# Airy Operator Calculus

MICHAEL TAYLOR

ABSTRACT. We study two classes of operators we call Airy operators, which arise in the construction of parametrices for boundary problems with grazing and gliding rays. In the grazing case these operators are pseudodifferential operators. In the gliding case they are of a more singular character. We analyze the latter class via a family of Airy operator identities. The derivation of such identities has some points in common with one proof of Egorov's theorem.

This is a TeXed version of the paper [T].

## 1. Introduction

This paper describes some joint work of the author and R. Melrose, which is more fully expounded in [8].

Airy operators arise in constructing parametrices for boundary problems with grazing and gliding rays. The basic Airy operators, $\Phi$ and $\Phi i$, are the following Fourier multipliers:

$$\widehat{\Phi u}(\xi) = \Phi(\zeta_0)\hat{u}(\xi), \quad \widehat{\Phi i\, u}(\xi) = \Phi i(\zeta_0)\hat{u}(\xi), \tag{1.1}$$

where

$$\Phi(\zeta_0) = \frac{A'}{A}(\zeta_0), \quad \Phi i(\zeta_0) = \frac{Ai'}{Ai}(\zeta_0), \tag{1.2}$$

and

$$\zeta_0 = \xi_1^{-1/3}(\xi_n + iT). \tag{1.3}$$

For $\Phi i$, we fix $T > 0$; for $\Phi$, we can take $T = 0$. Here, $Ai(\zeta)$ is the usual Airy function, the Fourier transform of $e^{is^3/3}$, and $A(\zeta) = A_\pm(\zeta) = Ai(e^{\pm 2\pi i/3}\zeta)$. The operator $\Phi$ appears in the analysis of grazing ray problems and is a pseudodifferential operator (microlocally near $\xi_n = 0,\ \xi_1 > 0$):

$$\Phi \in OPS^{1/3}_{1/3,0}. \tag{1.4}$$

The operator $\Phi i$ appears in the analysis of gliding ray problems and is not a pseudodifferential operator. In fact, it is a rather singular operator, a locally infinite sum of Fourier integral operators with singular phase. One thing $\Phi i$ has in common with $\Phi$ is its order on Sobolev spaces:

$$\Phi i : H^s \longrightarrow H^{s-1/3}. \tag{1.5}$$

This is a consequence of the estimate

$$|\Phi i(\zeta_0)| \le C_T \xi_1^{1/3}. \tag{1.6}$$

This estimate and other facts about $\Phi i(\zeta_0)$ we record here follow from elementary properties of the Airy function, and are proved in [8]. We have the following description of how $\Phi i$ affects the singularity of a distribution to which it is applied.

**Proposition 1.1.** *Let $u \in \mathcal{E}'(\mathbb{R}^n)$. Say $u = 0$ for $x_n < A$. Then $\Phi i\, u = 0$ for $x_n < A$. Also, the wave front set of $\Phi i\, u$ satisfies the following three conditions:*

$$\mathrm{WF}(\Phi i\, u) \cap \{\xi_n > 0\} \subset \mathrm{WF}(u), \tag{1.7}$$

$$\begin{aligned} \mathrm{WF}(\Phi i\, u) \cap \{\xi_n = 0\} \subset \{&x + (0,\dots,0,y),\xi) : \\ &(x,\xi) \in \mathrm{WF}(u),\ y \ge 0,\ \xi_n = 0\}, \end{aligned} \tag{1.8}$$

$$\mathrm{WF}(\Phi i\, u) \cap \{\xi_n < 0\} \subset \bigcup_{k=0}^{\infty} \mathcal{J}^k[\mathrm{WF}(u) \cap \{\xi_n < 0\}], \tag{1.9}$$

*where $\mathcal{J}$ is the singular canonical transformation*

$$\mathcal{J}(x,\xi) = \Big(x_1 + \frac{2}{3}\Big(-\frac{\xi_n}{\xi_1}\Big)^{3/2}, x_2, \dots, x_{n-1}, x_n + 2\Big(-\frac{\xi_n}{\xi_1}\Big)^{1/2}, \xi\Big). \tag{1.10}$$

This is proved in [8]; see also [1]. The inverse $\Phi i^{-1}$ also obeys Proposition 1.1, but whereas $\Phi^{-1}$ has order zero:

$$\Phi^{-1} \in OPS^0_{1/3,0}, \tag{1.11}$$

for $\Phi i^{-1}$ we only have the same behavior as (1.5):

$$\Phi i^{-1} : H^s \longrightarrow H^{s-1/3}. \tag{1.12}$$

This follows from the estimates

$$\begin{aligned} |\Phi i(\zeta_0)^{-1}| \le C_T \xi_1^{1/3} (1 + \xi_1^{-1/3}|\xi_n|)^{-1}, &\quad \text{for } \xi_n \le 0, \\ C_T (1 + \xi_1^{-1/3}|\xi_n|)^{-1/2}, &\quad \text{for } \xi_n \ge 0, \end{aligned} \tag{1.13}$$

proved in [8].

In the study of boundary problems, we are led to invert operators of the form

$$P\Phi + Q, \quad P\Phi i + Q, \tag{1.14}$$

with

$$P \in OPS^{m-1/3}, \quad Q \in OPS^m, \tag{1.15}$$

and certain generalizations. The inversion of $P\Phi + Q$, under certain natural hypotheses, can be carried out using the calculus of pseudodifferential operators, whereas other techniques are required to invert $P\Phi i + Q$. We will utilize certain Airy operator identities, derived in §4, to do this.

In §2 we will recall some facts about constructing parametrices for boundary problems in the case of grazing and gliding rays. Granted the material of this section, the following sections are self contained. Section 3 gives a brief discussion of Airy operators in the diffractive case, where they are pseudodifferential operators, and in §4 we get to the main new point, the development of a calculus of Airy operators in the gliding case.

## 2. Grazing and gliding ray parametrices

We look at parametrices for boundary problems

$$Pu = 0 \quad \text{on} \quad \Omega, \qquad Bu = f \quad \text{on} \quad \partial\Omega. \tag{2.1}$$

Recall that grazing rays are null bicharacteristics of $P$ that hit $\partial\Omega$ non-transversally, with exactly second order contact, remaining in $\overline{\Omega}$. If $\Omega \subset\subset \widetilde{\Omega}$ and $P$ has smooth coefficients on $\widetilde{\Omega}$, one has gliding rays on $\partial\Omega$ if $\widetilde{\Omega} \setminus \Omega$ has grazing rays. The basic form of the grazing ray parametrix is

$$u = \int [gA(\zeta) + ihA'(\zeta)]A(\zeta_0)^{-1} e^{i\theta} \hat{F}(\xi)\, d\xi, \tag{2.2}$$

and the gliding ray parametrix has the form

$$u = \int [g\, Ai(\zeta) + ih\, Ai'(\zeta)]A(\zeta_0) e^{i\theta} \hat{F}(\xi)\, d\xi. \tag{2.3}$$

Here, $(\zeta, \theta)$ is a pair of phase functions, obtained by solving certain eikonal equations, and $(g, h)$ is a pair of amplitudes, obtained by solving certain transport equations, $g \in S^0,\ h \in S^{-1/3}$. In the special case that $P$ is a second order scalar operator, with principal symbol $P_2(x, \xi)$, a quadratic form in $\xi$, the eikonal equations are of the form

$$\begin{aligned} \langle d\theta, d\theta\rangle + \zeta\langle d\zeta, d\zeta\rangle &= 0, \\ \langle d\theta, d\zeta\rangle &= 0, \end{aligned} \tag{2.4}$$

where $\langle \xi, \eta\rangle$ is the bilinear form polarizing $P_2$, $\langle \xi, \xi\rangle = P_2(x, \xi)$, and the transport equations are of the form

$$\begin{aligned} 2\langle d\theta, dg_\nu\rangle - 2\zeta\langle d\zeta, dh_\nu\rangle - \langle d\zeta, d\zeta\rangle h_\nu + (P^b\theta)g_\nu - \zeta(P^b\zeta)h_\nu &= iPg_{\nu-1}, \\ 2\langle d\zeta, dg_\nu\rangle - 2\langle d\theta, dh_\nu\rangle + (P^b\zeta)g_\nu - (P^b\theta)h_\nu &= -iPh_{\nu-1}, \end{aligned} \tag{2.5}$$

where $g \sim \sum_{\nu \geq 0} g_\nu$, etc. Here $P^b$ is obtained from $P$ by dropping the zero order part. (On $\xi_n \geq 0$, these equations are solved to infinite order on $\partial\Omega$; in $\xi_n < 0$ they are solved exactly.) See [8] or [11] for a discussion of the eikonal and transport equations in other contexts, e.g., for first order systems. In the gliding case, (2.3), we evaluate the amplitudes and phase phase functions in the complex domain, with $\xi_n$ replaced by $\xi_n + iT$, e.g.,

$$\zeta = \zeta(x, \xi_1, \dots, \xi_{n-1}, \xi_n + iT), \tag{2.6}$$

etc. One takes almost holomorphic extensions of these functions in the $\xi_n$ variable.

It is significant that one can arrange, on $\partial\Omega$,

$$\zeta\Big|_{\partial\Omega} = \zeta_0 = \xi_1^{-1/3}(\xi_n + iT). \tag{2.7}$$

One can also arrange

$$h\Big|_{\partial\Omega} = 0, \tag{2.8}$$

in the case of second order, scalar $P$.

The grazing ray parametrix was constructed by Melrose [3] and Taylor [9], and is also described in Chapter 10 of the book [10]. These constructions used cruder information, namely that one could arrange

$$\zeta\Big|_{\partial\Omega} - \zeta_0 = O\Big(\Big|\frac{\xi_n}{\xi_1}\Big|^\infty\Big), \quad h\Big|_{\partial\Omega} = O\Big(\Big|\frac{\xi_n}{\xi_1}\Big|^\infty\Big). \tag{2.9}$$

That one can arrange the stronger conditions (2.7) and (2.8) follows from the equivalence of glancing hypersurfaces [5], as shown in the unpublished manuscript [6] and also sketched in [11]; full details are given in [8], for both grazing and gliding ray parametrices. The gliding ray construction of [1] is similar to (2.3)–(2.9) in a number of respects, though different in detail.

The distribution $F$ in (2.2) or (2.3) is related to the boundary condition $Bu = f$ on $\partial\Omega$. For example, take the Dirichlet boundary condition $u|_{\partial\Omega} = f$; $P$ scalar and second order, say in the gliding case. From (2.3), (2.7), and (2.8), we obtain

$$\begin{aligned} u\Big|_{\partial\Omega} &= \int g_0 e^{i\theta_0} Ai(\zeta_0) A(\zeta_0) \hat{F}(\xi)\, d\xi \\ &= J(\mathcal{A}i\, \mathcal{A}\, F), \end{aligned} \tag{2.10}$$

where $\mathcal{A}i\, \mathcal{A}$ is the Fourier multiplier

$$\widehat{\mathcal{A}i\, \mathcal{A}\, F}(\xi) = Ai(\zeta_0) A(\zeta_0) \hat{F}(\xi), \tag{2.11}$$

and $J$ is an elliptic Fourier integral operator:

$$Jv = \int g_0 e^{i\theta_0} \hat{v}(\xi)\, d\xi; \tag{2.12}$$

here $\theta_0 = \theta|_{\partial\Omega}$ and $g_0 = g|_{\partial\Omega}$. In solving the transport equations, one can arrange $g_0$ nonvanishing, along with (2.8), and this makes $J$ elliptic. Thus, the Dirichlet problem is solved provided

$$F = (\mathcal{A}i\,\mathcal{A})^{-1}J^{-1}f. \tag{2.13}$$

Here, $J^{-1}$ is a Fourier integral operator giving a parametrix for $J$. The operator $(\mathcal{A}i\,\mathcal{A})^{-1}$ also obeys Proposition 1.1, as shown in [8].

In solving the transport equations, one is not forced to arrange (2.8). The following result shows what sort of freedom one has to choose convenient solutions to the transport equations.

**Proposition 2.1.** *For second order scalar $P$, one can obtain solutions $(g,h)$ to the transport equations so that, either*

$$h = \alpha g + \beta \quad \text{on} \quad \partial\Omega, \tag{2.14}$$

*with $\alpha \in S^{-1/3}$, $\beta \in S^{-1/3}$, given arbitrary, or*

$$g = g_0 \quad \text{on} \quad \partial\Omega, \tag{2.15}$$

*with $g_0 \in S^0$ given arbitrary. In case (2.14) we can also arange that $g|_{\partial\Omega}$ be elliptic.*

This result is proved in [8]. We remark that it is useful to allow $\alpha, \beta, g$, and $h$ to be $k \times k$ matrix valued, not necessarily scalar. In that case, we can generalize (2.14) to

$$h = \alpha_1 g + g\alpha_2 + \beta \quad \text{on} \quad \partial\Omega. \tag{2.14$'$}$$

If one does not arrange (2.8) but still wants to solve the Dirichlet problem, the computation (2.10) is changed to

$$u\Big|_{\partial\Omega} = \int [g_0 + ih_0\Phi i(\zeta_0)]e^{i\theta}Ai(\zeta_0)\hat{F}(\xi)\,d\xi, \tag{2.16}$$

where $h_0 = h|_{\partial\Omega}$. We get

$$u\Big|_{\partial\Omega} = J(I + R\Phi i)(\mathcal{A}i\,\mathcal{A}F), \tag{2.17}$$

where $R \in OPS^{-1/3}$ has principal symbol $ig_0^{-1}h_0$. Thus we must solve for $F$, via

$$F = (\mathcal{A}i\,\mathcal{A})^{-1}(I + R\Phi i)^{-1}J^{-1}f. \tag{2.18}$$

This is the first example of an Airy operator whose inversion requires some work. The following result is given in [8], (2.19) via a careful analysis of the formal expansion $I - R\Phi i + (R\Phi i)^2 - \cdots$, and (2.20) via some energy estimates.

**Proposition 2.2.** *Let $R \in OPS^{-1/3}$. Then, microlocally near $\xi_n = 0$, the Airy operator $I + R\Phi i$ has a parametrix $(I + R\Phi i)^{-1}$, satisfying the following conditions:*

$$u \in \mathcal{E}', \ u = 0 \text{ for } x_n < A \Longrightarrow (I + R\Phi i)^{-1} u \text{ is } C^\infty \text{ for } x_n < A, \tag{2.19}$$

*and*

$$\mathrm{WF}((I + R\Phi i)^{-1} u) \text{ satisfies conditions (1.7)–(1.9) of Proposition 1.1.} \tag{2.20}$$

The assertion (2.19) is a fairly straightforward consequence of a sufficiently detailed knowledge of the operator $\Phi i$. It is enough to allow one to construct a parametrix for the Dirichlet problem in the gliding case; one would exploit (2.20) to analyze the singularities of the solution. In §4 we will obtain very precise information on $(I + R\Phi i)^{-1}$, which will show that (2.20) is a consequence of Proposition 1.1. For this argument, we will need only (2.19).

Of course, we do not want to pick solutions $(g, h)$ of the transport equations, not satisfying (2.8), just to get a clumsy solution to the Dirichlet problem. Nor is this our only source of interest in studying parametrices for such operators as $I + R\Phi i$, or more generally $P\Phi i + Q$ as in (1.14). These operators naturally arise to be inverted in the study of various general classes of boundary problems, both for scalar second order $P$ and other sorts of operators, like first order systems. In [11] it is shown how boundary problems satisfying the Lopatinsky condition for strong well posedness, in the diffractive case, give rise to operators

$$P\Phi + Q, \quad Q \in OPS^{1/3} \text{ elliptic}, \ P \in OPS^0, \tag{2.21}$$

to invert. Note that, microlocally near $\xi_n = 0$, (2.21) is an elliptic operator in $OPS^{1/3}_{1/3,0}$, and hence $(P\Phi + Q)^{-1} \in OPS^{1/3}_{1/3,0}$ in this case. A finer analysis is given in §3. The same sort of calculations in the gliding case lead to the operators

$$P\Phi i + Q, \quad Q \in OPS^{1/3} \text{ elliptic}, \ P \in OPS^0, \tag{2.22}$$

to invert. Their inversion will be done in §4. Also, as shown in [11], in the diffractive case, for a class of boundary problems satisfying a variant of the Lopatinsky condition, analogous to the Neumann problem, one wants to invert $P\Phi + Q$ with

$$\begin{aligned} &Q \in OPS^{1/3} \text{ with vanishing principal symbol on } \xi_n = 0, \\ &P \in OPS^0 \text{ elliptic.} \end{aligned} \tag{2.23}$$

The same calculations in the gliding case lead one to invert $P\Phi i + Q$, under the conditions (2.23). We will also study this in §4.

## 3. Airy operators, the diffractive case

As we have stated, the operator $\Phi$ is a pseudodifferential operator in $OPS^{1/3}_{1/3,0}$. This follows from the fact that, as a function of one variable $\zeta$, in a strip about the real axis in $\mathbb{C}$, $A'(\zeta)/A(\zeta)$ satisfies the symbol estimates

$$\left|D^j \frac{A'}{A}(\zeta)\right| \leq C_j(1+|\zeta|)^{1/2-j}.$$

In fact, well known asymptotic expansions of $A(\zeta)$ and $A'(\zeta)$ imply

$$\frac{A'}{A}(\zeta) \sim \sum_{j\geq 0} a_j^{\pm} \zeta^{1/2-3j/2}, \quad \text{as} \quad \operatorname{Re}\zeta \to \pm\infty, \tag{3.1}$$

on a strip (even a pair of sectors) containing $\mathbb{R}$. It follows easily that, if

$$T = P\Phi + Q, \tag{3.2}$$

with

$$P \in OPS^{m-1/3}, \quad Q \in OPS^m, \quad Q \ \text{ elliptic,} \tag{3.3}$$

then, microlocally near $\xi_n = 0$,

$$T \in OPS^m_{1/3,0} \ \text{ is elliptic.} \tag{3.4}$$

The usual pseudodifferential calculus gives a parametrix

$$T^{-1} \in OPS^{-m}_{1/3,0} \tag{3.5}$$

in the case (3.2)–(3.3). Applying the usual symbol calculus provides an asymptotic expansion for the complete symbol of $T^{-1}$. A general term in such an expansion may involve products of a classical symbol and numerous factors of $\Phi(\zeta_0)$ and its derivatives $\Phi^{(i)}(\zeta_0)$. We will show that such an expression can be simplified considerably, using the differential equation

$$\Phi'(\zeta) = \Phi(\zeta)^2 - \zeta, \tag{3.6}$$

which follows from the Airy equation $A''(\zeta) - \zeta A(\zeta) = 0$. We will be brief in our discussion of this, as most of these results have been given in [4]; the discussion here contains a few simplifications.

We introduce the following class of operators. We say

$$T \in \mathcal{A}^{+,m} \tag{3.7}$$

provided $T$ is a pseudodifferential operator with symbol expansion (microlocally near $\xi_n = 0$)

$$T(x,\xi) \sim B + \sum_{j\geq 0} A_j(x,\xi)\xi_1^{-j/3}\Phi^{(j)}(\xi_1^{-1/3}\xi_n), \tag{3.8}$$

with

$$B \in S^m, \quad A_j \in S^{m-1/3}, \tag{3.9}$$

while $T \in OPS^m$ microlocally away from $\{\xi_n = 0\}$. Note that

$$\mathcal{A}^{+,m} \subset OPS^m_{1/3,0}. \tag{3.10}$$

We can represent elements of $\mathcal{A}^{+,m}$ by multiple symbols in a fashion which provides a characterization of $\mathcal{A}^{+,m}$ even simpler than (3.8)–(3.9).

**Proposition 3.1.** $\mathcal{A}^{+,m}$ *consists precisely of operators of the form*

$$Tu(x) = \int [a(x,y,\xi)\Phi(\xi_1^{-1/3}\xi_n) + b(x,y,\xi)]u(y)e^{i(x-y)\cdot\xi}\,dy\,d\xi, \tag{3.11}$$

*with*

$$a(x,y,\xi) \in S^{m-1/3}, \quad b(x,y,\xi) \in S^m. \tag{3.12}$$

*Proof.* Suppose $T$ has the form (3.8). Take a particular term

$$A_j(x,\xi)\xi_1^{-j/3}\Phi^{(j)}(\xi_1^{-1/3}\xi_n) = A_j(x,\xi)\partial^j_{\xi_n}\Phi = T_j^\#(x,\xi), \tag{3.13}$$

which belongs to $S^m_{1/3,0}$ if $j=0$, and to $S^{m-1/3-j/3}_{1/3,0}$ if $j \geq 1$. We see that

$$\begin{aligned} T_j^\# u &= \int A_j(x,\xi)\partial^j_{\xi_n}\Phi\, e^{i(x-y)\cdot\xi}u(y)\,dy\,d\xi \\ &= \int (-\partial_{\xi_n})^j[A_j(x,\xi)e^{i(x-y)\cdot\xi}]\Phi u(y)\,dy\,d\xi \\ &= (-1)^j \sum_{\ell=0}^{j} \binom{j}{\ell} \partial^{j-\ell}_{\xi_n} A_j(x,\xi)(x_n-y_n)^\ell \Phi e^{i(x-y)\cdot\xi}u(y)\,dy\,d\xi. \end{aligned} \tag{3.14}$$

Now in (3.11) we just have to choose $a(x,y,\xi)$ to have expansion at the diagonal $x=y$ dictated by (3.14). Note that we can take the leading term $a_0(x,y,\xi)$ of $a(x,y,\xi)$ to be

$$a_0(x,y,\xi) = A_0(x,\xi),$$

and we can set $b(x,y,\xi) = B(x,\xi)$. This shows that every operator in $\mathcal{A}^{+,m}$ is of the form (3.11). The converse statement follows from the standard method of obtaining the symbol of a pseudodifferential operator in $OPS^m_{1/3,0}$, from a multiple symbol representation such as (3.11), so the proposition is proved.

We now obtain a key result on the algebraic nature of $\mathcal{A}^{+,m}$.

**Proposition 3.2.** *If* $T_j \in \mathcal{A}^{+,m_j}$ *for* $j = 1, 2$, *then*

$$T_1 T_2 \in \mathcal{A}^{+,m_1+m_2}. \tag{3.15}$$

*Proof.* We have

$$T_1 T_1 \sim \sum A_j \Phi^{(j)} B_k \Phi^{(k)} + E \sim \sum C_{jk} \Phi^{(j)} \Phi^{(k)} + E, \tag{3.16}$$

with

$$E \in \mathcal{A}^{+,m_1+m_2}, \ A_j \in OPS^{m_1-j/3-1/3}, \ B_k \in OPS^{m_2-k/3-1/3},$$
$$C_{jk} \in OPS^{m_1+m_2-(j+k)/3-2/3}.$$

We want to show that $T_1 T_2$ can be asymptotically represented in the form

$$T_1 T_2 \sim \sum D_j \Phi^{(j)} + F, \quad D_j \in OPS^{m_1+m_2-1/3-j/3}, \ F \in OPS^{m_1+m_2}. \tag{3.17}$$

In order to achieve this, we use the identity (3.6), which implies (by induction)

$$\Phi^{(j-1)}(\zeta) = P_{j0}(\zeta)\Phi + \cdots + P_{jj}\Phi^j, \tag{3.18}$$

where $P_{jj}$ is a nonzero constant and more generally $P_{jk}(\zeta)$ is a polynomial belonging to the vector space $\mathcal{P}_{j-k}$, where $\mathcal{P}_\ell$ is the space of polynomials in $\zeta$ spanned by monomials $\zeta^\nu$, where

$$\nu \le \frac{\ell}{2}, \quad 2\nu \equiv \ell \ (\mathrm{mod}\ 3). \tag{3.19}$$

Thus we can invert the triangular system (3.18) to get

$$\Phi(\zeta)^j = r_{j0}(\zeta) + \sum_{k=1}^{j} r_{jk}(\zeta)\Phi^{(k-1)}, \quad r_{jk} \in \mathcal{P}_{j-k}. \tag{3.20}$$

Passing to (3.18) and back via (3.20) enables us to write

$$\Phi^{(j)}\Phi^{(k)} = \alpha_{0jk}(\zeta) + \sum_{\ell=1}^{j+k+2} \alpha_{\ell jk}(\zeta)\Phi^{(\ell-1)}, \tag{3.21}$$

where $\alpha_{jk}(\zeta)$ is a polynomial in $\zeta$ readily computible from (3.18), (3.20). We have

$$\alpha_{0jk}(\zeta) = \sum_{\lambda=0}^{j+k+2} \sum_{\mu+\nu=\lambda} r_{\lambda 0}(\zeta) P_{j+1,\mu}(\zeta) P_{k+1,\nu}(\zeta),$$

and, for $\ell \geq 1$,

$$\alpha_{\ell jk}(\zeta) = \sum_{\lambda=\ell}^{j+k+2} \sum_{\mu+\nu=\lambda} r_{\lambda\ell}(\zeta) P_{j+1,\mu}(\zeta) P_{k+1,\nu}(\zeta).$$

Note that, even though $\Phi^{(j)}\Phi^{(k)}$ is of order $1-j-k$ in $\zeta$, terms on the right side of (3.21) can have any order up to $(j+k+2)/2$ in $\zeta$. Particular examples of (3.21) are

$$\begin{aligned} \Phi\Phi = \zeta - \Phi', \quad \Phi\Phi' = \frac{1}{2} + \frac{1}{2}\Phi'', \quad \Phi'\Phi' = \frac{1}{6}(2\Phi - 4\zeta\Phi' + \Phi'''), \\ \Phi\Phi'' = \frac{1}{3}(-\Phi + 2\zeta\Phi' + \Phi'''). \end{aligned} \tag{3.22}$$

If we substitute (3.21) into (3.16), rearrangement produces a formal sum

$$\sum_{j,k} F_{jk}\Phi^{(j)} + E', \tag{3.23}$$

with

$$E' \in \mathcal{A}^{+,m_1+m_2}, \quad F_{jk} \in OPS^{m_1+m_2-1/3-j/3}.$$

However, for each $j$, there are infinitely many terms $F_{jk}$, and $\sum_k F_{jk}$ is not asymptotic in the usual sense. But as $k \to \infty$, terms homogeneous of a fixed degree vanish to increasingly high order at $\xi_n = 0$. Thus we can find $F_j \in OPS^{m_1+m_2-1/3-j/3}$ such that $\sum_{k=1}^{N} F_{jk} - F_j$ vanishes to arbitrarily high order at $\xi_n = 0$ for $N$ large. Now form

$$S \sim \sum_{j\geq 0} F_j\Phi^{(j)}. \tag{3.24}$$

We see that $S \in \mathcal{A}^{+,m_1+m_2}$ and

$$T_1T_2 - S \in OPS^{m_1+m_2}. \tag{3.25}$$

This proves the proposition.

It follows from (3.15) and the symbol calculus that

$$[T_1, T_2] \in \mathcal{A}^{+,m_1+m_2} \cap OPS^{m_1+m_2-2/3}_{1/3,0}. \tag{3.26}$$

The following gives a useful alternative characterization of this operator class.

**Proposition 3.3.** *An operator $T$ belongs to $\mathcal{A}^{+,m} \cap OPS^{m-2/3}_{1/3,0}$ if and only if its symbol has an expansion of the form (3.8) with*

$$B \in S^{m-1}, \quad A_0 \in S^{m-4/3}. \tag{3.27}$$

*Proof.* Let $b_0 \in S^m$ and $a_{00} \in S^{m-1/3}$ be the principal symbols of $B$ and $A_0$ in (3.8). Due to the asymptotic expansion

$$\Phi(\zeta) \sim c\sqrt{\zeta} + \cdots ,$$

we see that $T \in OPS^{m-2/3}_{1/3,0}$ implies the identity (on a conic neighborhood of $\xi_n = 0$)

$$b_0 = C\xi_1^{1/3} a_{00} \Big(\frac{\xi_n}{\xi_1}\Big)^{1/2}. \tag{3.28}$$

This identity implies that both $b_0$ and $a_{00}$ must vanish to infinite order at $\xi_n = 0$. This implies

$$B + A_0\Phi \in OPS^m, \tag{3.29}$$

so we can replace $B$ in (3.8) by $B + A_0\Phi$ and suppose without loss of generality that $a_{00} = 0$. Then (3.28) implies $b_0 = 0$, which proves our contention.

We now want to construct a parametrix for $I + A\Phi$, given $A \in OPS^{-1/3}$. We begin with the following simple result.

**Lemma 3.4.** *We have*

$$(I + A\Phi)(I - A\Phi) = B_1(I + C), \tag{3.30}$$

*with*

$$B_1 \in OPS^0 \ \textit{elliptic at}\ \xi_n = 0 \quad C \in \mathcal{A}^{+,0} \cap OPS^{-2/3}_{1/3,0}. \tag{3.31}$$

*Proof.* We have

$$(I + A\Phi)(I - A\Phi) = I - A\Phi A\Phi = I - A^2\Phi^2 + A[A,\Phi]\Phi. \tag{3.32}$$

Now by the identity (3.6) we have the right side of (3.32) equal to

$$I + A^2(D_1^{-1/3} D_n) - A^2\Phi' + A[A,\Phi]\Phi = (I + B_0) + C_0, \tag{3.33}$$

where

$$B_0 = A^2(D_1^{-1/3} D_N) \in OPS^0, \tag{3.34}$$

has symbol vanishing at $\xi_n = 0$ and

$$C_0 \in \mathcal{A}^{+,0} \cap OPS^{-2/3}_{1/3,0}, \tag{3.35}$$

by Proposition 3.2 and (3.26). Thus we have the lemma, with $B_1 = I + B_0$ and $C = B_1^{-1} C_0$.

We can now invert elliptic elements in $\mathcal{A}^{+,m}$.

**Proposition 3.5.** *If $T \in \mathcal{A}^{+,m}$ has the form (3.8) with $B \in OPS^m$ elliptic, then $T$ has a parametrix $T^{-1} \in \mathcal{A}^{+,-m}$, microlocally near $\xi_n = 0$.*

*Proof.* We know $T \in OPS^m_{1/3,0}$ is elliptic, so $T^{-1} \in OPS^{-m}_{1/3,0}$. To verify the proposition, factor out $B$ and write

$$T = B(I + A\Phi + R), \tag{3.36}$$

with

$$A \in OPS^{-1/3}, \quad R \in \mathcal{A}^{+,0} \cap OPS^{-2/3}_{1/3,0}. \tag{3.37}$$

By Lemma 3.4 we can write

$$(I + A\Phi + R)(I - A\Phi) = B_1(I + C), \tag{3.38}$$

with

$$B_1 \in OPS^0 \text{ elliptic}, \quad C \in \mathcal{A}^{+,0} \cap OPS^{-2/3}_{1/3,0},$$

and then the reasoning proving Proposition 3.2 shows that the Neumann series

$$(I + C)^{-1} \sim \sum_{k \geq 0} (-C)^k \tag{3.39}$$

produces an element

$$(I + C)^{-1} \in \mathcal{A}^{+,0}. \tag{3.40}$$

From here, the proposition follows from Proposition 3.2, since

$$T^{-1} = (I - A\Phi)(I + C)^{-1} B_1^{-1} B^{-1}. \tag{3.41}$$

As mentioned in §2, it is also important to invert operators of the form

$$T = P\Phi + Q, \tag{3.42}$$

with

$$\begin{aligned} &P \in OPS^m \text{ elliptic}, \\ &Q \in OPS^{m+1/3}_{1/3,0}, \text{ principal symbol vanishing on } \xi_n = 0. \end{aligned} \tag{3.43}$$

We can write

$$T = P\Phi(I + \Phi^{-1}P^{-1}Q) = P\Phi(I + \Phi^{-1}A),$$

with $A \in OPS^{1/3}$, having principal symbol vanishing on $\xi_n = 0$. Now since

$$\zeta\Phi(\zeta)^{-1} \sim \sum_{j\geq 0} b_j^{\pm}\zeta^{1/2-3j/2}, \quad \zeta \to \pm\infty,$$

we have $\Phi^{-1}A \in OPS^0_{1/3,0}$, and furthermore $I + \Phi^{-1}A$ is elliptic in $OPS^0_{1/3,0}$, microlocally near $\xi_n = 0$. Thus we obtain

$$T^{-1} = (I + \Phi^{-1}A)^{-1}\Phi^{-1}P^{-1} \in OPS^{-m}_{1/3,0}. \tag{3.44}$$

One can also produce a more precise class of Airy operators to which $T^{-1}$ belongs in this case. We refer to [4] for details on this. We note that the construction of a parametrix for $P\Phi i + Q$, under the hypothesis (3.43), which we will make in §4, has an analogue for (3.42). If $S \in OPS^{1/3}$ has principal symbol vanishing on $\xi_n = 0$, we have

$$\begin{aligned}(\Phi + S)^{-1} &= (I + R\Phi)^{-1}D(\Phi + C)^{-1}D_1 \\ &= (I + R\Phi)^{-1}D\Phi^{-1}(I + C\Phi^{-1})^{-1}D_1,\end{aligned} \tag{3.45}$$

where

$$R \in OPS^{-1/3}, \quad D \in OPS^0, \quad D_1 \in OPS^0, \tag{3.46}$$

and

$$C \in OPS^{-2/3}. \tag{3.47}$$

Compare (4.52). Proposition 3.5 gives $(I + R\Phi)^{-1} \in \mathcal{A}^{+,0}$. The term $(I + C\Phi^{-1})^{-1}$ is better behaved than the factor $(I + \Phi^{-1}A)^{-1}$ in (3.44). Indeed, (3.47) implies

$$C\Phi^{-1} \in OPS^{-2/3}_{1/3,0}, \tag{3.48}$$

so we have the Neumann series expansion

$$(I + C\Phi^{-1})^{-1} \sim \sum_{j\geq 0}(-C\Phi^{-1})^j \tag{3.49}$$

in this case.

## 4. Airy operators, the gliding case

Here we want to study operators of the form $P\Phi i+Q$, $P \in OPS^0$, $Q \in OPS^{1/3}$, and their inverses, and various generalizations. In case $Q$ is elliptic, we would hope to obtain inverses in $\mathcal{A}i^{+,-1/3}$, where, in analogy with (3.11), we set $\mathcal{A}i^{+,m}$ equal to the set of operators of the form

$$Cu(u) = \int [a(x,y,\xi)\Phi i(\zeta_0) + b(x,y,\xi)]u(y)e^{i(x-y)\cdot\xi}\,dy\,d\xi, \tag{4.1}$$

with

$$a(x,y,\xi) \in S^{m-1/3}, \quad b(x,y,\xi) \in S^m. \tag{4.2}$$

In fact, we have not obtained algebraic properties of $\mathcal{A}i^{+,m}$ analogous to those obtained for $\mathcal{A}^{+,m}$ in Proposition 3.2. However, we will obtain such properties for a subclass $\mathcal{A}i^{+,m}_{\sigma}$, which we will define shortly, and we will be fortunate to obtain parametrices for $P\Phi i+Q$ in this subclass, when $Q$ is elliptic. Let us note one simple property of $\mathcal{A}i^{+,m}$:

$$P \in OPS^{\mu},\ C \in \mathcal{A}i^{+,m} \Longrightarrow PC,\ CP \in \mathcal{A}i^{+,m+\mu}. \tag{4.3}$$

That $PC$ belongs to $\mathcal{A}i^{+,m+\mu}$ follows directly from (4.1), and one can analyze $CP$ as $(P^*C^*)^*$, noting that adjoints of elements of $\mathcal{A}i^{+,m}$ are precisely characterized by (4.1) with $\Phi i(\zeta_0)$ replaced by $\overline{\Phi i(\zeta_0)}$.

We define the class $\mathcal{A}i^{+,m}_{\sigma}$ to consist of all $T$ of the form

$$T \sim P + \sum_{j\ge 0} Q_j\,\Phi i\,R_j, \tag{4.4}$$

where

$$P \in OPS^m, \quad Q_j \in OPS^{\mu_j}, \quad R_j \in OPS^{\nu_j}, \quad \mu_j+\nu_j+\frac{1}{3} = m-\ell_j, \tag{4.5}$$

and

$$\ell_j \ge 0 \text{ is an integer, and } \ell_j \to \infty \text{ as } j \to \infty. \tag{4.6}$$

From (4.3) it is clear that

$$\mathcal{A}i^{+,m}_{\sigma} \subset \mathcal{A}i^{+,m}. \tag{4.7}$$

The following two results will be among the principal results of this section.

**Theorem 4.1.** *If* $T_j \in \mathcal{A}i_\sigma^{+,m_j}$*, then*

$$T_1 T_2 \in \mathcal{A}i_\sigma^{+,m_1+m_2}. \tag{4.8}$$

**Theorem 4.2.** *Let* $T \in \mathcal{A}i_\sigma^{+,m}$ *have the form (4.4) with* $P \in OPS^m$ *elliptic. Then, microlocally near* $\xi_n = 0$*,* $T$ *has a parametrix*

$$T^{-1} \in \mathcal{A}i_\sigma^{+,-m}. \tag{4.9}$$

A special case of Theorem 4.2 is that if $T = P + Q\Phi i$ with $P \in OPS^m$ elliptic, $Q \in OPS^{m-1/3}$, then, microlocally near $\xi_n = 0$, $T$ has a parametrix $T^{-1} \in \mathcal{A}i_\sigma^{+,-m}$. Actually, we will establish this result first, and deduce Theorem 4.2 from it, by a simple trick.

Now for any $\varepsilon > 0$, on $\xi_n \le -\varepsilon\xi_1 < 0$, the operator $\Phi i$ is locally a sum of a number $N(\varepsilon)$ of Fourier integral operators of order $1/3$, with disjoint canonical relations, and as $\varepsilon \to 0$, $N(\varepsilon)$ increases without bound. Thus the methods of §3 cannot work to analyze these Airy operators; the expansions used there would not be asymptotic in this situation. We will take an entirely different approach; we will deduce these theorems from certain identities among Airy operators.

The source of these Airy operator identities is the use of the gliding ray parametrix (2.3) for a scalar, second order operator $P$, with different choices of solutions $(g, h)$ to the transport equations. We will obtain representations of the Neumann operator (i.e., the Dirichlet-to-Neumann map) for different such choices, and since the Neumann operator is unique, comparing these different representations will yield Airy operator identities. The operator $P$ need have nothing to do with whatever boundary problem $(P_1, B_1)$ gave rise to the Airy operator (4.4) one wants to invert; typically $P_1$ need not be scalar and not necessarily second order. One may as well take for $P$ the model operator of Friedlander:

$$P = \partial_{n+1}^2 - x_{n+1}\partial_1^2 + \partial_1\partial_n, \quad \text{on} \ \ \{x : x_{n+1} \ge 0\}. \tag{4.10}$$

This simplifies the symplectic geometry, and one can immediately write out the phase functions $\theta, \zeta$ of (2.3) in this case:

$$\theta(x, \xi) = x' \cdot \xi, \quad \zeta(x, \xi) = \xi_1^{-1/3}(\xi_n + iT) - x_{n+1}\xi_1^{2/3}, \tag{4.11}$$

where $x' = (x_1, \dots, x_n)$.

The Neumann operator is defined as follows. For $f \in \mathcal{E}'(\partial\Omega)$, take the outgoing solution to the problem $Pu = 0$, $u|_{\partial\Omega} = f$, and set

$$Nf = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}. \tag{4.12}$$

Here, $\partial/\partial\nu$ is a vector field normal to $\partial\Omega$, with respect to the (Lorentz) metric on $\overline{\Omega}$ induced by the principal symbol of $P$. For (4.10), $\partial/\partial\nu = \partial/\partial x_{n+1}$ on $\partial\Omega$. For a solution $(g,h)$ of the transport equations, with $g \neq 0$ on $\partial\Omega$, $h=0$ on $\partial\Omega$, near $\xi_n = 0$, a straightforward calculation from (2.3) gives

$$N = J(A\Phi i + B)J^{-1}, \tag{4.13}$$

where $J$ is the zero order elliptic Fourier integral operator (2.12), and

$$A \in OPS^{2/3} \text{ has principal symbol } \zeta_\nu, \quad B \in OPS^0. \tag{4.14}$$

By construction, $\zeta_\nu \neq 0$ on $\partial\Omega$, near $\xi_n = 0, \xi_1 > 0$, so $A$ is elliptic; note that in case (4.11), we have $\zeta_\nu = \xi_1^{2/3}$ on $\partial\Omega$.

Now, for some other solution $(g', h')$ of the transport equations, with $g' \neq 0$, a similar calculation gives

$$N = JD(A'\Phi i + B')(I + R\Phi i)^{-1}D^{-1}J^{-1}, \tag{4.15}$$

where $J$ is the same Fourier integral operator as in (4.13). In analogy with (4.14), we have

$$A' \in OPS^{2/3} \text{ has principal symbol } \zeta_\nu, \tag{4.16}$$

and

$$B' \in OPS^1 \text{ has principal symbol } i(g')^{-1}h'\zeta_\nu\zeta \text{ on } \partial\Omega. \tag{4.17}$$

Furthermore,

$$D \in OPS^0 \text{ has principal symbol } g^{-1}g', \tag{4.18}$$

and

$$R \in OPS^{-1/3} \text{ has principal symbol } (g')^{-1}h'. \tag{4.19}$$

In the Friedlander case, we can take $g = 1$ on $\partial\Omega$ along with $h|_{\partial\Omega} = 0$; let us do this. If we compare (4.13) and (4.15), cancel the $J$s, and multiply through by $D(I + R\Phi i)$, we obtain

$$(A\Phi i + B)D(I + R\Phi i) = D(A'\Phi i + B'). \tag{4.20}$$

If we keep only the term quadratic in $\Phi i$ on the left, we can rewrite this identity as

$$\Phi i\, E\, \Phi i = (D + \delta_1)\Phi i - \Phi i\, D + C. \tag{4.21}$$

Here, $D \in OPS^0$ is as in (4.18), $\delta_1 \in OPS^{-1}$, and we have

$$E \in OPS^{-1/3}, \text{ principal symbol } h' \text{ on } \partial\Omega, \tag{4.22}$$

and

$$C \in OPS^{1/3}, \text{ principal symbol } ih'\zeta_0. \tag{4.23}$$

Now, if we make use of Proposition 2.1, we see that the principal symbol of $E$ can be specified arbitrarily on $\partial\Omega$, and one has the identity (4.21). Inductively, we can do the same with lower order terms in the expansion of any $E \in OPS^{-1/3}$. Since all Fourier multipliers commute with $\Phi i$, we have established the following.

**Proposition 4.3.** *Given any $E \in OPS^m$, there exist $D \in OPS^{m+1/3}$, $\delta_1 \in OPS^{m-2/3}$, and $C \in OPS^{m+2/3}$, such that the identity (4.21) holds.*

*Proof of Theorem 4.1.* Immediate from Proposition 4.3.

The following is an explicit special case of (4.21):

$$\Phi i^2 = D_1^{1/3} x_n \Phi i - \Phi i \, D_1^{1/3} x_n - D_1^{-1/3}(D_n + iT). \tag{4.24}$$

This special case is equivalent to the differential equation

$$\Phi i'(\zeta) = \Phi i(\zeta)^2 - \zeta, \tag{4.25}$$

analogous to (3.6).

We now prove the following special case of Theorem 4.2.

**Proposition 4.4.** *If $T = P + Q\Phi i$ with $P \in OPS^m$ elliptic and $Q \in OPS^{m-1/3}$, then, microlocally near $\xi_n = 0$, $T$ has a parametrix $T^{-1} \in \mathcal{A}i_\sigma^{+,-m}$.*

*Proof.* Factoring out $P$, we can suppose

$$T = I + S\Phi i, \quad S \in OPS^{-1/3}. \tag{4.26}$$

Now, using Proposition 2.1, let $(g', h')$ be a solution to the transport equations such that $h'(g')^{-1}$ is equal to the principal symbol of $S$ on $\partial\Omega$, $g'$ invertible. Let $D \in OPS^0$ have principal symbol $g'$, as in (4.18). Then, with $E = -SD$, we obtain the identity (4.21), with perhaps the lower order terms of $D$ altered. Consequently, we have

$$\begin{aligned} (I + S\Phi i)(I - SD\Phi i D^{-1}) &= I + S\Phi i - SD\Phi i D^{-1} - S\Phi i SD\Phi i D^{-1} \\ &= I + SCD^{-1} + S\delta_1 \Phi i D^{-1} \\ &= P_0(I + K), \end{aligned} \tag{4.27}$$

where

$$P_0 = I + SCD^{-1} \in OPS^0 \tag{4.28}$$

is elliptic at $\xi_n = 0$, by virtue of (4.23), and

$$K = P_0^{-1} S\delta_1 \, \Phi \, D^{-1} \in \mathcal{A}i_\sigma^{+,-1}. \tag{4.29}$$

Thus $K : H^s \to H^{s+1}$ and so the Neumann series is asymptotic. By virtue of Theorem 4.1,

$$(I + K)^{-1} \sim I - K + K^2 - \cdots \in \mathcal{A}i_\sigma^{+,0}. \tag{4.30}$$

Thus, microlocally near $\xi_n = 0$,

$$(I + S\Phi i)^{-1} = (I - SD\,\Phi i\,D^{-1})P_0^{-1}(I+K)^{-1} \in \mathcal{A}i_\sigma^{+,0}. \tag{4.31}$$

This proves the proposition.

In Proposition 4.5, $S$ could be a $k \times k$ matrix of operators. We can also invert

$$I + S_1\Phi i S_2, \tag{4.32}$$

with

$$S_j \in OPS^{m_j}, \quad m_1 + m_2 = -\frac{1}{3}, \tag{4.33}$$

and $A_1$ a $k \times \ell$ matrix and $S_2$ an $\ell \times k$ matrix. In fact, a simple calculation gives the parametrix

$$(I + S_1\Phi i S_2)^{-1} = I - S_1\Phi i(I + S_2 S_1 \Phi i)^{-1} S_2, \tag{4.34}$$

where, by Proposition 4.4, $(I + S_2 S_1 \Phi i)^{-1}$ exists in $\mathcal{A}i_\sigma^{+,0}$, microlocally near $\xi_n = 0$. By Theorem 4.1, we deduce that (4.34) belongs to $\mathcal{A}i_\sigma^{+,0}$.

We are now in a position to prove Theorem 4.2. If we factor out $P$, it suffices to invert

$$T_0 \sim I + \sum_{j \geq 0} Q_j \Phi i R_j, \tag{4.35}$$

with $Q_j \in OPS^{\mu_j}$, $R_j \in OPS^{\nu_j}$, $\mu_j + \nu_j = -1/3 - \ell_j$, where $\ell_j$ are nonnegative integers and $\ell_j \to \infty$ as $j \to \infty$. Suppose $\ell_j \geq 1$ for $j \geq K$, and that $Q_j$ and $R_j$ are all $k \times k$ matrices of operators. We can write

$$T_0 = I + S_1 \Phi i S_2 + \mathcal{K}, \tag{4.36}$$

where

$$S_2 : C^\infty(\partial\Omega, \mathbb{C}^k) \to C^\infty(\partial\Omega, \mathbb{C}^{kK}), \quad S_1 : C^\infty(\partial\Omega, \mathbb{C}^{kK}) \to C^\infty(\partial\Omega, \mathbb{C}^k) \tag{4.37}$$

are given by

$$S_2 u = (R_1 u, \dots, R_K u), \quad S_1(v_1, \dots, v_k) = Q_1 v_1 + \cdots + Q_K v_K, \tag{4.38}$$

and

$$\mathcal{K} \in \mathcal{A}i_\sigma^{+,-1}. \tag{4.39}$$

Now (4.34) gives $(I + S_1\Phi i S_2)^{-1} \in \mathcal{A}i^{+,0}$, and then

$$T_0(I + S_1\Phi i S_2)^{-1} = I + \mathcal{K}(I + S_1\Phi i S_2)^{-1} = I + \mathcal{K}_1, \tag{4.40}$$

with $\mathcal{K}_1 \in \mathcal{A}i_\sigma^{+,-1}$. Again the Neumann expansion gives a parametrix

$$(I + \mathcal{K}_1)^{-1} \sim I - \mathcal{K}_1 + \mathcal{K}_1^2 - \cdots \in \mathcal{A}i_\sigma^{+,0},$$

and so

$$T_0^{-1} = (I + S_1\Phi i S_2)^{-1}(I + \mathcal{K}_1)^{-1} \in \mathcal{A}i_\sigma^{+,0}. \tag{4.41}$$

The proof of Theorem 4.2 is complete.

It is useful to note the following result, complementary to Proposition 4.3. Rewrite (4.21) as

$$D\Phi i - \Phi i D = \Phi i E \Phi i - C - \delta_1 \Phi i. \tag{4.42}$$

Now, by Proposition 2.1, especially (2.15), we can take the principal symbol of $D$ to be arbitrary. By induction, we can let $D \in OPS^m$ be arbitrary, and there will exist $E \in OPS^{m-1/3}$, $C \in OPS^{m+1/3}$, and $\delta_1 \in OPS^{m-1}$ such that (4.42) holds. Another way we can write (4.42) is

$$\Phi i^{-1} D \Phi i = D + E\Phi i - \Phi i^{-1} C - \Phi i^{-1} \delta_1 \Phi i. \tag{4.43}$$

We can obtain such a formula for $\Phi i^{-1}\delta_1\Phi i$, and by an iterative process absorb the last term into the rest of the right side of (4.43), not affecting the principal symbols. Thus we have:

**Proposition 4.5.** *For any $D \in OPS^m$, there exist $E \in OPS^{m-1/3}$ and $C \in OPS^{m+1/3}$ such that*

$$\Phi i^{-1} D \Phi i = D_1 + E\Phi i - \Phi i^{-1} C, \tag{4.44}$$

*where $D_1$ has the same principal symbol as $D$. By (4.23), the principal symbol of $C$ vanishes on $\xi_n = 0$.*

This proposition leads to the following boundedness result.

**Proposition 4.6.** *For any $D \in OPS^0$, we have the continuous map on Sobolev spaces*

$$\Phi i^{-1} D \Phi i : H^s \longrightarrow H^s. \tag{4.45}$$

*Proof.* In light of (4.44) and (1.5), this is follows from the assertion that

$$\Phi i^{-1} C : H^s \longrightarrow H^s, \tag{4.46}$$

for any $C \in OPS^{1/3}$ whose principal symbol vanishes on $\xi_n = 0$. This in turn follows from a uniform bound on the Fourier multiplier:

$$|\langle \xi \rangle^{-1/3} \zeta_0 \Phi i(\zeta_0)^{-1}| \le M, \tag{4.47}$$

with $\zeta_0 = \xi_1^{-1/3}(\xi_n + iT)$. This follows from the estimate (1.13), which implies

$$\begin{aligned} |\Phi i(\zeta_0)^{-1}| \le C_T \langle \xi \rangle^{1/3} (1 + |\zeta_0|)^{-1}, &\quad \text{for } \ \xi_n \le 0, \\ C_T (1 + |\zeta_0|)^{-1/2}, &\quad \text{for } \ \xi_n \ge 0. \end{aligned} \tag{4.48}$$

This completes the proof.

Our next goal is to invert operators like $P\Phi i + Q$ under the hypothesis (2.23), or more generally

$$\begin{aligned} &Q \in OPS^m, \ \text{with vanishing principal symbol on } \xi_n = 0, \\ &P \in OPS^{m-1/3}, \ \text{elliptic.} \end{aligned} \tag{4.49}$$

Factoring out $P$, we may as well consider

$$\Phi i + S, \quad S \in OPS^{1/3} \ \text{ with vanishing principal symbol on } \xi_n = 0. \tag{4.50}$$

Now, utilizing Proposition 2.1, solve the transport equation so that

$$i(g')^{-1} h' = \zeta_{00}^{-1} \sigma(S), \quad \text{on } \ \partial\Omega, \tag{4.51}$$

where we set $\zeta_{00} = \xi_1^{-1/3}\xi_n$, i.e., the value at $T = 0$ of (1.3); make sure $g'$ is elliptic. In that case, the identity (4.20) holds with $B' = AS \bmod OPS^0$, which implies

$$(A')^{-1} D^{-1} A(\Phi i + A^{-1}B) D(I + R\Phi i) = \Phi i + S. \tag{4.52}$$

Thus inverting $\Phi i + S$ is reduced to inverting all the factors on the left side of (4.52). All the pseudodifferential factors are elliptic. The inversion of $I + R\Phi i$, in the operator class $\mathcal{A}i_\sigma^{+,0}$, was accomplished in Proposition 4.4. It remains to invert

$$\Phi i + A^{-1}B = (I + A^{-1}B\Phi i^{-1})\Phi i. \tag{4.53}$$

This is easy because, recall, $B \in OPS^0$, so $A^{-1}B \in OPS^{-2/3}$. It follows from (1.12) that

$$A^{-1} B \Phi i^{-1} : H^s \longrightarrow H^{s+1/3}, \tag{4.54}$$

for all $s$, so the expansion

$$(I + A^{-1}B\Phi i^{-1})^{-1} \sim I + \sum_{k \ge 1} (-A^{-1}B\Phi i^{-1})^k \tag{4.55}$$

is asymptotic and gives a parametrix. Thus we have

$$(\Phi i + A^{-1}B)^{-1} \sim \Phi i^{-1} \Big[ I + \sum_{k \ge 1} (-A^{-1}B\Phi i^{-1})^k \Big]. \tag{4.56}$$

From this and (4.52), we have the following conclusion.

**Proposition 4.7.** *Given* $\Phi i + S$ *as in (4.50),*

$$(\Phi i + S)^{-1} \sim \Big[I + \sum_{j\ge 0} R_j \Phi i R_j'\Big] D^{-1}\Phi i^{-1}\Big[I + \sum_{k\ge 1}(C\Phi i^{-1})^k\Big]E, \tag{4.57}$$

*with*

$$\begin{aligned} &R_j \in OPS^0, \quad R_j' \in OPS^{-1/3-\ell_j}, \quad \ell_j \in \mathbb{Z}^+, \ \ell_j \to \infty, \\ &D, \ E \in OPS^0, \quad \text{both elliptic}, \\ &C \in OPS^{-2/3}. \end{aligned} \tag{4.58}$$

While the microlocal behavior of the right side of (4.57) is clear, the expression itself is complicated. Some simplification is possible, via a variant of Proposition 4.5. It would be desirable to determine if further simplification is possible.

## A. Remark on Egorov's theorem

Our exploitation of different choices of solutions to transport equations in §4 is not the unique instance of exploiting such freedom to obtain operator identities. We will sketch here a proof of Egorov's theorem, based on this method; such a proof is a variant of a proof given in §3, Chapter 7, of [10].

Recall the method of geometrical optics applied to give a solution for small $t$ to the equation

$$\frac{\partial u}{\partial t} = i\lambda(t,x,D)u, \quad u(0) = f, \tag{A.1}$$

where $\lambda(t,x,D) \in OPS^1$ has real principal symbol. Call the solution operator $U(t)$, so $u(t) = U(t)f$. The method of geometrical optics writes, for small $|t|$,

$$U(t)f = \int a(t,x,\xi)e^{i\varphi(t,x,\xi)}\hat{f}(\xi)\,d\xi. \tag{A.2}$$

Here $\varphi$ solves the eikonal equation

$$\frac{\partial \varphi}{\partial t} = \lambda_1(t,x,\nabla_x\varphi), \tag{A.3}$$

where $\lambda_1$ is the principal symbol of $\lambda(t,x,D)$ and $a \sim \sum_{j\ge 0} a_j$, with $a_j \in S^{-j}$. We have the first transport equation for $a_0$:

$$\Big(\frac{\partial}{\partial t} - \sum \frac{\partial \lambda_1}{\partial \xi_j}\frac{\partial}{\partial x_j}\Big)a_0 - \Big(i\lambda_0 + \sum_{|\alpha|=2}\frac{1}{\alpha!}\lambda_1^{(\alpha)}\varphi_{(\alpha)}\Big)a_0 = 0, \tag{A.4}$$

and analogous higher transport equations for other $a_j$; see, e.g., Chapter 8 of [10]. A typical choice of initial conditions to take for the eikonal and transport equations is

$$\varphi(0,x,\xi) = x\cdot\xi, \quad a(0,x,\xi) = 1, \tag{A.5}$$

so at $t = 0$, (A.2) is just the Fourier inversion formula.

Egorov's theorem says that for each $p(x,D) \in OPS^m$ there is a $q(x,D) \in OPS^m$ such that (with $t$ fixed)

$$p(x,D)U(t) = U(t)q(x,D), \tag{A.6}$$

and the principal symbols of $p(x,D)$ and $q(x,D)$ are related by a canonical transformation. The most general form of Egorov's theorem replaces $U(t)$ by a general elliptic Fourier integral operator, but this can be deduced from the special case proved here by a few tricks, discussed in Chapter 8 of [10].

To evaluate the left side of (A.6), we can apply $p(x,D)$ under the integral sign in (A.2), utilizing the fundamental asymptotic lemma for pseudodifferential operators, which gives

$$p(x,D)\big(a(t,x,\xi)e^{i\varphi}\big) = b(t,x,\xi)e^{i\varphi}, \tag{A.7}$$

with

$$b\in S^m, \quad b_m(t,x,\xi) = a_0(t,x,\xi)p_m(x,\nabla_x\varphi). \tag{A.8}$$

Now to evaluate the right side of (A.6), we may write

$$U(t)q(x,D)f = \int c(t,x,\xi)e^{i\varphi(t,x,\xi)}\hat{f}(\xi)\,d\xi,$$

where the phase function $\varphi$ is as above, the amplitude $c(t,x,\xi)$ satisfies the transport equation (A.4) and analogous higher transport equations, but we choose the initial condition

$$c(0,x,\xi) = q(x,\xi) \tag{A.9}$$

on the amplitude. Comparing the process of obtaining $a(t,x,\xi)$ in (A.2), we see that

$$c_m(t,x,\xi) = a_0(t,x,\xi)q_m(\kappa_t(x,\xi),\xi), \tag{A.10}$$

where the point $\kappa_t(x,\xi)$ is defined by the property that $(\kappa_t(x,\xi),0)$ is on the same integral curve as $(x,t)$ for the ($\xi$-dependent) vector field $\partial/\partial t - \sum(\partial\lambda_1/\partial\xi_j)\partial/\partial x_j$. A few simple manipulations show that

$$\kappa_t(x,\xi) = \nabla_\xi\varphi(t,x,\xi). \tag{A.11}$$

(Compare the derivation of (3.30) in [10], p. 161.) Thus we have (A.6) to top order provided $q_m(x,\xi)$ is chosen so that

(A.12) $$q_m(\nabla_\xi \varphi, \xi) = p_m(x, \nabla_x \varphi).$$

The transformation

(A.13) $$(\nabla_\xi \varphi, \xi) \mapsto (x, \nabla_x \varphi)$$

is the canonical transformation associated with the Fourier integral operator (A.2) at a given $t$. With (A.6) arranged to top order, it is routine to continue the argument, choosing lower order terms of $q(x,\xi)$ to obtain (A.6) to all orders. We have shown that Egorov's theorem follows from the freedom to make an arbitrary choice in the initial condition (A.9) for the amplitude in the geometrical optics construction.

**References**


1. G. Eskin, Parametrix and propagation of singularities for the interior mixed problem, J. Anal. Math. 32 (1977), 17–62.
2. G. Eskin, General initial-boundary problems for second order hyperbolic equations, in *Sing. in Boundary Value Problems*, D. Reidel Publ. Co., Boston 1981, pp. 19–54.
3. R. Melrose, Microlocal parametrices for diffractive boundary value problems, Duke Math. J. 42 (1975), 605–635.
4. R. Melrose, Airy operators, Comm. PDE 3 (1978), 1–76.
5. R. Melrose, Equivalence of glancing hypersurfaces, Invent. Math. 37 (1976), 165–191.
6. R. Melrose, Parametrices for diffractive boundary problems, Unpublished manuscript, 1975.
7. R. Melrose and M. Taylor, Near peak scattering the the corrected Kirchhoff approximation for convex obstacles, Adv. in Math. 55 (1985), 242–315.
8. R. Melrose and M. Taylor, Boundary problems for wave equations with grazing and gliding rays, Monograph.
   http://www.unc.edu/math/Faculty/met/glide.pdf
9. M. Taylor, Grazing rays and reflection of singularities of solutions to wave equations, Comm. Pure Appl. Math. 29 (1976), 1–38.
10. M. Taylor, Pseudodifferential Operators, Princeton Univ. Press, Princeton NJ, 1981.
11. M. Taylor, Diffraction effects in the scattering of waves, in *Sing. in Boundary Value Problems*, D. Reidel Publ. Co., Boston 1981, pp. 271–316.
12. M. Taylor, Propagation, reflection, and diffraction of singularities of solutions to wave equations, Bull. AMS 84 (1978), 589–611.

[T] M. Taylor, Airy operator calculus, Contemp. Math 27 (1984), 169–192.