Variations on Gel'fand's Inverse Boundary Problem

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We discuss the equivalence of two inverse problems. In both cases, one wants to find a compact manifold with boundary $\overline{M}$ and a Riemannian metric tensor on $\overline{M}$. Here are two types of data:

(A) We are given ∂M and the trace on $\partial M \times \partial M$ of the solution operator to the wave equation, with Neumann boundary condition. We may as well say that we are given the Schwartz kernel of $\cos t\sqrt{-\Delta_N}$ on $(t, x, y) \in \mathbb{R} \times \partial M \times \partial M$.

(B) We are given the spectrum of Δ_N, with multiplicities, and the trace $\varphi_j|_{\partial M}$ for an orthonormal basis of eigenfunctions of Δ_N on $\overline{M}$.

Let us start with the first class of data. Given such data, for each even $F \in \mathcal{S}(\mathbb{R})$ we have the trace on $\partial M \times \partial M$ of the integral kernel of

$$F(\sqrt{-\Delta_N}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{F}(t)\ \cos t\sqrt{-\Delta_N}\, dt. \tag{1}$$

We next claim that this knowledge uniquely determines the spectrum of Δ_N. To see this, fix $\lambda \in [0, \infty)$, pick $f \in C_0^\infty(-1, 1)$ such that $f(0) = 1$, set $G_k(\xi) = f(k(\xi - \sqrt{\lambda}))$, and $F_k(\xi) = G_k(\xi) + G_k(-\xi)$. (Divide by 2 if $\lambda = 0$.) Note that the orthogonal projection Q of $L^2(M)$ onto the λ-eigenspace V_λ of $-\Delta_N$ is given by

$$Q = \lim_{k\to\infty} F_k(\sqrt{-\Delta_N}).$$

In fact, $Q = F_k(\sqrt{-\Delta_N})$ for all sufficiently large k. Hence the data of type (A) uniquely determine the trace on $\partial M \times \partial M$ of the integral kernel $\Phi(x, y)$ of Q, say $\Psi = \Phi|_{\partial M\times\partial M}$. We claim that

$$\Psi = 0 \Longrightarrow \Phi = 0, \tag{2}$$

which in turn yields

$$\Psi = 0 \Longleftrightarrow \lambda \notin \operatorname{Spec}(-\Delta_N). \tag{3}$$

To prove (2), say $N = \dim V_\lambda$, and note that if $\{\varphi_1, \dots, \varphi_N\}$ is an orthonormal basis of V_λ, then

$$\Phi(x, y) = \sum_{j=1}^{N} \varphi_j(x)\varphi_j(y). \tag{4}$$

Thus (2) is a consequence of the following:

Lemma 1. *If $\{\varphi_1, \dots, \varphi_N\}$ is an orthonormal basis of $V_\lambda \subset L^2(\overline{M})$, then*

$$\sum_{k,\ell} c_{k\ell}\varphi_k(x)\varphi_\ell(y) = 0 \quad on \quad \partial M \times \partial M \Longrightarrow c_{k\ell} \equiv 0. \tag{5}$$

Proof. For each fixed y, the left side of (5) is a function $\varphi(x)$ satisfying

$$\Delta\varphi = -\lambda\varphi \quad \text{on} \quad M, \quad \varphi, \partial_\nu\varphi = 0 \quad \text{on} \quad \partial M. \tag{6}$$

By uniqueness in the Cauchy problem (UCP), $\varphi \equiv 0$ on M, so under the hypotheses of the lemma,

$$\sum_{k,\ell} c_{k\ell}\varphi_k(x)\varphi_\ell(y) = 0 \tag{7}$$

for each $x \in \overline{M},\ y \in \partial M$. A repeat of this argument gives (7) for all $x \in \overline{M}, y \in \overline{M}$. Since $\{\varphi_k(x)\varphi_\ell(y) : 1 \le k, \ell \le N\}$ is an orthonormal set in $L^2(M \times M)$, this implies $c_{k\ell} \equiv 0$.

To continue, let λ be an eigenvalue of $-\Delta_N$. Define Φ, Ψ, and N as above. So we are given $\Psi = \Phi|_{\partial M \times \partial M}$, and we know that

$$\Psi(x, y) = \sum_{j=1}^{N} \varphi_j(x)\varphi_j(y), \quad x, y \in \partial M, \tag{8}$$

though we do not yet have $\varphi_j|_{\partial M}$, nor do we have N. (We will obtain N shortly.) Note that Ψ is a real-valued, symmetric function on $\partial M \times \partial M$, the integral kernel of a finite-rank, self-adjoint operator P on $L^2(\partial M)$. Given such Ψ, one can find an orthonormal set $\{\psi_j : 1 \le j \le M\} \subset L^2(\partial M)$ such that

$$\Psi(x, y) = \sum_{j=1}^{M} \psi_j(x)\psi_j(y). \tag{9}$$

We emphasize that $\{\psi_j\}$ is recoverable (not uniquely, perhaps) from Ψ. It remains to relate $\{\psi_j\}$ to $\{\varphi_j\}$.

First note that $\{\varphi_j|_{\partial M} : 1 \le j \le N\}$ spans the range of P and that $\{\psi_j : 1 \le j \le M\}$ forms a basis of the range of P. Furthermore, by UCP, the map $\varphi \mapsto \varphi|_{\partial M}$ is injective on the space V_λ. Hence $M = N$ and there is an $N \times N$ matrix $A = (a_{jk})$ such that

$$\psi_j = \sum_{k=1}^{N} a_{jk}\,\varphi_k\big|_{\partial M}, \quad j = 1, \dots, N. \tag{10}$$

Consequently,

$$\begin{aligned} \Psi(x, y) &= \sum_{j,k,\ell} a_{jk}a_{j\ell}\varphi_k(x)\varphi_\ell(y) \\ &= \sum_{k,\ell} b_{k\ell}\,\varphi_k(x)\varphi_\ell(y), \end{aligned} \tag{11}$$

where $b_{k\ell} = \sum_j a_{jk} a_{j\ell}$, i.e., $B = (b_{k\ell}) = A^t A$.

Comparing (8) and (11) gives, in light of Lemma 1,

$$b_{k\ell} = \delta_{k\ell}, \quad \text{i.e.,} \quad A^t A = I. \tag{12}$$

In other words, (10) holds with $A \in O(N)$. It follows that if we extend ψ_j into $\overline{M}$ by

$$\psi_j(x) = \sum_{k=1}^{N} a_{jk} \varphi_k(x), \quad x \in \overline{M}, \tag{13}$$

then $\{\psi_1, \dots, \psi_N\}$ is an orthonormal basis of V_λ, as desired. Hence the data (A) yield data of type (B), as asserted.

Remark. The equivalence of (A) and (B) was stated in [AK2LT]. For a proof, [KKL] was cited. It seems that [KKL] contains an argument similar to that given above, though the treatment there does ramble on.

References

[AK2LT] M. Anderson, A. Katsuda, Y. Kurylev, M. Lassas, and M. Taylor, Boundary regularity for the Ricci equation, geometric convergence, and Gel'fand's inverse boundary problem, Invent. Math. 158 (2004), 261–321.

[KKL] A. Katchalov, Y. Kurylev, and M. Lassas, Inverse Boundary Spectral Problems, Chapman Hall/CRC Press, Boca Raton, 2001.