Serendipitous Fourier Inversion

MICHAEL TAYLOR

ABSTRACT. It is known that pointwise behavior of multidimensional spherical Fourier inversion is more complicated on the torus than on Euclidean space. Figures presented here give graphical evidence of a certain choppiness of partial sums of the Fourier series of some functions with simple singularities on the 2-torus $\mathbb{T}^2$. This choppiness is provably absent for the analogous partial Fourier inversion on Euclidean space $\mathbb{R}^2$. Curiously, this choppiness clears up for a discrete set of partial sums. We call this phenomenon *serendipitous Fourier inversion.* The purpose of this paper is to introduce this notion and to produce estimates that establish its existence.

1. Introduction

The investigation discussed here began with a numerical study of partial sums

$$S_N f(x) = \sum_{k_1^2+k_2^2\le N^2} \hat{f}(k_1,k_2)\, e^{i(k_1x_1+k_2x_2)} \tag{1.1}$$

of the Fourier series of a function f on $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z}^2)$, with

$$\hat{f}(k_1,k_2) = (2\pi)^{-2} \iint_{\mathbb{T}^2} f(x_1,x_2)\, e^{-i(k_1x_1+k_2x_2)}\, dx_1\, dx_2. \tag{1.2}$$

We studied graphically the nature of convergence $S_N f \to f$, with particular attention to two cases. The first is

$$f(x) = \chi_a(x), \tag{1.3}$$

the characteristic function of the disk D_a of radius a centered at 0. We take $a = 2.5$. The second is

$$f(x) = R_a(x), \tag{1.4}$$

the fundamental solution to the wave equation on $\mathbb{R}\times\mathbb{T}^2$, evaluated at $t = a$; we take $a = 1.5$. Given $a \in (0,\pi)$, there is the formula

$$\begin{aligned} R_a(x) = \frac{1}{2\pi}\,(a^2-|x|^2)^{-1/2}, &\quad x \in D_a, \\ 0, &\quad x \notin D_a. \end{aligned} \tag{1.5}$$

Note that χ_a is piecewise smooth with a simple jump across ∂D_a, while R_a blows up at ∂D_a. We have $R_a \in L^p(\mathbb{T}^2)$ for $1 \le p < 2$, but $R_a \notin L^2(\mathbb{T}^2)$.

In each of these cases we can implement (1.1) using exact formulas for $\hat{f}(k),\ k = (k_1, k_2)$. We have $\hat{\chi}_a(0) = a^2/4\pi$ and

$$\hat{\chi}_a(k) = \frac{a}{2\pi|k|}\, J_1(a|k|), \quad k \neq 0. \tag{1.6}$$

Furthermore $\hat{R}_a(0) = a/4\pi^2$ and

$$\hat{R}_a(k) = \frac{\sin a|k|}{4\pi^2|k|}, \quad k \neq 0. \tag{1.7}$$

A number of graphs of $S_N\chi_a$ and $S_N R_a$ are presented at the end of this paper, and many more are given in [T4].

Various qualitative properties of $S_N f$, which apply to $f = \chi_a$ and to $f = R_a$, have been established in recent papers, particularly [BC1], [T1], and [T3]. It is shown that $S_N\chi_a(x) \to \chi_a(x)$ as $N \to \infty$ at each point $x \in \mathbb{T}^2$, where we set $\chi_a = 1/2$ on ∂D_a. In a neighborhood of ∂D_a, there is a uniform analysis of the behavior of $S_N\chi_a(x)$ analogous to the analysis of the Gibbs phenomenon for one-dimensional Fourier series. It is also shown, in [T1], that $S_N\chi_a(x)$ converges to $\chi_a(x)$ more slowly at $x = 0$ than at other points $x \in \mathbb{T}^2 \setminus \partial D_a$. (See the comments below (2.8) for a more precise statement.) Graphical depictions of these phenomena are given in Figure 1.

The nature of $S_N R_a(x)$ is a bit more exotic than that of $S_N\chi_a(x)$. There is an analogue of the Gibbs phenomenon, analyzed in a more general context in [T3]. In this case, one has pointwise convergence $S_N R_a(x) \to R_a(x)$ for each $x \notin \partial D_a \cup 0$. However, $S_N R_a(0)$ has an oscillatory divergence, manifesting a two-dimensional variety of the Pinsky phenomenon, discussed in the context of a piecewise smooth function on $\mathbb{R}^n$, for $n \ge 3$, in [P]. These behaviors are displayed in Figures 2A–2F.

One notable feature of these figures is the choppiness of the graphs of these functions $S_N f$, particularly apparent in Figures 2B–2E, and to a smaller extent in the 2nd–5th graphs in Figure 1. This choppiness is provably absent for the analogous partial Fourier inversion $\mathcal{S}_N f$ on $\mathbb{R}^2$, as we will discuss further in the next section. Also apparent in these figures, as part of the choppiness, is the obvious lack of rotational symmetry of these graphs, a symmetry that holds for $\mathcal{S}_N f$ in these cases but is broken when one passes from $\mathbb{R}^2$ to $\mathbb{T}^2$.

A further surprise, and the stimulus for this paper, is that for certain discrete values of N this choppiness magically clears up, and $S_N f$ behaves about as nicely on the torus as does $\mathcal{S}_N f$ on Euclidean space. This is illustrated in the first and last graphs in Figure 1 and, more strikingly, in Figures 2A and 2F. We call this phenomenon *serendipitous Fourier inversion*, and the main goal of this paper is to demonstrate its existence, for a natural class of functions, containing χ_a and R_a as special cases.

In §2 we recall some known results on partial Fourier inversion on $\mathbb{R}^2$ and on $\mathbb{T}^2$ (and in more general contexts), using a wave equation approach, as in [CV], [PT], and a number of subsequent papers. We show that $S_N f$ can be written as a sum $S_N^\beta f + T_N^\beta f$, and the behavior of $S_N^\beta f$ specified as precisely as that of $\mathcal{S}_N f$, when f is such a conormal distribution as χ_a or R_a. Known estimates on $T_N^\beta f$, while requiring substantial effort, have a cruder form, a fact that is consistent with the choppiness of $S_N f$ that we observe.

The Poisson summation formula is frequently effective in connecting Fourier analysis on $\mathbb{T}^n$ and $\mathbb{R}^n$. However, it is generally not so successful in the analysis of pointwise convergence. The simple reason for this is brought forth in §3. There the first key to the existence of serendipity in Fourier series on $\mathbb{T}^2$ arises as the disappearance of the obvious obstruction to applying this method to pointwise convergence. Once one sees what it is (cf. (3.4)), it is clear that this condition *might* lead to serendipity. However, the most familiar stationary phase estimates are not strong enough to show that this condition *does* lead to serendipity. Demonstrating this requires a new estimate on $\mathcal{S}_N f(x)$, uniformly valid for large N and $|x|$, when f is a compactly supported conormal distribution (satisfying geometrically natural conditions). The primary such estimate is stated in Proposition 3.1, and it is shown how this leads to the main result of this paper, stated as Theorem 3.2. We then begin to prove Proposition 3.1, and reduce it to a technical estimate, given in Lemma 3.3.

This lemma is proven in §4. The analysis there also uses the wave equation method discussed in §2, and the demonstration involves a careful uniform analysis of the large space and large time behavior of the solution to the wave equation on $\mathbb{R} \times \mathbb{R}^2$ with compactly supported, conormal initial data. The estimates established in §4 complete the proof of Proposition 3.1 and hence of the main result, Theorem 3.2.

When the analysis of §4 is specialized to the function R_a, its special structure gives rise to explicit formulas that make the desired uniform estimates on $\mathcal{S}_N R_a(x)$ relatively elementary, compared to the general case. We give these formulas in Appendix A, which one might want to read as a warm-up before tackling §4.

Appendix B analyzes the oscillation of the spike in $S_N R_a$. While this is not part of serendipity, it is a noteworthy part of the behavior of the Fourier series of R_a, whose study initiated this work.

2. Fourier inversion on $\mathbb{R}^2$ and $\mathbb{T}^2$

The behavior of $S_N f$ can be compared and contrasted with that of the Euclidean partial Fourier inversion

$$\mathcal{S}_N f(x) = \iint\limits_{|\xi|^2 \le N^2} \hat{f}(\xi)\, e^{i\xi\cdot x}\, d\xi, \tag{2.1}$$

of the Fourier transform of a function f on $\mathbb{R}^2$, with

$$\hat{f}(\xi) = (2\pi)^{-2} \iint\limits_{\mathbb{R}^2} f(x)\, e^{-ix\cdot\xi}\, dx. \tag{2.2}$$

The Gibbs and Pinsky phenomena were studied in this context in [CV], [P], and [PT] (also with $\mathbb{R}^2$ replaced by $\mathbb{R}^n$ for $n \ge 2$). Some of these papers use a wave equation approach to the analysis of $\mathcal{S}_N$. Namely, one writes

$$\mathcal{S}_N f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin Nt}{t}\, u(t,x)\, dt, \tag{2.3}$$

where $u(t,x)$ solves

$$u_{tt} - \Delta u = 0, \quad u(0,x) = f(x),\ u_t(0,x) = 0. \tag{2.4}$$

The large $|t|$ behavior in (2.3) is controlled by the fact that, if f has compact support, then, for any compact $K \subset \mathbb{R}^2$, there exists $T_0 < \infty$ such that $u(t,x)$ is smooth for $|t| > T_0$, and has at least an algebraic rate of decay as $|t| \to \infty$.

This decay phenomenon holds more generally when $\mathbb{R}^2$ is replaced by $\mathbb{R}^n$, but it has no analogue when $\mathbb{R}^n$ is replaced by $\mathbb{T}^n$, and this makes the analysis of S_N more subtle than the analysis of $\mathcal{S}_N$. Approaches taken in [BC1], [BC2], and [T1]–[T3] involve setting $S_N = S_N^\beta + T_N^\beta$, where we pick an even function $\beta \in C_0^\infty(\mathbb{R})$, with $\operatorname{supp} \beta \subset [-A, A]$ and $\beta(t) = 1$ for $|t| \le A/2$, and set

$$S_N^\beta f(x) = \frac{1}{\pi} \int \frac{\sin Nt}{t}\, \beta(t)\, u(t,x)\, dt. \tag{2.5}$$

The analysis of $S_N^\beta f(x)$ involves techniques well developed in [PT]. Analogues of Gibbs phenomena and Pinsky phenomena are accounted for by (2.5), provided β is chosen appropriately. There remains the task of estimating $T_N^\beta f$. Results of [BC1] give

$$\|T_N^\beta \chi_a\|_{L^\infty} \longrightarrow 0, \quad \text{as} \quad N \to \infty. \tag{2.6}$$

In (5.31) of [T1] it is shown that, if $A/2 > a$,

$$\|T_N^\beta \chi_a\|_{L^\infty} = o(N^{-1/2}). \tag{2.7}$$

This combines with the following result, established in [PT]:

$$\begin{aligned} S_N^\beta \chi_a(x) - \chi_a(x) = {} & O(N^{-1}), \quad x \in \mathbb{T}^2 \setminus (\partial D_a \cup 0), \\ & O(N^{-1/2}), \quad x = 0. \end{aligned} \tag{2.8}$$

In fact, this oscillates with an amplitude proportional to $N^{-1/2}$ at 0. This yields the result mentioned in §1, that $S_N\chi_a(x)$ tends to $\chi_a(x)$ more slowly at $x=0$ than at $x\notin\partial D_a\cup 0$, though information about how much more slowly is less precise for $S_N\chi_a$ on $\mathbb{T}^2$ than for $\mathcal{S}_N\chi_a$ on $\mathbb{R}^2$.

Regarding the Fourier series of R_a, it follows from (5.22) of [T1] that, if $A/2>a$,

$$\|T_N^\beta R_a\|_{L^\infty}\longrightarrow 0,\quad \text{as}\ \ N\to\infty, \tag{2.9}$$

which reduces convergence results on $S_N R_a$ mentioned in §1 to results on $S_N^\beta R_a$, obtainable by wave equation techniques as described above.

To close this section, we describe more precise results that have been shown to hold for $S_N^\beta f$, when f is a conormal distribution, such as χ_a or R_a. (The definition of conormal distributions will be recalled in (3.7).) On a neighborhood $\mathfrak{U}$ of ∂D_a in $\mathbb{T}^2$ we have the following expansion, from (4.11) of [T3].

$$\begin{aligned}S_N^\beta f(x)-f(x)&=A_0(x)\bigl[N^bF_b(N\psi(x))-\psi(x)_+^{-b}\bigr]\\&\qquad+\sum_{j=1}^K N^{b-j}A_j(x)\Phi_{a-j}(N\psi(x))+J_K(x,N),\end{aligned}\tag{2.10}$$

with

$$|J_K(x,N)|\le C_K\,N^{b-K-1},\quad K\ge 0. \tag{2.11}$$

Here $b=0$ for $f=\chi_a,\ b=1/2$ for $R_a,\ A_j(x)$ are certain smooth coefficients, and, for $b<1$,

$$F_b(s)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin t}{t}\,(s-t)_+^{-b}\,dt, \tag{2.12}$$

which is absolutely convergent if $b\in(0,1)$ and defined as an oscillatory integral for $b\le 0$. Also we set

$$\Phi_{b-j}(s)=F_{b-j}(s)-s_+^{j-b}. \tag{2.13}$$

As shown in §3 of [T3], one has, for $b<1$,

$$F_b(s)\sim s_+^{-b}+B(b)\frac{\sin(s-\pi(1-b)/2)}{s}+O(s^{-2}),\quad |s|\to\infty. \tag{2.14}$$

When f is conormal with singularity along ∂D_a, the function $\psi(x)$ solves the eikonal equation $|d\psi|=1,\ \psi|_{\partial D_a}=0$, so in fact

$$\psi(x)=a-|x|. \tag{2.15}$$

Finally we mention that the results described in this section have been established in more general contexts, including higher-dimensional compact manifolds. We say no more about this here, but refer to the cited papers.

3. Poisson summation and serendipity

As mentioned in the introduction, we can connect partial Fourier inversion on the torus and on Euclidean space via the Poisson summation formula, which in this context (stated more generally for n dimensions) takes the form

$$S_N f(x) = \sum_{\nu \in \mathbb{Z}^n} \mathcal{S}_N f(x + 2\pi\nu), \tag{3.1}$$

with $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n$. Assume f has support in $(-\pi, \pi)^n$, which we identify with $\mathbb{T}^n$. We identify the left side of (3.1) with a periodic function on $\mathbb{R}^n$, and a priori we can say that (3.1) converges in $\mathcal{S}'(\mathbb{R}^n)$. But of course this is rather weak convergence, and we desire something stronger.

However, there is a simple obstruction to using (3.1) to obtain pointwise results on $\mathbb{T}^n$ when corresponding results on $\mathbb{R}^n$ are known. To see this, note that

$$\mathcal{S}_N f(x) = \int \chi_N(\xi) \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi \tag{3.2}$$

is the inverse Fourier transform of $\chi_N(\xi)\hat{f}(\xi)$, which in the current setting is piecewise smooth, generally with a jump across $\{\xi : |\xi| = N\}$. By the stationary phase method one has, as $|x| \to \infty$,

$$\begin{aligned} \mathcal{S}_N f(x) = |x|^{-(n+1)/2} \big[A_0(N,\omega) e^{iN|x|} &+ B_0(N,\omega) e^{-iN|x|} \big] \\ &+ O(|x|^{-(n+3)/2}), \qquad \omega = \frac{x}{|x|}. \end{aligned} \tag{3.3}$$

Of course this implies that $\mathcal{S}_N f(x)$ is not integrable if $A_0(N,\omega) \neq 0$ or $B_0(N,\omega) \neq 0$. And that is bad news.

However, suppose f has the following property:

$$|\xi| = N \Longrightarrow \hat{f}(\xi) = 0. \tag{3.4}$$

Then we have

$$\mathcal{S}_N f(x) = |x|^{-(n+3)/2} \big[A_1(N,\omega) e^{iN|x|} + B_1(N,\omega) e^{-iN|x|} \big] + O(|x|^{-(n+5)/2}). \tag{3.5}$$

In particular, when (3.4) holds,

$$|\mathcal{S}_N f(x)| \le C(N,f)\,(1+|x|)^{-(n+3)/2}. \tag{3.6}$$

If $n = 2$, this implies that the series (3.1) is absolutely and uniformly convergent, for each such value of N. (From here on we stick to the setting $n = 2$.) When the condition (3.4) holds, we call $S_N f$ a *serendipitous* partial Fourier inversion of f on $\mathbb{T}^2$.

Consider the example $f = R_a$. By (1.7), we see that (3.4) holds for $aN/\pi \in \mathbb{Z}^+$. In Figures 2A–2F we graph $S_N R_a(x, y)$ for a range of values of N, namely $N = (k + \delta)\pi/a$, with $k = 6$ and $\delta = 0, 0.2, \dots, 0.8, 1$. Recall that we are taking $a = 1.5$. Note the qualitatively different appearances of Figures 2A and 2F, in which $k + \delta$ is an integer, compared to Figures 2B–2E, in which $k + \delta$ is not an integer.

The phenomenon of serendipitous Fourier inversion also arises for $S_N \chi_a$, though its appearance is more subtle than for $S_N R_a$, since χ_a is more regular than R_a. From (1.6) we see that $S_N \chi_a$ is serendipitous whenever $J_1(aN) = 0$. Figure 1 shows graphs of $S_N \chi_a(x, y)$ as aN ranges between the zeros $r_1 \approx 41.6171$ and $r_2 \approx 44.7593$ of J_1. Here $a = 2.5$.

To formulate our main result, we bring in the following class of conormal distributions. Let $\overline{\Omega} \subset \mathbb{R}^2$ be a compact set with smooth boundary $\partial\Omega = \Sigma$. For $\mu \in \mathbb{R}$, set

$$I^\mu(\mathbb{R}^2, \Sigma) = \{P\chi_\Omega : P \in OPS^\mu(\mathbb{R}^2)\}, \tag{3.7}$$

where OPS^μ denotes the space of classical pseudodifferential operators of order μ.

We will assume $\overline{\Omega} \subset (-\pi, \pi)^2$, and that Σ has strictly positive curvature. Then let K be a compact set containing both $\overline{\Omega}$ and all caustics formed by the Lagrangian flow-out of the conormal bundle of Σ. (In many cases, we can take $K = \overline{\Omega}$.)

Our main technical estimate of this section is the following.

Proposition 3.1. *Assume $\overline{\Omega}$ has the properties stated above, and let*

$$f \in I^\mu(\mathbb{R}^2, \Sigma) \quad \text{have compact support.} \tag{3.8}$$

Assume f satisfies the serendipity condition (3.4) for a set $\mathcal{N}$ of positive real numbers $N_\ell \to +\infty$. Let $\mathcal{O}$ be an open neighborhood of K. Then

$$|\mathcal{S}_N f(x)| \le C|x|^{-5/2}\, N^{-1+\mu}, \quad N \in \mathcal{N},\ x \in \mathbb{R}^2 \setminus \mathcal{O}. \tag{3.9}$$

This result is not an immediate consequence of (3.5)–(3.6); we need further control of the role of N in such estimates. Arguments given below, and in the next section, will provide such control.

Given Proposition 3.1, we can readily establish the main result of this paper. To formulate this result, let $\mathcal{A}$ be the minimal subset of $\mathbb{Z}^2$ such that

$$\bigcup_{\nu \in \mathcal{A}} [-\pi, \pi]^2 + 2\pi\nu \quad \text{contains a neighborhood of } \ K. \tag{3.10}$$

Of course, $\mathcal{A}$ is a finite set. (Frequently $\mathcal{A} = \{0\}$.) Then we have:

Theorem 3.2. *Under the hypotheses of Proposition 3.1,*

$$\Big|S_N f(x) - \sum_{\nu\in\mathcal{A}} \mathcal{S}_N f(x+2\pi\nu)\Big| \le CN^{-1+\mu}, \quad N \in \mathcal{N}, \quad x \in [-\pi,\pi]^2. \tag{3.11}$$

Example 1. Take $f = R_a$, which belongs to $I^{1/2}(\mathbb{R}^2, \partial D_a)$. In this case we can take $K = D_a$ and $\mathcal{A} = \{0\}$. The estimate (3.9) specializes to

$$|\mathcal{S}_N R_a(x)| \le C_\varepsilon |x|^{-5/2} N^{-1/2}, \quad |x| \ge a+\varepsilon,\ N \in (\pi/a)\mathbb{Z}^+. \tag{3.12}$$

The special nature of R_a allows for a fairly straightforward demonstration of the estimate (3.12), which we give in Appendix A.

Example 2. Similarly, for χ_a, which belongs to $I^0(\mathbb{R}^2, \partial D_a)$, we have

$$|\mathcal{S}_N \chi_a(x)| \le C_\varepsilon |x|^{-5/2} N^{-1}, \quad |x| \ge a+\varepsilon,\ J_1(aN) = 0. \tag{3.13}$$

In contrast to (3.12), we do not have an elementary direct proof of (3.13). Note that the corollary

$$\|S_N \chi_a - \mathcal{S}_N \chi_a\|_{L^\infty(\mathbb{T}^2)} \le CN^{-1}, \quad J_1(aN) = 0, \tag{3.14}$$

is much more precise than the estimate (2.7), valid for general $N \in [1,\infty)$, and interfaces quite nicely with (2.8) (or rather its analogue, with $S_N^\beta \chi_a$ replaced by $\mathcal{S}_N \chi_a$).

The first step in the proof of Proposition 3.1 is to note that, when (3.4) holds, we can write $e^{ix\cdot\xi} = (ix_j)^{-1}\partial_{\xi_j} e^{ix\cdot\xi}$ and integrate by parts without a boundary term to get

$$\begin{aligned} \mathcal{S}_N f(x) &= -\frac{1}{ix_j} \int\limits_{|\xi|\le N} \partial_{\xi_j} \hat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi \\ &= \frac{1}{x_j}\, \mathcal{S}_N(x_j f)(x). \end{aligned} \tag{3.15}$$

Of course $f \in I^\mu(\mathbb{R}^2, \Sigma) \Rightarrow x_j f \in I^\mu(\mathbb{R}^2, \Sigma)$. Hence Proposition 3.1 is a consequence of the following result.

Lemma 3.3. *In the setting of Proposition 3.1, assume we have a compactly supported*

$$g \in I^\mu(\mathbb{R}^2, \Sigma). \tag{3.16}$$

Then

$$|\mathcal{S}_N g(x)| \le C|x|^{-3/2}\, N^{-1+\mu}, \quad x \in \mathbb{R}^2 \setminus \mathcal{O}, \quad N \ge 1. \tag{3.17}$$

The proof of this lemma will occupy §4.

4. Proof of Lemma 3.3

To prove Lemma 3.3, we will use the wave equation approach to $\mathcal{S}_N g$, parallel to (2.3), i.e.,

$$\mathcal{S}_N g(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin Nt}{t}\, u(t,x)\, dt, \tag{4.1}$$

where $u(t,x)$ solves

$$u_{tt} - \Delta u = 0, \quad u(0,x) = g(x), \quad u_t(0,x) = 0. \tag{4.2}$$

For convenience we assume $\mu < 1$ in (3.16), which is the interesting case, after all. In particular, we assume

$$g \in L^1(\mathbb{R}^2), \quad \operatorname{supp} g \subset B_A = \{x : |x| \le A\}. \tag{4.3}$$

Then

$$u(t,x) = \partial_t R(t,\cdot) * g(x), \tag{4.4}$$

where, for $t > 0$,

$$R(t,x) = \frac{1}{2\pi} (t^2 - |x|^2)_+^{-1/2}. \tag{4.5}$$

We will be able to deduce the desired estimate (3.17) from sufficiently detailed estimates on $u(t,x)$.

Note that, by finite propagation speed, $u(t,x)$ vanishes for $|x| > t + A$, if (4.3) holds and $t > 0$. We will separately estimate $u(t,x)$ in the regions

$$|x| - A \le t \le |x| + 2A + 1, \quad t \ge |x| + 2A + 1, \tag{4.6}$$

assuming also (in the former case at least) that $x \in \mathbb{R}^2 \setminus \mathcal{O}$, as specified in (3.17).

For the first round of estimates we note from (4.5) that, for $s > 0$,

$$\begin{aligned} t = |x| + s \Rightarrow t^2 - |x|^2 &= s(2|x| + s) \\ \Rightarrow |R(t,x)| &\le C s^{-1/2} (|x| + s)^{-1/2}, \quad \text{and} \\ |\partial_t^\ell R(t,x)| &\le C s^{-\ell-1/2} (|x| + s)^{-1/2}. \end{aligned} \tag{4.7}$$

We then see from (4.4) that, if (4.3) holds, then, for $s > 0$,

$$
\begin{aligned}
t = |x| + 2A + s \Rightarrow |u(t,x)| &\le C s^{-3/2} (|x| + s)^{-1/2} \|g\|_{L^1}, \quad \text{and} \\
\Bigl| \partial_t^\ell \frac{u(t,x)}{t} \Bigr| &\le C s^{-\ell-3/2} (|x| + s)^{-3/2} \|g\|_{L^1}.
\end{aligned}
\tag{4.8}
$$

These estimates can be applied to a piece of $\mathcal{S}_N g(x)$, defined as follows. Pick $\varphi \in C^\infty(\mathbb{R})$ having the property that

$$
\begin{aligned}
\varphi(t) = 0 \quad &\text{for} \quad t \le 0, \\
1 \quad &\text{for} \quad t \ge 1,
\end{aligned}
\tag{4.9}
$$

and set

$$
\mathcal{S}_N^b g(x) = \frac{2}{\pi} \int_0^\infty \sin Nt \, \frac{\varphi(t - |x| - 2A - 1)}{t} \, u(t,x) \, dt. \tag{4.10}
$$

Integration by parts gives

$$
\mathcal{S}_N^b g(x) = (-1)^\ell N^{-2\ell} \frac{2}{\pi} \int_0^\infty \sin Nt \, \partial_t^{2\ell} \Bigl(\frac{\varphi(t - |x| - 2A - 1)}{t} \, u(t,x) \Bigr) dt, \tag{4.11}
$$

and using (4.8) we have that, as long as (4.3) holds,

$$
|\mathcal{S}_N^b g(x)| \le C_\ell N^{-2\ell} \, \|g\|_{L^1} \, |x|^{-3/2}. \tag{4.12}
$$

Using finite propagation speed, we can write

$$
\mathcal{S}_N g(x) = \mathcal{S}_N^\# g(x) + \mathcal{S}_N^b g(x), \tag{4.13}
$$

where $\mathcal{S}_N^\# g(x)$ is given as follows. Pick $\psi \in C_0^\infty(\mathbb{R})$ such that

$$
\operatorname{supp} \psi \subset [-3A - 1, 1], \quad \varphi(t) + \psi(t) = 1 \quad \text{for} \quad t \ge -3A - 1. \tag{4.14}
$$

Then set

$$
\mathcal{S}_N^\# g(x) = \frac{2}{\pi} \int_0^\infty \sin Nt \, \frac{\psi(t - |x| - 2A - 1)}{t} \, u(t,x) \, dt. \tag{4.15}
$$

It remains to estimate $\mathcal{S}_N^\# g(x)$, which we will do after estimating $u(t,x)$ in the first region specified in (4.6).

For this estimate, we need the full force of (3.16), in addition to (4.3). We examine (4.4) from the point of view of mapping properties on spaces of Lagrangian distributions of an operator κ_g, defined by

$$
\kappa_g S(t,x) = S(t,\cdot) * g(x), \tag{4.16}
$$

where $S(t,x)$ is a Lagrangian distribution of the same class as $\partial_t R(t,x)$. From (4.5) we see that

$$R \in I^{1/2}(\mathbb{R}^3, \mathcal{C}), \tag{4.17}$$

where $\mathcal{C} = \{(t,x) : |t| = |x|\}$, and this space is defined similarly to (3.7), at least away from $(t,x) = (0,0)$, which is sufficient for our purpose here. More precisely, the factorization

$$R(t,x) = \frac{1}{2\pi}(t+|x|)^{-1/2}(t-|x|)_+^{-1/2} \tag{4.18}$$

gives

$$R = O(|x|^{-1/2}) \ \text{ in } \ I^{1/2}(\mathbb{R}^3, \mathcal{C}), \tag{4.19}$$

in the sense of measuring seminorms of R on any given ball of radius 1 and center (t,x). Similarly

$$\partial_t R = O(|x|^{-1/2}) \ \text{ in } \ I^{3/2}(\mathbb{R}^3, \mathcal{C}). \tag{4.20}$$

Recall we are assuming

$$g \in I^\mu(\mathbb{R}^2, \Sigma), \quad \operatorname{supp} g \subset B_A. \tag{4.21}$$

Now under our hypotheses on Σ we can say that, for T_0 sufficiently large that all caustics disappear, on $\Omega_0 = \{(t,x) : t \geq T_0\}$ we have

$$\kappa_g : I^{3/2}(\Omega_0, \mathcal{C}) \longrightarrow I^\mu(\Omega_0, M^+) + I^\mu(\Omega_0, M^-), \tag{4.22}$$

where M^+ and M^- are two smooth surfaces, characteristic for $\partial_t^2 - \Delta$, formed by the Lagrangian flow-out of the conormal bundle to Σ. The mapping property (4.22) is a consequence of the general calculus of Lagrangian distributions; cf. [H], Chapter 25 (but note that the definition of the order of a conormal distribution used here is shifted from that used in [H]). It follows from (4.20)–(4.22) that

$$u\big|_{\Omega_0} = \kappa_g \partial_t R\big|_{\Omega_0} = O(|x|^{-1/2}) \ \text{ in } \ I^\mu(\Omega_0, M^+) + I^\mu(\Omega_0, M^-). \tag{4.23}$$

Hence

$$t^{-1}\psi(t-|x|-2A-1)u = O(|x|^{-3/2}) \ \text{ in } \ I^\mu(\Omega_0, M^+) + I^\mu(\Omega_0, M^-). \tag{4.24}$$

It then follows directly from (4.15) and (4.24) that

$$|\mathcal{S}_N^{\#} g(x)| \leq C\, N^{-1+\mu}\, |x|^{-3/2}, \tag{4.25}$$

for $x \in \mathcal{O},\ N \ge 1$. The proof of Lemma 3.3 is complete.

A. Specific formulas for $\mathcal{S}_N R_a$

In case $f = R_a$, we can derive a rather precise formula for $\mathcal{S}_N f$, as follows. With $\Lambda = \sqrt{-\Delta}$ on $\mathbb{R}^2$, we have

$$\mathcal{S}_N R_a(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin Nt}{t} \cos t\Lambda \, \frac{\sin a\Lambda}{\Lambda} \, \delta(x)\, dt. \tag{A.1}$$

Using $(\cos t\Lambda)(\sin a\Lambda) = \frac{1}{2}\sin(a+t)\Lambda + \frac{1}{2}\sin(a-t)\Lambda$, we obtain

$$\mathcal{S}_N R_a(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin Nt}{t} \big[R_{a+t}(x) + R_{a-t}(x)\big]\, dt. \tag{A.2}$$

Note that the quantity in square brackets is twice the even part (with respect to t) of $R_{a-t}(x)$. A change of variable yields

$$\begin{aligned} \mathcal{S}_N R_a(x) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin N(a-t)}{a-t}\, R_t(x)\, dt \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(\sin aN)\cos Nt - (\cos aN)\sin Nt}{a-t}\, R_t(x)\, dt. \end{aligned} \tag{A.3}$$

As mentioned in §3, serendipity in this case is expected when N is an integral multiple of π/a. In fact, the implication of $aN/\pi \in \mathbb{Z}^+$ on (A.3) is clear:

$$aN/\pi \in \mathbb{Z} \Longleftrightarrow \sin aN = 0. \tag{A.4}$$

Note that $(\sin Nt)R_t(x)$ is an even function of t, and hence it integrates to 0 against $1/t$. Since $(a-t)^{-1} + t^{-1} = -a/t(t-a)$, we have

$$aN/\pi \in \mathbb{Z}^+ \Rightarrow \mathcal{S}_N R_a(x) = -\frac{a}{\pi} \cos aN \int_{-\infty}^{\infty} \frac{\sin Nt}{t(t-a)}\, R_t(x)\, dt. \tag{A.5}$$

Recalling the formula (1.5) and making another change of variable, we have

$$\begin{aligned} \mathcal{S}_N R_a(x) &= \frac{a \cos aN}{\pi^2} \int_{|x|}^{\infty} \frac{\sin Nt}{t(t-a)} (t^2 - |x|^2)^{-1/2}\, dt \\ &= \frac{a \cos aN}{|\pi x|^2} \int_1^{\infty} \frac{\sin N|x|s}{s(s - a/|x|)} (s^2-1)^{-1/2}\, ds, \end{aligned} \tag{A.6}$$

whenever $aN/\pi \in \mathbb{Z}^+$. Elementary analysis of the last integral yields

$$|\mathcal{S}_N R_a(x)| \le C_\varepsilon |x|^{-5/2} N^{-1/2}, \quad \text{for } \ |x| \ge a + \varepsilon, \quad N \ge 1, \tag{A.7}$$

as asserted in (3.12).

B. The spike in $S_N R_a$

Graphs of $S_N R_a(x)$ indicate that, at least in the large N limit, $S_N R_a(0)$ oscillates between 0 and $2R_a(0)$. We make a calculation showing this is exactly true for $\mathcal{S}_N R_a(0)$, hence, by (2.9), asymptotically true for $S_N R_a(0)$.

In fact, using (A.2) and the formula (1.5) for $R_a(x, y)$, valid for $a > 0$, and noting R_a is odd in a, we have

$$\mathcal{S}_N R_a(0) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{\sin Nt}{t} \Big(\frac{1}{a+t} + \frac{1}{a-t} \Big)\, dt. \tag{B.1}$$

We can replace the quantity in parentheses by $2/(a-t)$ since $\sin Nt/t$ is even in t. A partial fraction decomposition of $1/t(a-t)$ then yields

$$\begin{aligned} \mathcal{S}_N R_a(0) &= \frac{1}{2\pi^2 a} \int_{-\infty}^{\infty} \big(\sin Nt\big) \Big(\frac{1}{t} + \frac{1}{a-t} \Big)\, dt \\ &= \frac{1 - \cos aN}{2\pi a}, \end{aligned} \tag{B.2}$$

giving precisely the asserted behavior.

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