**Airy functions and Airy quotients**

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Airy functions play important roles in the study of wave motion, particularly in diffraction theory. Here we establish basic properties of such functions, particularly $Ai(z)$ amd $A_\pm(z)$, and also results on various quotients of these functions and their derivatives. This material is taken from Appendix A of my monograph with R. Melrose, *Boundary Problems for Wave Equations on Domains with Grazing and Gliding Rays.*

For $s \in \mathbb{R}$, $Ai(s)$ is defined by:

$$Ai(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(st+t^3/3)}\, dt. \tag{A.0.1}$$

This integral is not absolutely convergent, but is well-defined as the Fourier transform of a tempered distribution. It follows directly that $Ai$ satisfies the second order differential equation (Airy's equation)

$$Ai''(s) - sAi(s) = 0. \tag{A.0.2}$$

From (A.0.2) it follows that $Ai(z)$ extends to an entire holomorphic function on $\mathbb{C}$. Set

$$A_\pm(z) = Ai(e^{\mp 2\pi i/3} z). \tag{A.0.3}$$

Thus, $A_\pm(z)$ also satisfy the differential equation (A.0.2). In fact we have

$$Ai(z) = e^{\pi i/3} A_+(z) + e^{-\pi i/3} A_-(z), \tag{A.0.4}$$

as we proceed to show.

Note that $Ai(z)$ is real for real $z$, so (A.0.3) implies that:

$$A_-(z) = \overline{A_+(\bar{z})}. \tag{A.0.5}$$

Thus we must have

$$Ai(z) = cA_+(z) + \bar{c}A_-(z). \tag{A.0.6}$$

Evaluating $Ai(0)$ and $Ai'(0)$ in two ways each, using (A.0.6) and (A.0.3), gives

$$c + \bar{c} = 1, \quad c\omega^{-2} + \bar{c}\omega^2 = 1,$$

where

$$\omega = e^{\pi i/3}, \tag{A.0.7}$$

and this in turn implies that $c = \omega^2/(1+\omega^2) = 1/(1-\omega) = \omega$, which proves (A.0.4).

## §A.1: Asymptotic expansion

An integral formula for $Ai(z)$ which is convergent for all $z \in \mathbb{C}$ can easily be obtained. Replace $t$ in (A.0.1) by $iv$ and deform the contour so that for real $z$,

$$Ai(z) = \frac{1}{2\pi i} \int_L e^{v^3/3 - zv}\, dv, \tag{A.1.1}$$

where $L$ is any contour that begins at a point at infinity in the sector $-\pi/2 \le \arg(v) \le -\pi/6$, and ends at infinity in the sector $\pi/6 \le \arg(v) \le \pi/2$. Since both sides of (A.1.1) are entire analytic, we have the identity for all $z \in \mathbb{C}$.

From (A.1.1) we can obtain a formula, valid in the region

$$\{z \in \mathbb{C}; |\arg(z)| \le (1-\delta)\pi\}, \quad \delta > 0, \tag{A.1.2}$$

i.e., in the complex plane $C$ with a small conic neighborhood of the closed negative real axis removed. Indeed, for $z \in \mathbb{R}^+$, set $v = z^{1/2} + it^{1/2}$ on the upper half of the path $L$ in (A.1.1) and $v = z^{1/2} - it^{1/2}$ on the lower half to obtain:

$$\begin{aligned} Ai(z) &= \frac{1}{2\pi}\, e^{-(2/3)z^{3/2}} \int_0^\infty \cos\Big(\frac{1}{3}t^{3/2}\Big) \exp(-tz^{1/2}) t^{-1/2}\, dt \\ &= \Psi(z)\, e^{-(2/3)z^{3/2}}. \end{aligned} \tag{A.1.3}$$

Since the right side is clearly holomorphic in the region (A.1.2), there is identity in that region. Well-known asymptotic methods can now be applied, in particular the method of steepest descents, to the integral defining $\Psi(z)$, giving

$$\Psi(z) \sim z^{-1/4} \sum_{j=0}^\infty a_j z^{-3j/2}, \quad a_0 = \frac{1}{4}\pi^{-3/2}, \tag{A.1.4}$$

as $|z| \to \infty$ within the region (A.1.2). Formal term by term differentiation yields valid asymptotic expansions in this region for the derivatives of $\Psi(z)$, see [Ol2].

The asymptotic expansion (A.1.3), (A.1.4) implies

$$A_\pm(z) = \Psi(\omega^{\mp 2} z) \exp\Big(\mp \frac{2}{3} i (-z)^{3/2}\Big) \tag{A.1.5}$$

in the regions

$$\Big\{z \in \mathbb{C}; \Big|\arg(z) \mp \frac{2}{3}\pi\Big| \le (1-\delta)\pi\Big\}, \quad \delta > 0, \tag{A.1.6}$$

and in these regions $\Psi(\omega^{\mp 2} z)$ has the same sort of asymptotic expansion as (A.1.4).

Another useful integral formula for $Ai(s), s > 0$, is obtained by writing the integral (A.0.1) as

$$Ai(s) = \frac{1}{\pi} \int_0^\infty \cos\Bigl(st + \frac{1}{3}t^3\Bigr)\, dt,$$

and making the change of variable $t = 2s^{1/2}\sinh(v/3)$. Since

$$4\sinh^3\Bigl(\frac{v}{3}\Bigr) + 3\sinh\Bigl(\frac{v}{3}\Bigr) = \sinh v,$$

it follows that:

$$Ai(z) = \frac{2}{\sqrt{3}\pi}\Bigl(\frac{z}{3}\Bigr)^{1/2} \int_0^\infty \cos\Bigl(\frac{2}{3}z^{3/2}\sinh v\Bigr)\cosh\Bigl(\frac{1}{3}v\Bigr)\, dv. \tag{A.1.7}$$

The integral on the right is a modified Hankel function. Generally, if $\xi > 0$ and $0 < \nu < 1$,

$$\begin{aligned} K_\nu(\xi) &= \frac{1}{\cos(\pi\nu/2)} \int_0^\infty \cos(\xi \sinh t)\cosh(\nu t)\, dt \\ &= \int_0^\infty e^{-\xi\cosh t}\cosh(\nu t)\, dt, \end{aligned} \tag{A.1.8}$$

the latter integral being convergent and holomorphic for $\mathrm{Re}(\xi) > 0$; see Erdelyi et al. [Er], Vol. 2, p. 82, or Lebedev, [Leb], pp. 119–140. Thus

$$Ai(z) = \frac{1}{\pi}\Bigl(\frac{z}{3}\Bigr)^{1/2} K_{1/3}\Bigl(\frac{2}{3}z^{3/2}\Bigr), \quad |\arg(z)| < \frac{1}{3}\pi. \tag{A.1.9}$$

Since $K_\nu(z)$ solves the modified Bessel equation

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} - \Bigl(1 + \frac{\nu^2}{z^2}\Bigr)w = 0, \tag{A.1.10}$$

it follows that $K_\nu(z)$ is holomorphic in $|\arg(z)| < \pi$, and (A.1.9) therefore holds in the larger region $|\arg(z)| < 2\pi/3$. In fact $K_\nu(z)$ can be continued to the logarithmic plane covering $\mathbb{C}\backslash 0$, and then (A.1.9) is valid globally.

The formula (A.1.8) implies that, for fixed $\nu > 0$, as $\xi \to 0$, $|\arg\ \xi| < \pi$,

$$K_\nu(\xi) \sim \frac{1}{2}\int_0^\infty e^{-(1/2)\xi e^t}\, e^{\nu t}\, dt \sim \frac{1}{2}\int_1^\infty e^{-\xi s/2}\, s^{\nu-1}\, ds \sim \frac{1}{2}\Gamma(\nu)\Bigl(\frac{2}{\xi}\Bigr)^\nu, \tag{A.1.11}$$

and hence the identity (A.1.9) implies

$$Ai(0) = \frac{1}{2\pi}\, 3^{-1/6}\, \Gamma\Bigl(\frac{1}{3}\Bigr) = \frac{3^{-2/3}}{\Gamma(2/3)}, \tag{A.1.12}$$

FIGURE A.1

the last identity in (A.1.12) following from $\Gamma(1/3)\Gamma(2/3) = \pi/(\sin \pi/3) = 2\pi/\sqrt{3}$. Further computation (cf. (A.2.12)) gives

$$Ai'(0) = -\frac{1}{2\pi}\, 3^{1/6}\, \Gamma\Big(\frac{2}{3}\Big) = -\frac{3^{-1/3}}{\Gamma(1/3)}. \tag{A.1.13}$$

Figure A.1 is a graph of $y = Ai(s)$, $s \in \mathbb{R}$, produced by numerically integrating (A.0.2), using the initial data (A.1.12)–(A.1.13).

## §A.2: Zeroes of $Ai$

The formulæ (A.1.3), (A.1.4) show that for any $\delta > 0$, there is some finite $R(\delta)$ such that $Ai(z)$ has no zeroes in (A.1.2) for $|z| > R(\delta)$. In this section we show that all the zeroes of $Ai(z)$ and all those of $Ai'(z)$ are real and negative. First we give a proof of an important special case of this.

**Proposition A.2.1.** *$A_\pm(s)$, $A'_\pm(s)$ are not zero for any $s \in \mathbb{R}$.*

*Proof.* This is a simple consequence of the Wronskian relation:

$$A'_+(z)A_-(z) - A_+(z)A'_-(z) = c_0 i = \frac{1}{2\pi i}. \tag{A.2.2}$$

By (A.0.5) and the same equation for the derivatives, the real zeroes of $A_+$ and $A_-$, or of their derivatives, must coincide. The existence of one such common zero would imply $c_0 = 0$ in (A.2.2). Disregarding our explicit computation of $c_0$, we see that this would imply $A_+(z) = c' A_-(z)$. This is not possible, since it would contradict (A.1.5).

The next result implies that

$$Ai(z) \neq 0, \quad |\arg(z)| \leq \frac{1}{3}\pi. \tag{A.2.3}$$

**Proposition A.2.4.** *$K_\nu(z) \neq 0$ for $|\arg(z)| \leq \pi/2$, if $\nu \in \mathbb{R}^+$.*

*Proof.* By (A.1.8) $K_\nu(z)$ is real for real $z$, so it is enough to consider $z$ in the fourth quadrant. We use the argument principle, and compute the change in the argument of $K_\nu(z)$ along a closed curve $ABCD$ as pictured in Fig. A.2. Along the piece $AB$ the change in argument can be computed approximately from the asymptotic expansion:

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \sum_{k=0}^{\infty} a_k(\nu) z^{-k}, \quad |z| \to \infty,$$

which can be obtained from (A.1.8). Thus:

$$\arg(K_\nu(B)) - \arg(K_\nu(A)) = -\frac{1}{4}\pi - iA + o(1) \text{ as } |A| = |B| \to \infty. \tag{A.2.5}$$

On $BC$ there is no change of argument since $K_\nu(z)$ is real and positive, by (A.1.8). On $CD$, we use the asymptotic expansion (A.1.11) for $K_\nu(z)$, as $z \to 0$, and conclude

$$\arg(K_\nu(D)) - \arg(K_\nu(C)) = \frac{1}{2}\nu\pi + o(1), \quad |C| = |D| \to 0. \tag{A.2.6}$$

FIGURE A.2

To find the change in argument from $D$ to $A$ we need to study $K_\nu(z)$ further. Consider the identity:

$$K_\nu(-it) = \frac{\pi i}{2}\, e^{\pi\nu i/2}\, \left[J_\nu(t) + iY_\nu(t)\right], \tag{A.2.7}$$

which can be obtained from (A.1.8) by transformation of the integrals (see Olver [Ol2]). The Bessel functions $J_\nu(t)$ and $Y_\nu(t)$ satisfy Bessel's equation:

$$\frac{d^2w}{dt^2} + \frac{1}{t}\frac{dw}{dt} + \left(1 - \frac{\nu^2}{t^2}\right) w = 0. \tag{A.2.8}$$

Both are real for $t > 0$ real. Hence their positive real zeroes intertwine:

$$0 < y_{\nu,1} < j_{\nu,1} < y_{\nu,2} < j_{\nu,2} < \dots .$$

Now we need to show that the $k$th positive zero of $J_\nu(t)$ is given by:

$$j_{\nu,k} = \pi(k + \tfrac{1}{2}\nu - \tfrac{1}{4}) + o(1) \text{ as } k \to \infty \quad (\nu \text{ fixed}). \tag{A.2.9}$$

In fact the asymptotic expansion:

$$\begin{aligned} J_\nu(t) \sim \left(\frac{2}{\pi t}\right)^{1/2} \Big[ &\cos\Big(z - \frac{1}{2}\pi\nu - \frac{1}{4}\pi\Big) \sum_{l=0}^{\infty} a_l(\nu) t^{-2l} \\ &- \sin\Big(z - \frac{1}{2}\pi\nu - \frac{1}{4}\pi\Big) \sum_{l=0}^{\infty} b_l(\nu) t^{-2l-1} \Big], \quad t \to \infty, \end{aligned} \tag{A.2.10}$$

which is readily obtained from an integral formula such as:

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \int_{-1}^{1} (1-t^2)^{\nu-1/2} \cos(zt)\, dt, \quad |\arg(z)| < \pi, \tag{A.2.11}$$

shows that $J_\nu(t)$ does have zeroes with the asymptotic behaviour (A.2.9), for large $k$. That the appropriate one is exactly the $k$th can be decided easily. For $\nu = 1/2$, $J_{1/2}(t) = \sqrt{(2/\pi t)} \sin t$, so (A.2.9) holds exactly in that case. For general $\nu$, (A.2.9) follows from the analyticity in $\nu$ and and the argument principle, there being no zeroes near $t = 0$.

Returning to the analysis of $K_\nu(z)$ on $DA$, we see from (A.2.9) that, if $A = -iy_{\nu,k}$ then the change of argument of $K_\nu(z)$ on $DA$ cancels out the change along the rest of the curve, up to a term which is $o(1)$ as $|A|, |B| \to \infty$, $|C|, |D| \to 0$. This proves Proposition A.2.4 and hence (A.2.3), since the change of argument must be an integer, hence zero.

In a fashion similar to (A.1.9) it can be shown that:

$$Ai'(z) = -\frac{z}{\sqrt{3}\pi} K_{2/3}\Big(\frac{2}{3} z^{3/2}\Big), \tag{A.2.12}$$

so Proposition A.2.4 also implies that:

$$Ai'(z) \neq 0, \quad |\arg(z)| \leq \frac{1}{3}\pi. \tag{A.2.13}$$

In order to show that all the zeroes of $Ai(z)$ and of $Ai'(z)$ are real it remains to demonstrate that

$$Ai(z), Ai'(z) \neq 0, \quad |\pi - \arg(z)| < \frac{2}{3}\pi, \quad z \notin \mathbb{R}^-. \tag{A.2.14}$$

To do this we follow the method of Lommel, as described by Olver [Ol2].

Pick $a, b \in \mathbb{C},\ a^3 \neq b^3$. From the identity:

$$\frac{d}{dz}\Big[bAi(az)Ai'(bz) - aAi(bz)Ai'(az)\Big] = z(b^3 - a^3)Ai(az)Ai(bz),$$

we conclude that:

$$\int_0^1 tAi(at)Ai(bt)\,dt = \frac{1}{a^3 - b^3}\Big[bAi(a)Ai'(b) - aAi(b)Ai'(a)\Big] - \frac{b-a}{a^3 - b^3}Ai(0)Ai'(0).$$

Similarly,

$$\int_0^1 Ai'(at)Ai'(bt)\,dt = \frac{1}{a^3 - b^3}\Big[a^2Ai(a)Ai'(b) - b^2Ai(b)Ai'(a)\Big] - \frac{a^2 - b^2}{a^3 - b^3}Ai(0)Ai'(0).$$

Suppose that $a = re^{i\theta}$ is a nonreal zero of $Ai(z)$ or of $Ai'(z)$. Then so is $b = re^{-i\theta}$ and from these formulæ, we get:

$$\int_0^1 tAi(at)Ai(bt)\,dt = -r^{-2}\,\frac{\sin\theta}{\sin 3\theta}\,Ai(0)Ai'(0), \tag{A.2.15}$$

$$\int_0^1 Ai'(at)Ai'(bt)\,dt = -r^{-1}\,\frac{\sin 2\theta}{\sin 3\theta}\,Ai(0)Ai'(0). \tag{A.2.16}$$

The integrals on the left are positive and $Ai(0)Ai'(0)$ is negative. This implies that both $\sin\theta/\sin 3\theta$ and $\sin 2\theta/\sin 3\theta$ must be positive and finite. This is not possible in the range $|\pi - \arg(a)| < 2\pi/3,\ a \notin \mathbb{R}^-$, so (A.2.14) holds. Together with (A.2.3) this gives:

**Theorem A.2.17.** *All the zeroes of $Ai(z)$ and $Ai'(z)$ are real and negative.*

Given that all the zeroes of $Ai(z)$ are real and negative, say:

$$Ai(s_j) = 0, \quad 0 > s_0 > s_1 \cdots \to -\infty, \tag{A.2.18}$$

FIGURE A.3

we can write:

$$\chi(z) = (1/2i)\log\left(\frac{A_+(z)}{A_-(z)}\right) \tag{A.2.19}$$

for $z$ in the plane $\mathbb{C}$ slit along the two rays starting from $e^{\pm 2\pi i/3}s_0$; see Figure A.3.

Also we shall denote by $\mathcal{K}$ the region:

$$\mathcal{K} = \left\{z \in \mathbb{C}; \mathrm{Re}(z) \leq \frac{1}{2}\mathrm{Re}(e^{2\pi i/3}s_0)\right\}.$$

Now, with

$$F(z) = [A_+(z)A_-(z)]^{1/2}, \tag{A.2.20}$$

we have

$$A_\pm(z) = F(z)e^{\pm i\chi(z)} \tag{A.2.21}$$

The asymptotic expansion (A.1.4), (A.1.5) gives:

$$F(z) \sim (-z)^{-1/4}\sum_{j=0}^{\infty} f_j(-z)^{-3j/2}, \quad z \in \mathcal{K},\ |z| \to \infty, \quad f_0 = \frac{1}{2\sqrt{\pi}} \tag{A.2.22}$$

and also for $z \in \mathcal{K}$,

$$\chi(z) \sim \frac{2}{3}(-z)^{3/2}\sum_{j=0}^{\infty} e_j(-z)^{-3j/2}, \quad e_0 = 1. \tag{A.2.23}$$

Thus (A.2.21), (A.2.22), (A.2.23) can be thought of as an asymptotic expansion for $A_{\pm}(z)$ which is in many ways more convenient than (A.1.4), (A.1.5). Note that (A.0.5) implies that

$$F(z) \text{ and } \chi(z) \text{ are real for } z \in \mathbb{R} \cap \mathcal{K}. \tag{A.2.24}$$

The definition (A.2.19) is equivalent to:

$$\frac{A_+(z)}{A_-(z)} = e^{2i\chi(z)}. \tag{A.2.25}$$

Differentiating and using the Wronskian relation (A.2.2) gives

$$2\chi'(z) = \frac{c_0}{F(z)^2}. \tag{A.2.26}$$

In terms of (A.2.21) a very convenient formula can be obtained for $Ai(z)$ for $z \in \mathcal{K}$ from (A.0.4). Namely,

$$Ai(z) = 2F(z)\cos\Big(\chi(z) - \frac{1}{3}\pi\Big) = 2F(z)\sin\Big(\chi(z) + \frac{1}{6}\pi\Big). \tag{A.2.27}$$

Since $F$ is non-vanishing in $\mathcal{K}$ the zeroes of $Ai(z)$ must occur at the points where $\chi(s_j) + \pi/6$ is an integral multiple of $\pi$. In view of (A.2.23) and (A.2.24) this gives good asymptotic control over the behaviour of the zeroes of $Ai(z)$. Also, the asymptotic behaviour of $Ai(z)$ as $|z| \to \infty$ is elucidated by (A.2.27).

## §A.3: Airy quotients

Next we record certain identities for Airy quotients. Formula (A.2.21) gives

$$\begin{aligned}\Phi_{\pm}(z) = \frac{A'_{\pm}(z)}{A_{\pm}(z)} &= \frac{F'(z)}{F(z)} \pm i\chi'(z) \\ &= \frac{F'(z)}{F(z)} \pm \frac{i}{2}\frac{c_0}{F(z)^2}.\end{aligned} \tag{A.3.1}$$

where the first equation is the definition of $\Phi_{\pm}(z)$. By (A.2.24) for real $z$ this decomposes $\Phi_{\pm}(z)$ into its real and imaginary parts. Differentiating (A.2.27) leads to:

$$\begin{aligned}\Phi i(z) = \frac{Ai'(z)}{Ai(z)} &= \frac{F'(z)}{F(z)} + \chi'(z)\cot\Big(\chi(z) + \frac{1}{6}\pi\Big) \\ &= \frac{F'(z)}{F(z)} + \frac{1}{2}\frac{c_0}{F(z)^2}\cot\Big(\chi(z) + \frac{1}{6}\pi\Big).\end{aligned} \tag{A.3.2}$$

Using the Wronskian relation

$$A'_\pm(z)Ai(z) - Ai'(z)A_\pm(z) = c_\pm, \tag{A.3.3}$$

one obtains

$$\Phi_\pm(z) - \Phi i(z) = c_\pm[A_\pm(z)Ai(z)]^{-1}. \tag{A.3.4}$$

From the formulæ above

$$\begin{aligned} A_\pm(z)Ai(z) &= \omega^{\mp 1}F(z)^2\big[e^{\pm 2i\chi(z)} + \omega^{\pm 2}\big] \\ &= \omega^{\pm 1}F(z)^2\big[e^{\pm 2i(\chi(z)-\pi/3)} + 1\big]. \end{aligned} \tag{A.3.5}$$

Directly from Airy's equation the Airy quotients satisfy a nonlinear differential equation of first order:

$$\Phi'(z) = z - \Phi(z)^2, \tag{A.3.6}$$

for $\Phi(z) = \Phi i(z)$ or $\Phi_\pm(z)$. Note that

$$\Phi_\pm(z) = \omega^{\mp 2}\Phi i(\omega^{\mp 2}z). \tag{A.3.7}$$

The poles of $\Phi_+(z)$ lie on the ray $e^{-i\pi/3}[-s_0, \infty)$ which is contained in the fourth quadrant. The poles of $\Phi_-(z)$ lie on the ray $e^{i\pi/3}[-s_0, \infty)$ in the first quadrant. Outside any conic neighborhood of the respective rays there are asymptotic expansions:

$$\Phi_\pm(z) \sim z^{1/2}\sum_{j=0}^{\infty} b_j^\pm z^{-3j/2}, \quad |z| \to \infty. \tag{A.3.8}$$

In particular, (A.3.8) holds for $\Phi_+(z)$ for $z$ in the upper half plane $\{\operatorname{Im} z \geq 0\}$, and a similar expansion holds for $\Phi_-(z)$ in the lower half plane since

$$\Phi_+(z) = \overline{\Phi_-(\overline{z})}. \tag{A.3.9}$$

The first constant is:

$$b_0^\pm = 1. \tag{A.3.10}$$

We wish to consider the manner in which $\Phi_+(z)$ maps the upper half plane into itself. The asymptotic expansion (A.3.8) shows that for $|z|$ large, and $\operatorname{Im} z \geq 0$, $\Phi_+(z)$ lies in an arbitrarily small conic neighborhood of the first quadrant,

$\operatorname{Re}\Phi_+ \geq 0$, $\operatorname{Im}\Phi_+ \geq 0$. In fact examination of (A.3.1) shows that for $|z|$ large, $\operatorname{Im} z \geq 0$, $\operatorname{Re}\Phi_+, \operatorname{Im}\Phi_+ > 0$. Indeed as $|z| \to \infty$ in a conic neighborhood of $\mathbb{R}^-$,

$$\frac{F'(z)}{F(z)} \sim z^{-1} \sum_{j=0}^{\infty} \alpha_j z^{-3j/2}, \quad F(z)^{-2} \sim (-z)^{1/2} \sum_{j=0}^{\infty} g_j z^{-3j/2}, \tag{A.3.11}$$

and as $|z| \to \infty$ in a conic neighborhood of $\mathbb{R}^+$,

(A.3.12)

$$\frac{F'(z)}{F(z)} \sim z^{1/2} \sum_{j=0}^{\infty} \tilde{\alpha}_j z^{-3j/2}, \quad F(z)^{-2} \sim z^{-1/2} \exp\Big(-\frac{4}{3} z^{3/2}\Big) \sum_{j=0}^{\infty} \tilde{g}_j z^{-3j/2},$$

where all the coefficients are real.

Next it will be shown that the closed upper half plane

$$\mathbb{C}^+ = \{z \in \mathbb{C}; \operatorname{Im}(z) \geq 0\}$$

is mapped by $\Phi_+$ into the open first quadrant

$$\mathcal{Q}_1 = \{0 < \arg(\Phi) < \frac{1}{2}\pi; |\Phi| > 0\}.$$

Since $F(s)$ is real for real $s$, (A.3.1) implies that $\operatorname{Im}\Phi_+(s) > 0$ for $s \in \mathbb{R}$. Thus, $\operatorname{Im}(\Phi_+(z))$ is positive for $z \in \mathbb{R}$ and near infinity in $\mathbb{C}+$. Hence it must be strictly positive for $z \in \mathbb{C}^+$ by the maximum principle, i.e.,

$$\operatorname{Im}\Phi_+(z) > 0, \quad z \in \mathbb{C}^+. \tag{A.3.13}$$

Next consider the real part of $\Phi_+(z)$. Certainly $\operatorname{Re}\Phi_+(z) > 0$ outside a compact subset $K \subset \mathbb{C}^+$. Let $z_0 = x_0 + iy_0$ be a point with maximal imaginary part at which $\operatorname{Re}\Phi_+(z)$ vanishes. From the differential equation (A.3.6), $\operatorname{Im}\Phi'_+(z_0) = y_0$, so if $y_0 > 0$,

$$\operatorname{Re}\Phi_+(z_0 + it) = -y_0 t + O(t^2) < 0,$$

if $t > 0$ is small. This contradicts the maximality of $y_0$, so the only possibility left is $y_0 = 0$. At such a point, $\Phi'_+(z_0)$ would be real, by (A.3.6) but since were have already shown $\operatorname{Re}\Phi_+(z) \geq 0$ this implies that $\Phi'_+(z_0) = 0$. Near such a zero of order two or higher the image of a half disc in $\mathbb{C}^+$ cannot satisfy $\operatorname{Re}\Phi_+(z) \geq 0$, so this possibility is eliminated; we have proved that:

$$\operatorname{Re}\Phi_+(z) > 0, \quad z \in \mathbb{C}^+. \tag{A.3.14}$$

One consequence of (A.3.14) and (A.3.1) is:

$$F'(s) > 0, \quad s \in \mathbb{R}, \tag{A.3.15}$$

which is equivalent to

$$A_+(s)A_-(s) = |A(s)|^2 \text{ is monotone increasing for } s \in \mathbb{R}. \tag{A.3.16}$$

From (A.2.26) it is clear that $\chi(s)$ is monotone for $s \in \mathbb{R}$. Again by (A.3.1)

$$\frac{d \operatorname{Im} \Phi_+(s)}{ds} < 0, \quad s \in \mathbb{R}, \tag{A.3.17}$$

so

$$\operatorname{Im} \Phi_+(s) \text{ is monotone decreasing for } s \in \mathbb{R}. \tag{A.3.18}$$

This shows that the curve $\mathbb{R} \ni s \longrightarrow \Phi_+(s)$, has no self-intersections and that its image in the Riemann sphere has winding number one about an interior point of $\Phi_+(\mathbb{C}^+)$. This completes the proof of:

**Theorem A.3.19.** $\Phi_+ : \mathbb{C}^+ \longrightarrow \mathcal{Q}_1$ *is a biholomorphism onto its image, which is contained in the open first quadrant.*

Assertions (A.3.13) and (A.3.14) were proved in [MeS2]. The fact that $\operatorname{Re} \Phi_+(s) > 0$, $\operatorname{Im} \Phi_+(s) > 0$ for $s \in \mathbb{R}$ was used by Imai and Shirota [ImSh], who show that this is equivalent to the monotonicity (A.3.16) of $|A_+(s)|^2$ and refer to Miller [Mil] for this result. Since $|A_+(s)|^2 = Ai(s)^2 + Bi(s)^2$, the graph on [Mi1], page B16 is consistent with (A.3.16) but an explicit proof does not seem to be given there. We present here a graph of the curve $\Phi_+(s)$ in $\mathbb{C}$, as $s$ runs over $\mathbb{R}$. See Fig. A.4. This graph was produced by numerically integrating the ODE (A.3.6) for $\Phi_+$, with initial data

$$\Phi_+(0) = -e^{-2\pi i/3}\ 3^{1/3}\ \frac{\Gamma(2/3)}{\Gamma(1/3)} = -e^{-2\pi i/3}\ \frac{\sqrt{\pi} 2^{2/3} 3^{1/3}}{\Gamma(1/6)}.$$

Note how rapidly the curve approaches the $x$-axis, which is to be expected, given (A.3.1) and the behavior (A.3.12) of $F(s)^{-2} = |A_+(s)|^{-2}$ as $s \to +\infty$. Of course, these formulas make it clear that $\Phi_+(s)$ has positive imaginary part for $s \in \mathbb{R}$; this is the simplest part of Theorem A.3.19.

We next consider how close $\Phi_+(z)$ is to $z^{1/2}$ by examining the difference between $\Phi_+(z)^2$ and $z$. From (A.3.6)

$$\Phi_+(z)^2 = z - \Phi_+'(z), \tag{A.3.20}$$

so

$$\Phi_+(z)^2 \sim z + \sum_{j=0}^{\infty} \gamma_j z^{-1/2-3j/2}, \quad \text{as } |z| \to \infty. \tag{A.3.21}$$

Combining (A.3.13), (A.3.14) with this and Theorem A.3.19 we have:

FIGURE A.4

**Corollary A.3.22.** $\Phi_+^2$ *is biholomorphic from* $\mathbb{C}^+$ *to its image, which is contained in the interior of* $\mathbb{C}^+$.

Note from (A.3.12) that for some positive constant $C$,

$$\operatorname{Im}\Phi_+(s)^2 \geq \begin{cases} C(1+|s|)^{-3/2}, & s \leq 0, \\ C\exp(-(4/3)s^{3/2}), s \geq 0. \end{cases}$$

Together with Corollary (A.3.22) this implies:

$$\operatorname{Im}\Phi i(x+iy)^2 \geq \begin{cases} C(1+|x|)^{-3/2} + Cy, & y \geq 0, x \leq 0, \\ C\exp(-(4/3)x^{3/2}) + Cy, & y \geq 0, x \geq 0. \end{cases} \tag{A.3.23}$$

Since $\operatorname{Re}\Phi i(x+iy)^2 = x + O((1+|x|^2+|y|^2)^{-1/4})$ we therefore have:

$$\operatorname{Re}\Phi_+(x+iy) \geq C(1+|x|)^{-1/2}\Big(y + (1+|x|)^{-3/2}\Big), \text{ if } y \geq 0, x \leq 0, \tag{A.3.24}$$

and

$$\operatorname{Im}\Phi_+(x+iy) \geq C(1+|x|)^{-1/2}\Big(y + \exp(-(4/3)x^{3/2})\Big) \text{ if } y \geq 0, x \geq 0. \tag{A.3.25}$$

We next turn to the examination of $\Phi i(z) = Ai'(z)/Ai(z)$. Note that $\Phi i(s)$ is real for real $s$. In fact, $\Phi i(s) > 0$ for $s > \sigma_0$, where

$$\{\sigma_j; j = 0, 1, 2, \dots\} = \{\sigma; Ai'(\sigma) = 0\}. \tag{A.3.26}$$

Thus, $\Phi i(\sigma_j) = 0$ and $\Phi i(z)$ has a simple pole at each of the zeroes, $z = s_j$, of $Ai(z)$. Note that

$$0 > \sigma_0 > s_0 > \sigma_1 > s_1 > \cdots . \tag{A.3.27}$$

For any fixed $\delta > 0$, the behaviour of $\Phi i(z)$ on the set

$$\mathfrak{A}_\delta = \{z \in \mathbb{C}; |\arg(z)| \leq \pi - \delta\} \tag{A.3.28}$$

is rather obvious. From the expansion (A.1.3), (A.1.4)

$$\Phi i(z) \sim z^{1/2} \sum_{j=0}^{\infty} \gamma_j z^{-3j/2}, \quad |z| \to \infty \text{ in } \mathfrak{A}_\delta. \tag{A.3.29}$$

Since $\Phi i(s)$ is real and positive for $s \in \mathbb{R}^+$, all the $\gamma_j$ in (A.3.29) are real with $\gamma_0 > 0$. From (A.3.7) and Theorem A.3.19 we obtain:

**Proposition A.3.30.** *$\Phi i$ maps $\mathfrak{A}_{\pi/3}$ biholomorphically onto a domain in $\{|\arg(z)| < \pi/3\}$.*

## §A.4: Behaviour of $\Phi i$ near $(-\infty, 0]$

It remains to examine $\Phi i(z)$ in detail in a conic neighborhood of the negative real axis. To do so it is useful to obtain formulae parallel to (A.2.21) and (A.2.27), using the functions:

$$G(z) = [A'_+(z)A'_-(z)]^{1/2}, \quad \psi(z) = \frac{1}{2i} \log \left[\frac{A'_+(z)}{A'_-(z)}\right], \tag{A.4.1}$$

for $z$ in the complex plane slit along two rays connecting, respectively, the zeroes of $A'_+(z)$ and those of $A'_-(z)$; cf. Figure A.3. Then

$$A'_\pm(z) = G(z)e^{\pm i\psi(z)}, \tag{A.4.2}$$

and

$$Ai'(z) = 2G(z) \sin\Big(\psi(z) + \frac{1}{6}\pi\Big). \tag{A.4.3}$$

Since $A'_+(z) = \overline{A'_-(\overline{z})}$,

$$G, \ \psi : \mathbb{R} \longrightarrow \mathbb{R}. \tag{A.4.4}$$

Differentiating the asymptotic expansion (A.1.3), (A.1.4), rotated to apply to $A'_\pm(z)$ we deduce that:

$$G(z) \sim (-z)^{1/4} \sum_{j=0}^{\infty} g_j (-z)^{-3j/2} \tag{A.4.5}$$

and

(A.4.6) $$\psi(z) \sim \frac{2}{3}(-z)^{3/2} \sum_{j=0}^{\infty} e_j (-z)^{-3j/2}$$

as $|z| \to \infty$ in $\operatorname{Re} z \leq 0$; cf. (A.2.22), (A.2.23).

In place of (A.2.26) we obtain

(A.4.7) $$2\psi'(z) = -c_0 \frac{z}{G(z)^2}.$$

Unlike $\chi(s)$, which is monotonic on the real line, $\psi(s)$ is monotonic increasing for $s < 0$ and monotonic decreasing for $s > 0$. In fact in $s < 0, \psi(s)$ is closely related to $\chi(s)$. From (A.2.27) and (A.4.3) and noting that the zeroes of $Ai(s)$ and $Ai'(s)$ are interlaced, it follows that $\chi(s)+\pi/6$ and $\psi(s)+\pi/6$ alternately assume values which are integer multiples of $\pi$, so the difference must be bounded. In fact, (A.2.23), (A.4.6) together give:

(A.4.8) $$\chi(z) - \psi(z) \sim \frac{1}{2}\pi - \sum_{j=1}^{\infty} \sigma_j z^{-3j/2},$$

as $|z| \to \infty$ in $\{\operatorname{Re} z \leq 0\}$.

Differentiating (A.4.2) and proceeding as in the derivation of (A.3.1) yields

(A.4.9) $$\Phi_{\pm}(z)^{-1} = \frac{1}{z}\frac{G'(z)}{G(z)} \mp \frac{c_0 i}{2} G(z)^2.$$

Then, (A.3.14) and (A.3.15) imply that $\Phi_+^{-1}(s)$ lies in the first quadrant, so:

(A.4.10) $$G'(s) \text{ has the same sign as } s, \;\; s \in \mathbb{R}.$$

Comparison of (A.3.1) and (A.4.9) also gives

(A.4.11) $$G^2 = \left(\tfrac{1}{2}c_0\right)^2 F^{-2} + (F')^2.$$

To resume the discussion of the behaviour of $\Phi i(z)$ for $z$ in a conic neighborhood of $\mathbb{R}^-$, consider (A.2.27) and (A.4.3), which show:

(A.4.12) $$\Phi i(z) = \frac{G \sin(\psi + \pi/6)}{F \sin(\chi + \pi/6)}.$$

From the definitions of F and G,

(A.4.13) $$\frac{G}{F}(z) = [\Phi_+(z)\Phi_-(z)]^{1/2}.$$

Figure A.5

The formula (A.4.12) can be used to describe $\Phi i(z)$ in the set

$$\mathcal{D} = \left\{ z \in \mathbb{C}; \operatorname{Re}(z) \leq -C, 0 \leq \operatorname{Im}(z) \leq C(1+|z|)^{-1/2} \right\}. \tag{A.4.14}$$

Divide $\mathcal{D}$ as follows. Pick the half-way points between the zeroes and the poles of $\Phi i(z)$,

$$\alpha_j = \frac{1}{2}(\sigma_j + s_j), \quad \beta_j = \frac{1}{2}(s_j + \sigma_{j+1}), \quad j \geq 0.$$

Then consider the parts:

$$\begin{aligned} \mathcal{E}_j &= \{z \in \mathcal{D}; \beta_j \leq \operatorname{Re} z \leq \alpha_j\}, \quad j \geq 0, \\ \mathcal{F}_j &= \{z \in \mathcal{D}; \alpha_j \leq \operatorname{Re} z \leq \alpha_{j-1}\}, \quad j \geq 1, \end{aligned} \tag{A.4.15}$$

as illustrated in Figure A.5.

The lower boundary of $\mathcal{E}_j$ is roughly centered at $s_j$, that of $\mathcal{F}_j$ at $\sigma_j$. Note that

$$s_j - s_{j+1} \sim \sigma_j - \sigma_{j+1} \sim c(-s_j)^{-1/2}.$$

By (A.2.27) and (A.4.3), $\chi + \pi/6$ maps $[s_{j+1}, s_j]$ to $[-(j+1)\pi, -j\pi]$. Thus the map:

$$\chi_j = \chi + \frac{1}{6}\pi + j\pi$$

maps $s_j$ to the origin. From the asymptotic expansion for $\chi$, it follows that

$$\chi_j(\mathcal{E}_j) \subset \mathcal{R},$$

where $\mathcal{R}$ is a rectangle in the upper half plane with base on the real axis centered at the origin. In fact for large $j$ each $\chi_j$ has inverse, $\kappa_j$, holomorphic in a neighborhood of $\mathcal{R}$ with range containing $\mathcal{E}_j$. Set

$$v_j(z) = j^{-1/3}\Phi i\big(\kappa_j(z)\big). \tag{A.4.16}$$

From (A.4.12), (A.4.13), the asymptotic expansions (A.4.6) and (A.4.8), it follows that as $j \to \infty$, for some constant $v$,

$$v_j(z) \to v\tan(z) \tag{A.4.17}$$

uniformly on $\mathcal{R}$. Similar arguments apply to the function $\psi$ defined on $\mathcal{F}_j$, their normalizations $\psi + (1/6 + j)\pi$ with inverses $\lambda_j$ so that the functions:

$$w_j(z) = \frac{j^{1/3}}{\Phi i\big(\lambda_j(z)\big)} \to w\tan(z), \tag{A.4.18}$$

uniformly on $\mathcal{R}$ for some constant $w$.

From (A.4.16) it follows that, for large $j$,

$$|\Phi i(z)| \le cj^{1/3} \le C(1 + |z|)^{1/2}, \quad z \in \mathcal{F}_j, \tag{A.4.19}$$

and

$$\operatorname{Im}\Phi i(z) \ge cj^{1/3}\operatorname{Im}(j^{1/3}z) \ge C(1 + |z|)\operatorname{Im} z, \quad z \in \mathcal{F}_j, \tag{A.4.20}$$

with the constants positive and independent of $j$. Simlarly from (A.4.17),

$$|\Phi i(z)|^{-1} \le cj^{-1/3} \le C(1 + |z|)^{-1/2}, \quad z \in \mathcal{E}_j \tag{A.4.21}$$

and

$$\operatorname{Im}\Phi i(z)^{-1} \ge cj^{-1/3}\operatorname{Im}(j^{1/3}z) = C\operatorname{Im} z, \quad z \in \mathcal{E}_j. \tag{A.4.22}$$

These last inequalities give in particular:

$$|\Phi i(z)| \le C|\operatorname{Im}(z)|^{-1}, \quad z \in \mathcal{E}_j, \tag{A.4.23}$$

and

$$\operatorname{Im}\Phi i(z) \ge Cj^{1/3} \ge C(1 + |z|)^{1/2}, \quad z \in \mathcal{E}_j. \tag{A.4.24}$$

These inequalities have been proved uniformly for large $j$, but of course are simple to demonstrate for any finite value of $j$ so hold uniformly, with different constants, for all $j$.

Combining (A.4.19) and (A.4.23) gives

$$|\Phi i(z)| \le C|\operatorname{Im}(z)|^{-1}, \quad z \in \mathcal{D}, \tag{A.4.25}$$

and combining (A.4.20) and (A.4.24) gives:

$$\operatorname{Im} \Phi i(z) \ge C\big(1+|z|\big)\ |\operatorname{Im}(z)|, \quad z \in \mathcal{D}. \tag{A.4.26}$$

Note also that

$$\operatorname{Im}\{\Phi i(z)^{-1}\} \ge C \operatorname{Im} z, \quad z \in \mathcal{D}. \tag{A.4.27}$$

It is useful to get similar bounds for the Airy function $Ai(z)$ and its derivative $Ai'(z)$, for $z \in \mathcal{D}$. Indeed, starting from (A.2.27) and using reasoning similar to that in the derivation of (A.4.25) and (A.14.26) one finds that:

$$\operatorname{Im} Ai(z) \ge C(1+|z|)^{1/4} \operatorname{Im} z, \quad z \in \mathcal{D}, \tag{A.4.28}$$

and

$$|Ai(z)^{-1}| \le C(1+|z|)^{-1/4}|\operatorname{Im}(z)|^{-1}, \quad z \in \mathcal{D}. \tag{A.4.29}$$

Further estimation of the same type leads to

$$\operatorname{Im} Ai'(z) \ge C(1+|z|)^{3/4} \operatorname{Im} z, \quad z \in \mathcal{D}, \tag{A.4.30}$$

and

$$|Ai'(z)^{-1}| \le C(1+|z|)^{-3/4}|\operatorname{Im}(z)|^{-1}, \quad z \in \mathcal{D}. \tag{A.4.31}$$

The region $\mathcal{D}$ used above is particularly convenient for such estimates but there is in fact no difficulty in extending the same type of argument to a larger region such as:

$$\mathcal{D}^{\#} = \{z \in \mathbb{C}; \operatorname{Re} z \le 0,\ 0 \le \operatorname{Im} z \le C\}. \tag{A.4.32}$$

We leave to the reader the details, and only note that the estimate $\operatorname{Im} z \le C(1+|z|)^{-1/2}$ valid in $\mathcal{D}$ can no longer be used, so one arrives at estimates such as:

$$|\Phi i(z)| \le C\big(|\operatorname{Im}(z)|^{-1} + |z|^{1/2}\big), \quad z \in \mathcal{D}^{\#}. \tag{A.4.33}$$

Finally, we mention estimates of $\Phi i(z)$ and $\Phi i(z)^{-1}$ on

$$\mathfrak{U}^{\#} = \{z \in \mathbb{C} : \operatorname{Im} z \ge B\}, \tag{A.4.34}$$

given $B > 0$, which follow from (A.3.29) for $z \in \mathfrak{U}^{\#} \cap \mathfrak{A}_\delta$ and from (A.4.12) and the analysis of its ingredients, via (A.4.13) and (A.4.6)–(A.4.8), for $z \in \mathfrak{U}^{\#} \setminus \mathfrak{A}_\delta$. We have

(A.4.35) $$|\Phi i(z)| \le C|z|^{1/2}, \quad |\Phi i(z)^{-1}| \le C|z|^{-1/2}, \quad z \in \mathfrak{U}^{\#}.$$

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