Fourier Integral Operators and Harmonic Analysis On Compact Manifolds

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Harmonic analysis was invented by Daniel Bernoulli for his solution of the simplest hyperbolic equation, the equation of vibrating strings, though his achievement was not understood and followed through at the time and the subject had to be re-invented by Fourier as a tool to treat the heat equation. Meanwhile, the theory of spherical harmonics was invented by Laplace for use in his study of the Laplace equation. Thus the use of various types of harmonic analysis in the study of the three main types of classical partial differential equations, in particular the use of orthogonal expansions, goes back to the beginning of the subject. For more information on this, see the beautiful book by Kline [22] on the history of mathematics.

More recent tools include singular integral operators (pseudodifferential operators) to treat elliptic equations, and also parametrices to study the heat equation, and most recently Fourier integral operators, which apply to the study of hyperbolic equations, amongst other things.

These tools in turn have found application to the study of eigenfunction expansions. The use of parametrices for elliptic and parabolic equations in eigenvalue asymptotics is well known; see Agmon [2] or McKean and Singer [26]. Stein in [38] made nice use of the heat equation to develop a Littlewood-Paley theory on compact Lie groups, and Hörmander [16] used pseudodifferential operators to study the behavior of Riesz means of eigenfunction expansions. The theory of Fourier integral operators was used as a tool in the study of eigenvalue asymptotics by Hörmander in [17] in order to get a sharp estimate on the remainder term.

Here we discuss how Fourier integral operators can be used to get a systematic generalization of many topics in harmonic analysis on the torus (or $\mathbb{R}^n$) to arbitrary compact manifolds, on which an elliptic self-adjoint operator (or perhaps a family of commuting operators) is given. Our starting point is an analysis of functions of such self-adjoint operators, as defined by the spectral theorem. We use the Fourier inversion formula to define such operators, and Fourier integral operators enter quite naturally, giving one an explicit hold on the symbols of the resulting operators. In §2 we list some applications to problems in harmonic analysis. We refer to Chapter 12 of [41] for the details on most of these results. In §3 we apply the machinery to a classical problem in hyperbolic PDE, the scattering of waves by a sphere. In doing this, we turn the theory of hyperbolic equations upon itself, using the method of saparation of variables in a standard fashion to reduce the scattering problem to a problem in spherical harmonics, and applying results developed in §§1–2 to the compact manifold $S^1 \times S^2$, to solve this problem.

The basic class of pseudodifferential operators we deal with consists of operators defined by

$$p(x,D)u = \int p(x,\xi)e^{ix\cdot\xi}\hat{u}(\xi)\,d\xi,$$

where $\hat{u}(\xi)$ is the Fourier transform:

$$\hat{u}(\xi) = (2\pi)^{-n}\int u(y)e^{-iy\cdot\xi}\,dy,$$

and $p(x,\xi)$ belongs to some symbol class. For example, $S^m_{\rho,\delta}$, defined by Hörmander [18], consists of functions $p(x,\xi)$ such that

$$|D^\beta_x D^\alpha_\xi p(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}.$$

We denote by S^m the set of symbols $p(x,\xi)\in S^m_{1,0}$ having the asymptotic behavior

$$p(x,\xi)\sim p_m(x,\xi)+p_{m-1}(x,\xi)+\cdots, \quad |\xi|\to\infty,$$

where $p_{m-j}(x,\xi)$ is homogeneous of degree $m-j$ in ξ. If Σ is some symbol class, we write $OP\Sigma$ for the associated set of operators. Thus we have $OPS^m_{\rho,\delta}$, etc.

The Fourier integral operators we will use will be of the form

$$Ju(x) = \int p(x,\xi)e^{i\varphi(x,\xi)}\hat{u}(\xi)\,d\xi,$$

where $p(x,\xi)$ belongs to a symbol class, as above, and $\varphi(x,\xi)$ is a smooth real-valued function, homogeneous of degree 1 in ξ, and $\nabla_x\varphi\neq 0$ on the support of $p(x,\xi)$. We refer to the paper of Hörmander [18] for the theory of Fourier integral operators, or to the exposition of Duistermaat [10], or of Nirenberg [32], or to Chapter 8 of [41].

Addendum. This is a revision of the paper [T]. Part of the purpose was to present a version in TeX, which is more readable than the original, produced by an old fashioned typewriter. In the process, we have updated some references, and added some new references (appearing after the publication of [T]). We have also made some minor mathematical improvements, some of which we comment on in the text.

1. Functions of an elliptic self-adjoint operator

Let $A\in OPS^1$ be an elliptic self-adjoint operator on a compact Riemannian manifold M, with principal symbol $a_1(x,\xi)$, real valued. The main example is $\sqrt{-\Delta}$, where Δ is the Laplace operator on M. In this section we want to analyze $p(A)$ as a pseudodifferential operator, when $p(\lambda)$ belongs to various symbol classes. More generally, we are interested in $p(A_1,\dots,A_k)$ when $A_j\in OPS^1$ are commuting self-adjoint operators with the property that $A_1^2+\cdot+A_k^2$ is elliptic, a situation that involves no extra difficulty, so we mainly discuss the case $k = 1$. The operator $p(A)$ is defined by the spectral theorem.

Our point of departure is to use the Fourier inversion formula:

$$p(A)u = \int_{-\infty}^{\infty} \hat{p}(s)e^{isA}u\,ds. \tag{1.1}$$

The unitary operator e^{isA} is the solution operator to the hyperbolic equation

$$\frac{\partial v}{\partial s} = iAv, \tag{1.2}$$

so Fourier integral operators arise naturally as a tool to analyze (1.1).

Suppose $p(\lambda) \in S^m_{\rho,0}(\mathbb{R})$, for some $\rho > 0$. This implies that $\hat{p}(s)$ is C^∞ except at $s = 0$, and all its derivatives are rapidly decreasing as $|s| \to \infty$. If $\varphi_1 \in C_0^\infty((-\varepsilon, \varepsilon))$, $\varphi(s) = 1$ for $|s| \leq \varepsilon/2$, and $\varphi_2(s) = 1 - \varphi_1(s)$, we can write

$$p(A)u = \int_{-\infty}^{\infty} \hat{p}(s)\varphi_1(s)e^{isA}u\,ds + (iA)^{-N} \int_{-\infty}^{\infty} \frac{\partial^N}{\partial s^N}(\varphi_2\hat{p})e^{isA}\,ds. \tag{1.3}$$

Taking N large we see that the second term is smoothing, so it is only necessary to study the operator e^{isA} for $|s|$ small. In particular, since e^{isA} propagates singularities only by an amount $\leq C|s|$, we have:

Proposition 1.1. *If $p(\lambda) \in S^m_{\rho,0}(\mathbb{R})$, $\rho > 0$, then $p(A)$ is a microlocal operator, i.e., $WF(p(A)u) \subset WF(u)$.*

To study $p(A)$ more precisely, we apply the method of geometrical optics to e^{isA}. Write (for $|s|$ small)

$$e^{isA}u(x) = \int b(s,x,\xi)e^{i\varphi(s,x,\xi)}\hat{u}(\xi)\,d\xi, \tag{1.4}$$

where φ is homogeneous of degree 1 in ξ, $\varphi(0,x,\xi) = x \cdot \xi$, $b \sim \sum_{j \geq 0} b_j$, with each b_j homogeneous of degree $-j$ in ξ. We want $b(0,x,\xi) = 1$, so that at $s = 0$ (1.4) becomes the Fourier inversion formula. One is led to the eikonal equation

$$\frac{\partial \varphi}{\partial s} = a_1(x, \nabla_x \varphi),$$

and various transport equations for b_j, by considering the asymptotic expansion of $(\partial_s - iA)(be^{i\varphi})$. For details on this construction, see [10], [18], or Chapter 8 of [41]. Then (1.1) yields

$$\begin{aligned} p(A)u &= \iint \hat{p}(s)b(s,x,\xi)e^{i\varphi(s,x,\xi)}\hat{u}(\xi)\,d\xi\,ds \\ &= \int p(D_s)\Big(b(s,x,\xi)e^{i\varphi(s,x,\xi)}\Big)\Big|_{s=0}\hat{u}(\xi)\,d\xi. \end{aligned} \tag{1.5}$$

The fundamental asymptotic expansion lemma for pseudodifferential operators yields

$$p(D_s)(be^{i\varphi}) = c(s,x,\xi)e^{i\varphi}, \quad c(s,x,\xi) \in S^m_{\rho,1-\rho}, \tag{1.6}$$

provided $\rho > 1/2$. In such a case,

$$c(s,x,\xi) = b(s,x,\xi)p(\varphi_s) \mod S^{m-(2\rho-1)}_{\rho,1-\rho}. \tag{1.7}$$

Indeed, one can read off a complete asymptotic expansion of $c(s,x,\xi)$ in this case. This is often proven via the stationary phase principle. For another approach, see [T3], Chapter 7, §7, Exercise 2. Recalling that $b(0,x,\xi) = 1$ and $\varphi_s(0,x,\xi) = a_1(x,\nabla_x\varphi(0,x,\xi)) = a_1(x,\xi)$, we see that

$$p(A)u(x) = \int c(0,x,\xi)e^{ix\cdot\xi}\hat{u}(\xi)\,d\xi \tag{1.8}$$

$$c(0,x,\xi) = p(a_1(x,\xi)) \mod S^{m-(2\rho-1)}_{\rho,1-\rho}. \tag{1.9}$$

We have proved the following result on functions of A:

Theorem 1.2. *If $p(\lambda) \in S^m_{\rho,0}$, $1/2 < \rho \le 1$, then $p(A) \in OPS^m_{\rho,1-\rho}$ and*

$$\sigma_{p(A)} = p(a_1(x,\xi)) \quad \textit{mod} \quad S^{m-(2\rho-1)}_{\rho,1-\rho}. \tag{1.10}$$

Special cases of this result include the following. Seeley [36] considered $p(\lambda) = \lambda^\sigma$, which belongs to $S^{\mathrm{Re}\,\sigma}_{1,0}$, and Strichartz [40] considered $p(\lambda) \in S^m_{1,0}$. Their derivations were quite different from that given above.

We mention that when $p(\lambda) \in S^m_{1/2,0}$, (1.6) continues to hold, with $c \in S^m_{1/2,1/2}$, but (1.7) becomes vacuous. Thus $p(A)$ is a pseudodifferential operator in $OPS^m_{1/2,1/2}$ in this case, but one loses the formula for its symbol.

For applications in §3, we are interested in understanding $p(A)$ for certain symbols $p(\lambda)$ of more degenerate type than are handled in Theorem 1.2. Here functions of several operators are of particular interest, so $p(\lambda) = p(\lambda_1,\dots,\lambda_k)$. One family of symbols is that obtained by imposing the following Marcinkiewicz condition.

Definition 1.3. *We say $q(x,\xi) \in \mathcal{M}^m_\rho$ if and only if*

$$\xi^\alpha D^\alpha_\xi q(x,\xi) \in S^m_{\rho,0},$$

for all $\alpha \ge 0$.

Typically ξ is related to λ by a linear change of coordinates. A consequence of the Marcinkiewicz multiplier theorem (Theorem 6′ in chapter 4 of [39]) is that

$$q(x,\xi) \in \mathcal{M}^m_\rho \Longrightarrow q(x,D) : H^{s,p} \to H^{s-m,p},$$

for $p \in (1,\infty)$, where $H^{s,p}$ denote L^p-Sobolev spaces.

A notable subclass of $\mathcal{M}^m_\rho$ is described as follows. Let Σ be a linear subspace of $\mathbb{R}^k$ given by $\{\eta = 0\}$, in (ξ,η) coordinates

Definition 1.4. *We say* $q(x, \xi, \eta) \in \mathcal{N}_\rho^m(\Sigma)$ *if and only if* $q \in S_{1,0}^m$ *off* $|\eta| < |\xi|$, *and, for* $|\eta| < |\xi|$,

$$|D_\eta^\gamma D_\xi^\alpha D_x^\beta q| \le C_{\alpha\beta\gamma} |\xi|^{m-|\alpha|} (|\xi|^\rho + |\eta|)^{-|\gamma|}.$$

It is easy to see that $\mathcal{N}_\rho^m(\Sigma) \subset \mathcal{M}_\rho^m$. Also $\mathcal{N}_\rho^m(\Sigma)$ depends only on Σ, not upon the linear coordinates chosen.

If $p(\lambda) \in \mathcal{N}_\rho^m(\Sigma)$, we can estimate the Sobolev $H^{\sigma,p}$-norm of $p(D_s)(be^{ik\varphi})$ for $\sigma > 0$ small, $p < \infty$ large, and deduce that

$$|p(D_s)(be^{ik\varphi})| = O(k^{m+\varepsilon}).$$

From this one can obtain the following replacement for (1.6), valid for $0 < \rho < 1/2$.

$$p(D_s)(be^{i\vartheta}) = c\, e^{i\vartheta}, \quad c \in S_{\rho,1-\rho}^{m+\varepsilon}. \tag{1.11}$$

Details on this are given in [41], Chapter 11. Plugging this result into the analysis leading to Theorem 1.2 yields the following.

Theorem 1.5. *If* $p(\lambda) \in \mathcal{N}_\rho^m(\Sigma)$, $0 < \rho < 1/2$, *then*

$$p(A_1, \dots, A_k) \in OPS_{\rho,1-\rho}^{m+\varepsilon}.$$

There are other contexts in which it is desirable to go beyond the boundaries of Theorem 1.2. Here is one. Pick $\sigma \in (0,1)$ and consider $e^{iA^\sigma} = f_\sigma(A)$ where $f_\sigma(\lambda) = e^{i\lambda^\sigma} \in S_{1-\sigma,0}^0(\mathbb{R})$. Theorem 1.2 implies that $e^{iA^\sigma} \in OPS_{1-\sigma,\sigma}^0$ for $0 < \sigma < 1/2$. Now we can express e^{itA^σ}, the solution operator to

$$\frac{\partial u}{\partial t} = iA^\sigma u,$$

as a Fourier integral operator with inhomogeneous phase function

$$e^{itA^\sigma} v(x) = \int b(t, x, \xi) e^{i\psi(t,x,\xi) + ix\cdot\xi} \hat{v}(\xi)\, d\xi, \tag{1.12}$$

where ψ solves the "eikonal equation"

$$\frac{\partial \psi}{\partial t} = a_1(x, \nabla_x \psi + \xi)^\sigma, \quad \psi(0, x, \xi) = 0, \tag{1.13}$$

and $b \in S_{1,0}^0$ is asymptotic to a sum of terms obtained by solving a sequence of transport equations. From (1.13) we see that $\psi \in S_{1,0}^\sigma$ and is real valued. It easily follows that $e^{i\psi} \in S_{1-\sigma,\sigma}^0$, for $0 < \sigma < 1$. Consequently, we see that

$$e^{itA^\sigma} = q_\sigma(t, x, D_x) \in OPS_{1-\sigma,\sigma}^0,$$

for $0 < \sigma < 1$, where $q(t,x,\xi) = be^{i\psi}$. However, if one wants to make use of this operator, it would perhaps be most convenient to stick to its representation (1.12) as a Fourier integral operator with inhomogeneous phase function.

A primary application of these results is to $A = \sqrt{-\Delta}$. There are several ways to show that

$$\sqrt{-\Delta} \in OPS^1.$$

One goes as follows. We can formally construct what should be the symbol of $\sqrt{-\Delta}$ using a simple iterative procedure. For example if $A_2(x,\xi)$ is the principal symbol of $-\Delta$, that of $\sqrt{-\Delta}$ should be $A(x,\xi)^{1/2}$. Having carried through the construction of such a symbol, we claim that if B is the associated pseudodifferential operator (arranged to be self-adjoint and positive), we have $B^2 + \Delta = R \in OPS^{-\infty}$, and then we can show that $B - \sqrt{-\Delta}$ is a smoothing operator, using

$$\begin{aligned} (\sqrt{-\Delta})^{-1} &= \frac{1}{2\pi i} \int\limits_{\gamma} \lambda^{-1/2} (\lambda + \Delta)^{-1} \, d\lambda \\ B^{-1} &= \frac{1}{2\pi i} \int\limits_{\gamma} \lambda^{-1/2} (\lambda + \Delta - R)^{-1} \, d\lambda, \end{aligned}$$

where λ is the obvious curve. There are other, arguably better ways to see that $\sqrt{-\Delta} \in OPS^1$. See Part V of §3 for another approach. For yet other approaches, see [T3], Chapter 7, Exercises at the end of §11 and of §12.

We close this section by indicating a connection between Theorem 1.2 and Egorov's Theorem, one of the fundamental results in the theory of Fourier integral operators, which states that if $P = p(x,D) \in OPS^m_{\rho,1-\rho}$, $1/2 < \rho \le 1$, and if J is an elliptic Fourier integral operator, then

$$JPJ^{-1} \in OPS^m_{\rho,1-\rho},$$

and, if $\mathcal{J}$ is the canonical transformation associated with J, then

$$q(x,\xi) = p(\mathcal{J}(x,\xi)) \mod S^{m-(2\rho-1)}_{\rho,1-\rho}.$$

In the case of functions of one operator A, pick a canonical transformation $\mathcal{J}$ such that, at least on a small conic subset Γ of $T^*M \setminus 0$, $a_1(\mathcal{J}(x,\xi)) = \xi_1$. Find a unitary FIOP J whose canonical transformation agrees with $\mathcal{J}$ on Γ. Thus JAJ^{-1} agrees with $(1/i)\partial/\partial x_1 = D_1$ modulo a smoothing operator, at least on distributions with wave front set in $\mathcal{J}(\Gamma)$. One is tempted to conclude that $p(A)$ agrees with $J^{-1}p(D_1)J$ modulo a smoothing operator, on distributions with wave front set in Γ. But $p(D_1)$ is manifestly a pseudodifferential operator, on $\mathcal{J}^{-1}(\Gamma)$, and Egorov's theorem applies to $J^{-1}p(D_1)J$. All this is true, and perhaps the easiest way to prove it is via (1.1), and one needs all the results on propagation of singularities for e^{isA} used above, except for the specific geometric optics formulae, whose role gets replaced by Egorov's theorem. In the case of functions of several commuting operators $A_1, \ldots, A_k$, the same considerations hold, provided the symbols $a_1(x,\xi), \ldots, a_k(x,\xi)$ have linearly independent gradients (also independent of the form $\xi \cdot dx$). This condition is not satisfied by all sets of operators of interest (the Casimir operators, for example), but it is satisfied by the pair of operators we consider in §3.

The analogue of (1.11) is

$$J\,OP\mathcal{M}^m_\rho\,J^{-1} \subset OPS^{m+\varepsilon}_{\rho,1-\rho}, \quad 0 < \rho < \frac{1}{2}.$$

2. Eigenfunction expansions and spectral theory

In this section we give a brief indication of how the results of §1 can be applied in a very simple was to obtain information about eigenfunction expansions. The asymptotic expansion for (1.6), truncated at some point and the error estimated crudely, gives excellent information on $p(A)$, and this can be applied to a *bounded* family of symbols $p_t(\lambda),\ 0 < t \le 1$. In particuler, if $p(\lambda) \in S^m_{1,0}(\mathbb{R})$ if fixed, $m \le 0$, and $p_t(\lambda) = p(t\lambda)$, then $\{p_t : 0 < t \le 1\}$ is bounded in $S^0_{1,0}(\mathbb{R})$. This simple class will serve most of our needs. We will consider five types of applications. There will be no room for proofs. Details can be found in Chapter 12 of [41].

I. Convergence of eigenfunction expansions

Suppose $p(\lambda) \in S^0_{1,0}(\mathbb{R}),\ p(0) = 1$. Then $p(tA)$ is a bounded subset of $OPS^0_{1,0}$, and so is a bounded set of operators on $L^p(M),\ 1 < p < \infty$, and also on $C^{k+\alpha}(M),\ 0 < \alpha < 1,\ k = 0, 1, 2, \ldots$. Clearly $p(tA)u \to u$ in $C^\infty(M)$ if $u \in C^\infty(M)$. Thus $p(tA)u \to u$ in $L^p(M)$ is $u \in L^p(M)$. Since $C^\infty(M)$ is not dense in $C^{k+\alpha}(M)$, we only get convergence in $C^{k+\alpha-\varepsilon}(M)$, which has an unsatisfactory flavor. A better result is the following.

Theorem 2.1. *Suppose $p(\lambda) \in S^{-\sigma}_{1,0}(\mathbb{R}),\ \sigma > 0$, and $p(0) = 1$. Let $u \in C^{k+\alpha}(M)$ and let $v(t,x) = p(tA)u(x)$. Then $v \in C^{k+\alpha}([0,1] \times M)$.*

This type of regularity is analogous to the Schauder estimates for solutions to elliptic boundary problems (see [3]). One also has estimates on the maximal function

$$M_p(u) = \sup_{0<t\le 1} |p(tA)u|.$$

Namely, if $p(\lambda) \in S^{-\sigma}_{1,0}(\mathbb{R}),\ \sigma > 0$, then $\|M_p(u)\|_{L^q(M)} \le C_q \|u\|_{L^q(M)},\ 1 < q < \infty$, and one has the appropriate weak type $(1,1)$ estimates. Pointwise convergence of $p(tA)u$ to u almost everywhere follows, provided $p(0) = 1$.

We remark that one can get by with conditions on only finitely many derivatives on $p(\lambda)$. In particular, results on Riesz means are obtained, but straightforward tightening of the methods of §1 seems to yield weaker results on Riesz means than those obtained by Hörmander [16].

Remark. Progress on such matters as Riesz means, made after the original version of this paper appeared, is covered in [Sog].

II. Approximation properties

Consider $p(\lambda) \in C_0^\infty(\mathbb{R})$, with $p(\lambda) = 1$ for $|\lambda| \le 1$. If $u \in C^\infty(M)$ belongs to V_η, the linear span of the eigenspaces of A with eigenvalue $\le \eta$ in absolute value, it follows that $u = p(tA)u$ for $t = 1/\eta$. Analyzing

$$p(tA)u(x) = \int\limits_M K_t(x, y)u(y)\, dV(y),$$

it is a simple matter to obtain

$$\|u\|_{C^k(M)} \le C\eta^k \|u\|_{L^\infty}, \quad \forall\, u \in V_\eta,$$

which generalizes Bernstein's inequality. From there, one can imitate the usual proof of Bernstein's theorem, as given in [4], for example.

Conversely, one can generalize Jackon's theorem, to obtian that if $u \in C^k(M)$, there exists $v \in V_\eta$ such that

$$\|u - v\|_{L^\infty} \le C\eta^{-k}\|u\|_{C^k}.$$

To do this, pick $p(\lambda) \in C_0^\infty(\mathbb{R})$, supported in $(-1, 1)$, with $p)(\lambda) = 1$ for $|\lambda| \le 1/2$, and let $v = v(\eta) = p(\eta^{-1}A)u$. Clearly $v \in V_\eta$. It remains to show that

$$\|u - p(\eta^{-1}A)u\|_{L^\infty} \le C\eta^{-k}\|u\|_{C^k}.$$

The proof of this is a little harder than the proof of Bernstein's inequality, but not much harder.

III. Compact Lie groups

If G is a compact Lie group of rank k, there is a natural basis of bi-invariant differential operators $C_1, \dots, C_k$ (with $C_1 = -\Delta$), of order $m_1, \dots, m_k$. Set $A_j = (-\Delta)^{1/2 - m_j/2} C_j \in OPS^1$. Using a well known theorem of Chevelley (cf. Zelobenko [47]) plus a theorem of Mather that allows one to write a smooth function invariant under the action of a compact group (in this case, the Weyl group) in terms of a smooth function of the invariant polynomials satisfying the conclusion of the Hilbert basis theorem, one obtains the following.

Theorem 2.2. *Let* $p(\lambda) \in S^m_{1,0}(\mathbb{R}^k)$ *be invariant under the Weyl group, and let* $P : \mathcal{D}'(G) \to \mathcal{D}'(G)$ *be defined by*

$$P\pi^{ij}_\lambda = p(\lambda+\delta)\pi^{ij}_\lambda.$$

Then $P \in OPS^m_{1,0}(G)$.

Here $\{\pi_\lambda\}$ is a complete set of irreducible unitary representations of G, indexed by λ belonging to a lattice in $\mathbb{R}^k$, intersected with a Weyl chamber. One can read off the principal symbol of P. Applications of this to some questions in group representation theory, such as the asymptotic behavior of multiplicities of weights and Clebsch-Gordon coefficients, are given in Cahn and Taylor [5].

Theorem 2.2 yields some of the L^p continuity theorems for multipliers on compact Lie groups obtained by Clerc [7], Weiss [46], and others, but not all of them. It also yields Hölder continuity results, which these authors did not consider. Some special cases of Theorem 2.2 were pointed out by Strichartz [40].

IV. Eigenvalue asymptotics

Obtaining information on the asymptotic behavior of the eigenvalues of A was the reason Hörmander wrote his paper [17], so we need hardly go into this topic here, except to say that, by analyzing the asymptotic behavior of $p(tA)$, $t \searrow 0$, with $p(\lambda) = 1$ for $|\lambda| \le 1$, $p(\lambda) = 0$ for $|\lambda| \ge 1+\varepsilon$, and letting $\varepsilon \searrow 0$, one obtains quickly information on the asymptotic behavior of the spectral function and eigenvalues of A, without need of any Tauberian theorems. Constructing a family $p_\eta(\lambda)$ of symbols that decrease from 1 to 0 as λ goes from η to with a bounded family in $S^0_{\rho,0}(\mathbb{R})$, and the operators $p_\eta(A)$ analyzed as elements of $OPS^0_{\rho,1-\rho}$, if $1/2 < \rho \le 1$. For $0 < \rho \le 1/2$, the analysis of the operators is more difficult, but the trace can still be evaluated, since one understands $\operatorname{Tr} e^{isA}$. One can even take $\rho = 0$, and if C is sufficiently large $\operatorname{Tr} p_\eta(A)$ can be obtained from a knowledge of $\operatorname{Tr} e^{isA}$ for $|s|$ small. This leads to Hörmander's estimate. If C is somewhat smaller, one needs to know $\operatorname{Tr} e^{isA}$ over a larger interval, and the closed orbits of H_{a_1} (the closed geodesics of M if $A = \sqrt{-\Delta}$) play a role. See Chazarain [6] and Duistermaat and Guillemin [11] for treatments of this phenomenon.

It would be nice to be able to analyze $p_\eta(A)$ for ρ *negative.* For example, in the lattice point problem, with $M = \mathbb{T}^2$, one gets good estimates with $\rho = -1/3$. If we were to use formula (1.1) in this analysis, it would be necessary to understand e^{isA} uniformly as $|s| \to \infty$. Thus, the wave equation seems to be the wrong tool for the job here. If one had a good parametrix for the solution operator e^{itA^2} to the Schrödinger equation

$$\frac{\partial v}{\partial t} = -A^2 v,$$

even for small $|t|$, one could write $p_\eta(A) = q_\eta(A^2)$ where $q_\eta(\lambda)$ dips from 1 to 0 as λ goes from η^2 to $\eta^2 + C\eta^{1+\rho}$ (i.e., from τ to $\tau + C\tau^{1/2+\rho/2}$, with $\tau = \eta^2$) and consequently there would be some chance of analyzing $\operatorname{Tr} p_\eta(A) = \operatorname{Tr} q_\eta(A^2)$, knowing

$\operatorname{Tr} e^{itA^2}$ for $|t|$ small. Since e^{itA^2} blows singularities all over the place immediately, constructing such a parametrix could not be reduced to a local problem, and it looks very difficult. Developments along these lines pose interesting problems for the future.

V. Non compact spaces

Although this topic falls outside the scope indicated by the title of this paper, it is hard not to mention that an analysis via (1.1) yields results for non-compact M, under certain circumstances, when A^2 is a differential operator, e.g., $A^2 = \tau^2 - \Delta,\ \tau \geq 0$. Let us assume $p(\lambda)$ is an *even* function on $\mathbb{R}$. We then write

$$p(A)u = \int_{-\infty}^{\infty} \hat{p}(t) \cos tA\, u\, dt.$$

The advantage of using this formula is that $v(t,x) = \cos tA\, u(x)$ solves the wave equation

$$\frac{\partial v}{\partial t^2} - (\Delta - \tau^2)v = 0, \quad v(0) = u,\ \partial_t v(0) = 0,$$

and finite propagation speed holds, as well as a parametrix construction for solutions to the wave equation.

REMARK. The formula above is slightly different from that appearing in the original version of this paper. The simple but useful change arose in work on [CGT].

In case M is a homogeneous space, the volume of balls of radius R is bounded by Ce^{KR}. If $p(\lambda)$ is holomorphic on the strip $\{\lambda \in \mathbb{C} : |\operatorname{Im} \lambda| \leq \gamma\}$ and $|D_\lambda^j p(\lambda)| \leq C_j(1+|\lambda|)^{-j}$ on this strip, then $p(\sqrt{\tau^2 - \Delta})$ has a kernel $K(x,y)$ that looks like the kernel of a pseudodifferential operator near the diagonal $x = y$, and that decays like $e^{-\sigma d(x,y)}$ as $d(x,y) \to \infty$, for each $\sigma < \gamma$. Hence, if $\gamma > K$, one gets $p(\sqrt{\tau^2 - \Delta}) : L^p(M) \to L^p(M)$, for $1 < p < \infty$. In the special case when M is a rank 1 symmetric space, such a result was obtained by Stanton and Tomas [37], by a different method. Also the case of symmetric spaces $M = G/K$ where G is a complex semisimple Lie group has been treated by Clerc and Stein [8]. These works can use a narrower complex strip than indicated above.

REMARK. L^p-boundedness results for manifolds with bounded geometry, using such narrower complex strips, were obtained in [T2].

3. Scattering of waves by a sphere

In this section we will examine a classical problem in scattering theory. Namely, given a distribution $f \in \mathcal{E}'(\mathbb{R} \times S^2)$, we want to find the outgoing solution to the wave equation, on $\mathbb{R}^3 \setminus \{x : |x| \le 1\}$, with boundary value f. Thus we look for u such that

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \quad |x| > 1, \tag{3.1}$$

$$u\big|_{|x|=1} = f, \tag{3.2}$$

$$u = 0 \;\text{ for }\; t << 0. \tag{3.3}$$

As is well known, there is a unique solution to (3.1)–(3.3). We search for an explicit parametrix that will, for example, enable one to read off the singularities of u from the singularities of f.

This problem was actively tackled in the early 1900s and led to some developments in special function theory. In particular Watson [45] introduced a variant of the Poisson summation formula, known as the Watson transform, in a effort ot solve the problem, but he had difficulty handling the remainder terms, and perhaps the first successful treatment was given by Nussensweig [33]. His treatment uses a great deal of special function theory, and is fairly long. Here we shall treat the problem within the framework of harmonic analysis on a compact manifold. The general problem of constructing a parametrix for solutions to the wave equation on the exterior of any smooth strictly convex obstacle, or more general equations in regions whose boundaries are convex with respect to the null bichacteristics, has been solved by Melrose [30] and Taylor [42], the latter author using work of Ludwig [25]. This construction is fairly complicated and we refer to [30], [41], [42] for further discussion.

If we take the partial Fourier transform with respect to t:

$$\begin{aligned} v(x,\lambda) &= \int_{-\infty}^{\infty} u(t,x)e^{i\lambda t}\,dt, \\ g(x,\lambda) &= \int_{-\infty}^{\infty} f(t,x)e^{i\lambda t}\,dt, \end{aligned} \tag{3.4}$$

the system (3.1)–(3.3) becomes the reduced wave equation

$$(\Delta + \lambda^2)v = 0 \;\text{ for }\; |x| > 1, \tag{3.5}$$

$$v\big|_{|x|=1} = g(\lambda), \tag{3.6}$$

and (3.3) translates into the "Sommerfeld radiation condition"

$$v = o(r^{-1}), \quad \frac{\partial v}{\partial r} - i|\lambda|v = o(r^{-1}), \quad \text{as } \; r \to \infty. \tag{3.7}$$

We will use the method of separation of variables, so we write

$$\Delta v = \frac{1}{r^2}\Big(r^2\frac{\partial^2}{\partial r^2} + 2r\frac{\partial}{\partial r} + \Delta_S\Big)v,$$

where Δ_S is the Laplace operator on S^2 and $r = |x|$. Then (3.5) becomes

$$r^2 \frac{\partial^2 v}{\partial r^2} + 2r \frac{\partial v}{\partial r} + (\lambda^2 r^2 + \Delta_S) v = 0. \tag{3.8}$$

This is easily converted into Bessel's equation, and the outgoing radiation condition (3.7) requires that we use the Hankel function $H^{(1)}$. We get

$$v(x,\lambda) = r^{-1/2} H^{(1)}_{(-\Delta_S + 1/4)^{1/2}}(|\lambda| r) h(\lambda), \tag{3.9}$$

where $h(\lambda)$ is specified by the boundary condition. In fact, setting $r = 1$ in (3.9) yields $v|_{|x|=1} = H_A^{(1)}(|\lambda|)h(\lambda)$, and so to obtain (3.6), we set

$$v(x,\lambda) = r^{-1/2} H_A^{(1)}(r|\lambda|) H_A^{(1)}(|\lambda|)^{-1} g(\lambda). \tag{3.10}$$

Here we have set

$$A = \Big(-\Delta_S + \frac{1}{4} \Big)^{1/2}. \tag{3.11}$$

Note that $H_\nu^{(1)}(\mu) \neq 0$ for $\nu, \mu > 0$.

We want to avoid the task of analyzing (3.10) directly, uniformly in λ, so we use a few basic facts about the solution to the wave equation to simplify the analysis. First, we recall that the solution to (3.1)–(3.3) is represented by the Green formula

$$u(t,x) = \iint\limits_{\mathbb{R}\times S^2} \Big(u(s,y) \frac{\partial G}{\partial n}(t-s, x-y) - \frac{\partial}{\partial n} u(s,y) G(t-s,x-y) \Big)\, ds\, dS(y), \tag{3.12}$$

where

$$G(t,x) = \frac{\delta(|x|-t)}{4\pi t} \quad \text{for} \quad t > 0, \quad 0 \quad \text{for} \quad t < 0,$$

is the fundamental solution to the wave equation. We are given that $u|_{\mathbb{R}\times S^2} = f$. If we can find $(\partial u/\partial n)|_{\mathbb{R}\times S^2}$, or a parametrix for this expression, the behavior of u can be read off from (3.12). Thus we need only analyze the *Neumann operator*

$$Nf = \frac{\partial u}{\partial n}\Big|_{|x|=1}. \tag{3.13}$$

From (3.10) we have

$$\frac{\partial}{\partial r} v(x,\lambda)\Big|_{|x|=1} = |\lambda| H_A^{(1)\prime}(|\lambda|) H_A^{(1)}(|\lambda|)^{-1} g(\lambda) - \frac{1}{2} g(\lambda). \tag{3.14}$$

Thus, passing to the inverse Fourier transform with respect to λ, we obtain a formula for $(\partial u/\partial n)|_{|x|=1}$, i.e., for the Neumann operator:

$$Nf = F(A, |D_t|) f - \frac{1}{2} f, \tag{3.15}$$

where

$$F(\nu,\mu)=\mu\frac{H_\nu^{(1)\prime}(\mu)}{H_\nu^{(1)}(\mu)}. \tag{3.16}$$

Before we consider to what symbol class F belongs, we need to overcome two minor problems in order to put (3.15) into the framework of §1.

(i) N is defined on distributions on $\mathbb{R}\times S^2$, which is not compact.
(ii) A and $|D_t|$ are not pseudodifferential operators on the Cartesian product space.

To get around (i), we employ the following device. Suppose the support of f is contained in a t-interval of length T_0. Add together all translates of $u(t,x)$ by integral multiples of $2T_0$:

$$u_1(t,x)=\sum_{k=-\infty}^{\infty}u(t+2kT_0,x). \tag{3.17}$$

This sum converges weakly because of local energy decay, which for $f\in\mathcal{E}'\cap H^1(\mathbb{R}\times S^2)$, says

$$\int_B\big(|u(t,x)|^2+|\nabla_x u(t,x)|^2+|u_t(t,x)|^2\big)\,dx\le ce^{-\alpha t},\quad t\nearrow\infty. \tag{3.18}$$

For more singular f one can deduce an exponential decay rate in a weaker topology. The estimate (3.18) is a nontrivial result; it was first proved by Morawetz (cf. [24]), with the obstacle $\{x:|x|\le 1\}$ replaced by a more general class of "star shaped" obstacles. However, the proof of such exponential decay is simpler than the solution to the diffraction problem (from which such energy decay can easily be deduced; see [28]). Now u_1 solves the wave equation, with boundary data $f_1=\sum_{-\infty}^{\infty}f(t+2kT_0,x)$, and the representation (3.12) holds. We can regard u_1 as a distribution on $(\mathbb{R}/2T_0\mathbb{Z})\times(\mathbb{R}^2\setminus\{x:|x|\le 1\})$. We can also regard f_1 and $\partial u_1/\partial n=Nf_1$ as distributions on $\mathbb{R}/2T_0\mathbb{Z})\times S^2$, where the last identity defines N. Then N is given by the same formula as (3.15), except now N operates on distributions on $M=S^1\times S^2$, with $S^1=\mathbb{R}/2T_0\mathbb{Z}$.

Problem (ii) arises because the symbol of A is singular on $\mathcal{N}_1$, the union of the normal bundles to $\{t=\text{const.}\}$, while the symbol of $|D_t|$ is singular on $\mathcal{N}_2$, the union of the normal bundles to $\{x=\text{const.}\}$. However, if $WF(f)\cap\mathcal{N}_j=\emptyset,\ j=1,2$, than A and $|D_t|$ act on f like pseudodifferential operators, and the analysis of §1 for $F(A,|D_t|)f$ goes through. For general $f\in\mathcal{D}'(M)$, write $f=f_1+f_2$ with $WF(f_1)\cap\mathcal{N}_j=\emptyset$, and $WF(f_2)$ contained in a small conic neighborhood of $\mathcal{N}_1\cup\mathcal{N}_2$. Now $\mathcal{N}_1\cup\mathcal{N}_2$ is bounded away from the variety $\Sigma_G\subset T^*M\setminus 0$ over which the tangential bicharacteristics of $\partial_t^2-\Delta$ pass, so the simplest constructions of geometrical optics yield a parametrix for the solution to the wave equation with boundary data f_2, and one easily analyzes Nf_2 as a classical pseudodifferential operator of order 1. Thus we restrict our attention to distributions f_1 with wave front set contained in a small conic neighborhood of Σ_G.

We return to the task of analyzing $F(\nu,\mu)$, and we see that it suffices to analyze this function, as a symbol, on a small conic neighborhood of $\{\nu=\mu, \mu,\nu>0\}$. The key to this analysis is the uniform asymptotic expansion of Bessel functions, for large order and argument, derived by Langer [23] and Olver [34]; see also [1], pp. 368-9. For our purposes, the relevant formulae are (as $\nu\to\infty$):

$$\begin{aligned} H^{(1)}(\nu z) \sim 2e^{-\pi i/3}\Big(\frac{4\zeta}{1-z^2}\Big)^{1/4}\Big(&\nu^{-1/3}A(\nu^{2/3}\zeta)\sum_{k\geq 0} a_k(\zeta)\nu^{-2k} \\ &+\nu^{-5/3}A'(\nu^{2/3}\zeta)\sum_{k\geq 0} b_k(\zeta)\nu^{-2k}\Big), \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} H_\nu^{(1)\prime}(\nu z) \sim \frac{4}{z}e^{2\pi i/3}\Big(\frac{1-z^2}{4\zeta}\Big)^{1/4}\Big(&\nu^{-2/3}A'(\nu^{2/3}\zeta)\sum_{k\geq 0} d_k(\zeta)\nu^{-2k} \\ &+\nu^{-4/3}A(\nu^{2/3}\zeta)\sum_{k\geq 0} c_k(\zeta)\nu^{-2k}\Big). \end{aligned} \tag{3.20}$$

Here $A(s)=Ai(e^{2\pi i/3}s)$, where $Ai(s)$ is the Airy function, defined by

$$Ai(s)=\frac{1}{\pi}\int_{-\infty}^{\infty} e^{i(st-t^3/3)}\,dt.$$

The function $\zeta=\zeta(z)$ is defined by

$$\frac{2}{3}\zeta^{3/2}=\int_z^1\frac{\sqrt{1-t^2}}{t}\,dt=\log\frac{1+\sqrt{1-z^2}}{z}-\sqrt{1-z^2}.$$

We note that ζ is analytic in z, even at $z=1$, and $\zeta'(1)<0$. Also, at $z=1$, $(1-z^2)^{-1}\zeta=2^{-2/3}$. The expansions (3.19) and (3.20) are uniformly valid on compact subsets of the region $-\pi+\delta<\arg z<\pi-\delta$, and in particular for real z close to 1. Furthermore, formal differentiation leads to correct asymptotic expansions in these cases. As for the Airy function $A(s)$, it has the following asymptotic behavior (cf. [13], or Appendix A of [MT2]):

$$\begin{aligned} A(s) &\sim s^{-1/4}(\alpha_0+\alpha_1 s^{-3/2}+\cdots)e^{-(2/3)is^{3/2}}, \quad s\to+\infty, \\ &\ s^{-1/4}(\beta_0+\beta_1 s^{-3/2}+\cdots)e^{(2/3)(-s)^{3/2}}, \quad s\to-\infty. \end{aligned} \tag{3.21}$$

$A'(s)$ has the asymptotic behavior obtained by formally differentiating (3.21), as do higher derivatives. In consequence, one easily verifies that

$$\frac{A'}{A}(\lambda) \in S^{1/2}(\mathbb{R}), \quad \frac{A}{A'}(\lambda) \in S^{-1/2}_{1,0}(\mathbb{R}), \tag{3.22}$$

granted that the denominators are nonvanishing for real λ. In fact, the only zeros of $Ai(s)$ or $Ai'(s)$ are real and negative. As for the coefficients in (3.19)–(3.20), suffice it to say that they are smooth as functions of z and $a_0(0)$ and $d_0(0)$ are nonvanishing. From these facts, we deduce that

$$\frac{H^{(1)}_\nu{}'(\nu z)}{H^{(1)}_\nu(\nu z)} \sim \alpha(z)\nu^{-1/3}\frac{A'}{A}(\nu^{2/3}\zeta)\frac{d(z,\nu) + \nu^{-2/3}(A/A')(\nu^{2/3}\zeta)c(z,\nu)}{a(z,\nu) + \nu^{-4/3}(A'/A)(\nu^{2/3}\zeta)b(z,\nu)}, \tag{3.23}$$

where $\alpha(z)$ is smooth, $\alpha(1) \neq 0$, $a(z,\nu) \sim \sum_{k\geq 0} a_k(\zeta)\nu^{-k}$, etc. Taking $\mu = z\nu$ in (3.22), one verifies that, in a conic neighborhood of $\mu = \nu$,

$$\frac{A'}{A}(\nu^{2/3}\zeta(\mu,\nu)) \in S^{1/3}_{1/3,0}, \quad \frac{A}{A'}(\nu^{2/3}\zeta(\mu,\nu)) \in S^{0}_{1/3,0}.$$

From (3.23) it follows that

$$F(\nu,\mu) \in S^1_{1/3,0} \tag{3.24}$$

on a conic neighborhood of $\mu = \nu > 0$.

This result is sufficient to guarantee that the Neumann operator is microlocal, i.e., $WF(Nf) \subset WF(f)$. Combined with (3.12) this solves the problem of specifying the singularities of u in terms of the singularities of f. But for many applications to scattering theory it is useful to have a more detailed description, which fortunately can be provided by the symbol class $\mathcal{N}^m_\rho$ defined in §1. In fact, (3.23) yields

$$F(\nu,\mu) \in \mathcal{N}^1_{1/3}(\Sigma), \quad \Sigma = \{\mu = \nu > 0\},$$

in a conic neighborhood of $\mu = \nu$. Furthermore, using a cutoff $\varphi_1(\nu,\mu) = \varphi(\nu^{-a}(\mu - \nu))$, with $1/2 < a < 1$, $\varphi \in C^\infty_0(\mathbb{R})$, $\varphi(\lambda) = 1$ for $|\lambda| \leq 1$, we can write

$$\begin{aligned} F(\nu,\mu) &= \varphi_1(\nu,\mu)F(\nu,\mu) + (1 - \varphi_1(\nu,\mu)F(\nu,\mu) \\ &= F_1(\nu,\mu) + F_0(\nu,\mu), \end{aligned}$$

with

$$F_0 \in S^1_{a,0}, \quad F_1 \in \mathcal{N}^{1/2+a/2}_{1/3}(\Sigma).$$

Consequently, writing

$$N = F_0(A, |D_t|) + F_1(A, |D_t|) - \frac{1}{2} = N_0 + N_1,$$

we have

$$N_0 \in OPS^1_{a,1-a}, \quad N_1 \in OPS^{1/2+a/2+\varepsilon}_{1/3,2/3},$$

and consequently

$$N \in OPS^1_{1/3,2/3}, \tag{3.25}$$

and we have some control over its symbol.

For another interpretation of N, choose a canonical transformation of $T^*(S^1 \times S^2) \setminus 0$, near some point $p_0 \in \Sigma_G$, to a conic open subset of $T^*\mathbb{R}^3 \setminus 0$, taking p_0 to $(0; 1, 1, 0)$ and taking the symbol of A to ξ_1 and the symbol of $|D_t|$ to ξ_2, and implement this by en elliptic Fourier integral operator J in such a fashion that, modulo a smoothing operator, we have

$$N = JF(D_1, D_2)J^{-1} - \frac{1}{2}. \tag{3.26}$$

Then (3.25) can be obtained from (3.26) by a variant of Egorov's theorem mentioned in §1. Here $(F(D_1, D_2)u)\hat{}(\xi) = F(\xi_1, \xi_2)\hat{u}(\xi)$, so $F(D_1, D_2)$ acts as an element of $OP\mathcal{N}^1_{1/3}$ on distributions with wave front set near $\{\xi_1 = \xi_2 > 0, \xi_3 = 0\} = S$. If one examines the behavior of $F(D_1, D_2)$ on analytic wave front sets, one discovers the operator is not analytically microlocal near S; the analytic wave front set gets smeared out in x-space, though not in ξ-space. Since the canonical transformation can be taken analytic and J can be taken to be a quantized contact transformation, in the sense of Sato et al. [35], one can read off the analytic wave front set of Nf from (3.26) and thus describe the "creeping waves" in the shadow region, so named by Keller [20]. Analyzing $F(D_1, D_2)$ in this context involves calculations similar to those done in [15], describing the propagation of the analytic wave front set in another example.

Remark. See [Sj] for general results on progation of analytic wave front sets.

If the obstacle $\{|x| \le 1\}$ is replaced by a more general smooth convex obstacle K, with positive curvature, the results of Melrose [30] and Taylor [42] imply that the Neumann operator is of the form

$$N = J(AQ + B)J^{-1},$$

where J is an elliptic Fourier integral operator, $A, B \in OPS^1_{1,0}$, with A elliptic, and $Q \in OP\mathcal{N}^0_{1/3}$ has symbol

$$q(x,\xi) = |\xi|^{-1/3}\frac{A'}{A}(|\xi|^{-1/3}\eta)$$

on $|\xi| < |\eta|$. Also the symbol of B vanishes on $\{\eta = 0\}$. The canonical transformation associated with J takes Σ_G to $\{\eta = 0\}$. From this we can verify (3.25) and get some control over the symbol of N. This analysis has enabled the author to obtain a uniform bound on the error in the Kirchhoff approximation, given by G. R. Kirchhoff in [21], which specifies that the solution to the reduced wave equation

$$(\Delta + \lambda^2)u = 0 \ \text{ on } \ \mathbb{R}^3 \setminus K, \quad u\big|_{\partial K} = e^{i\lambda x\cdot\omega},$$

has normal derivative given approximately by

$$\frac{\partial u}{\partial n}\Big|_{\partial K} \approx i\lambda |n\cdot\omega| e^{i\lambda x\cdot\omega}.$$

This is a useful rule for computations in scattering theory, whose justification had to await the solution to the grazing ray problem. In Chapter 10 of [41] it is shown that

$$\frac{\partial u}{\partial n}\Big|_{\partial K} = K(x,\lambda) e^{i\lambda x\cdot\omega},$$

with

$$|K(x,\lambda) - i\lambda|n\cdot\omega|| \le C\lambda^{3/4+\varepsilon}(1+\lambda^{1/6}|n\cdot\omega|)^{-9/2}.$$

We mention that if one wants to treat the Neumann boundary condition, where (3.2) is replaced by

$$\frac{\partial u}{\partial n}\Big|_{|x|=1} = f,$$

one studies N^{-1}, which by (3.15) reduces to a study of

$$F(\nu,\mu)^{-1} \in \mathcal{N}^{-2/3}_{1/3}(\Sigma).$$

Remark. Both for more details on the analysis described here on (3.1)–(3.3), and for a treatment of considerable advances of the study of more general obstacles, see [MT1] and [MT2].

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