)$. Now, with $u = (I + T^*T)^{-1}f$ and $v = (I + T^*T)^{-1}g$, we easily compute

$$\begin{aligned} \big(f, (I + T^*T)^{-1}g\big) &= \big((I + T^*T)u, v\big) \\ &= (u, v) + (Tu, Tv) = \big((I + T^*T)^{-1}f, g\big), \end{aligned} \tag{8.13}$$

so $(I + T^*T)^{-1}$ is a symmetric operator on H. Since its domain is H, we have $(I + T^*T)^{-1}$ bounded and self-adjoint, and thus Proposition 8.2 finishes the proof.

If T is symmetric, note that

$$\|(T \pm i)u\|^2 = \|Tu\|^2 + \|u\|^2, \quad \text{for } u \in \mathcal{D}(T). \tag{8.14}$$

If T is closed, it follows that the ranges $\mathcal{R}(T \pm i)$ are closed. The following result provides an important criterion for self-adjointness.

Proposition 8.5. *Let $T : H \to H$ be symmetric. The following three conditions are equivalent:*

$$T \text{ is self-adjoint}, \tag{8.15}$$

$$T \text{ is closed and } \operatorname{Ker}(T^* \pm i) = 0, \tag{8.16}$$

$$\mathcal{R}(T \pm i) = H. \tag{8.17}$$

Proof. Assume (8.17) holds, that is, both ranges are all of H. Let $u \in \mathcal{D}(T^*)$; we want to show that $u \in \mathcal{D}(T)$. $\mathcal{R}(T-i) = H$ implies there exists $v \in \mathcal{D}(T)$ such that $(T-i)v = (T^*-i)u$. Since $\mathcal{D}(T) \subset \mathcal{D}(T^*)$, this implies $u - v \in \mathcal{D}(T^*)$ and $(T^* - i)(u-v) = 0$. Now the implication (8.17) $\Rightarrow$ $\operatorname{Ker}(T^* \mp i) = 0$ is clear from (8.6), so we have $u = v$; hence $u \in \mathcal{D}(T)$, as desired. The other implications of the proposition are straightforward.

In particular, if T is self-adjoint on H, $T \pm i : \mathcal{D}(T) \to H$ bijectively. Hence

$$U = (T-i)(T+i)^{-1} : H \longrightarrow H, \tag{8.18}$$

bijectively. By (8.14) this map preserves norms; we say U is unitary. The association of such a unitary operator (necessarily bounded) with any self-adjoint operator (perhaps unbounded) is J. von Neumann's unitary trick. Note that $I - U = 2i(T+i)^{-1}$, with range equal to $\mathcal{D}(T)$. We can hence recover T from U as

$$T = i(I+U)(I-U)^{-1}, \tag{8.19}$$

both sides having domain $\mathcal{D}(T)$.

We next give a construction of a self-adjoint operator due to K. O. Friedrichs, which is particularly useful in PDE. One begins with the following set-up. There are two Hilbert spaces H_0 and H_1, with inner products $(\ ,\ )_0$ and $(\ ,\ )_1$, respectively, and a continuous injection

$$J : H_1 \longrightarrow H_0, \tag{8.20}$$

with dense range. We think of J as identifying H_1 with a dense linear subspace of H_0; given $v \in H_1$, we will often write v for $Jv \in H_0$. A linear operator $A : H_0 \to H_0$ is defined by the identity

$$(Au, v)_0 = (u, v)_1, \tag{8.21}$$

for all $v \in H_1$, with domain

$$\begin{aligned}\mathcal{D}(A) = \{u \in H_1 \subset H_0 : v \mapsto (u,v)_1 &\text{ extends from } H_1 \to \mathbb{C} \text{ to a} \\ &\text{continuous, conjugate-linear functional } H_0 \to \mathbb{C}\}.\end{aligned} \tag{8.22}$$

Thus the graph of A is described as

$$\begin{aligned}G_A = \{(u,w) \in H_0 \oplus H_0 : u \in H_1 &\text{ and} \\ (u,v)_1 = (w,v)_0 &\text{ for all } v \in H_1\}.\end{aligned} \tag{8.23}$$

We claim that G_A is closed in $H_0 \oplus H_0$; this comes down to establishing the following.

Lemma 8.6. *If $(u_n, w_n) \in G_A$, $u_n \to u$, $w_n \to w$ in H_0, then $u \in H_1$ and $u_n \to u$ in H_1.*

Proof. Let $u_{mn} = u_m - u_n$, $w_{mn} = w_m - w_n$. We know that $(u_{mn}, v)_1 = (w_{mn}, v)_0$, for each $v \in H_1$. Taking $v = u_{mn}$ gives $\|u_{mn}\|_1^2 = (w_{mn}, u_{mn})_0 \to 0$ as $m, n \to \infty$. This implies that (u_n) is Cauchy in H_1, and the rest follows.

Actually, we could have avoided writing down this last short proof, as it will not be needed to establish our main result:

Proposition 8.7. *The operator A defined above is a self-adjoint operator on H_0.*

Proof. Consider the adjoint of J, $J^* : H_0 \to H_1$. This is also injective with dense range, and the operator JJ^* is a bounded, self-adjoint operator on H_0, that is injective with dense range. To restate (8.22), $\mathcal{D}(A)$ consists of elements $u = J\tilde{u}$ such that $v \mapsto (\tilde{u}, v)_1$ is continuous in Jv, in the H_0-norm, that is, there exists $w \in H_0$ such that $(\tilde{u}, v)_1 = (w, Jv)_0$, hence $\tilde{u} = J^* w$. We conclude that

$$\mathcal{D}(A) = \mathcal{R}(JJ^*) \tag{8.24}$$

and, for $u \in H_0$, $v \in H_1$,

$$(AJJ^*u, Jv)_0 = (J^*u, v)_1 = (u, Jv)_0. \tag{8.25}$$

It follows that

$$A = (JJ^*)^{-1}, \tag{8.26}$$

and Proposition 8.2 finishes the proof.

We remark that, given a closed, densely defined operator T on H_0, one can make $\mathcal{D}(T) = H_1$ a Hilbert space with inner product $(u, v)_1 = (Tu, Tv)_0 + (u, v)_0$. Thus Friedrichs' result, Proposition 8.7, contains von Neumann's result, Proposition 8.4. This construction of Friedrichs is used to good effect in Chapter 5.

We next discuss the resolvent and spectrum of a general closed, densely defined operator $T : V \to V$. By definition, $\zeta \in \mathbb{C}$ belongs to the resolvent set $\rho(T)$ if and only if $\zeta - T : \mathcal{D}(T) \to V$, bijectively. Then the inverse

$$R_\zeta = (\zeta - T)^{-1} : V \longrightarrow \mathcal{D}(T) \subset V \tag{8.27}$$

is called the resolvent of T; clearly, $R_\zeta \in \mathcal{L}(V)$. As in §5, the complement of $\rho(T)$ is called the spectrum of T and denoted $\sigma(T)$.

Such an operator may have an empty resolvent set. For example, the unbounded operator on $L^2(\mathbb{R}^2)$ defined by multiplication by $x_1 + ix_2$, with domain consisting of all $u \in L^2(\mathbb{R}^2)$ such that $(x_1 + ix_2)u \in L^2(\mathbb{R}^2)$, has this property. There are also examples of unbounded operators with empty spectrum. Note that Proposition 8.5 implies that $\pm i \in \rho(T)$ whenever T is self-adjoint. The same argument shows that any $\zeta \in \mathbb{C} \setminus \mathbb{R}$ belongs to $\rho(T)$, hence $\sigma(T)$ is contained in $\mathbb{R}$, when T is self-adjoint.

We note some relations between $\sigma(T)$ and $\sigma(R_\zeta)$, given that $\zeta \in \rho(T)$. Clearly, 0 belongs to $\rho(R_\zeta)$ if and only if $\mathcal{D}(T) = V$. Since R_ζ is bounded, we know that its spectrum is a nonempty, compact subset of $\mathbb{C}$. If $\lambda \in \rho(R_\zeta)$, write $S_\lambda = (\lambda - R_\zeta)^{-1}$. It follows easily that S_λ and R_ζ commute, and both preserve $\mathcal{D}(T)$. A computation gives

$$\begin{aligned} I = (\lambda - R_\zeta)S_\lambda &= \lambda(\zeta - T)S_\lambda(\zeta - T)^{-1} - S_\lambda(\zeta - T)^{-1} \\ &= \lambda(\zeta - \lambda^{-1} - T)S_\lambda(\zeta - T)^{-1} \quad \text{on } V, \end{aligned} \tag{8.28}$$

and similarly,

$$\begin{aligned} I &= \lambda(\zeta - T)^{-1}S_\lambda(\zeta - T) - (\zeta - T)^{-1}S_\lambda \\ &= \lambda S_\lambda(\zeta - T)^{-1}(\zeta - \lambda^{-1} - T) \quad \text{on } \mathcal{D}(T). \end{aligned} \tag{8.29}$$

This establishes the following:

Proposition 8.8. *Given $\zeta \in \rho(T)$, if $\lambda \in \rho(R_\zeta)$ and $\lambda \neq 0$, then $\zeta - \lambda^{-1} \in \rho(T)$. Hence $\rho(T)$ is open in $\mathbb{C}$. We have, for such λ,*

$$(\zeta - \lambda^{-1} - T)^{-1} = \lambda(\lambda - R_\zeta)^{-1}(\zeta - T)^{-1}. \tag{8.30}$$

The second assertion follows from the fact that $\lambda \in \rho(R_\zeta)$ provided $|\lambda| > \|R_\zeta\|$.

If there exists $\zeta \in \rho(T)$ such that R_ζ is compact, we say T has compact resolvent. By Proposition 8.8 it follows that when T has compact resolvent, then $\sigma(T)$ is a discrete subset of $\mathbb{C}$. Every resolvent in (8.30) is compact in this case. If T is self-adjoint on H with compact resolvent, there exists $z \in \rho(T) \cap \mathbb{R}$, and $(z - T)^{-1}$ is a compact, self-adjoint operator, to which Proposition 6.6 applies. Thus H has an orthonormal basis of eigenvectors of T:

$$v_j \in \mathcal{D}(T), \quad Tv_j = \lambda_j v_j, \tag{8.31}$$

where $\{\lambda_j\}$ is a sequence of real numbers with no finite accumulation point. Important examples of unbounded operators with compact resolvent arise amongst differential operators; cf. Chapter 5.

Exercises

1. Consider the following operator, which is densely defined on $L^2(\mathbb{R})$:

$$Tf(x) = f(0)e^{-x^2}, \quad \mathcal{D} = C_0^\infty(\mathbb{R}).$$

Show that T is unbounded and also that T has no closure.

9. Semigroups

If V is a Banach space, a one-parameter semigroup of operators on V is a set of bounded operators

$$P(t) : V \longrightarrow V, \quad t \in [0, \infty), \tag{9.1}$$

satisfying

$$P(s+t) = P(s)P(t), \tag{9.2}$$

for all $s, t \in \mathbb{R}^+$, and

$$P(0) = I. \tag{9.3}$$

We also require strong continuity, that is,

$$t_j \to t \Longrightarrow P(t_j)v \to P(t)v, \tag{9.4}$$

for each $v \in V$, the convergence being in the V-norm. A semigroup of operators will by definition satisfy (9.1)–(9.4). If $P(t)$ is defined for all $t \in \mathbb{R}$ and satisfies these conditions, we say it is a one-parameter group of operators.

A simple example is the translation group

$$T_p(t) : L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R}), \quad 1 \le p < \infty, \tag{9.5}$$

defined by

$$T_p(t)f(x) = f(x-t). \tag{9.6}$$

The properties (9.1)–(9.3) are clear in this case. Note that $\|T_p(t)\| = 1$ for each t. Also, $\|T_p(t) - T_p(t')\| = 2$ if $t \neq t'$; to see this, apply the difference to a function f with support in an interval of length $|t-t'|/2$. To verify the strong continuity (9.4), we make the following observation. As noted in §1, the space $C_{00}(\mathbb{R})$ of compactly supported, continuous functions on $\mathbb{R}$ is dense in $L^p(\mathbb{R})$ for $p \in [1, \infty)$. Now, if $f \in C_{00}(\mathbb{R})$, $t_j \to t$, then $T_p(t_j)f(x) = f(x - t_j)$ have support in a fixed compact set and converge uniformly to $f(x-t)$, so clearly we have convergence in (9.4) in L^p-norm for each $f \in C_{00}(\mathbb{R})$. The following simple but useful lemma completes the proof of (9.4) for T_p.

Lemma 9.1. *Let $T_j \in \mathcal{L}(V, W)$ be uniformly bounded. Let L be a dense, linear subspace of V, and suppose*

$$T_j v \to T_0 v, \quad \text{as } j \to \infty, \tag{9.7}$$

in the W-norm, for each $v \in L$. Then (9.7) holds for all $v \in V$.

Proof. Given $v \in V$ and $\varepsilon > 0$, pick $w \in L$ such that $\|v - w\| < \varepsilon$. Suppose $\|T_j\| \le M$ for all j. Then

$$\begin{aligned}\|T_j v - T_0 v\| &\le \|T_j v - T_j w\| + \|T_j w - T_0 w\| + \|T_0 w - T_0 v\| \\ &\le \|T_j w - T_0 w\| + 2M\|v - w\|.\end{aligned}$$

Thus

$$\limsup_{j\to\infty} \|T_j v - T_0 v\| \le 2M\varepsilon,$$

which proves the lemma.

Many examples of semigroups appear in the main text, particularly in Chapters 3, 6, and 9, so we will not present further examples here.

We note that a uniform bound on the norm

$$\|P(t)\| \le M, \quad \text{for } |t| \le 1 \tag{9.8}$$

for some $M \in [1, \infty)$, holds for any strongly continuous semigroup, as a consequence of the uniform boundedness principle. From (9.8) we deduce that, for all $t \in \mathbb{R}^+$,

$$\|P(t)\| \le M\ e^{Kt}, \tag{9.9}$$

for some K; for a group, one would use $M\ e^{K|t|}$, $t \in \mathbb{R}$.

Of particular interest are unitary groups—strongly continuous groups of operators $U(t)$ on a Hilbert space H such that

$$U(t)^* = U(t)^{-1} = U(-t). \tag{9.10}$$

Clearly, in this case $\|U(t)\| = 1$. The translation group T_2 on $L^2(\mathbb{R})$ is a simple example of a unitary group.

A one-parameter semigroup $P(t)$ of operators on V has an *infinitesimal generator* A, which is an operator on V, often unbounded, defined by

$$Av = \lim_{h\to 0} h^{-1}\big(P(h)v - v\big), \tag{9.11}$$

on the domain

$$\mathcal{D}(A) = \{v \in V : \lim_{h\to 0} h^{-1}\big(P(h)v - v\big) \text{ exists in } V\}. \tag{9.12}$$

The following provides some basic information on the generator.

Proposition 9.2. *The infinitesimal generator A of $P(t)$ is a closed, densely defined operator. We have*

$$P(t)\mathcal{D}(A) \subset \mathcal{D}(A), \tag{9.13}$$

for all $t \in \mathbb{R}^+$, and

$$AP(t)v = P(t)Av = \frac{d}{dt}\,P(t)v, \quad \text{for } v \in \mathcal{D}(A). \tag{9.14}$$

If (9.9) holds and $\operatorname{Re}\zeta > K$, then ζ belongs to the resolvent set of A, and

$$(\zeta - A)^{-1}v = \int_0^\infty e^{-\zeta t}\, P(t)v\, dt, \quad v \in V. \tag{9.15}$$

Proof. First, if $v \in \mathcal{D}(A)$, then for $t \in \mathbb{R}^+$,

$$h^{-1}\bigl(P(h)P(t)v - P(t)v\bigr) = P(t)\; h^{-1}\bigl(P(h)v - v\bigr), \tag{9.16}$$

which gives (9.13), and also (9.14), if we replace $P(h)P(t)$ by $P(t+h)$ in (9.16). To show that $\mathcal{D}(A)$ is dense in V, let $v \in V$, and consider

$$v_\varepsilon = \varepsilon^{-1} \int_0^\varepsilon P(t)v\, dt.$$

Then

$$\begin{aligned} h^{-1}\bigl(P(h)v_\varepsilon - v_\varepsilon\bigr) &= \varepsilon^{-1}\Bigl[h^{-1}\int_\varepsilon^{\varepsilon+h} P(t)v\, dt - h^{-1}\int_0^h P(t)v\, dt\Bigr] \\ &\to \varepsilon^{-1}\bigl(P(\varepsilon)v - v\bigr), \quad \text{as } h \to 0, \end{aligned}$$

so $v_\varepsilon \in \mathcal{D}(A)$ for each $\varepsilon > 0$. But $v_\varepsilon \to v$ in V as $\varepsilon \to 0$, by (9.4), so $\mathcal{D}(A)$ is dense in V.

Next we prove (9.15). Denote the right side of (9.15) by R_ζ, clearly a bounded operator on V. First we show that

$$R_\zeta(\zeta - A)v = v, \quad \text{for } v \in \mathcal{D}(A). \tag{9.17}$$

In fact, by (9.14) we have

$$\begin{aligned} R_\zeta(\zeta - A)v &= \int_0^\infty e^{-\zeta t} P(t)(\zeta v - Av)\, dt \\ &= \int_0^\infty \zeta e^{-\zeta t} P(t)v\, dt - \int_0^\infty e^{-\zeta t}\, \frac{d}{dt}\, P(t)v\, dt, \end{aligned}$$

and integrating the last term by parts gives (9.17). The same sort of argument shows that $R_\zeta : V \to \mathcal{D}(A)$, that $(\zeta - A)R_\zeta$ is bounded on V, and that

$$(\zeta - A)R_\zeta v = v, \tag{9.18}$$

for $v \in \mathcal{D}(A)$. Since $(\zeta - A)R_\zeta$ is bounded on V and $\mathcal{D}(A)$ is dense in V, (9.18) holds for all $v \in V$. This proves (9.15). Finally, since the resolvent set of A is nonempty, and $(\zeta - A)^{-1}$, being continuous and everywhere defined, is closed, so is A. The proof of the proposition is complete.

We write, symbolically,

$$P(t) = e^{tA}. \tag{9.19}$$

In view of the following proposition, the infinitesimal generator determines the one-parameter semigroup with which it is associated uniquely. Hence we are justified in saying "A generates $P(t)$."

Proposition 9.3. *If $P(t)$ and $Q(t)$ are one-parameter semigroups with the same infinitesimal generator, then $P(t) = Q(t)$ for all $t \in \mathbb{R}^+$.*

Proof. Let $v \in V$ and $w \in V'$. Then, for Re ζ large enough,

$$\begin{aligned} \int_0^\infty e^{-\zeta t} \langle P(t)v, w\rangle \, dt &= \langle (\zeta - A)^{-1} v, w\rangle \\ &= \int_0^\infty e^{-\zeta t} \langle Q(t)v, w\rangle \, dt. \end{aligned} \tag{9.20}$$

Uniqueness for the Laplace transform of a scalar function implies $\langle P(t)v, w\rangle = \langle Q(t)v, w\rangle$ for all $t \in \mathbb{R}^+$ and for any $v \in V$ and $w \in V'$. Then the Hahn-Banach theorem implies $P(t)v = Q(t)v$, as desired.

We note that if $P(t)$ is a semigroup satisfying (9.9) and if we have a function $\varphi \in L^1(\mathbb{R}^+, e^{Kt}dt)$, we can define $P(\varphi) \in \mathcal{L}(V)$ by

$$P(\varphi)v = \int_0^\infty \varphi(t)\ P(t)v \ dt. \tag{9.21}$$

In particular, this works if $\varphi \in C_0^\infty(0, \infty)$. In such a case, it is easy to verify that, for all $v \in V$, $P(\varphi)v$ belongs to the domain of all powers of A and

$$A^k P(\varphi)v = (-1)^k \int_0^\infty \varphi^{(k)}(t) P(t)v \ dt. \tag{9.22}$$

This shows that all the domains $\mathcal{D}(A^k)$ are dense in V, refining the proof of denseness of $\mathcal{D}(A)$ in V given in Proposition 9.2.

A general characterization of generators of semigroups, due to Hille and Yosida, is briefly discussed in the exercises. Here we mention two important special cases, which follow from the spectral theorem, established in Chapter 8.

Proposition 9.4. *If A is self-adjoint and positive (i.e., $(Au, u) \geq 0$ for $u \in \mathcal{D}(A)$), then $-A$ generates a semigroup $P(t) = e^{-tA}$ consisting of positive, self-adjoint operators of norm ≤ 1.*

Proposition 9.5. *If A is self-adjoint, then iA generates a unitary group, $U(t) = e^{itA}$.*

In both cases it is easy to show that the generator of such (semi)groups must be of the form hypothesized. For example, if $U(t)$ is a unitary group and we denote by iA the generator, the identity

$$h^{-1}\big([U(h) - I]u, v\big) = h^{-1}\big(u, [U(-h) - I]v\big) \tag{9.23}$$

shows that A must be symmetric. By Proposition 9.2, all $\zeta \in \mathbb{C} \setminus \mathbb{R}$ belong to the resolvent set of A, so by Proposition 8.5, A is self-adjoint. If A is self-adjoint, iA is said to be skew-adjoint.

We now give a criterion for a symmetric operator to be essentially self-adjoint, that is, to have self-adjoint closure. This is quite useful in PDE; see Chapter 8 for some applications.

Proposition 9.6. *Let A_0 be a linear operator on a Hilbert space H, with domain $\mathcal{D}$, assumed dense in H. Let $U(t)$ be a unitary group, with infinitesimal generator iA, so A is self-adjoint, $U(t) = e^{itA}$. Suppose $\mathcal{D} \subset \mathcal{D}(A)$ and $A_0 u = Au$ for $u \in \mathcal{D}$, or equivalently*

$$\lim_{h\to 0} h^{-1}\big(U(h)u - u\big) = A_0 u, \quad \text{for all } u \in \mathcal{D}. \tag{9.24}$$

Also suppose $\mathcal{D}$ is invariant under $U(t)$:

$$U(t)\mathcal{D} \subset \mathcal{D}. \tag{9.25}$$

Then A_0 is essentially self-adjoint, with closure A. Suppose, furthermore, that

$$A_0 : \mathcal{D} \longrightarrow \mathcal{D}. \tag{9.26}$$

Then A_0^k, with domain $\mathcal{D}$, is essentially self-adjoint for each positive integer k.

Proof. It follows from Proposition 8.5 that A_0 is essentially self-adjoint if and only if the range of $i + A_0$ and the range of $i - A_0$ are dense in H. So suppose $v \in H$ and (for one choice of sign)

$$\big((i \pm A_0)u, v\big) = 0, \quad \text{for all } u \in \mathcal{D}. \tag{9.27}$$

Using (9.25) together with the fact that $A_0 = A$ on $\mathcal{D}$, we have

$$\big((i \pm A_0)u, U(t)v\big) = 0, \quad \text{for all } t \in \mathbb{R},\ u \in \mathcal{D}. \tag{9.28}$$

Consequently, $\int \rho(t)U(t)v\, dt$ is orthogonal to the range of $i \pm A_0$, for any $\rho \in L^1(\mathbb{R}^+)$. Choosing $\rho \in C_0^\infty(0,\infty)$ an approximate identity, we can approximate v by elements of $\mathcal{D}(A)$, indeed of $\mathcal{D}(A^k)$ for all k. Thus we can suppose in (9.27) that $v \in \mathcal{D}(A)$. Hence, taking adjoints, we have

$$\big(u, (-i \pm A)v\big) = 0, \quad \text{for all } u \in \mathcal{D}. \tag{9.29}$$

Since $\mathcal{D}$ is dense in H and $\operatorname{Ker}(-i \pm A) = 0$, this implies $v = 0$. This yields the first part of the proposition. Granted (9.26), the same proof works with A_0 replaced by A_0^k (but $U(t)$ unaltered), so the proposition is proved.

This result has an extension to general semigroups which is of interest.

Proposition 9.7. *Let $P(t)$ be a semigroup of operators on a Banach space B, with generator A. Let $\mathcal{L} \subset \mathcal{D}(A)$ be a dense, linear subspace of B, and suppose $P(t)\mathcal{L} \subset \mathcal{L}$ for all $t \geq 0$. Then A is the closure of its restriction to $\mathcal{L}$.*

Proof. By Proposition 9.2, it suffices to show that $(\lambda - A)(\mathcal{L})$ is dense in B provided $\operatorname{Re} \lambda$ is sufficiently large, namely, $\operatorname{Re} \lambda > K$ with $\|P(t)\| \leq Me^{Kt}$. If $w \in B'$ annihilates this range and $w \neq 0$, pick $u \in \mathcal{L}$ such that $\langle u, w \rangle \neq 0$. Now

$$\frac{d}{dt}\langle P(t)u, w\rangle = \langle AP(t)u, w\rangle = \langle \lambda P(t)u, w\rangle$$

since $P(t)u \in \mathcal{L}$. Thus $\langle P(t)u, w\rangle = e^{\lambda t}\langle u, w\rangle$. But if $\operatorname{Re} \lambda > K$ as above, this is impossible unless $\langle u, w\rangle = 0$. This completes the proof.

We illustrate some of the preceding results by looking at the infinitesimal generator A_p of the group T_p given by (9.5)–(9.6). By definition, $f \in L^p(\mathbb{R})$ belongs to $\mathcal{D}(A_p)$ if and only if

$$h^{-1}\big(f(x-h) - f(x)\big) \tag{9.30}$$

converges in L^p-norm as $h \to 0$, to some limit. Now the limit of (9.30) always exists in the space of distributions $\mathcal{D}'(\mathbb{R})$ and is equal to $-(d/dx)u$, where d/dx is applied in the sense of distributions. In fact, we have the following.

Proposition 9.8. *For $p \in [1, \infty)$, the group T_p given by (9.5)–(9.8) has infinitesimal generator A_p given by*

$$A_p f = -\frac{df}{dx}, \tag{9.31}$$

for $f \in \mathcal{D}(A_p)$, with

$$\mathcal{D}(A_p) = \{f \in L^p(\mathbb{R}) : f' \in L^p(\mathbb{R})\}, \tag{9.32}$$

where $f' = df/dx$ is considered a priori as a distribution.

Proof. The argument above shows that (9.31) holds, with $\mathcal{D}(A_p)$ contained in the right side of (9.32). The reverse containment can be derived as a consequence of the following simple result, taking $\mathcal{L} = C_0^\infty(\mathbb{R})$.

Lemma 9.9. *Let $P(t)$ be a one-parameter semigroup on B, with infinitesimal generator A. Let $\mathcal{L}$ be a weak*-dense, linear subspace of B'. Suppose that $u, v \in B$ and that*

$$\lim_{h\to 0} h^{-1}\langle P(h)u - u, w\rangle = \langle v, w\rangle, \quad \forall\, w \in \mathcal{L}. \tag{9.33}$$

Then $u \in \mathcal{D}(A)$ and $Au = v$.

Proof. The hypothesis (9.33) implies that $\langle P(t)u, w\rangle$ is differentiable and that

$$\frac{d}{dt}\langle P(t)u, w\rangle = \langle P(t)v, w\rangle, \quad \forall\, w \in \mathcal{L}.$$

Hence $\langle P(t)u - u, w\rangle = \int_0^t \langle P(s)v, w\rangle\, ds$, for all $w \in \mathcal{L}$. The weak* denseness of $\mathcal{L}$ implies $P(t)u - u = \int_0^t P(s)v\, ds$, and the convergence in the B-norm of $h^{-1}\big(P(h)u - u\big) = h^{-1}\int_0^h P(s)v\, ds$ to v as $h \to 0$ follows.

The space (9.32) is the Sobolev space $H^{1,p}(\mathbb{R})$ studied in Chapter 13; in case $p = 2$, it is the Sobolev space $H^1(\mathbb{R})$ introduced in Chapter 4.

Note that if we define

$$A_0 : C_0^\infty(\mathbb{R}) \longrightarrow C_0^\infty(\mathbb{R}), \quad A_0 f = -\frac{df}{dx}, \tag{9.34}$$

then Proposition 9.7 applies to T_p, $p \in [1, \infty)$, with $B = L^p(\mathbb{R})$, $\mathcal{L} = C_0^\infty(\mathbb{R})$, to show that, as a closed operator on $L^p(\mathbb{R})$,

$$A_p \text{ is the closure of } A_0, \quad \text{for } p \in [1, \infty). \tag{9.35}$$

This amounts to saying that $C_0^\infty(\mathbb{R})$ is dense in $H^{1,p}(\mathbb{R})$ for $p \in [1, \infty)$, which can easily be verified directly.

The fact that a semigroup $P(t)$ satisfies the operator differential equation (9.14) is central. We now establish the following converse.

Proposition 9.10. *Let A be the infinitesimal generator of a semigroup. If a function $u \in C\big([0, T), \mathcal{D}(A)\big) \cap C^1\big([0, T), V\big)$ satisfies*

$$\frac{du}{dt} = Au, \quad u(0) = f, \tag{9.36}$$

then $u(t) = e^{tA} f$, for $t \in [0, T)$.

Proof. Set $v(s, t) = e^{sA}u(t) \in C^1(Q, V)$, $Q = [0, \infty) \times [0, T)$. Then (9.36) implies that $(\partial_s - \partial_t)v = e^{sA}Au(t) - e^{sA}Au(t) = 0$, so $u(t) = v(0, t) = v(t, 0) = e^{tA} f$.

We can thus deduce that, given $g \in C\big([0, T), \mathcal{D}(A)\big)$, $f \in \mathcal{D}(A)$, the solution $u(t)$ to

$$\frac{\partial u}{\partial t} = Au + g(t), \quad u(0) = f, \tag{9.37}$$

is unique and is given by

$$u(t) = e^{tA}f + \int_0^t e^{(t-s)A}g(s)\,ds. \tag{9.38}$$

This is a variant of Duhamel's principle.

We can also define a notion of a "weak solution" of (9.37) as follows. If A generates a semigroup, then $\mathcal{D}(A')$ is a dense, linear subspace of V'. Suppose that, for every $\psi \in \mathcal{D}(A')$, $\langle u(t), \psi\rangle \in C^1([0,T))$; if $f \in V$, $g \in C([0,T),V)$, and

$$\frac{d}{dt}\langle u(t),\psi\rangle = \langle u(t), A'\psi\rangle + \langle g(t),\psi\rangle, \quad u(0) = f, \tag{9.39}$$

we say $u(t)$ is a weak solution to (9.37).

Proposition 9.11. *Given $f \in V$ and $g \in C\big([0,T),V\big)$, (9.37) has a unique weak solution, given by (9.38).*

Proof. First, consider (9.38), with $f \in V$, $g \in C(J,V)$, and $J = [0,T)$. Let $f_j \to f$ in V and $g_j \to g$ in $C(J,V)$, where $f_j \in \mathcal{D}(A)$ and $g_j \in C^1(J,V) \cap C(J,\mathcal{D}(A))$. Then, by Proposition 9.10,

$$u_j(t) = e^{tA}f_j + \int_0^t e^{(t-s)A}g_j(s)\,ds \tag{9.40}$$

is the unique solution in $C^1(J,V)\cap C(J,\mathcal{D}(A))$ to

$$\frac{\partial u_j}{\partial t} = Au_j + g_j, \quad u_j(0) = f_j.$$

Thus, for any $\psi \in \mathcal{D}(A')$, u_j solves (9.39), with g and f replaced by g_j and f_j, respectively, and hence

$$\langle u_j(t),\psi\rangle = \langle f_j,\psi\rangle + \int_0^t \langle u_j(s), A'\psi\rangle\,ds + \int_0^t \langle g_j(s),\psi\rangle\,ds. \tag{9.41}$$

Passing to the limit, we have

$$\langle u(t),\psi\rangle = \langle f,\psi\rangle + \int_0^t \langle u(s), A'\psi\rangle\,ds + \int_0^t \langle g(s),\psi\rangle\,ds, \tag{9.42}$$

which implies (9.39).

For the converse, suppose that $u \in C(J,V)$ is a weak solution, satisfying (9.39), or equivalently, that (9.42) holds. Set $\varphi(t) = j$ for $0 \le t \le 1/j$, 0 elsewhere, and consider $P(\varphi_j)$, defined by (9.21). We see that $\langle Av, P(\varphi_j)'\psi\rangle = \langle AP(\varphi_j)v,\psi\rangle$. Hence $P(\varphi_j)' : V' \to \mathcal{D}(A')$, and also $\langle v, A'P(\varphi_j)'\psi\rangle = \langle AP(\varphi_j)v,\psi\rangle$ for $v \in \mathcal{D}(A)$, $\psi \in V'$. If you replace ψ by $P(\varphi_j)'\psi$ in (9.41), then $u_j(t) = P(\varphi_j)u(t)$ satisfies (9.41), with $f_j = P(\varphi_j)f$, $g_j(t) = P(\varphi_j)g(t)$; hence $u_j \in C^1(J,V)\cap C(J,\mathcal{D}(A))$ is given by (9.40), and passing to the limit gives (9.38) for u.

We close this section with a brief discussion of when we can deduce that, given a generator A of a semigroup and another operator B, then $A+B$ also generates a semigroup. There are a number of results on this, to the effect that $A+B$ works if B is "small" in some sense, compared to A. These results are part of the "perturbation theory" of semigroups. The following simple case is useful.

Proposition 9.12. *If A generates a semigroup e^{tA} on V and B is bounded on V, then $A+B$ also generates a semigroup.*

Proof. The idea is to solve the equation

$$\frac{\partial u}{\partial t} = Au + Bu, \quad u(0) = f, \tag{9.43}$$

by solving the integral equation

$$u(t) = e^{tA} f + \int_0^t e^{(t-s)A} Bu(s)\, ds. \tag{9.44}$$

In other words, we want to solve

$$(I - \mathcal{N})u(t) = e^{tA} f \in C\big([0,\infty), V\big), \tag{9.45}$$

where

$$\mathcal{N}u(t) = \int_0^t e^{(t-s)A} Bu(s)\, ds, \quad \mathcal{N} : C(\mathbb{R}^+, V) \to C(\mathbb{R}^+, V). \tag{9.46}$$

Note that

(9.47)

$$\mathcal{N}^k u(t) = \int_0^t \int_0^{t_{k-1}} \cdots \int_0^{t_1} e^{(t-t_{k-1})A} B e^{(t_{k-1}-t_{k-2})A} \cdots$$
$$\cdots B e^{(t_1-t_0)A} Bu(t_0)\, dt_0 \cdots dt_{k-1}.$$

Hence, if e^{tA} satisfies the estimate (9.9),

$$\sup_{0\le t\le T} \|\mathcal{N}^k u(t)\| \le \big(M\|B\|\big)^k e^{tK} \cdot \big(\mathrm{vol}\ S_k^T\big) \cdot \sup_{0\le t\le T} \|u(t)\|, \tag{9.48}$$

where vol S_k^T is the volume of the k-simplex

$$S_k^T = \{(t_0, \dots, t_{k-1}) : 0 \le t_0 \le \cdots \le t_{k-1} \le T\}.$$

Looking at the case $A = 0$, $B = b$ (scalar) of (9.43), with solution $u(t) = e^{tb} f$, we see that

$$\mathrm{vol}\ S_k^T = \frac{T^k}{k!}. \tag{9.49}$$

It follows that

$$\mathcal{S}g(t) = g(t) + \sum_{k=1}^{\infty} \mathcal{N}^k g(t) \tag{9.50}$$

is convergent in $C(\mathbb{R}^+, V)$, given $g(t) \in C(\mathbb{R}^+, V)$. Now consider

$$Q(t)f = e^{tA}f + \sum_{k=1}^{\infty} \mathcal{N}^k \bigl(e^{tA}f\bigr). \tag{9.51}$$

It is straightforward to verify that $Q(t)$ is a strongly continuous semigroup on V, with generator $A + B$.

An extension of Proposition 9.12—part of the perturbation theory of R. Phillips—is given in the exercises. We mention another perturbation result, due to T. Kato. A semigroup $P(t)$ is called a *contraction semigroup* on V if $\|P(t)\| \le 1$ for all $t \ge 0$.

Proposition 9.13. *If A generates a contraction semigroup on V, then $A + B$ generates a contraction semigroup, provided $\mathcal{D}(B) \supset \mathcal{D}(A)$, B is "dissipative," and*

$$\|Bf\| \le \vartheta \|Af\| + C_1 \|f\|, \tag{9.52}$$

for some $C_1 < \infty$ and $\vartheta < 1/2$. If V is a Hilbert space, we can allow any $\vartheta < 1$.

To say that B is dissipative means that if $u \in \mathcal{D}(B) \subset V$ and $u^\# \in V'$ satisfies $\langle u, u^\# \rangle = \|u\|^2$, then

$$\operatorname{Re} \langle Bu, u^\# \rangle \le 0. \tag{9.53}$$

If V is a Hilbert space with inner product $(\ ,\ )$, this is equivalent to

$$\operatorname{Re}(Bu, u) \le 0, \quad \text{for } u \in \mathcal{D}(B). \tag{9.54}$$

Proofs of Proposition 9.13 typically use the Hille-Yosida characterization of which A generate a contraction semigroup. See the exercises for further discussion.

Exercises

In Exercises 1–3, define, for $I = (0, 1)$,

$$A_0 : C_0^\infty(I) \longrightarrow C_0^\infty(I), \quad A_0 f = -\frac{df}{dx}. \tag{9.55}$$

1. Given $f \in L^2(I)$, define Ef on $\mathbb{R}$ to be equal to f on I and to be periodic of period 1, and define $U(t) : L^2(I) \to L^2(I)$ by

$$U(t)f(x) = (Ef)(x - t)|_I. \tag{9.56}$$

Show that $U(t)$ is a unitary group whose generator D is a skew-adjoint extension of A_0. Describe the domain of D.

2. More generally, for $e^{i\theta} \in S^1$, define Ef on $\mathbb{R}$ to equal f on I and to satisfy

$$(Ef)(x+1) = e^{i\theta} f(x).$$

Then define $U_\theta(t) : L^2(I) \to L^2(I)$ by (9.56), with this E. Show that $U_\theta(t)$ is a unitary group whose generator D_θ is a skew-adjoint extension of A_0. Describe the domain of D_θ.

3. This time, define Ef on $\mathbb{R}$ to equal f on I and zero elsewhere. For $t \ge 0$, define $P(t) : L^2(I) \to L^2(I)$ by (9.56) with this E. Show that $P(t)$ is a strongly continuous semigroup. Show that $P(t) = 0$ for $t \ge 1$. Show that the infinitesimal generator B of $P(t)$ is a closed extension of A_0 which has empty spectrum. Describe the domain of B.

4. Let P^t be a strongly continuous semigroup on the Banach space X, with infinitesimal generator A. Suppose A has compact resolvent. If K is a closed bounded subset of X, show that K is compact if and only if $P^t \to I$ uniformly on K. (*Hint*: Let $T_j = h^{-1} \int_0^h P^t \, dt$, $h = 1/j$, and use Exercise 4 of §6.)

Exercises 5–8 deal with the case where $P(t)$ satisfies (9.1)–(9.3) but the strong continuity of $P(t)$ is replaced by weak continuity, that is, convergence in (9.4) holds in the $\sigma(V, V')$-topology on V. We restrict attention to the case where V is reflexive.

5. If $\varphi \in C_0^\infty(\mathbb{R}^+)$, show that $P(\varphi)v$ is well defined in V, satisfying

$$\langle P(\varphi)v, \omega\rangle = \int_0^\infty \varphi(t)\langle P(\varphi)v, \omega\rangle \, dt, \quad v \in V, \ \omega \in V'.$$

6. Show that $V_0 = \operatorname{span}\{P(\varphi)v : v \in V, \varphi \in C_0^\infty(\mathbb{R}^+)\}$ is *dense* in V. (*Hint*: Suppose $\omega \in V'$ annihilates V_0.)

7. Show that $P(t_j)P(\varphi)v = P(\varphi_j)v$, where $\varphi_j(\tau) = \varphi(\tau - t_j)$ for $\tau \ge t_j$, 0 for $\tau < t_j$. Deduce that as $t_j \to t$,

$$P(t_j)P(\varphi)v \to P(t)P(\varphi)v, \quad \text{in } V\text{-norm},$$

for $v \in V$, $\varphi \in C_0^\infty(\mathbb{R}^+)$. (*Hint*: Estimate $\|P(\varphi_j - \varphi_0)v\|$, with $\varphi_0(\tau) = \varphi(\tau - t)$. To do this, show that (9.9) continues to hold.)

8. Deduce that the hypotheses on $P(t)$ in Exercises 5–7 imply the strong continuity (9.4).
(*Hint*: Use Lemma 9.1.)

9. If $P(t)$ is a strongly continuous semigroup on V, then $Q(t) = P(t)'$, acting on V', satisfies (9.1)–(9.3), with weak* continuity in place of (9.4). Deduce that if V is reflexive, $Q(t)$ is a strongly continuous semigroup on V'. Give an example of $P(t)$ on a (nonreflexive) Banach space V for which $P(t)'$ is *not* strongly continuous in $t \in [0, \infty)$.

10. Extend Proposition 9.12 to show that if A generates a semigroup e^{tA} on V and if $\mathcal{D}(B) \supset \mathcal{D}(A)$ is such that Be^{tA} is bounded for $t > 0$, satisfying

$$\|Be^{tA}\|_{\mathcal{L}(V)} \le C_0 t^{-\alpha}, \quad t \in (0, 1],$$

for some $\alpha < 1$, then $A + B$ also generates a semigroup.
(*Hint*: Show that (9.51) still works. Note that the integrand in the formula (9.57) for $\mathcal{N}^k(e^{tA}f)$ is of the form $\cdots Be^{(t_1-t_0)A}Be^{t_0A}f$.)

11. Recall that $P(t)$ is a contraction semigroup if it satisfies (9.1)–(9.4) and $\|P(t)\| \le 1$ for all $t \ge 0$. Show that the infinitesimal generator A of a contraction semigroup has the following property:

$$\lambda > 0 \Longrightarrow \lambda \in \rho(A), \quad \text{and} \quad \|(\lambda - A)^{-1}\| \le \frac{1}{\lambda}. \tag{9.57}$$

12. The Hille-Yosida theorem states that whenever $\mathcal{D}(A)$ is dense in V and there exist $\lambda_j > 0$ such that

$$\lambda_j \nearrow +\infty, \quad \lambda_j \in \rho(A), \quad \|(\lambda_j - A)^{-1}\| \le \frac{1}{\lambda_j}, \tag{9.58}$$

then A generates a contraction semigroup. Try to prove this. (*Hint*: With $\lambda = \lambda_j$, set $A_\lambda = \lambda A(\lambda - A)^{-1}$, which is in $\mathcal{L}(V)$. Define $P_\lambda(t) = e^{tA_\lambda}$ by the power-series expansion. Show that

$$\|P_\lambda(t)\| \le 1, \quad \Big\|\Big(P_\lambda(t) - P_\mu(t)\Big)f\Big\| \le t\Big\|(A_\lambda - A_\mu)f\Big\|, \tag{9.59}$$

and construct $P(t)$ as the limit of $P_{\lambda_j}(t)$.)

13. If $P(t)$ satisfies (9.9), set $Q(t) = e^{-Kt}P(t)$, so $\|Q(t)\| \le M$ for $t \ge 0$. Show that $|||f||| = \sup_{t\ge 0} \|Q(t)f\|$ defines an equivalent norm on V, for which $Q(t)$ is a contraction semigroup. Then, using Exercisess 11 and 12, produce a characterization of generators of semigroups.
14. Show that if $P(t)$ is a contraction semigroup, its generator A is dissipative, in the sense of (9.53).
15. Show that if $\mathcal{D}(A)$ is dense, if $\lambda_0 \in \rho(A)$ for some λ_0 such that Re $\lambda_0 > 0$, and if A is dissipative, then A generates a contraction semigroup. (*Hint*: First show that the hypotheses imply $\lambda \in \rho(A)$ whenever Re $\lambda > 0$. Then apply the Hille-Yosida theorem.)
Deduce Propositions 9.4 and 9.5 from this result.
16. Prove Proposition 9.13. (*Hint*: Show that $\lambda \in \rho(A + B)$ for some $\lambda > 0$, and apply Exercise 15. To get this, show that when A is dissipative and $\lambda > 0$, $\lambda \in \rho(A)$, then

$$\|A(\lambda - A)^{-1}\| \le \kappa,$$

where $\kappa = 2$ for a general Banach space V, while $\kappa = 1$ if V is a Hilbert space.)

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