Boundary Regularity
For the Ricci Equation,
Geometric Convergence,
And Gelfand's Inverse Boundary Problem

Determining a Region
By How Its Boundary Vibrates

Michael Anderson, Atsushi Katsuda,
Yaroslav Kurylev, Matti Lassas,
Michael Taylor

Boundary spectral data for compact Riemannian manifold $(\overline{M}, g)$:

$$\{\lambda_j, \varphi_j|_{\partial M}\}_{j=1}^{\infty}.$$

Orthonormal basis of Neumann eigenfunctions φ_j:

$$\Delta\varphi_j = -\lambda_j\varphi_j, \quad N\varphi_j\big|_{\partial M} = 0.$$

Alternative data (equivalent): trace on $\mathbb{R} \times \partial M \times \partial M$ of

$$\cos t\sqrt{-\Delta}.$$

Theorem. *Given $g \in C^2(\overline{M})$, the boundary spectral data determine $(\overline{M}, g)$ uniquely, up to isometry.*

First step. Given f on $[0, T] \times \partial M$, define u^f on $[0, T] \times \overline{M}$ by

$$(\partial_t^2 - \Delta)u^f = 0, \quad u(0) = \partial_t u(0) = 0, \quad Nu\big|_{\partial M} = f.$$

Claim: Gelfand data determines Fourier coefficients

$$u_k^f(t) = (u^f(t), \varphi_k).$$

Proof: Derive ODE:

$$\begin{aligned}\partial_t^2 u_k^f(t) &= (\Delta u^f(t), \varphi_k)\\ &= -\lambda_k (u^f(t), \varphi_k) + \int\limits_{\partial M} f(t,x)\varphi_k(x)\, dS(x)\\ &= -\lambda_k u_k^f(t) + \text{ KNOWN STUFF},\end{aligned}$$

while

$$u_k^f(0) = \partial_t u_k^f(0) = 0.$$

Solve ODE explicitly.

Second step. Take $\Gamma \subset \partial M$ and set

$$M(\Gamma, t^+) = \{x \in M : \mathrm{dist}(x, \partial M) \le t^+\}.$$

See Figure 1. Claim: Image of $L^2(M(\Gamma, t^+))$ under Fourier transform

$$\mathcal{F} : L^2(M) \longrightarrow \ell^2$$

is determined by Gelfand data.

Proof uses Lemma:

$$\{u^f(t^+) : f \in \mathrm{Lip}([0, t^+] \times \Gamma)\}$$

is dense in $L^2(M(\Gamma, t^+))$.
Proof of lemma uses unique continuation result of D. Tataru.

Third step. Hence image in ℓ^2 of $L^2(M(\Gamma, t^+, t^-))$ under $\mathcal{F}$ is determined, where

$$M(\Gamma, t^+, t^-) = M(\Gamma, t^+) \setminus M(\Gamma, t^-).$$

See Figure 2.
Hence image of $L^2(M(\underline{\Gamma}, \underline{t}^+, \underline{t}^-))$ under $\mathcal{F}$ is determined, where

$$M(\underline{\Gamma}, \underline{t}^+, \underline{t}^-) = \bigcap_j M(\Gamma_j, t_j^+, t_j^-).$$

See Figure 3.

Fourth step. Gelfand data enable one to answer question:
Given $f \in C(\partial M)$, does there exist $x \in \overline{M}$ such that

$$f(y) = \operatorname{dist}(x, y), \quad \forall\, y \in \partial M?$$

Thus the image in $C(\partial M)$ of the boundary distance representation of $\overline{M}$:

$$R : \overline{M} \to C(\partial M), \quad R(x)(y) = \operatorname{dist}(x, y),$$

is determined by the Gelfand data.
Hence $\overline{M}$ is determined as a topological space, **provided** R can be proven to be one-to-one. This holds provided geodesics do not branch. See Figure 4.

Fifth step. Let $P(\underline{\Gamma}, \underline{t}^+, \underline{t}^-)$ denote the orthogonal projection of ℓ^2 onto the image $\mathcal{F}\, L^2(M(\underline{\Gamma}, \underline{t}^+, \underline{t}^-))$. Then

$$(P(\underline{\Gamma}, \underline{t}^+, \underline{t}^-)e_i, e_j)_{\ell^2} = \int\limits_{M(\underline{\Gamma}, \underline{t}^+, \underline{t}^-)} \varphi_i(x)\varphi_j(x)\, dV(x).$$

Note $\varphi_1 \equiv (\mathrm{Vol}\ M)^{-1/2}$. Taking $e_i = e_j = e_1$ determines $\mathrm{Vol}\, M(\underline{\Gamma}, \underline{t}^+, \underline{t}^-)$. Then taking $e_i = e_1$ determines

$$\int\limits_{M(\underline{\Gamma}, \underline{t}^+, \underline{t}^-)} \varphi_j(x)\, dV(x).$$

So we recover $\varphi_j(x)$ on M, for each $x \in M$. This determines the differentiable structure and metric tensor on M.

Interior Regularity

Assume

$$g_{jk} \in C^r(\Omega) \cap H^{1,2}(\Omega), \quad r > 0. \tag{1}$$

$$\mathrm{Ric}_{jk} \in L^\infty(\Omega). \tag{2}$$

Then, in local harmonic coordinates,

$$\partial^2 g_{jk} \in \mathrm{bmo}. \tag{3}$$

Note: $g_{jk} \in C^r,\ r > 0 \Rightarrow \exists$ local harmonic coordinates.

Ricci Equation

$$\Delta g_{\ell m} + B_{\ell m}(g_{jk}, \nabla g_{jk}) = -2\,\mathrm{Ric}_{\ell m}\,. \tag{4}$$

Holds in local harmonic coordinates. Laplace-Beltrami operator Δ acts componentwise.

Early work by DeTurk and Kazdan:

$$g_{jk} \in C^2, \quad \mathrm{Ric}_{jk} \in C^r \Rightarrow g_{jk} \in C^{r+2},$$

in local harmonic coordinates, $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$.

Example

$$M \subset \mathbb{R}^n, \quad C^2 \text{ surface, dimension } k.$$

In graph coordinates, $g_{jk} \in C^1$.

Gauss map is C^1, curvature is continuous.

Hence, in harmonic coordinates, $\partial^2 g_{jk} \in \mathrm{bmo}$.

Existence of curvature

Connection 1-form

$$\Gamma^a{}_{bj} = \frac{1}{2} g^{am}(\partial_j g_{bm} + \partial_b g_{jm} - \partial_m g_{jb}).$$

Curvature 2-form

$$\mathcal{R} = d\Gamma + \Gamma \wedge \Gamma.$$

Have

$$\begin{aligned} g_{jk} \in C(\Omega) \cap H^{1,2}(\Omega) &\Rightarrow \Gamma^a{}_{bj} \in L^2(\Omega) \\ &\Rightarrow R^a{}_{bjk} \in H^{-1,2}(\Omega) + L^1(\Omega) \\ &\Rightarrow \mathrm{Ric}_{jk} \in H^{-1,2}(\Omega) + L^1(\Omega) \\ &\Rightarrow \mathrm{Ric}^j{}_k,\ S \in H^{-1,p'}(\Omega), \end{aligned}$$

where

$$p' < \frac{n}{n-1}, \quad n = \dim \Omega.$$

Also,

$$\begin{aligned} & g_{jk} \in H^{1,p}(\Omega), \quad p > n \\ & \quad \Rightarrow R^a{}_{bjk},\ \mathrm{Ric}_{jk},\ \mathrm{Ric}^j{}_k,\ S \in H^{-1,p}(\Omega). \end{aligned}$$

Boundary Regularity

Take $\overline{\Omega} = \Omega \cup \Sigma$, with

$$\begin{aligned} \Omega &= \{x \in \mathbb{R}^n : |x| < 1,\ x_n > 0\}, \\ \Sigma &= \{x \in \mathbb{R}^n : |x| < 1,\ x_n = 0\}. \end{aligned}$$

Assume

$$\begin{aligned} g_{jk} &\in H^{1,p}(\Omega), \quad p > n, \\ h_{jk} &\in H^{1,2}(\Sigma), \quad 1 \le j, k \le n-1, \\ \mathrm{Ric}^\Omega &\in L^\infty(\Omega), \\ \mathrm{Ric}^\Sigma &\in L^\infty(\Sigma), \\ H &\in \mathrm{Lip}(\Sigma). \end{aligned}$$

Here $h_{jk} = g_{jk}|_\Sigma$, and

$$H = \text{ mean curvature of } \Sigma \hookrightarrow \overline{\Omega}.$$

Theorem. *In local boundary harmonic coordinates,*

$$g_{jk} \in C^2_*(\overline{\Omega}).$$

Hence ∇g_{jk} has a log-Lipschitz modulus of continuity, so there is no branching of geodesics.

Boundary Conditions for Ricci Equation

In boundary harmonic coordinates, for $1 \le j, k \le n-1$,

(1)
$$\begin{aligned} \Delta g_{jk} &= B_{jk}(g, \nabla g) - 2\,\mathrm{Ric}^\Omega_{jk}, \\ g_{jk}\big|_\Sigma &= h_{jk} \in \mathfrak{h}^{2,\infty}(\Sigma). \end{aligned}$$

Regularity of (h_{jk}) from previous result.
Results on Dirichlet problem due to Morrey apply.

Need to treat g_{jn}, $1 \le j \le n$. Actually, we directly treat g^{jn}, via Neumann boundary problems:

$$\Delta g^{jn} = B^{jn}(g, \nabla g) + 2(\mathrm{Ric}^{\Omega})^{jn}, \tag{2}$$

$$N g^{nn} = -2(n-1)H\, g^{nn}, \tag{3}$$

on Σ, and, for $1 \le j \le n-1$,

$$N g^{jn} = -(n-1)H\, g^{jn} + \frac{1}{2}\frac{1}{\sqrt{g^{nn}}} g^{jk}\, \partial_k g^{nn}. \tag{4}$$

The boundary conditions (3)–(4) make straightforward sense provided

$$g_{jk} \in C^{1+s}(\overline{\Omega}), \quad \text{for some} \quad s > 0.$$

Otherwise, we must deal with a weak formulation of

$$\Delta w = F, \quad Nw\big|_{\Sigma} = G,$$

namely

$$\int_{\Omega} \langle \nabla w, \nabla \psi \rangle \, dV = -\int_{\Omega} F\psi \, dV - \int_{\Sigma} G\psi \, dS,$$

for all $\psi \in C^{\infty}(\overline{\Omega})$ with compact support (intersecting Σ but not the rest of $\partial\Omega$).

Another boundary regularity result:
Assume

$$\begin{aligned} g_{jk} &\in H^{1,p}(\Omega), \quad p > n, \\ h_{jk} &\in H^{1,2}(\Sigma), \quad 1 \le j, k \le n-1, \\ \mathrm{Ric}^{\Omega} &\in L^{p_1}(\Omega), \quad p_1 > n, \\ \mathrm{Ric}^{\Sigma} &\in L^{p_2}(\Sigma), \quad p_2 > n-1, \\ H &\in C^{\sigma}(\Sigma), \quad \sigma > 0. \end{aligned}$$

Then, in boundary harmonic coordinates,

$$g_{jk} \in C^{1+s}(\overline{\Omega}),$$

for some $s > 0$.

Weak Solutions to Neumann Problem

Assume

$$g_{jk} \in C^r(\overline{\Omega}), \quad \partial\Omega \text{ class } C^{1+r}, \quad 0 < r < 1.$$

$$u \in H^{1,2}(\Omega), \quad \text{weak solution to}$$

$$\Delta u = f, \quad Nu\big|_{\partial\Omega} = g.$$

Proposition. *Given* $s \in (0, r)$, $p \geq n/(1-s)$,

$$f \in L^p(\Omega),\ g \in C^s(\partial\Omega) \Rightarrow u \in C^{1+s}(\overline{\Omega}).$$

First reduce to $f = 0$. Then use single layer potential

$$\mathcal{S}h(x) = \int_{\partial\Omega} E(x, y)h(y)\, dS(y).$$

One obtains

$$u = \mathcal{S}h, \quad h \in C^s(\partial\Omega),$$

with h solving

$$\Big(-\frac{1}{2}I + K^*\Big)h = g.$$

Geometric Convergence

Class of compact Riemannian manifolds with boundary,

$$\mathcal{M}(R_0, i_0, S_0, d_0):$$

Ricci tensor bounds:

(1) $$\|\operatorname{Ric}_M\|_{L^\infty(M)} \leq R_0, \quad \|\operatorname{Ric}_{\partial M}\|_{L^\infty(\partial M)} \leq R_0,$$

Injectivity radius bounds:

(2) $$i_M \geq i_0, \quad i_{\partial M} \geq i_0, \quad i_b \geq 2i_0.$$

Mean curvature bound:

(3) $$\|H\|_{\mathrm{Lip}(\partial M)} \leq S_0,$$

Diameter bound:

(4) $$\operatorname{diam}(\overline{M}, g) \leq d_0.$$

Fix the dimension n.

Theorem. *Given* $n \in \mathbb{N}$, $R_0, i_0, S_0, d_0 \in (0, \infty)$,

$$\mathcal{M}(R_0, i_0, S_0, d_0) \quad \textit{is precompact}$$

in the C^r *topology, for each* $r < 2$, *i.e., any sequence has a convergent subsequence*

$$(\overline{M}_k, g_k) \longrightarrow (\overline{M}, g), \tag{5}$$

in the C^r *topology. Furthermore,*

$$g \in C^2_*(\overline{M}). \tag{6}$$

Meaning of C^r-convergence in (5):
For k large, have diffeomorphisms

$$F_k : \overline{M} \longrightarrow \overline{M}_k$$

such that

$$F_k^* g_k \to g \quad \text{in} \quad C^r\text{-topology}.$$

Key to proof of theorem:
Find lower bound on **harmonic radius**, i.e., size of balls on which there are harmonic coordinates, in which the metric tensor satisfies good bounds.

Harmonic radius estimate has following ingredients:

- (*) Blow-up argument
- (*) Fundamental equations of surface theory
- (*) Cheeger-Gromoll splitting theorem
- (*) Boundary regularity result

Stabilization of Inverse Problems

Direct problem (general set-up)

$$\mathcal{D} : \mathcal{M} \longrightarrow \mathcal{B}. \tag{1}$$

Task: Identify object $M \in \mathcal{M}$ by measurement of data $\mathcal{D}(M)$.
Common case: $\mathcal{D}$ continuous in (1).

First task: uniqueness.
Success: $\mathcal{D}$ is one-to one.
Ill posedness of inverse problem: no continuous inverse.

Key to stabilization:
A priori knowledge that $M \in \mathcal{M}_0$, with

$$\overline{\mathcal{M}}_0 \subset \mathcal{M} \quad \text{compact.} \tag{2}$$

Then $\mathcal{D}$ maps $\overline{\mathcal{M}}_0$ **homeomorphically** onto its image.

Stabilization of inverse boundary spectral problem:

$$\mathcal{M}_0 = \mathcal{M}(R_0, i_0, S_0, d_0).$$