Asymptotics for Some Non-Classical Conormal Distributions Whose Symbols Contain Negative Powers of $\log |\xi|$

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Abstract

We examine distributions on $\mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z})$ whose Fourier coefficients are of the form $(\log n)^{-1}$, and variants. These distributions are smooth except at $\theta = 0$, and the nature of their singularities at $\theta = 0$ turns out to be much more complex than those of their counterparts that involve positive powers of $\log n$. We also study related Fourier transforms. We move from one dimension to higher dimensions, where a wider variety of phenomena arise, and more subtle analytical techniques are called for.

Contents

*2010 Math Subject Classification: 35S05, 42A16, 42B10, 42B20. *Key words.* Fourier series, Fourier transform, conormal distribution, asymptotics, symbols, pseudodifferential operator, Work supported by NSF grant DMS-1500817.

1 Introduction

The series

$$\sum_{n=2}^{\infty} \frac{1}{\log n} \sin n\theta \tag{1.1}$$

appears in [3] as an example of a trigonometric series that converges pointwise for each $\theta \in \mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z})$ but is not the Fourier series of an L^1 function. The convergence can be demonstrated using the Dirichlet test for convergence of an infinite series (cf. [11], §2.31), and also as a special case of Riemann localization (given (1.2) below, cf. [9]). In the parlance of that time, it was said that (1.1) was not a Fourier series, a conclusion repeated in [11] and in [14]. Since the work of L. Schwartz, we say (1.1) is the Fourier series of a *distribution*, call it u_L, and we can say quite a bit about this distribution. For example, as we will see below,

$$u_L \in C^\infty(\mathbb{T}^1 \setminus 0). \tag{1.2}$$

Furthermore, methods from Chapter 5 of [14] yield that, for small $|\theta|$,

$$u_L(\theta) = -\frac{1}{\theta \log |\theta|} + O\Big(\frac{1}{|\theta|(\log |\theta|)^2}\Big). \tag{1.3}$$

The principal term on the right side of (1.3) is not in L^1, but it defines a principal value distribution, and we can say of the distribution u_L that

$$u_L + \mathrm{PV}\, \frac{1}{\theta \log |\theta|} \in L^1\Big(\Big[-\frac{1}{2}, \frac{1}{2}\Big]\Big). \tag{1.4}$$

Our primary goal in this paper centers about obtaining a much more precise description of the asymptotic behavior of $u_L(\theta)$ as $\theta \to 0$, as well as that of related distributions. We will see that the behavior is quite complex, much more complex than the behavior of the slightly more singular distribution

$$v_L = \sum_{n=2}^{\infty} (\log n) \sin n\theta, \tag{1.5}$$

which also belongs to $C^\infty(\mathbb{T}^1 \setminus 0)$.

To describe the behavior of $u_L(\theta)$ as $\theta \to 0$, we use the following class of special functions, of a sort introduced by B. Ziemian (cf. [12], [13], and also [6]), whose work was brought to our attention by R. Mazzeo. We set

$$K_\Phi(z) = \int_0^1 \Phi(s) z^s \, ds, \quad \text{for} \quad \mathrm{Re}\, z > 0, \quad \Phi \in C^\infty([0,1]), \tag{1.6}$$

and, more generally, for $b \in (0, \infty)$,

$$K_{\Phi,b}(z) = \int_0^b \Phi(s) z^s \, ds, \quad \operatorname{Re} z > 0, \quad \Phi \in C^\infty([0, b]). \tag{1.7}$$

As this introduction proceeds, we will discuss successive further generalizations, from integrating along paths in the complex domain to allowing $\Phi(s)$ to take values in a topological vector space. Integration by parts in (1.6) yields

$$K_\Phi(z) = -\frac{1}{\log z}\Big(\Phi(0) - \Phi(1)z\Big) - \frac{1}{\log z} K_{\Phi'}(z), \tag{1.8}$$

and more generally

$$K_{\Phi,b}(z) = -\frac{1}{\log z}\Big(\Phi(0) - \Phi(b)z^b\Big) - \frac{1}{\log z} K_{\Phi',b}(z). \tag{1.9}$$

These operations can be iterated, producing asymptotic expansions involving powers of $(\log z)^{-1}$, as $z \to 0$.

Our first improvement of (1.3) is

$$u_L(\theta) = \frac{1}{\theta} K_\psi(|\theta|) + O\big((\log |\theta|)^{-1}\big), \tag{1.10}$$

where

$$\psi(s) = \Gamma(1 - s) \sin \frac{\pi}{2}(1 - s) = \Gamma(1 - s) \cos \frac{\pi s}{2}. \tag{1.11}$$

Note that $\psi \in C^\infty([0, 1])$. Further improvements will be described below.

We find it convenient to work with Fourier integrals instead of Fourier series, so take

$$\begin{aligned} a \in C^\infty(\mathbb{R}), \quad a(\xi) = -a(-\xi), \quad a(\xi) &= \frac{1}{\log \xi} \ \text{ for } \ \xi \geq 2, \\ a(\xi) &= 0 \ \text{ for } \ |\xi| \leq 1, \end{aligned} \tag{1.12}$$

and consider

$$U_L(x) = \int_0^\infty a(\xi) \sin x\xi \, d\xi, \tag{1.13}$$

i.e.,

$$U_L(x) = \frac{1}{2i} \int_{-\infty}^\infty a(\xi) e^{ix\xi} \, d\xi. \tag{1.14}$$

This integral is not absolutely convergent, but it exists as an *oscillatory integral.* In more detail, the function $a(\xi)$ in (1.12) is a tempered distribution, i.e., $a \in \mathcal{S}'(\mathbb{R})$, and the Fourier transform maps $\mathcal{S}'(\mathbb{R})$ to itself, so (1.14) defines $U_L \in \mathcal{S}'(\mathbb{R})$. Further structure follows from the fact that

$$|a^{(k)}(\xi)| \le C_k(1+|\xi|)^{-k}, \tag{1.15}$$

and

$$x^k U_L(x) = \frac{i^k}{2i} \int_{-\infty}^{\infty} a^{(k)}(\xi) e^{ix\xi}\, d\xi, \tag{1.16}$$

and more generally

$$\Bigl(\frac{d}{dx}\Bigr)^{\ell} x^k U_L(x) = \frac{i^{k+\ell}}{2i} \int_{-\infty}^{\infty} \xi^{\ell} a^{(k)}(\xi) e^{ix\xi}\, d\xi, \tag{1.17}$$

so

$$k \ge \ell + 2 \Longrightarrow \Bigl|\Bigl(\frac{d}{dx}\Bigr)^{\ell} x^k U_L(x)\Bigr| \le C_{k\ell} < \infty. \tag{1.18}$$

It follows that U_L is C^∞ on $\mathbb{R} \setminus 0$ and rapidly decreasing, with all its derivatives, as $|x| \to \infty$.

With these estimates in hand, we can use the Poisson summation formula to write

$$u_L(\theta) = \sum_{k=-\infty}^{\infty} U_L(\theta + 2\pi k), \tag{1.19}$$

and see that(1.2) holds and that the singularity of u_L at $\theta = 0$ coincides with that of U_L. In particular, (1.10) is equivalent to

$$U_L(x) = \frac{1}{x} K_\psi(|x|) + O\bigl((\log|x|)^{-1}\bigr), \tag{1.20}$$

as $x \to 0$, with ψ as in (1.11).

A key ingredient in the proof of (1.20) is the identity

$$\int_0^\infty \frac{1}{\log\xi}\Bigl(1 - \frac{1}{\xi}\Bigr) \sin x\xi\, d\xi = \frac{1}{x} K_\psi(|x|), \tag{1.21}$$

with ψ as in (1.11), which we prove in §2. A tempting approach from here is to replace $1 - \xi^{-1}$ by $1 - \xi^{-k}$ for large k. A difficulty arises because the resulting integrand is then not Lebesgue integrable on $[0, 1]$.

One way around this difficulty is to integrate over $\xi \in [1, \infty)$. It is of interest to consider more generally, for $0 \le a < b$,

$$\begin{aligned} F_{ab}(x) &= \int_1^\infty \frac{1}{\log\xi}\Bigl(\xi^{-a} - \xi^{-b}\Bigr) e^{-ix\xi}\, d\xi \\ &= C_{ab}(x) - i S_{ab}(x), \end{aligned} \tag{1.22}$$

where

$$C_{ab}(x) = \int_1^\infty \frac{1}{\log \xi}\Big(\xi^{-a} - \xi^{-b}\Big) \cos x\xi \, d\xi,$$
$$S_{ab}(x) = \int_1^\infty \frac{1}{\log \xi}\Big(\xi^{-a} - \xi^{-b}\Big) \sin x\xi \, d\xi. \tag{1.23}$$

As with (1.13), these integrals exist as oscillatory integrals, which are C^∞ on $\mathbb{R} \setminus 0$. Note that $S_{01}(x)$ differs from (1.21) by the Fourier transform of a distribution with compact support, hence by a C^∞ function. We also replace (1.7) by

$$K_{\Phi,a,b}(z) = \int_a^b \Phi(s) z^s \, ds, \tag{1.24}$$

for $\operatorname{Re} z > 0$, $\Phi \in C^\infty([a,b])$. In such a case, integration by parts yields

$$K_{\Phi,a,b}(z) = -\frac{1}{\log z}\Big(\Phi(a)z^a - \Phi(b)z^b\Big) - \frac{1}{\log z} K_{\Phi',a,b}(z), \tag{1.25}$$

and this can be iterated, producing asymptotic expansions involving powers of $(\log z)^{-1}$, as $z \to 0$. We will show the following.

Theorem 1.1 *Assume* $0 \le a < b$ *and* $a, b \notin \{1,2,3,\dots\}$. *Then*

$$F_{ab}(x) \equiv \frac{1}{|x|} \int_a^b \Gamma(1-s) e^{-\pi i (\operatorname{sgn} x)(1-s)/2} |x|^s \, ds, \tag{1.26}$$

i.e.,

$$S_{ab}(x) \equiv \frac{1}{x} K_{\psi,a,b}(|x|),$$
$$C_{ab}(x) \equiv \frac{1}{|x|} K_{\varphi,a,b}(|x|), \tag{1.27}$$

with ψ *as in* (1.11) *and*

$$\varphi(s) = \Gamma(1-s) \cos \frac{\pi(1-s)}{2} = \Gamma(1-s) \sin \frac{\pi s}{2}. \tag{1.28}$$

Here we use the notation

$$f(x) \equiv g(x) \tag{1.29}$$

to mean that $f - g$ is C^∞ on a neighborhood of $x = 0$.

Note the following complication. Namely ψ and φ are not smooth on $[a,b]$ in general. In fact, ψ and φ are meromorphic in s, with simple poles at

$$\{2,4,6,8,\dots\} \quad \text{and} \quad \{1,3,5,7,\dots\}, \tag{1.30}$$

respectively. In light of this, we take (1.24) to mean we integrate from a to b along a path γ_{ab} from a to b in $\mathbb{C}$ that avoids these poles. If $\tilde{\gamma}_{ab}$ is another such path, a residue calculation shows that the two integrals differ by a function that is a polynomial in x. In this more general setting, integration by parts still works to produce (1.25), with Φ given by φ or ψ.

Recalling our initial interest in (1.1) and its associate (1.13), we see that an analysis of

$$\begin{aligned} F_a(x) &= \int_2^\infty \frac{1}{\log \xi} \xi^{-a} e^{-ix\xi}\, d\xi \\ &= C_a(x) - iS_a(x) \end{aligned} \tag{1.31}$$

is of primary interest, so of course it is useful to realize that

$$F_a - F_{ab} \in C^k(\mathbb{R}), \quad \text{provided} \quad b > k + 1. \tag{1.32}$$

Hence (1.27) for large b captures the behavior of the singularities of $S_a(x)$ and $C_a(x)$ near $x = 0$. Clearly U_L in (1.13) satisfies

$$U_L(x) \equiv S_0(x). \tag{1.33}$$

Thus the result for $S_{0b}(x)$ in (1.27) refines (1.20). Also, (1.27) with $a = 0$ complements this with

$$\begin{aligned} \int_1^\infty \frac{1}{\log \xi} \Big(1 - \xi^{-b}\Big) \cos x\xi\, d\xi &= C_{0b}(x) \\ &\equiv \frac{1}{|x|} K_{\varphi,b}(|x|), \end{aligned} \tag{1.34}$$

with φ as in (1.28) and $K_{\varphi,b}$ as in (1.7), again interpreted as an integral over a path from 0 to b in $\mathbb{C}$ that avoids the poles of $\varphi(s)$. Note that $\varphi(0) = 0$, while

$$\varphi'(0) = -\frac{\pi}{2}, \tag{1.35}$$

so an iteration of (1.9) gives

$$C_{0b}(x) = \frac{\pi/2}{|x|(\log |x|)^2} + O\Big(\frac{1}{|x|(\log |x|)^3}\Big), \tag{1.36}$$

accompanied by a further asymptotic expansion involving higher powers of $(\log |x|)^{-1}$.

The reason for the restriction on a and b in Theorem 1.1 is that since the path γ_{ab} needs to avoid the poles of $\psi(s)$ and $\varphi(s)$, given by (1.11) and (1.28), its endpoints must also avoid these poles. More precisely, the result

(1.27) for $S_{ab}(x)$ requires a and b to avoid $\{2,4,6,8,\dots\}$, and its counterpart for $C_{ab}(x)$ requires a and b to avoid $\{1,3,5,7,\dots\}$. For this reason, (1.21) fits into (1.27), but the analysis for $C_{01}(x)$ needs more work. Of course, in view of our discussion about (1.31)–(1.32), we generally envision taking b large, and do not care about whether it is an integer.

On the other hand, we do care about $S_{kb}(x)$ and $C_{kb}(x)$, for positive integers k (and $b > k$). To get useful information on these functions, we can use the identities

$$S'_{a+1,b+1}(x) = C_{ab}(x), \quad C'_{a+1,b+1}(x) = -S_{ab}(x), \tag{1.37}$$

integrate, and then proceed iteratively from $C_{0b}(x)$ and $S_{0b}(x)$ to $C_{k,b+k}(x)$ and $S_{k,b+k}(x)$, where $b > 0$ (and is not an integer). Examples start with

$$\begin{aligned} S_{1,b+1}(x) &\equiv (\operatorname{sgn} x) K_{\varphi_1,b}(|x|), \\ C_{1,b+1}(x) &\equiv \log\big|\log|x|\big| - K_{\psi_1,b}(|x|) - \operatorname{li}(|x|^b), \end{aligned} \tag{1.38}$$

where

$$\varphi_1(s) = \frac{\varphi(s)}{s}, \quad \psi_1(s) = \frac{\psi(s)-1}{s}, \quad \operatorname{li}(x) = \int_0^x \frac{dt}{\log t}, \tag{1.39}$$

and ψ and φ are as in (1.11) and (1.28). See §6 for more on this. We mention parenthetically that if $a \ge 0$ and $\ell > a$ is a positive integer, one can analyze $C_{a\ell}(x)$ and $S_{a\ell}(x)$ by picking a non-integer $b > \ell$ and using

$$F_{a\ell}(x) = F_{ab}(x) - F_{\ell b}(x). \tag{1.40}$$

As advertized, we establish the identity (1.21) in §2. More generally, we show that

$$\int_0^\infty \frac{1}{\log \xi}\Big(1 - \xi^{-b}\Big) \sin x\xi \, d\xi = \frac{1}{x} K_{\psi,b}(|x|) \quad \text{if} \quad 0 < b < 2, \tag{1.41}$$

and that

$$\int_0^\infty \frac{1}{\log \xi}\Big(1 - \xi^{-b}\Big) \cos x\xi \, d\xi = \frac{1}{|x|} K_{\varphi,b}(|x|) \quad \text{if} \quad 0 < b < 1, \tag{1.42}$$

where ψ and φ are as in (1.11) and (1.28). We also show that (1.41) implies (1.10).

Methods of §2 need to be modified for larger b, since then the integrands in (1.41) and (1.42) are not Lebesgue integrable on $[0,1]$. As mentioned, this motivates us to switch attention to the functions in (1.22)–(1.23), obtained by integrating over $\xi \in [1,\infty)$.

One approach to extending (1.41)–(1.42) is to work with the identities

$$
\begin{aligned}
C_{ab}(x) &= \frac{1}{|x|}\int_a^b \Gamma(1-s,ix)\cos\frac{\pi}{2}(1-s)\,|x|^s\,ds,\\
S_{ab}(x) &= \frac{1}{x}\int_a^b \Gamma(1-s,ix)\sin\frac{\pi}{2}(1-s)\,|x|^s\,ds,
\end{aligned}
\tag{1.43}
$$

where $\Gamma(z,ix)$ is the complementary incomplete gamma function, given by

$$
\Gamma(z,ix) = \int_{ix}^{\infty} e^{-t}t^{z-1}\,dt, \tag{1.44}
$$

which is an entire holomorphic function of z for each $x \in \mathbb{R}\setminus 0$. In this approach, the major task is to pass from (1.43) to (1.27). The approach via (1.43) can be made to work, but it is not the approach we take here.

One reason we do not use (1.43) is that this approach seems not to generalize beyond one dimension, while, as described below, we also aim for higher dimensional results in this paper. The approach we take is amenable to higher dimensional extensions. The key is to make use of the fact that the functions

$$
|\xi|^{-s} \quad \text{and} \quad (\operatorname{sgn}\xi)|\xi|^{-s}, \tag{1.45}
$$

which belong to $L^1_{\text{loc}}(\mathbb{R})$ for $\operatorname{Re} s < 1$, have meromorphic extensions to functions of s with values in $\mathcal{S}'(\mathbb{R})$, with a discrete set of poles, namely

$$
\begin{aligned}
|\xi|^{-s} &\ \text{ holomorphic for } \ s \notin \{1,3,5,\dots\},\\
(\operatorname{sgn}\xi)|\xi|^{-s} &\ \text{ holomorphic for } \ s \notin \{2,4,6,\dots\}.
\end{aligned}
\tag{1.46}
$$

These results are discussed in §3, together with higher dimensional extensions, including the following, with $r(x) = |x|,\ x \in \mathbb{R}^n$. Namely, given $a \in C^\infty(S^{n-1})$, $\omega = x/|x|$, $a(\omega)r^{-s}$, which is in L^1_{loc} for $\operatorname{Re} s < n$, has a meromorphic extnsion. We have

$$
\begin{aligned}
r^{-s} &\ \text{ holomorphic for } \ s \notin \{n, n+2, n+4,\dots\},\\
\frac{x_j}{r}r^{-s} &\ \text{ holomorphic for } \ s \notin \{n+1, n+3, n+5,\dots\}.
\end{aligned}
\tag{1.47}
$$

More generaly, if $h_\ell(x)$ is a harmonic polynomial on $\mathbb{R}^n$, homogeneous of degree ℓ, then we have a meromorphic extension of $h_\ell(\omega)r^{-s}$:

$$
h_\ell(\omega)r^{-s} \ \text{ holomorphic for } \ s \notin \{n+\ell, n+\ell+2, n+\ell+4,\dots\}, \tag{1.48}
$$

in the sense that $s \mapsto h_\ell(\omega)r^{-s}$ is a meromorphic function of s with values in $\mathcal{S}'(\mathbb{R}^n)$, with such poles. It follows that the Fourier transform $\mathcal{F}$, defined on $\mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}u(\xi) = (2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} u(x)e^{-ix\cdot\xi}\, dx, \tag{1.49}$$

and extended by duality to $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, yields a meromorphic family of tempered distributions $\mathcal{F}(h_\ell(\omega)r^{-s})$, with poles as in (1.48). Further results, established in §3, yield that, for each $a \in C^\infty(S^{n-1})$, $a(\omega)r^{-s}$ is meromorphic in s with values in $\mathcal{S}'(\mathbb{R}^n)$, and with the exception of such poles (and another technical exception),

$$\mathcal{F}(a(\omega)r^{-s}) = A_n(s)a(\omega)\, r^{s-n}, \tag{1.50}$$

where $A_n(s)$ is a meromorphic function of s with values in the space of linear operators on $C^\infty(S^{n-1})$, of a sort given a rather precise analysis in §3. We present $A_n(s)$ as a product of a unitary Fourier integral operator (independent of s) and a pseudodifferential operator on S^{n-1} of order $-s + n/2$.

In §3 we also consider cut-offs,

$$(1 - \varphi(x))a(\omega)r^{-s}, \tag{1.51}$$

with $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi(x) = 1$ for $|x|$ small. Then (1.51) is an entire holomorphic function of s with values in $\mathcal{S}'(\mathbb{R}^n)$, and so is its Fourier transform. Away from the poles that arise for $a(\omega)r^{-s}$, this Fourier transform has the same singularity at the origin as (1.50), and at the poles $\log r$ factors arise. We derive a uniform analysis of the singularity for $\mathcal{F}((1 - \varphi(x))a(\omega)r^{-s})$, for s in a neighborhood of such a pole, bringing in

$$q_\sigma = \frac{r^\sigma - 1}{\sigma}, \tag{1.52}$$

an entire function of σ with values in $\mathcal{S}'(\mathbb{R}^n)$, satisfying $q_0 = \log r$, which will be useful in §8.

Results of §§2–3 are used in §4 to prove Theorem 1.1. We continue to obtain one dimensional results in §§5–7. In §5 we derive a number of useful properties of the functions $K_{\Phi,b}$, in preparation for §6, which extends the analysis of F_{ab} to include the cases $a = k \in \mathbb{N}$, bringing in additional special functions, such as seen in (1.38). In §7 we extend the scope of our study of F_a to

$$F_{[k]a}(x) = \int_2^\infty \frac{1}{(\log \xi)^k} \xi^{-a} e^{-ix\xi}\, d\xi, \tag{1.53}$$

for $k \in \mathbb{N}$. The techniques brought to bear in §§6–7 strongly use the one-dimensional structure, and different techniques are developed in the following sections to handle higher dimensions.

Section 8 provides higher dimensional extensions of Theorem 1.1 and subsequent results, such as (1.38). Theorem 8.1 establishes a formula that reveals the small $|x|$ behavior of

$$(2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} \frac{h_\ell(\hat{\xi})}{\log|\xi|} \Bigl(|\xi|^{-a} - |\xi|^{-b}\Bigr) e^{-ix\cdot\xi}\, d\xi, \tag{1.54}$$

where $\hat{\xi} = \xi/|\xi|$, in terms of a function of the form (1.24), provided a and b avoid the poles described in (1.48). A key ingredient is the identity of (1.54) with

$$\int_a^b A_n(s) p(\hat{x}) |x|^{s-n}\, ds, \tag{1.55}$$

with $A_n(s)$ as in (1.50), $p = h_\ell$, and the integral taken along a path in $\mathbb{C}$ that avoids such poles. The behavior of (1.54), with a cut-off $1 - \varphi(\xi)$ thrown in, for $a = n + \ell + 2k,\ k \in \{0, 1, 2, \dots\}$, is given in Theorem 8.2, and, as in (1.38), brings in also $\log\bigl|\log|x|\bigr|$ and $\mathrm{li}(|x|^\beta)$, this time with $\beta = b - a$. Whereas (1.38) was derived from Theorem 1.1 by integrating in x, the proof of Theorem 8.2 requires a different technique. We use the uniform analysis of the Fourier transform of (1.51) for s near a pole of (1.50), mentioned above.

More generally than (1.54), we analyze

$$(2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} \frac{p(\hat{\xi})}{\log|\xi|} \Bigl(|\xi|^{-a} - |\xi|^{-b}\Bigr) e^{-ix\cdot\xi}\, d\xi, \tag{1.56}$$

for general $p \in C^\infty(S^{n-1})$, in §8. This is also given by (1.55), assuming a and b are not poles of $A_n(s)p$ and we integrate over a path from a to b in $\mathbb{C}$ that avoids such poles. This leads to one further expansion of the notion of $K_{\Phi,a,b}(z)$, from that given in (1.24). Namely, $\Phi(s)$ can be a meromorphic function of s with values in some Frechet space, in this case $C^\infty(S^{n-1})$ (one might imagine other classes of complete, locally convex, topological vector spaces), assuming a and b are not poles of Φ, and we integrate from a to b along a path in $\mathbb{C}$ that avoids these poles. In this setting, (1.56) is equal to

$$|x|^{-n} K_{\Phi,a,b}(|x|), \quad \Phi(s) = A_n(s)p. \tag{1.57}$$

Making use of Theorem 8.2, we also treat the case when a is a pole of $A_n(s)p$, in Proposition 8.3. This involves throwing a factor of $1 - \varphi(\xi)$ into (1.56).

In §9 we move from (1.56) to

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} (1-\varphi(\xi)) \frac{p(\hat{\xi})}{(\log|\xi|)^k} \Bigl(|\xi|^{-a} - |\xi|^{-b}\Bigr) e^{-ix\cdot\xi}\, d\xi, \tag{1.58}$$

with $k \geq 2$. We show that this has the form

$$\frac{|x|^{-n}}{(k-1)!} K_{\Psi_k,a,b}(|x|), \quad \text{mod} \ \ C^m(\mathbb{R}^n), \tag{1.59}$$

with

$$\Psi_k(s) = (s-a)^{k-1} A_n(s) p, \tag{1.60}$$

provided $b > n+m$ and $b \notin \mathbb{Z}$. Here $K_{\Psi_k,a,b}(|x|)$ is treated in the framework of (1.57).

In §10 we replace $\log|\xi|$ by $\log\lambda(\xi)$, where λ is smooth, positive, and homogeneous of degree one on $\mathbb{R}^n \setminus 0$, so $\lambda(\xi) = q(\hat{\xi})|\xi|$ for a positive $q \in C^\infty(S^{n-1})$. We analyze

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} \frac{p(\hat{\xi})}{\log\lambda(\xi)} \Bigl(\lambda(\xi)^{-a} - \lambda(\xi)^{-b}\Bigr) e^{-ix\cdot\xi}\, d\xi, \tag{1.61}$$

and show this is

$$|x|^{-n} K_{\Psi,a,b}(|x|), \quad \Psi(s) = A_n(s)(pq^{-s}), \tag{1.62}$$

for $a, b \notin \mathcal{E}_{p,q}$, the set of poles of $A_n(s)(pq^{-s})$, which is a subset of $\{n, n+1, n+2, \dots\}$. Methods of §9 can be extended to replace $(\log\lambda(\xi))^{-1}$ by $(1-\varphi(\xi))(\log\lambda(\xi))^{-k}$ in (1.61), but we omit the details.

Results of this paper put one in a position to treat variable coefficients, allowing $p(\hat{\xi})$ and $\lambda(\xi)$ to be replaced by $p(x,\hat{\xi})$ and $\lambda(x,\xi)$ in (1.56), (1.58), and (1.61). From here, we can envisage deriving precise asymptotics near the diagonal for the Schwartz kernels of operators such as

$$\Bigl(\log\sqrt{2-\Delta_M}\Bigr)^{-1}, \tag{1.63}$$

and related operators, where Δ_M is the Laplace-Beltrami operator on a compact Riemannian manifold M. We expect to be able to derive from such analysis results such as the following. Let L denote the operator in (1.63). We claim that

$$L : \mathcal{M}(M) \longrightarrow L^1(M), \tag{1.64}$$

where $\mathcal{M}(M)$ denotes the space of finite Borel measures on M. For general $A \in OPS^0(M)$, AL fails to have such a mapping property, but we claim that AL^2 does have it. In case $M = \mathbb{T}^n$ is a flat torus, and A is translation-invariant, such assertions follow from results of §§8–10. We plan to take these issues up in a future work.

This paper ends with some appendices. Appendix A gives a direct proof of the weak asymptotic result (1.3), using an argument adapted from Chapter 5 of [14].

Appendix B derives the asymptotic behavior of (1.5) for $\theta \to 0$. We show that

$$v_L - \Big(\mathrm{PV}\,\frac{1}{\theta}\log|\theta| - \gamma\,\mathrm{PV}\,\frac{1}{\theta}\Big) \in C^\infty((-\pi,\pi)), \tag{1.65}$$

where γ is Euler's constant.

Appendix C provides some technical analysis of the operators $A_n(s)$, introduced in (1.50), of use in the proof of the results in §8.

We make some further comments about how (1.65) contrasts with the behavior of u_L. Such v_L as in (1.65) belongs to a space of *polyhomogeneous distributions*, which can be defined as follows. We will work on $\mathbb{R}$ and use the variable x. First, if $\alpha \in \mathbb{C}$ and $\operatorname{Re}\alpha > -1$, we denote by $\mathbf{H}_\alpha(\mathbb{R})$ the space of finite linear combinations of functions of the form

$$(x_+)^\alpha(\log|x|)^\ell, \quad ((-x)_+)^\alpha(\log|x|)^\ell, \quad \ell \in \mathbb{Z}^+ = \{0,1,2,\dots\}, \tag{1.66}$$

where $x_+ = x$ for $x > 0$, 0 for $x < 0$. We then say $u \in \mathbf{H}_{\alpha-k}(\mathbb{R})$ if $u = d^k v/dx^k$ for some $v \in \mathbf{H}_\alpha(\mathbb{R})$. We say $u \in \mathcal{S}'(\mathbb{R})$ is polyhomogeneous if

$$u \sim \sum_{j\ge 0} u_j, \quad u_j \in \mathbf{H}_{\alpha_j}(\mathbb{R}), \tag{1.67}$$

where $\operatorname{Re}\alpha_j \nearrow +\infty$ as $j \to \infty$. (We might require $\alpha_j = \alpha_0 + j$.) Here "$\sim$" means that, for each k, there exists n such that

$$u - \sum_{j=0}^n u_j \in C^k(\mathbb{R}). \tag{1.68}$$

The two terms in (1.65) belong to $\mathbf{H}_{-1}(\mathbb{R})$, since

$$\mathrm{PV}\,\frac{1}{x} = \frac{d}{dx}\log|x|, \quad \mathrm{PV}\,\frac{1}{x}\log|x| = \frac{1}{2}\frac{d}{dx}(\log|x|)^2. \tag{1.69}$$

The phrase "polyhomogeneous symbol" appears in §18.1 of [1]. These symbols have the form

$$a(x,\xi) \sim \sum_{j\ge 0} a_j(x,\xi), \tag{1.70}$$

with $a_j(x,\xi)$ smooth on $\mathbb{R}^n \times \mathbb{R}^n$, and homogeneous of degree $m - j/k$ for $|\xi| \geq 1$, for some $m \in \mathbb{C}$, $k \in \mathbb{N}$. These have also been called "classical symbols," especially when $k = 1$. Note that log terms do not appear here. However, log terms can appear in the Schwartz kernels of the associated pseudodifferential operators, i.e., in the Fourier transforms of such $a(\xi)$, considered as elements of $\mathcal{S}'(\mathbb{R}^n)$, in the x-independent case. This occurs, for example, is $P(\xi)$ is an elliptic polynomial on $\mathbb{R}^n$, of order $m \leq n$, and $a(\xi) = (1 - \varphi(\xi))P(\xi)^{-1}$, for an appropriate cut-off $\varphi(\xi)$. Then one has (cf. [8], Chapter 3, Proposition 9.2)

$$E = \hat{a}(x) \sim \sum_{\ell \geq 0} (E_\ell + p_\ell(x) \log |x|), \tag{1.71}$$

with $E_\ell \in \mathcal{S}'(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus 0)$ homogeneous of degree $m - n + \ell$ and $p_\ell(x)$ a polynomial homogeneous of degree $m - n + \ell$, the latter making an appearance only for $\ell \geq n - m$. The extended notion of polyhomogeneous cononormal distributions appears in [4].

Distributions such as u_L, U_L, F_a, C_a, and S_a are not polyhomogeneous. The analysis of their singularities, discussed above, requires expansions much different from (1.67). An appropriate parallel to (1.67) is

$$S_a(x) \sim \frac{1}{x} K_{\psi,a,b}(|x|), \quad b \to \infty. \tag{1.72}$$

In this formulation, the special function $(1/x)K_{\psi,a,b}(|x|)$ is analogous to $\sum_{j=0}^n u_j$ in (1.68). On the other hand, the asymptotic expansion of $K_{\psi,a,b}(|x|)$ derivable from (1.25), when substituted in, leads to a relatively weak result, for example

$$U_L(x) \sim -\frac{1}{x \log |x|} + \sum_{j \geq 2} \frac{a_j}{x (\log |x|)^j}. \tag{1.73}$$

in this case, one has a result about the difference between U_L and

$$-\frac{1}{x \log |x|} + \sum_{j=2}^{n} \frac{a_j}{x (\log |x|)^j} \tag{1.74}$$

that is much weaker than (1.68), even much weaker than (1.20). In light of this, we recognize (1.72) as a different paradigm for the sort of asymptotic expansion relevant to our study of such a "nonclassical" conormal distribution.

2 First key identities, and proof of (1.10)

We start with the elementary identity

$$\begin{aligned}\int_a^b \xi^{-s}\,ds &= \int_a^b e^{-s\log\xi}\,ds \\ &= \frac{\xi^{-a}-\xi^{-b}}{\log\xi},\end{aligned} \tag{2.1}$$

valid for $a < b,\ \xi > 0$ (suitably interpreted for $\xi = 1$), and insert it into the identity

$$\begin{aligned}\int_0^\infty \xi^{-s} e^{-\varepsilon\xi} e^{-ix\xi}\,d\xi &= \int_0^\infty e^{-(\varepsilon+ix)\xi}\xi^{-s}\,d\xi \\ &= (\varepsilon+ix)^{s-1}\Gamma(1-s),\end{aligned} \tag{2.2}$$

valid for $\varepsilon > 0,\ s < 1$, to get

$$\int_0^\infty \frac{1}{\log\xi}\Big(\xi^{-a}-\xi^{-b}\Big)e^{-\varepsilon\xi}e^{-ix\xi}\,d\xi = \int_a^b \Gamma(1-s)(\varepsilon+ix)^{s-1}\,ds, \tag{2.3}$$

provided also $b < 1$. We next pass to the limit $\varepsilon \searrow 0$. Note that, for $x \in \mathbb{R}\setminus 0$,

$$(\varepsilon+ix)^{s-1} = (\varepsilon^2+x^2)^{(s-1)/2} e^{i(s-1)\tan^{-1}(x/\varepsilon)}, \tag{2.4}$$

so

$$\lim_{\varepsilon\searrow 0}(\varepsilon+ix)^{s-1} = (ix+0)^{s-1} = |x|^{s-1}e^{\pi i(\operatorname{sgn} x)(s-1)/2}, \tag{2.5}$$

and hence

$$\begin{aligned}&\int_0^\infty \frac{1}{\log\xi}\Big(\xi^{-a}-\xi^{-b}\Big)e^{-ix\xi}\,d\xi \\ &\qquad = \frac{1}{|x|}\int_a^b \Gamma(1-s)e^{\pi i(\operatorname{sgn} x)(1-s)/2}|x|^s\,ds,\end{aligned} \tag{2.6}$$

for $0 \le a < b < 1$, the left side being a priori a tempered distribution on $\mathbb{R}$. Taking real and imaginary parts yields

$$\begin{aligned}&\int_0^\infty \frac{1}{\log\xi}\Big(\xi^{-a}-\xi^{-b}\Big)\cos x\xi\,d\xi \\ &\qquad = \frac{1}{|x|}\int_a^b \Gamma(1-s)\cos\frac{\pi}{2}(1-s)\,|x|^s\,ds \\ &\qquad = \frac{1}{|x|}K_{\varphi,a,b}(|x|),\end{aligned} \tag{2.7}$$

and

$$\int_0^\infty \frac{1}{\log \xi}\Big(\xi^{-a}-\xi^{-b}\Big)\sin x\xi\, d\xi$$
$$= \frac{1}{x}\int_a^b \Gamma(1-s)\sin\frac{\pi}{2}(1-s)\,|x|^s\, ds \tag{2.8}$$
$$= \frac{1}{x}K_{\psi,a,b}(|x|),$$

for $0 \le a < b < 1$, with ψ and φ as in (1.11) and (1.28). Taking $a = 0$ yields (1.41)–(1.42). We can pass to the limit $b \nearrow 1$ in (2.8), obtaining (1.21). However, when $b \nearrow 1$ in (2.7), both sides diverge.

Taking a closer look at (2.8), we note that both the first and the second integrals there are convergent, near $\xi = 0$ and on $s \in [a, b]$, respectively, as long as $0 \le a < b < 2$. To pass from our demonstration of their equality when $0 \le a < b \le 1$, to equality in this more general case, we can note that both integrals are well defined for complex b, with $a < \operatorname{Re} b < 2$, and are holomorphic in b, so the identity (2.8) analytically continues. Specializing to real b, we have it for $0 \le a < b < 2$.

Using this, we can establish (1.20) (hence (1.10)) and some refinements, as follows. The function $U_L(x)$, defined by (1.14), differs by a smooth, odd function of x from

$$\int_2^\infty \frac{1}{\log \xi}\sin x\xi\, d\xi, \tag{2.9}$$

and (2.8), (with $a = 0$) differs by a smooth, odd function from

$$\int_2^\infty \frac{1}{\log \xi}\Big(1-\xi^{-b}\Big)\sin x\xi\, d\xi, \tag{2.10}$$

whenever $0 < b < 2$. Hence

$$U_L(x) = \frac{1}{x}K_{\psi,b}(|x|) + \int_2^\infty \frac{\xi^{-b}}{\log \xi}\sin x\xi\, d\xi + O(|x|), \tag{2.11}$$

for each $b \in (1, 2)$. Meanwhile, taking $b = 1+\beta,\ 0 < \beta < 1$, and using $|\sin x\xi| \le |x\xi|$ for $|x\xi| \le 1$, we have

$$\Big|\int_2^\infty \frac{\xi^{-b}}{\log \xi}\sin x\xi\, d\xi\Big|$$
$$\le |x|\int_2^{1/|x|} \frac{\xi^{-\beta}}{\log \xi}\, d\xi + \int_{1/|x|}^\infty \frac{\xi^{-1-\beta}}{\log \xi}\, d\xi \tag{2.12}$$
$$\le C|x|^\beta,$$

for $|x| \le 1/2$. Consequently, for $|x| \le 1/2$,

$$\Big|U_L(x) - \frac{1}{x}K_{\psi,b}(|x|)\Big| \le C_b|x|^{b-1}, \quad \text{for} \ \ 1 < b < 2. \tag{2.13}$$

To go from here to (1.20), we merely note that, for $1 < b < 2$,

$$K_{\psi,b}(|x|) = K_\psi(|x|) + K_{\psi,1,b}(|x|), \tag{2.14}$$

and, by (1.25),

$$\big|K_{\psi,1,b}(|x|)\big| \le \frac{C|x|}{\big|\log|x|\big|}. \tag{2.15}$$

3 Meromorphic families of tempered distributions

Here we study some classes of functions $u(s)$, taking values in the Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions, that depend holomorphically on s, except for some poles. We start with some families that are homogeneous, of a degree that varies with s.

We recall some well known facts about homogeneous distributions, which can be found, for example, in Chapter 3 of [8].

The dilation group $D(t)f(x) = f(tx)$ $(t > 0)$ extends to distributions, and we say $u \in \mathcal{D}'(\mathbb{R}^n)$ is homogeneous of degree m if $D(t)u = t^m u$ for all $t > 0$. Here, $m \in \mathbb{C}$. We set

$$\begin{aligned} \mathcal{H}_m(\mathbb{R}^n) &= \{u \in \mathcal{D}'(\mathbb{R}^n) : u \text{ is homogeneous of degree } m\}, \\ \mathcal{H}_m^{\#}(\mathbb{R}^n) &= \{u \in \mathcal{H}_m(\mathbb{R}^n) : u \in C^\infty(\mathbb{R}^n \setminus 0)\}. \end{aligned} \tag{3.1}$$

We have $\mathcal{H}_m(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, and

$$\begin{aligned} \mathcal{F} &: \mathcal{H}_m(\mathbb{R}^n) \longrightarrow \mathcal{H}_{-m-n}(\mathbb{R}^n), \\ \mathcal{F} &: \mathcal{H}_m^{\#}(\mathbb{R}^n) \longrightarrow \mathcal{H}_{-m-n}^{\#}(\mathbb{R}^n). \end{aligned} \tag{3.2}$$

If we set $r^{-s}(x) = |x|^{-s}$, then $r^{-s} \in L^1_{\text{loc}}(\mathbb{R}^n)$ for $\operatorname{Re} s < n$ defines a holomorphic function from $\{s \in \mathbb{C} : \operatorname{Re} s < n\}$ to $\mathcal{S}'(\mathbb{R}^n)$, satisfying $r^{-s} \in \mathcal{H}_{-s}^{\#}(\mathbb{R}^n)$ for such s. By (3.2), $\mathcal{F}(r^{-s}) \in \mathcal{H}_{s-n}^{\#}(\mathbb{R}^n)$ for such s. The fact that $\mathcal{F}$ commutes with the natural action of the orthogonal group $O(n)$ on $\mathcal{S}'(\mathbb{R}^n)$ inplies $\mathcal{F}(r^{-s}) = c(s)r^{n-n}$ provided $\operatorname{Re} s < n$ and $\operatorname{Re}(n-s) < n$, and a computation that involves testing against a Gaussian (cf. [8], Chapter 3, (8.32)–(8.35)) yields

$$\mathcal{F}(r^{-s}) = 2^{n/2-s}\Gamma\Big(\frac{n-s}{2}\Big)\Gamma\Big(\frac{s}{2}\Big)^{-1} r^{s-n}, \tag{3.3}$$

for $0 < \operatorname{Re} s < n$. Equivalently,

$$2^{s/2}\Gamma\Big(\frac{n-s}{2}\Big)^{-1}\mathcal{F}(r^{-s}) = 2^{(n-s)/2}\Gamma\Big(\frac{s}{2}\Big)^{-1} r^{s-n}, \tag{3.4}$$

for such s. Now $\Gamma(z)$ is meromorphic in z, with poles at $\{0,-1,-2,\dots\}$ and no zeros, so $\Gamma(z)^{-1}$ is entire, with zeros at $\{0,-1,-2,\dots\}$. It follows that the left side of (3.4) is holomorhic in $\{\operatorname{Re} s < n\}$ and the right side is holomorphic in $\{\operatorname{Re} s > 0\}$. Thus both sides have entire analytic continuations, and this defines

$$r^{-s} \in \mathcal{H}^{\#}_{-s}(\mathbb{R}^n), \quad \text{for} \quad s \notin \{n, n+2, n+4, \dots\}, \tag{3.5}$$

as a meromorphic function of s with values in $\mathcal{S}'(\mathbb{R}^n)$. We note parenthetically that if $s = -2k$ is an even, non-positive integer, then r^{-s} is a polynomial, so the left side of (3.4) is a linear combination of derivatives of the delta function δ and consequently so is the limit as $s \to -2k$ of $\Gamma(s/2)^{-1} r^{s-n}$.

Turning to the case $n = 1$, we note that (2.2), (2.4), and (2.5) yield

$$\int_{\mathbb{R}} |\xi|^{-s} e^{-ix\xi}\, d\xi = 2\Big(\sin\frac{\pi s}{2}\Big)\Gamma(1-s)|x|^{s-1}, \tag{3.6}$$

and

$$\int_{\mathbb{R}} (\operatorname{sgn}\xi)|\xi|^{-s} e^{-ix\xi}\, d\xi = -2\Big(\cos\frac{\pi s}{2}\Big)\Gamma(1-s)(\operatorname{sgn} x)|x|^{s-1}, \tag{3.7}$$

for $0 < \operatorname{Re} s < 1$. The equivalence of (3.6) to (3.3) when $n = 1$ (and $\operatorname{Re} s < 1$) follows from standard identities for the gamma function. As for (3.7), the left side is holomorphic for $\operatorname{Re} s < 1$, and the right side is holomorphic for $\operatorname{Re} s > 0$, except for poles of $(\cos \pi s/2)\Gamma(1-s)$ at $s = 2, 4, 6, \dots$ (the poles of $\Gamma(1-s)$ at $s = 1, 3, 5, \dots$ being cancelled by zeros of $\cos \pi s/2$). The upshot is that we get a meromorphic continuation of $(\operatorname{sgn}\xi)|\xi|^{-s}$, yielding

$$(\operatorname{sgn}\xi)|\xi|^{-s} \in \mathcal{H}^{\#}_{-s}(\mathbb{R}), \quad \text{for} \quad s \notin \{2, 4, 6, \dots\}. \tag{3.8}$$

This result can also be deduced from

$$\frac{d}{d\xi}|\xi|^{-s+1} = (1-s)(\operatorname{sgn}\xi)|\xi|^{-s}, \tag{3.9}$$

and the $n = 1$ case of (3.5) (with s replaced by $s - 1$), except for the case $s = 1$, where we have instead that $(d/d\xi)\log|\xi| = (\operatorname{sgn}\xi)|\xi|^{-1} = \mathrm{PV}\, 1/\xi$.

Moving to higher dimensions, we claim that $(x_j/r)r^{-s}$, which is in $L^1_{\rm loc}(\mathbb{R}^n)$ for $\operatorname{Re} s < n$, has a meromorphic continuation satisfying

$$\frac{x_j}{r} r^{-s} \in \mathcal{H}^{\#}_{-s}(\mathbb{R}^n), \quad \text{for} \quad s \notin \{n+1, n+3, n+5, \dots\}. \tag{3.10}$$

In fact, parallel to (3.9), we have

$$\partial_j r^{-s+1} = (1-s)\frac{x_j}{r} r^{-s}, \tag{3.11}$$

which yields (3.10) as a consequence of (3.5), except for $s = 1$ and $s = n-1$, and both of these cases are elementary since then $(x_j/r)r^{-s} \in L^1_{\rm loc}(\mathbb{R}^n)$ (given $n > 1$). Note that

$$\mathcal{F}(x_j r^{-s-1}) = i\partial_j \mathcal{F}(r^{-s-1}), \tag{3.12}$$

and if we apply ∂_j to (3.3), with s replaced by $s+1$, we get

$$\mathcal{F}\Bigl(\frac{x_j}{r} r^{-s}\Bigr) = i2^{n/2-s-1}\Gamma\Bigl(\frac{n-s-1}{2}\Bigr)\Gamma\Bigl(\frac{s+1}{2}\Bigr)^{-1}\partial_j r^{s+1-n}, \tag{3.13}$$

and, since

$$(s+1-n)\Gamma\Bigl(\frac{n-s-1}{2}\Bigr) = -2\Gamma\Bigl(\frac{n-s+1}{2}\Bigr), \tag{3.14}$$

we get

$$\mathcal{F}\Bigl(\frac{x_j}{r} r^{-s}\Bigr) = -i2^{n/2-s}\Gamma\Bigl(\frac{n-s+1}{2}\Bigr)\Gamma\Bigl(\frac{s+1}{2}\Bigr)^{-1}\frac{x_j}{r} r^{s-n}. \tag{3.15}$$

As a check, (3.10) implies the left side of (3.15) has poles at $\{s = n+1, n+3, n+5, \dots\}$. As for the right side, the numerator in the quotient of gamma functions has poles at $\{s = n+1, n+3, n+5, \dots\}$, and the factor $(x_j/r)r^{s-n}$ has poles at $\{s = -1, -3, -5, \dots\}$, which are cancelled by the poles of $\Gamma((s+1)/2)$.

Without using (3.10), we see that it is elementary that both sides of (3.15) are holomorphic in s for $0 < \operatorname{Re} s < n$, that the left side is holomorphic on $\operatorname{Re} s < n$, and the right side is holomorphic for $\operatorname{Re} s > 0$, except for the poles at $\{s = n+1, n+3, n+5, \dots\}$, so we are again led to (3.10).

One can continue along this line, using

$$\partial_j \partial_k r^{-s+2} = (2-s)(1-s)\frac{x_j x_k}{r^2} r^{-s}, \quad j \neq k. \tag{3.16}$$

The left side is holomorphic on $s \notin \{n+2, n+4, n+6, \dots\}$, so if $j \neq k$

$$\frac{x_j x_k}{r^2} r^{-s} \in \mathcal{H}^{\#}_{-s}(\mathbb{R}^n) \ \ \text{for} \ \ s \notin \{n+2, n+4, n+6, \dots\}, \tag{3.17}$$

except possibly for $s = 1$ and $s = 2$. However $(x_j x_k / r^2) r^{-s}$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for $s = 1$ if $n > 1$ and for $s = 2$ if $n > 2$. The case $s = n = 2$ will be taken care of below. One can also compute the Fourier transform of (3.17) using a device parallel to that applied to compute (3.15). Rather than pursue this, we will move up to a greater level of generality.

Namely, we let $h_\ell(x)$ be a harmonic polynomial, homogeneous of degree ℓ, and consider

$$\frac{h_\ell(x)}{r^\ell} r^{-s} = h_\ell(\omega) r^{-s}, \quad \omega = \frac{x}{|x|}. \tag{3.18}$$

This is in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for $\operatorname{Re} s < n$, and it is a holomorphic function of s with values in $\mathcal{S}'(\mathbb{R}^n)$, and we have

$$h_\ell(\omega) r^{-s} \in \mathcal{H}^{\#}_{-s}(\mathbb{R}^n), \tag{3.19}$$

for such s. If also $\operatorname{Re} s > 0$, then its Fourier transform, which is in $\mathcal{H}^{\#}_{s-n}(\mathbb{R}^n)$, is in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. In particular, it has the form

$$a_\ell(\omega) r^{n-s}, \tag{3.20}$$

for some $a_\ell \in C^\infty(S^{n-1})$. We may as well assume $n \geq 2$, since the case $n = 1$ is thoroughly covered by (3.6)–(3.8).

Let us denote by $\mathcal{H}^{\#}_{-s,\ell}(\mathbb{R}^n)$ the space of elements of $\mathcal{H}^{\#}_{-s}(\mathbb{R}^n)$ of the form (3.19), where h_ℓ is some harmonic polynomial, homogeneous of degree ℓ. For $\operatorname{Re} s < n$, we have a direct sum decomposition

$$\mathcal{H}^{\#}_{-s}(\mathbb{R}^n) = \bigoplus_{\ell \geq 0} \mathcal{H}^{\#}_{-s,\ell}(\mathbb{R}^n), \tag{3.21}$$

coming from

$$C^\infty(S^{n-1}) = \bigoplus_{\ell \geq 0} E_{n,\ell}, \tag{3.22}$$

where $E_{n,\ell}$ is the eigenspace of the Laplace-Beltrami operator Δ_S on S^{n-1} with eigenvalue $\lambda_\ell = -\ell(\ell + n - 2)$ (cf. [8], Chapter 8). Now the orthogonal group $O(n)$ acts on $C^\infty(S^{n-1})$, commuting with Δ_S and hence preserving each eigenspace $E_{n,\ell}$, so it acts on $\mathcal{H}^{\#}_{-s}(\mathbb{R}^n)$, preserving each space $\mathcal{H}^{\#}_{-s,\ell}(\mathbb{R}^n)$. Furthermore, as we have noted before, $\mathcal{F}$ commutes with the action of $O(n)$. Therefore

$$\mathcal{F} : \mathcal{H}^{\#}_{-s,\ell}(\mathbb{R}^n) \longrightarrow \mathcal{H}^{\#}_{s-n,\ell}(\mathbb{R}^n). \tag{3.23}$$

In addition, $O(n)$ acts irreducibly on each eigenspace $E_{n,\ell}$, provided $n \geq 2$. It follows via Schur's lemma (at least for $0 < \operatorname{Re} s < n$, for now) that

$$\mathcal{F}(h_\ell(\omega)r^{-s}) = c_{n,\ell}(s)h_\ell(\omega)r^{s-n}, \quad \forall\, h_\ell \in E_{n,\ell}. \tag{3.24}$$

To compute the coefficients $c_{n,\ell}(s)$, we take

$$h_\ell(x) = (x_1 + ix_2)^\ell, \tag{3.25}$$

and use the identity

$$\begin{aligned}\mathcal{F}(h_\ell(\omega)r^{-s}) &= \mathcal{F}(h_\ell(x)r^{-s-\ell}) \\ &= i^\ell(\partial_1 + i\partial_2)^\ell \mathcal{F}(r^{-s-\ell}),\end{aligned} \tag{3.26}$$

together with (3.3) with s replaced by $s+\ell$, i.e.,

$$\mathcal{F}(r^{-s-\ell}) = 2^{n/2-s-\ell}\Gamma\Big(\frac{n-s-\ell}{2}\Big)\Gamma\Big(\frac{s+\ell}{2}\Big)^{-1} r^{s+\ell-n}. \tag{3.27}$$

Note that

$$(\partial_1 + i\partial_2)r^\sigma = \sigma(x_1 + ix_2)r^{\sigma-2}, \tag{3.28}$$

and $(\partial_1 + i\partial_2)(x_1 + ix_2)^j = 0$, so, inductively,

$$(\partial_1 + i\partial_2)^k r^\sigma = \sigma(\sigma-2)\cdots(\sigma - 2(k-1))(x_1 + ix_2)^k r^{\sigma-2k}. \tag{3.29}$$

Hence

$$\begin{aligned}&(\partial_1 + i\partial_2)^\ell r^{s+\ell-n} \\ &= (s+\ell-n)(s+\ell-n-2)\cdots(s+\ell-2-2(\ell-1))\Big(\frac{x_1+ix_2}{r}\Big)^\ell r^{s-n}.\end{aligned} \tag{3.30}$$

We can use this to apply $(\partial_1 + i\partial_2)^\ell$ to (3.27). Note that

$$\begin{aligned}&\Gamma\Big(\frac{n-s-\ell}{2}\Big)\Big(\frac{n-\ell-s}{2}\Big)\Big(\frac{n-\ell-s}{2}+1\Big)\cdots\Big(\frac{n-\ell-s}{2}+\ell-1\Big) \\ &= \Gamma\Big(\frac{n-s+\ell}{2}\Big).\end{aligned} \tag{3.31}$$

It follows that

$$c_{n,\ell}(s) = (-i)^\ell 2^{n/2-s}\Gamma\Big(\frac{n-s+\ell}{2}\Big)\Gamma\Big(\frac{s+\ell}{2}\Big)^{-1}. \tag{3.32}$$

To summarize, for all $h_\ell \in E_{n,\ell}$, we have

$$\mathcal{F}(h_\ell(\omega)r^{-s}) = (-i)^\ell 2^{n/2-s}\Gamma\Big(\frac{n-s+\ell}{2}\Big)\Gamma\Big(\frac{s+\ell}{2}\Big)^{-1} h_\ell(\omega)r^{s-n}, \tag{3.33}$$

at least for $0 < \operatorname{Re} s < n$. Now the left side of (3.33) is holomorphic in $\operatorname{Re} s < n$. As for the right side, $h_\ell(\omega) r^{s-n}$ is clearly holomorphic in $\operatorname{Re} s > 0$. The factor $\Gamma((n-s+\ell)/2)$ is meromorphic in s with poles at $\{s = n+\ell, n+\ell+2, n+\ell+4, \dots\}$. It follows that $h_\ell(\omega) r^{-s}$ analytically continues to be meromorphic in s, and

$$h_\ell(\omega) r^{-s} \in \mathcal{H}^{\#}_{-s,\ell}(\mathbb{R}^n), \quad \text{for} \quad s \notin \{n+\ell, n+\ell+2, n+\ell+4, \dots\}. \tag{3.34}$$

Also the identity (3.33) analytically continues (with the standard adjustment when $s+\ell$ is an even non-positive integer). These results contain (3.3)–(3.5) for $\ell = 0$, (3.10) and (3.15) for $\ell = 1$, and (3.17) for $\ell = 2$ (also treating the case $n = s = 2$).

For each n and s, as $\ell \to +\infty$, Stirling's formula gives

$$\Gamma\Big(\frac{n-s+\ell}{2}\Big)\Gamma\Big(\frac{s+\ell}{2}\Big)^{-1} \sim \Big(\frac{\ell}{2}\Big)^{n/2-s}. \tag{3.35}$$

See Appendix C for a proof. Consequently, if we have

$$a(\omega) r^{-s} \in \mathcal{H}^{\#}_{-s}(\mathbb{R}^n), \tag{3.36}$$

so $a \in C^\infty(S^{n-1})$, we get

$$\mathcal{F}(a(\omega) r^{-s}) = b_s(\omega) r^{s-n}, \tag{3.37}$$

with

$$\begin{aligned} b_s(\omega) = A_n(s) a(\omega) &\in C^\infty(S^{n-1}), \\ \text{for} \quad s &\notin \{n, n+1, n+2, \dots\} \cup \{0, -1, -2, \dots\}, \end{aligned} \tag{3.38}$$

where

$$A_n(s) : C^\infty(S^{n-1}) \longrightarrow C^\infty(S^{n-1}), \quad \text{for} \quad s \notin \{n, n+1, n+2, \dots\} \tag{3.39}$$

is defined by

$$A_n(s) h_\ell(\omega) = c_{n,\ell}(s) h_\ell(\omega). \tag{3.40}$$

In fact, by (3.35), if $H^{k,2}(S^{n-1})$ denotes the L^2-Sobolev space of functions on S^{n-1}, of regularity degree k,

$$A_n(s) : H^{k,2}(S^{n-1}) \longrightarrow H^{k+s-n/2}(S^{n-1}), \quad s \notin \{n, n+1, n+2, \dots\}. \tag{3.41}$$

Parenthetically, we note that (3.32) implies

$$A_n(n-s) A_n(s) h_\ell(\omega) = (-1)^\ell h_\ell(\omega), \tag{3.42}$$

for s as in (3.38), which is consistent with the identity

$$\mathcal{F}^2 u(x) = u(-x). \tag{3.43}$$

For another perspective on the family of operators $A_n(s)$, let us set

$$\Lambda = \Bigl(-\Delta_S + \frac{(n-2)^2}{4}\Bigr)^{1/2} - \frac{n-2}{2} \in OPS^1(S^{n-1}), \tag{3.44}$$

an elliptic, self-adjoint, pseudodifferential operator satisfying

$$\Lambda h_\ell = \ell h_\ell, \quad \forall\, h_\ell \in E_{n,\ell}. \tag{3.45}$$

Then

$$A_n(s) = 2^{n/2-s} e^{-\pi i \Lambda/2} \Phi_{n,s}(\Lambda), \tag{3.46}$$

with

$$\Phi_{n,s}(\ell) = \Gamma\Bigl(\frac{n-s+\ell}{2}\Bigr)\Gamma\Bigl(\frac{s+\ell}{2}\Bigr)^{-1}, \tag{3.47}$$

which, by (3.35), or more precisely (C.17), (cf. [7], Chapter 12) yields

$$\Phi_{n,s}(\Lambda) \in OPS^{-s+n/2}(S^{n-1}), \quad \text{for} \quad s \notin \{n, n+1, n+2, \dots\}, \tag{3.48}$$

elliptic, and invertible if also $s \notin \{0, -1, -2, \dots\}$. Also, $e^{-\pi i \Lambda/2}$ is an elliptic, unitary, Fourier integral operator, of order 0.

We return to (3.37)–(3.38) and discuss what can happen when $s = -k,\ k \in \{0, 1, 2, \dots\}$. Of course, the left side of (3.37) converges as $s \to -k$ to an element of $\mathcal{H}^{\#}_k(\mathbb{R}^n)$, and the right side converges to an element of $\mathcal{H}^{\#}_{-k-n}(\mathbb{R}^n)$, equal to $\mathcal{F}(a(\omega)r^k)$. Also, as $s \to -k$, it follows from (3.46)–(3.47) that $A_n(s)a(\omega) \to A_n(-k)a(\omega)$ in $C^\infty(S^{n-1})$. Furthermore, taking into account the analysis behind (3.33)–(3.34), we have $A_n(-k)a(\omega)r^{-k-n} \in \mathcal{H}^{\#}_{-k-n}(\mathbb{R}^n)$. However, instead of equality, we can in general just conclude that

$$\mathcal{F}(a(\omega)r^k) - A_n(-k)a(\omega)r^{-k-n} \ \text{ is supported on } \ \{0\}, \tag{3.49}$$

i.e., it is a linear combination of derivatives of the delta function. The case $k = 0$ is of particular interest for developments in §8, so we mention that the analysis in (3.33)–(3.48) yields for $a \in C^\infty(S^{n-1})$ that

$$\int\limits_{S^{n-1}} a(\omega)\, dS(\omega) = 0 \Longrightarrow \mathcal{F}(a(\omega)) = A_n(0)a(\omega)r^{-n}. \tag{3.50}$$

In this case, (3.46) holds with $s = 0$, and

$$\Phi_{n,0}(\ell) = \Gamma\Bigl(\frac{n+\ell}{2}\Bigr)\Gamma\Bigl(\frac{\ell}{2}\Bigr)^{-1}. \tag{3.51}$$

The hypothesis of (3.50) implies that $b(\omega) = A_n(0)a(\omega)$ integrates to 0, and the conclusion can be written

$$\mathcal{F}(a(\omega)) = \mathrm{PV}\, b(\omega)r^{-n}, \quad b = A_n(0)a. \tag{3.52}$$

For another perspective on the right side of (3.52), take

$$b \in C^\infty(S^{n-1}), \qquad \int\limits_{S^{n-1}} b(\omega)\, dS(\omega) = 0, \tag{3.53}$$

and consider $b(\omega)r^{-s}$, which is in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for $\mathrm{Re}\, s < n$. If we take $\varphi \in \mathcal{S}(\mathbb{R}^n)$ to be radial, with $\varphi(0) = 1$, then, for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\int\limits_{\mathbb{R}^n} b(\omega)r^{-s} f(x)\, dx = \int\limits_{\mathbb{R}^n} b(\omega)r^{-s}[f(x) - f(0)\varphi(x)]\, dx, \tag{3.54}$$

for $\mathrm{Re}\, s < n$. However, the right side of (3.54) is absolutely integrable for $\mathrm{Re}\, s < n+1$, hence extends $b(\omega)r^{-s}$ to being holomorphic on $\mathrm{Re}\, s < n+1$. This is independent of the choice of such φ as described above, and from this one gets $b(\omega)r^{-s} \in \mathcal{H}^{\#}_{-s}(\mathbb{R}^n)$, for $\mathrm{Re}\, s < n+1$, given that b satisfies (3.53). Compare (3.10) for the case $b(\omega) = \omega_j$.

This is part of the $\ell = 0$ case of the following result, which follows from (3.34) and the analysis of $A_n(\omega)$. Assume

$$a \in C^\infty(S^{n-1}), \quad a \perp E_{n,j} \ \text{ for } \ 0 \le j \le \ell. \tag{3.55}$$

Then

$$\begin{aligned} A_n(s)a \in C^\infty(S^{n-1}) \ &\text{ is holomorphic for} \\ s \notin \{n+\ell+1, &n+\ell+2, n+\ell+3, \dots\}. \end{aligned} \tag{3.56}$$

Furthermore,

$$a(\omega)r^{-s} \ \text{ is holomorphic for } \ s \notin \{n+\ell+1, n+\ell+2, n+\ell+3, \dots\}, \tag{3.57}$$

and

$$\begin{aligned} &\mathcal{F}(a(\omega)r^{-s}) = A_n(s)a(\omega)r^{s-n} \\ &\text{for } s \notin \{n+\ell+1, n+\ell+2, \dots\} \cup \{-\ell-1, -\ell-2, -\ell-3, \dots\}. \end{aligned} \tag{3.58}$$

Having studied various meromorphic families of homogeneous distributions, we now introduce cut-offs. Take

$$\varphi \in C_0^\infty(\mathbb{R}^n), \quad \varphi(x) = \varphi(-x), \quad \varphi(x) = 1 \text{ for } |x| \le 1, \tag{3.59}$$

and consider

$$(1 - \varphi(x))a(\omega)r^{-s}, \tag{3.60}$$

with $a \in C^\infty(S^{n-1})$. This is clearly an entire holomorphic function of s, with values in $\mathcal{S}'(\mathbb{R}^n)$. In fact, at each $s \in \mathbb{C}$, we get a symbol in $S_{1,0}^{-\operatorname{Re} s}(\mathbb{R}^n)$. It follows that applying $\mathcal{F}$ to (3.60) yields an entire function of s with values in $\mathcal{S}'(\mathbb{R}^n)$, and standard arguments, parallel to (1.15)–(1.18), imply that these are all C^∞ on $\mathbb{R}^n \setminus 0$, and rapidly decreasing as $|x| \to \infty$. We have

$$\mathcal{F}\big((1 - \varphi(x))a(\omega)r^{-s}\big) = (I - \varphi(D))A_n(s)a(\omega)r^{s-n}, \tag{3.61}$$

whenever s satisfies (3.38). Now we want to see how applying $I - \varphi(D)$ removes the poles of $A_n(s)a(\omega)r^{s-n}$ that might arise at $s \in \{n, n+1, n+2, \dots\}$ (depending on the nature of $a(\omega)$).

We start with the case $a(\omega) = 1$, for which (3.67) specializes to

$$\mathcal{F}\big((1 - \varphi(x))r^{-s}\big) = 2^{n/2-s}\Gamma\Big(\frac{n-s}{2}\Big)\Gamma\Big(\frac{s}{2}\Big)^{-1}(I - \varphi(D))r^{s-n}, \tag{3.62}$$

for $s \notin \{n, n+2, n+4, \dots\}$. To see what happens near $s = n$, let us set $s = n + \sigma$, so (3.62) becomes

$$\mathcal{F}\big((1 - \varphi(x))r^{-n-\sigma}\big) = 2^{-n/2-\sigma}\Gamma\Big(-\frac{\sigma}{2}\Big)\Gamma\Big(\frac{n+\sigma}{2}\Big)^{-1}(I - \varphi(D))r^{\sigma}. \tag{3.63}$$

Now $\varphi(D)1 = 1$ (in fact $\varphi(D)p(x) = p(x)$ for every polynomial $p(x)$), so

$$\frac{1}{\sigma}(I - \varphi(D))r^\sigma = (I - \varphi(D))\frac{r^\sigma - 1}{\sigma}. \tag{3.64}$$

We see that

$$q_\sigma = \frac{r^\sigma - 1}{\sigma} \tag{3.65}$$

is an entire holomorphic function of σ, with values in $\mathcal{S}'(\mathbb{R}^n)$, having the convergent power series

$$q_\sigma(x) = \log r + \sum_{k=1}^\infty \frac{\sigma^k}{(k+1)!}(\log r)^{k+1}. \tag{3.66}$$

Hence we can rewrite (3.63) as

$$\mathcal{F}\big((1-\varphi(x))r^{-n-\sigma}\big) = 2^{-n/2-\sigma}\sigma\Gamma\Big(-\frac{\sigma}{2}\Big)\Gamma\Big(\frac{n+\sigma}{2}\Big)^{-1}(I-\varphi(D))q_\sigma, \tag{3.67}$$

manifestly without a pole at $\sigma = 0$. Also $\varphi(D)q_\sigma$ is an entire holomorphic function of σ with values in $C^\infty(\mathbb{R}^n)$.

More generally, to examine (3.62) near $s = n+2k$, we set $s = n+2k+\sigma$, and rewrite (3.62) as

$$\begin{aligned}&\mathcal{F}\big((1-\varphi(x))r^{-n-2k-\sigma}\big)\\ &= 2^{-n/2-2k-\sigma}\Gamma\Big(-k-\frac{\sigma}{2}\Big)\Gamma\Big(\frac{n+2k+\sigma}{2}\Big)^{-1}(I-\varphi(D))r^{2k+\sigma}.\end{aligned} \tag{3.68}$$

We have $\varphi(D)r^{2k} = r^{2k}$, so

$$\begin{aligned}\frac{1}{\sigma}(I-\varphi(D))r^{2k+\sigma} &= (I-\varphi(D))\Big(r^{2k}\frac{r^\sigma-1}{\sigma}\Big)\\ &= (I-\varphi(D))(r^{2k}q_\sigma),\end{aligned} \tag{3.69}$$

again with q_σ as in (3.65)–(3.66). Thus, parallel to (3.67), we have

$$\begin{aligned}&\mathcal{F}\big((1-\varphi(x))r^{-n-2k-\sigma}\big)\\ &= 2^{-n/2-2k-\sigma}\sigma\Gamma\Big(-k-\frac{\sigma}{2}\Big)\Gamma\Big(\frac{n+2k+\sigma}{2}\Big)^{-1}(I-\varphi(D))(r^{2k}q_\sigma),\end{aligned} \tag{3.70}$$

again manifestly without a pole at $\sigma = 0$. As with (3.67), $\varphi(D)(r^{2k}q_\sigma)$ is an entire function of σ with values in $C^\infty(\mathbb{R}^n)$.

We move on to the case $a(\omega) = h_\ell(\omega)$, where h_ℓ is a harmonic polynomial, homogeneous of degree ℓ. Then (3.61) specializes to

$$\begin{aligned}&\mathcal{F}\big((1-\varphi(x))h_\ell(\omega)r^{-s}\big)\\ &= (-i)^\ell 2^{n/2-s}\Gamma\Big(\frac{n-s+\ell}{2}\Big)\Gamma\Big(\frac{s+\ell}{2}\Big)^{-1}(I-\varphi(D))h_\ell(\omega)r^{s-n},\end{aligned} \tag{3.71}$$

for $s \notin \{n+\ell, n+\ell+2, n+\ell+4, \dots\}$. To see what happens for s near $n+\ell+2k$, we set $s = n+\ell+2k+\sigma$, and rewrite (3.71) as

$$\begin{aligned}&\mathcal{F}\big((1-\varphi(x))h_\ell(\omega)r^{-n-\ell-2k-\sigma}\big)\\ &\quad= (-i)^\ell 2^{-n/2-\ell-2k-\sigma}\Gamma\Big(-k-\frac{\sigma}{2}\Big)\Gamma\Big(\frac{n+\ell+2k+\sigma}{2}\Big)^{-1}\\ &\qquad\times (I-\varphi(D))(h_\ell(\omega)r^{\ell+2k+\sigma}).\end{aligned} \tag{3.72}$$

Note that

$$h_\ell(\omega)r^{\ell+2k+\sigma} = h_\ell(x)r^{2k+\sigma}, \quad \text{and} \quad (I-\varphi(D))(h_\ell(x)r^{2k}) = 0, \tag{3.73}$$

hence

$$\frac{1}{\sigma}(I-\varphi(D))(h_\ell(\omega)r^{\ell+2k+\sigma}) = (I-\varphi(D))(h_\ell(x)r^{2k}q_\sigma). \tag{3.74}$$

Thus, parallel to (3.67) and (3.70), we have

$$\begin{aligned} &\mathcal{F}\bigl((1-\varphi(x))h_\ell(\omega)r^{-n-\ell-2k-\sigma}\bigr) \\ &\quad = (-i)^\ell 2^{-n/2-\ell-2k-\sigma}\sigma\Gamma\Bigl(-k-\frac{\sigma}{2}\Bigr)\Gamma\Bigl(\frac{n+\ell+2k+\sigma}{2}\Bigr)^{-1} \\ &\qquad \times (I-\varphi(D))(h_\ell(x)r^{2k}q_\sigma), \end{aligned} \tag{3.75}$$

yet again manifestly without a pole at $\sigma = 0$. As in (3.67) and (3.70), $\varphi(D)(h_\ell(x)r^{2k}q_\sigma)$ is an entire function of σ with values in $C^\infty(\mathbb{R}^n)$. Consequently we have an explicit description of the singularity at $x = 0$ of the left side of (3.75), valid uniformly for σ near 0:

$$(-i)^\ell 2^{-n/2-\ell-2k-\sigma}\sigma\Gamma\Bigl(-k-\frac{\sigma}{2}\Bigr)\Gamma\Bigl(\frac{n+\ell+2k+\sigma}{2}\Bigr)^{-1} h_\ell(x)r^{2k}q_\sigma. \tag{3.76}$$

We return to the general setting (3.60), with $a \in C^\infty(S^{n-1})$, take $M \in (0,\infty)$, and we want to obtain a precise analysis of the singularity of $\mathcal{F}((1-\varphi(x))a(\omega)r^{-s})$, valid locally uniformly for $0 \le \operatorname{Re} s \le M$. To get this, pick $\ell \in \mathbb{Z},\ \ell > M$, and write

$$\begin{gathered} a(\omega) = h_1(\omega) + \cdots + h_\ell(\omega) + a_\ell(\omega), \\ h_j \in E_{n,j}, \quad a_\ell \perp E_{n,j}, \text{ for } 0 \le j \le \ell. \end{gathered} \tag{3.77}$$

Then (3.56)–(3.58) hold for a_ℓ, and, as in (3.61),

$$\mathcal{F}\bigl((1-\varphi(x))a_\ell(\omega)r^{-s}\bigr) = (I-\varphi(D))A_n(s)a_\ell(\omega)r^{s-n}, \tag{3.78}$$

for $0 \le \operatorname{Re} s \le M$, and $\varphi(D)A_n(s)a_\ell(\omega)r^{s-n}$ is holomorphic in s with values in $C^\infty(\mathbb{R}^n)$ for such s, so the singularity of the left side of (3.78) is given by

$$A_n(s)a_\ell(\omega)r^{s-n}, \tag{3.79}$$

locally uniformly for $0 \le \operatorname{Re} s \le M$. On the other hand, for $0 \le j \le \ell$, $\mathcal{F}(h_j(\omega)r^{-s})$ is meromorphic in s, with poles in $\{n+j, n+j+2, n+j+4, \dots\}$, and we have (3.33), with j in place of ℓ. We have

$$\mathcal{F}\bigl((1-\varphi(x))h_j(\omega)r^{-s}\bigr) = (I-\varphi(D))A_n(s)h_j(\omega)r^{s-n}, \tag{3.80}$$

away from these poles, and, away from such poles, $\varphi(D)A_n(s)h_j(\omega)r^{s-n}$ is holomorphic in s with values in $C^\infty(\mathbb{R}^n)$, so again the singular of $\mathcal{F}((1-\varphi(x))h_j(\omega)r^{-s})$ is given by

$$A_n(s)h_j(\omega)r^{s-n}, \tag{3.81}$$

locally uniformly in s, away from these poles. Meanwhile, the behavior near each such pole $n+j+2k$ is given by (3.75)–(3.76), with ℓ replaced by j.

4 Proof of Theorem 1.1

Our first order of business is to extend the scope of the identities (2.7)–(2.8), beyond $0 \le a < b < 1$, and $0 \le a < b < 2$, respectively. To do this, we use material from §3, including the facts that $|\xi|^{-s}$ and $(\operatorname{sgn}\xi)|\xi|^{-s}$ have meromorphic continuations, yielding

$$\begin{aligned} |\xi|^{-s} \ &\text{ holomorphic for } \ s \notin \{1,3,5,\dots\}, \\ (\operatorname{sgn}\xi)|\xi|^{-s} \ &\text{ holomorphic for } \ s \notin \{2,4,6,\dots\}. \end{aligned} \tag{4.1}$$

Thus we can define tempered distributions

$$\begin{gathered} \frac{1}{\log|\xi|}\Big(|\xi|^{-a} - |\xi|^{-b}\Big) = \int_a^b |\xi|^{-s}\,ds, \\ a,b \notin \{1,3,5,\dots\}, \end{gathered} \tag{4.2}$$

and

$$\begin{gathered} \frac{\operatorname{sgn}\xi}{\log|\xi|}\Big(|\xi|^{-a} - |\xi|^{-b}\Big) = \int_a^b (\operatorname{sgn}\xi)|\xi|^{-s}\,ds, \\ a,b \notin \{2,4,6,\dots\}. \end{gathered} \tag{4.3}$$

The integral in (4.2) is taken along any path γ_{ab} from a to b in $\mathbb{C}$ that avoids $\{1,3,5,\dots\}$, and the integral in (4.3) is taken along any path σ_{ab} from a to b that avoids $\{2,4,6,\dots\}$. If $\tilde{\gamma}_{ab}$ and $\tilde{\sigma}_{ab}$ are two other such paths, the resulting integrals differ by distributions supported at $\{0\}$, i.e., by finite linear combinations of derivatives of δ. Thus (4.2) and (4.3) are defined and holomorphic on the universal covering spaces of $\mathbb{C}\setminus\{1,3,5,\dots\}$ and $\mathbb{C}\setminus\{2,4,6,\dots\}$, respectively.

While it would suffice to work with these compound objects, it is natural to take $a=0$ and use the resulting identity to define

$$\frac{1}{\log|\xi|}|\xi|^{-b} = \frac{1}{\log|\xi|} - \int_0^b |\xi|^{-s}\,ds, \tag{4.4}$$

for $b \notin \{1, 3, 5, \dots\}$, and

$$\frac{\operatorname{sgn}\xi}{\log|\xi|}|\xi|^{-b} = \frac{\operatorname{sgn}\xi}{\log|\xi|} - \int_0^b (\operatorname{sgn}\xi)|\xi|^{-s}\,ds, \tag{4.5}$$

for $b \notin \{2, 4, 6, \dots\}$. Both sides of (4.4) and (4.5) have classical PV singularities at $\xi = \pm 1$.

Now we apply the Fourier transform $\mathcal{F} : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ to both sides of (4.2) and to both sides of (4.3), using (3.6)–(3.7), which, as noted in §3, are valid for $s \notin \{1, 3, 5, \dots\}$ and for $s \notin \{2, 4, 6, \dots\}$, respectively, with due attention to taking the limit of $(\sin \pi s/2)|x|^{s-1}$ as $s \to -2k \in \{0, -2, -4, \dots\}$ and taking the limit of $(\cos \pi s/2)(\operatorname{sgn} x)|x|^{s-1}$ as $s \to -2k - 1 \in \{-1, -3, -5, \dots\}$. Applying $\mathcal{F}$ to (4.2) yields

$$\begin{aligned} &\int_{\mathbb{R}} \frac{1}{\log|\xi|}\Big(|\xi|^{-a} - |\xi|^{-b}\Big)e^{-ix\xi}\,d\xi \\ &\qquad = 2\int_a^b \Big(\sin\frac{\pi s}{2}\Big)\Gamma(1-s)|x|^{s-1}\,ds \\ &\qquad = \frac{2}{|x|}K_{\varphi,a,b}(|x|), \end{aligned} \tag{4.6}$$

with $\varphi(s) = \Gamma(1-s)\sin \pi s/2$, as in (1.28), provided

$$a, b \notin \{1, 3, 5, \dots\}. \tag{4.7}$$

Meanwhile, applying $\mathcal{F}$ to (4.3) yields

$$\begin{aligned} &\int_{\mathbb{R}} \frac{\operatorname{sgn}\xi}{\log|\xi|}\Big(|\xi|^{-a} - |\xi|^{-b}\Big)e^{-ix\xi}\,d\xi \\ &\qquad = -2\frac{\operatorname{sgn} x}{|x|}\int_a^b \Big(\cos\frac{\pi s}{2}\Big)\Gamma(1-s)|x|^{s}\,ds \\ &\qquad = -\frac{2}{x}K_{\psi,a,b}(|x|), \end{aligned} \tag{4.8}$$

with $\psi(s) = \Gamma(1-s)\cos \pi s/2$, as in (1.11), provided

$$a, b \notin \{2, 4, 6, \dots\}. \tag{4.9}$$

Now the left sides of (4.6) and (4.8) differ from $2C_{ab}(x)$ and $2S_{ab}(x)$, respectively, by Fourier transforms of compactly supported distributions on $\mathbb{R}$, so the respective differences are C^∞. Consequently the results (4.6)–(4.9) give (1.27) and hence prove Theorem 1.1.

In fact, we have extended Theorem 1.1 a bit. The conditions (4.7) and (4.9) on a and b are more precise than the conditions stated in Theorem 1.1. Also, here we do not require $0 \le a < b$; in fact a and b can be complex. As is natural, the definition (1.24) of $K_{\Phi,a,b}(z)$ extends to the case of complex a and b, provided Φ is meromorphic on $\mathbb{C}$ and the integral is taken along a path from a to b that avoids its poles. As long as these poles are contained in $\mathbb{N} = \{1, 2, 3, \dots\}$, results of choosing two different such paths give versions of $K_{\Phi,a,b}(z)$ that differ by a polynomial in z. In the case (4.4), this difference is an odd polynomial in $|x|$, which when multiplied by $|x|^{-1}$ yields a polynomial in x^2. In the case (4.6), the difference is an even polynomial in $|x|$, i.e., a polynomial in x^2 (with no constant term).

5 Useful properties of $K_{\Phi,b}$

Recall that for $b > 0$, $K_{\Phi,b}(z)$ is defined by

$$K_{\Phi,b}(z) = \int_0^b \Phi(s) z^s \, ds, \quad \operatorname{Re} z > 0. \tag{5.1}$$

We work in the following setting, to accomodate the functions ψ and φ given by (1.11) and (1.28). We assume Φ is meromorphic on some complex neighborhood $\mathcal{O}$ of $[0, b]$, with a finite number of poles, all contained in $(0, b)$, and the integral is taken along a path γ_{0b} in $\mathcal{O}$ from 0 to b that avoids these poles. If Φ has no poles in $[0, b]$, we can simply integrate over the interval $[0, b]$. As mentioned in §1, asymptotic behavior as $z \to 0$ is derived from

$$K_{\Phi,b}(z) = -\frac{1}{\log z}\Big(\Phi(0) - \Phi(b) z^b\Big) - \frac{1}{\log z} K_{\Phi',b}(z) \tag{5.2}$$

and iterations. Here we record some further results, which will prove useful in §6.

First, applying d/dz to (5.1) gives

$$\frac{d}{dz} K_{\Phi,b}(z) = \frac{1}{z} K_{s\Phi,b}(z). \tag{5.3}$$

Thus integrating gives

$$\int_0^x \frac{1}{r} K_{s\Phi,b}(r) \, dr = K_{\Phi,b}(x), \quad x > 0. \tag{5.4}$$

Note that, by (5.2), with $\Phi(s)$ replaced by $s\Phi(s)$, we have

$$\begin{aligned} K_{s\Phi,b}(r) &= -\frac{1}{\log r} K_{\Phi+s\Phi',b}(r) + \frac{\Phi(b)}{\log r} r^b \\ &= \frac{\Phi(0)}{(\log r)^2} + O\Big(\frac{1}{(\log r)^3}\Big), \quad r \in \Big(0, \frac{1}{2}\Big), \end{aligned} \tag{5.5}$$

guaranteeing integrability of the left side of (5.4).

A useful companion to (5.4) is

$$\begin{aligned} \int_0^x K_{\Phi,b}(r)\,dr &= \int_0^x \int_0^b \Phi(s) r^s \,ds\,dr \\ &= \int_0^b \Phi(s) \frac{x^{s+1}}{s+1}\,ds \\ &= x K_{\Phi/(s+1),b}(x), \end{aligned} \tag{5.6}$$

for $x > 0$. Going further, for $j \in \mathbb{N}$,

$$\begin{aligned} \int_0^x r^j K_{\Phi,b}(r)\,dr &= \int_0^x \int_0^b \Phi(s) r^{s+j} \,ds\,dr \\ &= \int_0^b \Phi(s) \frac{x^{s+j+1}}{s+j+1}\,ds \\ &= x^{j+1} K_{\Phi/(s+j+1),b}(x), \end{aligned} \tag{5.7}$$

for $x > 0$.

6 Analysis of F_{ab} for $a = k \in \mathbb{N}$

As mentioned in §1, to analyze $F_{kb}(x) = C_{kb}(x) - iS_{kb}(x)$ for $k \in \mathbb{N}$, we make use of the identities

$$C'_{k+1,b+1}(x) = -S_{kb}(x), \quad S'_{k+1,b+1}(x) = C_{kb}(x), \tag{6.1}$$

and then integrate, working up from the formulas for $C_{0b}(x)$ and $S_{0b}(x)$ established in Theorem 1.1, to wit

$$S_{0b}(x) \equiv \frac{1}{x} K_{\psi,b}(|x|), \quad C_{0b}(x) \equiv \frac{1}{|x|} K_{\varphi,b}(|x|), \tag{6.2}$$

with ψ and φ as in (1.11) and (1.28), i.e.,

$$\psi(s) = \Gamma(1-s) \sin\frac{\pi}{2}(1-s), \quad \varphi(s) = \Gamma(1-s)\cos\frac{\pi}{2}(1-s). \tag{6.3}$$

As noted in (1.36), thanks to the fact that $\varphi(0) = 0$, C_{0b} is absolutely integrable on a neighborhood of $x = 0$, so

$$\begin{aligned} S_{1,b+1}(x) &= \int_0^x C_{0b}(r)\,dr \\ &\equiv \int_0^x \frac{1}{r} K_{\varphi,b}(r)\,dr \\ &= (\operatorname{sgn} x) K_{\varphi_1,b}(|x|), \end{aligned} \tag{6.4}$$

the last identity by (5.4) and the oddness of $S_{1,b+1}(x)$. Here

$$\varphi_1(s) = \frac{\varphi(s)}{s} = \Gamma(1-s)\frac{\sin \pi s/2}{s}. \tag{6.5}$$

This establishes the first part of (1.38). For the other half, since S_{0b} is not integrable near $x = 0$, we pick a small $x_0 > 0$ and write, for $x > 0$,

$$\begin{aligned} C_{1,b+1}(x) &\equiv -\int_{x_0}^x S_{0,b}(r)\,dr \\ &\equiv -\int_{x_0}^x \frac{1}{r} K_{\psi,b}(r)\,dr. \end{aligned} \tag{6.6}$$

Now $\psi(0) = 1$, so we can write

$$\psi(s) = 1 + s\psi_1(s), \quad \psi_1(0) = \Gamma'(1). \tag{6.7}$$

Then

$$\begin{aligned} K_{\psi,b}(r) &= K_{1+s\psi_1(s),b}(r) \\ &= \frac{r^b - 1}{\log r} + K_{s\psi_1,b}(r), \end{aligned} \tag{6.8}$$

so, for small $x > 0$,

$$C_{1,b+1}(x) \equiv \int_{x_0}^x \frac{dr}{r \log r} - \int_{x_0}^x \frac{r^{b-1}}{\log r}\,dr - K_{\psi_1,b}(x). \tag{6.9}$$

Now,

$$\int_{x_0}^x \frac{dr}{r \log r} = \log|\log x| - \log|\log x_0|. \tag{6.10}$$

As for the second integral on the right side of (6.9), we have, for $b > 0$,

$$\int_{x_0}^x \frac{r^{b-1}}{\log r}\,dr = \lambda_b(x) - \lambda_b(x_0), \tag{6.11}$$

where

$$\begin{aligned}\lambda_b(x) &= \int_0^x \frac{r^{b-1}}{\log r}\,dr \\ &= \int_0^{x^b} \frac{dt}{\log t} \\ &= \mathrm{li}(x^b),\end{aligned} \tag{6.12}$$

where $\mathrm{li}(x)$ is the logarithmic integral:

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\log t}. \tag{6.13}$$

Putting these results together, and keeping in mind that $C_{1,b+1}(x)$ is even, we have

$$C_{1,b+1}(x) \equiv \log\bigl|\log|x|\bigr| - K_{\psi_1,b}(|x|) - \mathrm{li}(|x|^b), \tag{6.14}$$

which is the second half of (1.38).

Note that, for small $|x|$,

$$\mathrm{li}(|x|) = -\int_{\log 1/|x|}^{\infty} t^{-1}e^{-t}\,dt. \tag{6.15}$$

Integration by parts yields

$$\int_u^\infty t^{-1}e^{-t}\,dt = \frac{e^{-u}}{u} - \int_u^\infty t^{-2}e^{-t}\,dt, \tag{6.16}$$

and iterating shows that, as $x \to 0$, $\mathrm{li}(|x|)$ has an asymptotic expansion of the form

$$\mathrm{li}(|x|) \sim \frac{|x|}{\log|x|}\Bigl(1 + \sum_{j\geq 1} \alpha_j (\log|x|)^{-j}\Bigr). \tag{6.17}$$

Having taken care of $k = 1$, we proceed to $k = 2$, again using (6.1). We have, for $x > 0$,

$$\begin{aligned}C_{2,b+2}(x) &\equiv -\int_0^x K_{\varphi_1,b}(r)\,dr \\ &= -xK_{\varphi_1/(s+1),b}(x),\end{aligned} \tag{6.18}$$

by (5.6). Recalling that $C_{2,b+2}(x)$ is even in x, we get

$$C_{2,b+2}(x) \equiv -|x|K_{\varphi_1/(s+1),b}(|x|). \tag{6.19}$$

Next,

$$S_{2,b+2}(x) = \int_0^x C_{1,b+1}(r)\,dr, \tag{6.20}$$

which leads us to integrate each term on the right side of (6.14). First, for small $x > 0$,

$$\begin{aligned}\int_0^x \log|\log r|\,dr &= \int_{\log 1/x}^{\infty} (\log t)e^{-t}\,dt \\ &= x\log|\log x| + \int_{\log 1/x}^{\infty} t^{-1}e^{-t}\,dt \\ &= x\log|\log x| - \mathrm{li}(x),\end{aligned} \tag{6.21}$$

the second identity by integration by parts. Next, by (5.6),

$$\int_0^x K_{\psi_1,b}(r)\,dr = xK_{\psi_1/(s+1),b}(x). \tag{6.22}$$

Finally,

$$\begin{aligned}\int_0^x \mathrm{li}(r^b)\,dr &= \int_0^x \int_0^r \frac{y^{b-1}}{\log y}\,dy\,dr \\ &= \int_0^x \int_y^x \frac{y^{b-1}}{\log y}\,dr\,dy \\ &= \int_0^x \frac{(x-y)y^{b-1}}{\log y}\,dy \\ &= x\int_0^x \frac{y^{b-1}}{\log y}\,dy - \int_0^x \frac{y^b}{\log y}\,dy \\ &= x\,\mathrm{li}(x^b) - \mathrm{li}(x^{b+1}),\end{aligned} \tag{6.23}$$

the first and last identities by (6.12). Putting together ((6.21)–(6.23) and recalling that $S_{2,b+1}(x)$ is odd, we have

$$\begin{aligned}S_{2,b+2}(x) \equiv{}& x\log\big|\log|x|\big| - (\operatorname{sgn} x)\,\mathrm{li}(|x|) - xK_{\psi_1/(s+1),b}(|x|) \\ &- x\,\mathrm{li}(|x|^b) + (\operatorname{sgn} x)\,\mathrm{li}(|x|^{b+1}).\end{aligned} \tag{6.24}$$

Proceeding to the case $k = 3$, we see that (6.19) gives, for small $x > 0$,

$$S_{3,b+3}(x) \equiv -\int_0^x rK_{\varphi_1/(s+1),b}(r)\,dr, \tag{6.25}$$

and hence, by (5.7) and oddness,

$$S_{3,b+3}(x) \equiv (\operatorname{sgn} x)x^2K_{\varphi_1/(s+1)(s+2),b}(|x|). \tag{6.26}$$

Meanwhile

$$C_{3,b+3}(x) \equiv -\int_0^x S_{2,b+2}(r)\,dr, \tag{6.27}$$

leading to integrating the terms in (6.24). The primary term is (for small $x > 0$)

$$\begin{aligned}
\int_0^x r \log|\log r|\, dr &= \int_{\log 1/x}^{\infty} (\log t) e^{-2t}\, dt \\
&= -\frac{1}{2}(\log t)e^{-2t}\Big|_{t=\log 1/x}^{\infty} + \frac{1}{2}\int_{\log 1/x}^{\infty} t^{-1} e^{-2t}\, dt \\
&= \frac{1}{2}x^2 \log|\log x| + \frac{1}{2}\int_0^x \frac{r}{|\log r|}\, dr \\
&= \frac{1}{2}x^2 \log|\log x| - \frac{1}{2}\,\mathrm{li}(x^2),
\end{aligned} \tag{6.28}$$

leading to

$$C_{3,b+3}(x) \sim \frac{1}{2}x^2 \log\big|\log|x|\big| - \frac{1}{2}\,\mathrm{li}(x^2) + \cdots . \tag{6.29}$$

Another approach to the asymptotics of $C_{kb}(x)$ and $S_{kb}(x)$ for small x is contained in the analysis in §8; see Theorem 8.2.

7 Replacing $(\log \xi)^{-1}$ by $(\log \xi)^{-k}$

The analysis of $F_a(x)$, given by (1.31), can be extended to

$$F_{[k]a}(x) = \int_2^{\infty} \frac{1}{(\log \xi)^k} \xi^{-a} e^{-ix\xi}\, d\xi. \tag{7.1}$$

Here we concentrate on the case $k = 2$, and for notational simplicity we set

$$G_a(x) = F_{[2]a}(x) = \int_2^{\infty} \frac{1}{(\log \xi)^2} \xi^{-a} e^{-ix\xi}\, d\xi. \tag{7.2}$$

To start, we see that $F_a(x) = F_{[1]a}(x)$ satifsies

$$\begin{aligned}
ixF_a(x) &= -\int_2^{\infty} \frac{1}{\log \xi} \xi^{-a} \frac{d}{d\xi} e^{-ix\xi}\, d\xi \\
&= \int_2^{\infty} \frac{d}{d\xi}\Big(\frac{1}{\log \xi}\xi^{-a}\Big) e^{-ix\xi}\, d\xi - \frac{2^{-a}}{\log 2} e^{-2ix}.
\end{aligned} \tag{7.3}$$

Now

$$\frac{d}{d\xi}\frac{1}{\log \xi} = -\frac{1}{\xi(\log \xi)^2}, \tag{7.4}$$

so

$$
\begin{aligned}
ixF_a(x) \equiv -\int_2^\infty \frac{1}{(\log\xi)^2}\xi^{-a-1}e^{-ix\xi}\,d\xi \\
- a\int_2^\infty \frac{1}{\log\xi}\xi^{-a-1}e^{-ix\xi}\,d\xi.
\end{aligned}
\tag{7.5}
$$

In other words,

$$
G_{a+1}(x) \equiv -ixF_a(x) - aF_{a+1}(x). \tag{7.6}
$$

On the other hand,

$$
G'_{a+1}(x) = -iG_a(x), \tag{7.7}
$$

so we can differentiate (7.6) and get

$$
G_a(x) \equiv F_a(x) + xF'_a(x) - iaF'_{a+1}(x). \tag{7.8}
$$

Recall from (1.32) that $F_a(x)$ differs from $F_{ab}(x)$ by a fairly smooth function if b is large. Meanwhile, by Theorem 1.1,

$$
F_{ab}(x) \equiv \frac{1}{|x|}K_{\varphi,a,b}(|x|) - \frac{i}{x}K_{\psi,a,b}(|x|). \tag{7.9}
$$

Hence

$$
\begin{aligned}
F'_{ab}(x) \equiv \frac{1}{x}K'_{\varphi,a,b}(|x|) - \frac{1}{x|x|}K_{\varphi,a,b}(|x|) \\
- \frac{i}{|x|}K'_{\psi,a,b}(|x|) + \frac{i}{x^2}K_{\psi,a,b}(|x|),
\end{aligned}
\tag{7.10}
$$

so

$$
\begin{aligned}
xF'_{ab}(x) \equiv K'_{\varphi,a,b}(|x|) - \frac{1}{|x|}K_{\varphi,a,b}(|x|) \\
- i(\operatorname{sgn} x)K'_{\psi,a,b}(|x|) + \frac{i}{x}K_{\psi,a,b}(|x|),
\end{aligned}
\tag{7.11}
$$

and therefore

$$
F_{ab}(x) + xF'_{ab}(x) \equiv K'_{\varphi,a,b}(|x|) - i(\operatorname{sgn} x)K'_{\psi,a,b}(|x|). \tag{7.12}
$$

Now (5.3) generalizes to

$$
\frac{d}{dz}K_{\Phi,a,b}(z) = \frac{1}{z}K_{s\Phi,a,b}(z), \tag{7.13}
$$

so

$$
F_{ab}(x) + xF'_{ab}(x) \equiv \frac{1}{|x|}K_{s\varphi,a,b}(|x|) - \frac{i}{x}K_{s\psi,a,b}(|x|). \tag{7.14}
$$

Note that the right side of (7.14) is obtained from the right side of (7.9) simply by taking $\varphi \mapsto s\varphi$ and $\psi \mapsto s\psi$.

If we specialize to $a = 0$, we get, for

$$G_0(x) = \int_2^\infty \frac{1}{(\log \xi)^2} e^{-ix\xi}\, d\xi, \tag{7.15}$$

the result

$$G_0(x) \approx \frac{1}{|x|} K_{s\varphi,b}(|x|) - \frac{i}{x} K_{s\psi,b}(|x|), \tag{7.16}$$

where we use

$$f(x) \approx g_b(x) \tag{7.17}$$

to indicate that the difference is as smooth as one likes near $x = 0$, provided b is sufficiently large. Note that the factor s in $s\varphi$ and $s\psi$ makes $G_0(x)$ more regular at $x = 0$, by a factor of $(\log |x|)^{-1}$, than $F_0(x)$.

One can continue along these lines. For example,

$$\begin{aligned} ixG_0(x) &= -\int_2^\infty \frac{1}{(\log \xi)^2} \frac{d}{d\xi} e^{-ix\xi}\, d\xi \\ &\equiv -2 \int_2^\infty \frac{1}{\xi (\log \xi)^3} e^{-ix\xi}\, d\xi, \end{aligned} \tag{7.18}$$

so

$$\begin{aligned} 2 \int_2^\infty \frac{1}{(\log \xi)^3} e^{-ix\xi}\, d\xi &\equiv G_0(x) + xG_0'(x) \\ &\approx \frac{1}{|x|} K_{s^2\varphi,b}(|x|) - \frac{i}{x} K_{s^2\psi,b}(|x|). \end{aligned} \tag{7.19}$$

Inductively,

$$k! \int_2^\infty \frac{1}{(\log \xi)^{k+1}} e^{-ix\xi}\, d\xi \approx \frac{1}{|x|} K_{s^k\varphi,b}(|x|) - \frac{i}{x} K_{s^k\psi,b}(|x|). \tag{7.20}$$

See §9 for another approach to this asymptotic analysis, valid in higher dimension.

8 Asymptotics in higher dimensions

To begin, we recall from §3 that, parallel to (4.1), if $p \in C^\infty(S^{n-1})$, then $p(\omega) r^{-s}$, which is in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for $\operatorname{Re} s < n$, has a meromorphic continuation,

$$p(\omega) r^{-s} \ \text{ holomorphic for } \ s \notin \mathcal{E}_p, \tag{8.1}$$

with values in $\mathcal{S}'(\mathbb{R}^n)$, where $\mathcal{E}_p \subset \mathbb{N}$ depends on the choice of p. Specific examples include (3.5), (3.10), and (3.34), with further results described in (3.55)–(3.58). We can then define tempered distributions on $\mathbb{R}^n$,

$$\frac{p(\omega)}{\log r}(r^{-a} - r^{-b}) = \int_a^b p(\omega) r^{-s}\, ds, \quad a, b \notin \mathcal{E}_p. \tag{8.2}$$

The integral in (8.2) is taken along a path γ_{ab} from a to b in $\mathbb{C}$ that avoids $\mathcal{E}_p$. If $\tilde{\gamma}_{ab}$ is another such path, the resulting integrals differ by a distribution supported at $\{0\}$, i.e., by a finite linear combination of derivatives of δ. Thus (8.2) is defined and holomorphic on the universal covering surface of $\mathbb{C} \setminus \mathcal{E}_p$, with values in $\mathcal{S}'(\mathbb{R}^n)$. We can take $a = 0$ and define

$$\frac{p(\omega)}{\log r} r^{-b} = \frac{p(\omega)}{\log r} - \int_0^b p(\omega) r^{-s}\, ds, \quad b \notin \mathcal{E}_p. \tag{8.3}$$

Both sides of (8.3) have a classical PV singularity on $S^{n-1} = \{r = 1\}$.

Now we can apply the Fourier transform $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ to both sides of (8.2), using (3.37)–(3.40). We get

$$\begin{aligned}(2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} \frac{p(\hat{\xi})}{\log|\xi|} \Bigl(|\xi|^{-a} - |\xi|^{-b}\Bigr) e^{-ix\cdot\xi}\, d\xi \\ = \int_a^b A_n(s) p(\hat{x}) |x|^{s-n}\, ds,\end{aligned} \tag{8.4}$$

for $a, b \notin \mathcal{E}_p$, where $A_n(s) : C^\infty(S^{n-1}) \to C^\infty(S^{n-1})$ is given by (3.43)–(3.47). Here,

$$\hat{\xi} = \frac{\xi}{|\xi|}, \quad \hat{x} = \frac{x}{|x|}. \tag{8.5}$$

In (8.4), the integral on the right side is taken over any path γ_{ab} from a to b in $\mathbb{C}$ that avoids $\mathcal{E}_p$.

If we specialize (8.4) to $p(\omega) = h_\ell(\omega)$, where $h_\ell(x)$ is a harmonic polynomial on $\mathbb{R}^n$, homogeneous of degree ℓ, then, by (3.40) and (3.32),

$$A_n(s) h_\ell(\omega) = (-i)^\ell 2^{n/2-s} \Gamma\Bigl(\frac{n-s+\ell}{2}\Bigr) \Gamma\Bigl(\frac{s+\ell}{2}\Bigr)^{-1} h_\ell(\omega), \tag{8.6}$$

for

$$s \notin \mathcal{E}_{h_\ell} = \{n+\ell, n+\ell+2, n+\ell+4, \dots\}. \tag{8.7}$$

We have the following.

Theorem 8.1 *If h_ℓ is a harmonic polynomial on $\mathbb{R}^n$, homogeneous of degree ℓ, then*

$$\begin{aligned}(2\pi)^{-n/2} &\int_{\mathbb{R}^n} \frac{h_\ell(\hat{\xi})}{\log|\xi|}\Big(|\xi|^{-a}-|\xi|^{-b}\Big)e^{-ix\cdot\xi}\,d\xi \\ &= (-i)^\ell 2^{n/2}\int_a^b 2^{-s}\Gamma\Big(\frac{n-s+\ell}{2}\Big)\Gamma\Big(\frac{s+\ell}{2}\Big)^{-1} h_\ell(\hat{x})|x|^{s-n}\,ds \\ &= (-i)^\ell 2^{n/2} h_\ell(\hat{x})|x|^{-n}K_{\psi_{n,\ell},a,b}(|x|),\end{aligned} \tag{8.8}$$

where

$$\psi_{n,\ell}(s) = 2^{-s}\Gamma\Big(\frac{n-s+\ell}{2}\Big)\Gamma\Big(\frac{s+\ell}{2}\Big)^{-1}, \tag{8.9}$$

and we assume

$$a, b \notin \mathcal{E}_{h_\ell}. \tag{8.10}$$

In case $\ell = 0$, we can take $h_0(x) = 1$, and then the result of (8.8) is

$$2^{n/2}|x|^{-n}K_{\psi_{n,0},a,b}(|x|), \tag{8.11}$$

where

$$\psi_{n,0}(s) = 2^{-s}\Gamma\Big(\frac{n-s}{2}\Big)\Gamma\Big(\frac{s}{2}\Big)^{-1}. \tag{8.12}$$

We see that $\psi_{n,0}(0) = 0$, so if $a = 0$, we get (via (1.25))

$$2^{n/2}|x|^{-n}K_{\psi_{n,0},b}(|x|) = \frac{C_n}{|x|^n(\log|x|)^2} + O\Big(\frac{1}{|x|^n(\log|x|)^3}\Big), \tag{8.13}$$

which is therefore in $L^1_{\text{loc}}(\mathbb{R}^n)$. Compare (1.36) for the case $n = 1$.

In case $\ell = 1$, we can take $h_1(x) = x_j$, and then the result of (8.8) is

$$-i2^{n/2}\,\frac{x_j}{|x|}\,|x|^{-n}K_{\psi_{n,1},a,b}(|x|), \tag{8.14}$$

where

$$\psi_{n,1}(s) = 2^{-s}\Gamma\Big(\frac{n-s+1}{2}\Big)\Gamma\Big(\frac{s+1}{2}\Big)^{-1}. \tag{8.15}$$

This time $\psi_{n,1}(0) = \pi^{-1/2}\Gamma((n+1)/2) \neq 0$, so, if $a = 0$, we get

$$-i2^{n/2}\,\frac{x_j}{|x|}\,|x|^{-n}K_{\psi_{n,1},b}(|x|) = C_n'\frac{x_j}{|x|}\,\frac{1}{|x|^n\log|x|} + O\Big(\frac{1}{|x|^n(\log|x|)^2}\Big). \tag{8.16}$$

The remainder term belongs to L^1_{loc}, but the principal term does not. The left side (which, recall, is in $C^\infty(\mathbb{R}^n\setminus 0)$) is a PV distribution near the origin. Compare (1.3) for the case $n = 1$.

We now bring in cut-offs. Take $\varphi \in C_0^\infty(\mathbb{R}^n)$ as in (3.59). Then, given $p \in C^\infty(S^{n-1})$,

$$(1-\varphi(x))\frac{p(\omega)}{\log r} r^{-s} \tag{8.17}$$

is an entire holomorphic function of s, with values in $\mathcal{S}'(\mathbb{R}^n)$, and

$$(1-\varphi(x))\frac{p(\omega)}{\log r}(r^{-a}-r^{-b}) = \int_a^b (1-\varphi(x))p(\omega)r^{-s}\,ds, \tag{8.18}$$

for all $a, b \in \mathbb{C}$. We can apply the Fourier transform $\mathcal{F}$ to both sides of this identity. If $a, b \notin \mathcal{E}_p$ and the integral on the right side of (8.18) is taken along a path that avoids $\mathcal{E}_p$, then we can deduce from Theorem 8.1 that

$$\begin{aligned}(2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} (1-\varphi(\xi))\frac{h_\ell(\hat{\xi})}{\log|\xi|}\Big(|\xi|^{-a}-|\xi|^{-b}\Big)e^{-ix\cdot\xi}\,d\xi \\ = (-i)^\ell 2^{n/2}(I-\varphi(D))h_\ell(\hat{x})|x|^{-n}K_{\psi_{n,\ell,a,b}}(|x|).\end{aligned} \tag{8.19}$$

Furthermore, $\varphi(D)h_\ell(\hat{x})|x|^{-n}K_{\psi_{n,\ell,a,b}}(|x|)$ is holomorphic in $a, b \in \mathbb{C}\setminus\mathcal{E}_{h_\ell}$, with values in $C^\infty(\mathbb{R}^n)$, which retrieves the elementary fact that the left side of (8.19) has the same singularity as the left side of (8.8), for such a and b.

We now want to focus on the situation that

$$a = n+\ell+2k \in \mathcal{E}_{h_\ell}, \quad k \in \{0,1,2,\dots\}. \tag{8.20}$$

We also assume

$$b > a, \quad b \notin \mathcal{E}_{h_\ell}, \tag{8.21}$$

and that the path γ_{ab} from a to b over which we integrate in (8.18) avoids $\mathcal{E}_{h_\ell}$, except at its initial point a. Setting $s = n+\ell+2k+\sigma$, we have (8.18) (with $p = h_\ell$) equal to

$$\int_0^\beta (1-\varphi(x))h_\ell(\omega)r^{-n-\ell-2k-\sigma}\,d\sigma, \quad \beta = b-a. \tag{8.22}$$

By (8.21), $\beta > 0$. Now applying (3.75)–(3.76) gives

$$\begin{aligned}(2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} (1-\varphi(\xi))\frac{h_\ell(\hat{\xi})}{\log|\xi|}\Big(|\xi|^{-a}-|\xi|^{-b}\Big)e^{-ix\cdot\xi}\,d\xi \\ \equiv (-i)^\ell 2^{-n/2-\ell-2k}h_\ell(x)|x|^{2k}\int_0^\beta \sigma\Gamma\Big(-k-\frac{\sigma}{2}\Big)\Gamma\Big(\frac{n+\ell+2k+\sigma}{2}\Big)^{-1}q_\sigma\,d\sigma,\end{aligned} \tag{8.23}$$

where, recall, $f \equiv g$ means $f - g$ is C^∞ on a neighborhood of $0 \in \mathbb{R}^n$, and

$$q_\sigma(x) = \frac{|x|^\sigma - 1}{\sigma}, \quad q_0(x) = \log|x|, \tag{8.24}$$

is an entire holomorphic function of σ with values in $\mathcal{S}'(\mathbb{R}^n)$. The change of variable from s to σ takes a path γ_{ab} that avoids $\mathcal{E}_{h_\ell}$ (except at $s = a$) to a path from 0 to β in $\mathbb{C}$ that avoids the poles of the integrand on the right side of (8.23). To analyze (8.23) further, we look at

$$\Psi_{n,k,\ell}(\sigma) = \sigma\Gamma\Big(-k - \frac{\sigma}{2}\Big)\Gamma\Big(\frac{n+\ell+2k+\sigma}{2}\Big)^{-1}, \tag{8.25}$$

which has poles at

$$\sigma = 2k - 2j, \quad j \in \{0, 1, 2, \dots\}, \quad j \neq k. \tag{8.26}$$

Note that

$$\Psi_{n,k,\ell}(0) = \alpha_{n,k,\ell} = -\frac{1}{2}\Gamma\Big(\frac{n+\ell+2k}{2}\Big)^{-1} \frac{d}{dz}\frac{1}{\Gamma(z)}\Big|_{z=-k} \tag{8.27}$$

is nonzero. We can write

$$\Psi_{n,k,\ell}(s) = \alpha_{n,k,\ell} + s\zeta_{n,k,\ell}(s), \tag{8.28}$$

where $\zeta_{n,k,\ell}$ has the same poles as $\Psi_{n,k,\ell}$. Then (8.23) is

$$\begin{aligned} &\equiv (-i)^\ell 2^{-n/2-\ell-2k} h_\ell(x)|x|^{2k} \\ &\quad \times \Big[\alpha_{n,k,\ell}\int_0^\beta q_s\,ds + \int_0^\beta \zeta_{n,k,\ell}(s)(r^s - 1)\,ds\Big]. \end{aligned} \tag{8.29}$$

The quantity in square brackets is a sum of two terms, the second of which is

$$\equiv K_{\zeta_{n,k,\ell},\beta}(|x|). \tag{8.30}$$

The first is $\alpha_{n,k,\ell}$ times

$$Q_\beta(r) = \int_0^\beta \frac{r^s - 1}{s}\,ds. \tag{8.31}$$

Note that

$$Q'_\beta(r) = \int_0^\beta r^{s-1}\,ds = \frac{r^\beta - 1}{r\log r}, \tag{8.32}$$

and $Q_\beta(1) = 0$, so

$$Q_\beta(r) = \int_1^r \frac{\rho^\beta - 1}{\rho \log \rho}\, d\rho. \tag{8.33}$$

As in (6.12),

$$\int_0^r \frac{\rho^{\beta-1}}{\log \rho}\, d\rho = \mathrm{li}(r^\beta), \tag{8.34}$$

given $\beta > 0,\ r < 1$. Also

$$\int_{1/e}^r \frac{d\rho}{\rho \log \rho} = \int_1^{\log 1/r} \frac{du}{u} = \log|\log r|, \tag{8.35}$$

so

$$Q_\beta(r) = -\log|\log r| + \mathrm{li}(r^\beta) + \tau(\beta), \tag{8.36}$$

with $\tau(\beta)$ independent of r. We have established the following.

Theorem 8.2 *If h_ℓ is a harmonic polynomial on $\mathbb{R}^n$, homogeneous of degree ℓ, and*

$$a = n + \ell + 2k \in \mathcal{E}_{h_\ell}, \quad b \notin \mathcal{E}_{h_\ell}, \quad b > a, \tag{8.37}$$

and $\varphi(\xi)$ is as in (3.59)*, then*

$$\begin{aligned}(2\pi)^{-n/2} &\int_{\mathbb{R}^n} (1 - \varphi(\xi)) \frac{h_\ell(\hat{\xi})}{\log|\xi|} \Big(|\xi|^{-a} - |\xi|^{-b} \Big) e^{-ix\cdot\xi}\, d\xi \\ &\equiv (-i)^\ell 2^{-n/2-\ell-2k} h_\ell(x) |x|^{2k} \\ &\quad \times \Big[\alpha_{n,k,\ell} \big(-\log\big|\log|x|\big| + \mathrm{li}(|x|^{b-a}) \big) + K_{\zeta_{n,k,\ell}, b-a}(|x|) \Big],\end{aligned} \tag{8.38}$$

with $\alpha_{n,k,\ell}$ given by (8.27) *and $\zeta_{n,k,\ell}(s)$ given by* (8.28).

For $\ell = k = 0$ and $n = 1$, (8.38) has the same form as (6.14). For $\ell = 1$, $k = 0$, and $n = 1$, (6.24) appears to have two terms of a form different from those in (8.38). These sum to

$$-(\mathrm{sgn}\, x)\big(\mathrm{li}(|x|) - \mathrm{li}(|x|^{\beta+1})\big), \quad \beta = b - a. \tag{8.39}$$

Note, however, that

$$\begin{aligned}\mathrm{li}(|x|) - \mathrm{li}(|x|^{\beta+1}) &= \int_0^{|x|} \frac{1 - \rho^\beta}{\log \rho}\, d\rho \\ &= -\int_0^{|x|} \int_0^\beta \rho^s\, ds\, d\rho \\ &= -\int_0^\beta \frac{|x|^{s+1}}{s+1}\, ds \\ &= -|x| K_{1/(s+1),\beta}(|x|),\end{aligned} \tag{8.40}$$

and $(\operatorname{sgn} x)|x| = x$, so (8.39) can be absorbed into the last term in square brackets in (8.38).

Let us return to the general formula (8.4), under the hypothesis that $a, b \notin \mathcal{E}_p$. Then there is a path from a to b in $\mathbb{C}$ such that

$$A_n(s)p \ \text{ has no poles on } \ \gamma_{ab}. \tag{8.41}$$

where we treat $A_n(s)p$ as a meromorphic function of s with values in $C^\infty(S^{n-1})$. In such a case, (8.4) can be written as

$$(2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} \frac{p(\hat{\xi})}{\log|\xi|} \Bigl(|\xi|^{-a} - |\xi|^{-b}\Bigr) e^{-ix\cdot\xi}\, d\xi = |x|^{-n} K_{\Phi,a,b}(|x|), \tag{8.42}$$

where Φ is a meromorphic function of s with values in $C^\infty(S^{n-1})$, namely

$$\Phi(s)(\omega) = A_n(s)p(\omega). \tag{8.43}$$

As in §1, we define

$$K_{\Phi,a,b}(z) = \int_a^b \Phi(s) z^s\, ds, \quad \operatorname{Re} z > 0, \tag{8.44}$$

the integral over a path γ_{ab} that avoids the poles of $\Phi(s)$, this time taking this as the integral of a function with values in a Frechet space (namely $C^\infty(S^{n-1})$). As in §1, we can integrate by parts in (8.44), obtaining

$$K_{\Phi,a,b}(z) = -\frac{1}{\log z}\Bigl(\Phi(a)z^a - \Phi(b)z^b\Bigr) - \frac{1}{\log z} K_{\Phi',a,b}(z), \tag{8.45}$$

and iterate this, to produce an asymptotic expansion involving powers of $(\log z)^{-1}$, as $z \to 0$. For example, if $a = 0$ and $b \in \mathbb{R}^+ \setminus \mathcal{E}_p$ in (8.41), we get

$$-\frac{1}{|x|^n \log|x|} A_n(0)p(\hat{x}) + O\Bigl(\frac{1}{|x|^n (\log|x|)^2}\Bigr), \tag{8.46}$$

as the leading part of an expansion in powers of $(\log|x|)^{-1}$. By (8.6), $A_n(0)$ annihilates the constant term in the spherical harmonic expansion of p, but not the higher terms.

We now turn to a treatment of

$$(2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} (1 - \varphi(\xi)) \frac{p(\hat{\xi})}{\log|\xi|} \Bigl(|\xi|^{-a} - |\xi|^{-b}\Bigr) e^{-ix\cdot\xi}\, d\xi, \tag{8.47}$$

with φ as in (3.59), when $a \in \mathcal{E}_p$ (still assuming $b \notin \mathcal{E}_p$), say

$$a = n + \ell, \quad \ell \in \{0, 1, 2, \dots\}. \tag{8.48}$$

We can write

$$p = p^{\#} + \sum_{j=0}^{\ell} h_j, \quad h_j \in E_{n,j}, \quad p^{\#} \perp E_{n,j} \ \text{ for } \ 0 \le j \le \ell. \tag{8.49}$$

We can apply Theorem 8.2 to each term arising from (8.47) with p replaced by $h_j,\ 0 \le j \le \ell$. As for what one gets with $p^{\#}$ in place of p in (8.47), we can apply the following (with $p^{\#}$ in place of p).

Proposition 8.3 *Assume $p \in C^\infty(S^{n-1})$ satisfies*

$$p \perp E_{n,j} \ \textit{ for } \ 0 \le j \le \ell. \tag{8.50}$$

Then (8.42) *holds for*

$$a, b \notin \{n + \ell + 1, n + \ell + 2, n + \ell + 3, \dots\}. \tag{8.51}$$

Proof. The argument used to prove Theorem 8.1 applies, suplemented by (3.55)–(3.58). ☐

9 Passing to $(\log|\xi|)^{-k}$ in multi-D

We start with the identity

$$\begin{aligned}
\int_\alpha^\beta & \frac{p(\hat{\xi})}{\log|\xi|}\Bigl(|\xi|^{-a} - |\xi|^{-\beta}\Bigr)\, da \\
&= \frac{p(\hat{\xi})}{\log|\xi|}\Bigl(\int_\alpha^\beta |\xi|^{-a}\, da - (\beta - \alpha)|\xi|^{-\beta}\Bigr) \\
&= \frac{p(\hat{\xi})}{(\log|\xi|)^2}\Bigl(|\xi|^{-\alpha} - |\xi|^{-\beta}\Bigr) - (\beta - \alpha)\frac{p(\hat{\xi})}{\log|\xi|}|\xi|^{-\beta}.
\end{aligned} \tag{9.1}$$

On the last line, each term has a classical PV singularity at $|\xi| = 1$, which could be erased by multiplying by $1 - \varphi(\xi)$, with φ as in (3.59) and (8.17). Here, as usual, $p \in C^\infty(S^{n-1})$. We assume

$$\alpha, \beta \notin \mathcal{E}_p, \tag{9.2}$$

parallel to (8.2), and the integral in (9.1) is taken along a path from α to β in $\mathbb{C}$ that avoids $\mathcal{E}_p$. By (8.4), the Fourier transform of the left side of (9.1) is

$$\begin{aligned}\int_\alpha^\beta \int_a^\beta & A_n(s) p(\hat{x}) |x|^{s-n}\, ds\, da \\ &= \int_\alpha^\beta \int_\alpha^s A_n(s) p(\hat{x}) |x|^{s-n}\, da\, ds \\ &= \int_\alpha^\beta (s-a) A_n(s) p(\hat{x}) |x|^{s-n}\, ds.\end{aligned} \tag{9.3}$$

It follows that

$$\begin{aligned}(2\pi)^{-n/2} \int_{\mathbb{R}^n} & \frac{p(\hat{\xi})}{(\log|\xi|)^2} \Big(|\xi|^{-\alpha} - |\xi|^{-\beta}\Big) e^{-ix\cdot\xi}\, d\xi \\ &= \int_\alpha^\beta (s-\alpha) A_n(s) p(\hat{x}) |x|^{s-n}\, ds \\ &\qquad + (\beta-\alpha) \mathcal{F}\Big(\frac{p(\omega)}{\log r} r^{-\beta}\Big)(x),\end{aligned} \tag{9.4}$$

when (9.2) holds. As for the last term, we have, by (8.3),

$$\mathcal{F}\Big(\frac{p(\omega)}{\log r} r^{-\beta}\Big) \in C^k(\mathbb{R}^n), \quad \text{for } \ \beta > n+k,\ \beta \notin \mathcal{E}_p. \tag{9.5}$$

so the first term on the right side of (9.4) reveals the nature of the singularity of the left side of (9.4), if β is taken large enough, assuming (9.2).

In fact, we can loosen the hypothesis (9.2) on α. Of the two terms on the right side of (9.4), the second is holomorphic in $\beta \notin \mathcal{E}_p$ and linear in α. We know the first term (the integral) is holomorphic in $\alpha \notin \mathcal{E}_p$ for each $\beta \notin \mathcal{E}_p$. Now suppose $\alpha \in \mathcal{E}_p$, and the integral from α to β is taken along a path from α to β that avoids $\mathcal{E}_p$ except at $s = \alpha$. Then the pole of $A_n(s)$ at $s = \alpha$ is cancelled by the factor $s - \alpha$ in the integrand, so in fact this is a removable singularity.

One can iterate this process. To do this, it is convenient to bring in the factor $1 - \varphi(\xi)$. Extending (9.1), we have

$$\begin{aligned}(1-\varphi(\xi)) \int_\alpha^\beta & \frac{p(\hat{\xi})}{(\log|\xi|)^{k-1}} \Big(|\xi|^{-a} - |\xi|^{-\beta}\Big)\, da \\ &= (1-\varphi(\xi)) \frac{p(\hat{\xi})}{(\log|\xi|)^k} \Big(|\xi|^{-\alpha} - |\xi|^{-\beta}\Big) \\ &\qquad - (1-\varphi(\xi)) \frac{(\beta-\alpha) p(\hat{\xi})}{(\log|\xi|)^{k-1}} |\xi|^{-\beta},\end{aligned} \tag{9.6}$$

and, from here, inductively,

$$\begin{aligned}(2\pi)^{-n/2}&\int_{\mathbb{R}^n}(1-\varphi(\xi))\frac{p(\hat{\xi})}{(\log|\xi|)^k}\Big(|\xi|^{-\alpha}-|\xi|^{-\beta}\Big)e^{-ix\cdot\xi}\,d\xi\\ &=\frac{1}{(k-1)!}(I-\varphi(D))\int_\alpha^\beta(s-\alpha)^{k-1}A_n(s)p(\hat{x})|x|^{s-n}\,ds+(\beta-\alpha)R_{k,\beta}(x),\end{aligned}\tag{9.7}$$

with

$$R_{k,\beta}=\mathcal{F}\Big((1-\varphi)\frac{p(\omega)}{(\log r)^{k-1}}r^{-\beta}\Big)\tag{9.8}$$

as smooth as one likes, if $\beta\in\mathbb{R}^+$ is sufficiently large, provided (9.2) holds. As in our analysis of (9.4), the singularities at points $\alpha\in\mathcal{E}_p$ are removable in the integral from α to β on the right side of (9.7), as long as $k\geq 2$.

In case $p(\omega)=h_\ell(\omega)$, where h_ℓ is a harmonic polynomial, homogeneous of degree ℓ, by (8.6) the integral in the right side of (9.7) is equal to

$$\begin{aligned}(-i)^\ell 2^{n/2}&\int_\alpha^\beta(s-\alpha)^{k-1}2^{-s}\Gamma\Big(\frac{n-s+\ell}{2}\Big)\Gamma\Big(\frac{s+\ell}{2}\Big)^{-1}h_\ell(\hat{x})|x|^{s-n}\,ds\\ &=(-i)^\ell 2^{n/2}h_\ell(\hat{x})|x|^{-n}K_{\Psi_{n,\ell,k},\alpha,\beta}(|x|),\end{aligned}\tag{9.9}$$

where

$$\Psi_{n,\ell,k}(s)=(s-\alpha)^{k-1}\psi_{n,\ell}(s),\tag{9.10}$$

with $\psi_{n,\ell}(s)$ as in (8.9).

Generally, for $p\in C^\infty(S^{n-1})$, we can represent the integral on the right side of (9.7) as

$$|x|^{-n}K_{\Phi_k,\alpha,\beta}(|x|),\tag{9.11}$$

where

$$\Phi_k(s)=(s-\alpha)^{k-1}A_n(s)p\tag{9.12}$$

is a meromorphic function of s with values in $C^\infty(S^{n-1})$, and results parallel to (8.45) apply.

10 Replacing $\log|\xi|$ by $\log\lambda(\xi)$

Let $\lambda\in C^\infty(\mathbb{R}^n\setminus 0)$ be >0 and homogeneous of degree 1. Then there exists $q\in C^\infty(S^{n-1})$ such that $q>0$ on S^{n-1} and

$$\lambda=q(\omega)r.\tag{10.1}$$

Hence $\log\lambda = \log r + \log q$, so if $\log\lambda$ were in the numerator of a symbol, there would be no additional problem. Since for us it is in the denominator, we need to work a little harder. Parallel to (2.1), we can take

$$\frac{1}{\log\lambda(\xi)}\Bigl(\lambda(\xi)^{-a} - \lambda(\xi)^{-b}\Bigr) = \int_a^b \lambda(\xi)^{-s}\, ds, \tag{10.2}$$

for $\operatorname{Re} a,\ \operatorname{Re} b < n$. We want to extend the scope of this, since we particularly want to take b large, so we need to extend results of §3, obtaining a meromorphic continuation of λ^{-s} from $\operatorname{Re} s < n$. Clearly $\lambda^{-s} = q^{-s}r^{-s}$ for $\operatorname{Re} s < n$, and, as in (3.37)–(3.40),

$$\mathcal{F}\bigl(q(\omega)^{-s}r^{-s}\bigr) = A_n(s)q(\omega)^{-s}r^{s-n}, \tag{10.3}$$

first for $0 < \operatorname{Re} s < n$. Now the left side of (10.3) is holomorphic for $\operatorname{Re} s < n$, and the right side is holomorphic for $\operatorname{Re} s > 0$, except for the poles of $A_n(s)q(\omega)^{-s}$, which is a subset of $\{n, n+1, n+2, \dots\}$. More generally, if $p \in C^\infty(S^{n-1})$,

$$\mathcal{F}\bigl(p(\omega)\lambda^{-s}\bigr) = A_n(s)(pq^{-s})r^{s-n}, \tag{10.4}$$

first for $0 < \operatorname{Re} s < n$, and then both sides analytically continue to

$$s \in \mathbb{C} \setminus \mathcal{E}_{p,q}, \tag{10.5}$$

where $\mathcal{E}_{p,q}$ is the set of poles of $A_n(s)(pq^{-s})$, a subset of $\{n, n+1, n+2, \dots\}$.

In particular, we have a meromorphic continuation of $p(\omega)\lambda^{-s}$, holomorphic for s as in (10.5), all of whose poles are simple, and we can define

$$\frac{p(\omega)}{\log\lambda}\Bigl(\lambda^{-a} - \lambda^{-b}\Bigr) = \int_a^b p(\omega)\lambda^{-s}\, ds, \quad a, b \notin \mathcal{E}_{p,q}, \tag{10.6}$$

for λ as in (10.1). As in (8.2), the integral in (10.6) is taken along a path γ_{ab} from a to b in $\mathbb{C}$ that avoids $\mathcal{E}_{p,q}$, and if $\tilde{\gamma}_{ab}$ is another such path, the resulting integrals differ by a distribution supported at $\{0\}$. Thus (10.6) is defined and holomorphic for a and b in the universal covering surface of $\mathbb{C} \setminus \mathcal{E}_{p,q}$, with values in $\mathcal{S}'(\mathbb{R}^n)$. Parallel to (8.3), we can take $a = 0$ and define

$$\frac{p(\omega)}{\log\lambda}\lambda^{-b} = \frac{p(\omega)}{\log\lambda} - \int_0^b p(\omega)\lambda^{-s}\, ds, \quad b \notin \mathcal{E}_{p,q}. \tag{10.7}$$

Both sides have a classical PV singularity on $\{\lambda = 1\}$.

Now we can apply the Fourier transform to both sides of (10.6), using (10.4). We get

$$\begin{aligned}(2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} \frac{p(\hat{\xi})}{\log \lambda(\xi)} \Big(\lambda(\xi)^{-a} - \lambda(\xi)^{-b}\Big) e^{-ix\cdot\xi}\, d\xi \\ = \int_a^b A_n(s) \big(p(\hat{x}) q(\hat{x})^{-s}\big) |x|^{s-n}\, ds,\end{aligned} \tag{10.8}$$

for $a, b \notin \mathcal{E}_{p,q}$, again the last integral taken over a path γ_{ab} from a to b that omits $\mathcal{E}_{p,q}$.

Remark. The left side of (10.8) can be written

$$(2\pi)^{-n/2} \int\limits_{\mathbb{R}^n} \frac{p(\hat{\xi}) q(\hat{\xi})^{-a}}{\log \lambda(\xi)} \Big(|\xi|^{-a} - q(\hat{\xi})^{a-b} |\xi|^{-b}\Big) e^{-ix\cdot\xi}\, d\xi. \tag{10.9}$$

Note that $a \notin \mathcal{E}_{p,q}$ provided that $A_n(s)(pq^{-a})$ does not have a pole at $s = a$.

Now, given $a, b \notin \mathcal{E}_{p,q}$, we can write the right side of (10.8) as

$$|x|^{-n} K_{\Psi,a,b}(|x|), \tag{10.10}$$

where Ψ is a meromorphic function with values in $C^\infty(S^{n-1})$, given by

$$\Psi(s) = A_n(s)(pq^{-s}), \tag{10.11}$$

and results parallel to (8.45) apply. For example, if $a = 0$ and $b \in \mathbb{R}^+ \setminus \mathcal{E}_{p,q}$, we get

$$\begin{aligned}-\frac{1}{|x|^n \log |x|} A_n(0) p(\hat{x}) + \frac{1}{|x|^n (\log |x|)^2} \Big(A_n'(0) p(\hat{x}) - A_n(0)\big(p(\hat{x}) \log q(\hat{x})\big)\Big) \\ + O\Big(\frac{1}{|x|^n (\log |x|)^3}\Big),\end{aligned} \tag{10.12}$$

The leading term is just as in (8.46). The behavior of $q \in C^\infty(S^{n-1})$ figures into the next term in the asymptotic expansion.

A Alternative proof of (1.3)

Here we provide a direct proof of (1.3). The argument we use is parallel to that used for the proof of Theorem 2.17 in Chapter 5 of [14], except that we deal with Fourier integrals, rather than Fourier series, which allows for some simplifications of the details.

In fact, the comparison of u_L with U_L in (1.13)–(1.19) readily implies that (1.3) is equivalent to the result

$$S_0(x) = -\frac{1}{x \log |x|} + O\Big(\frac{1}{|x|(\log |x|)^2}\Big), \tag{A.1}$$

as $x \to 0$, with

$$S_0(x) = \int_2^\infty \frac{1}{\log \xi} \sin x\xi \, d\xi, \tag{A.2}$$

introduced in (1.31). Since $S_0(x)$ is odd, it suffices to treat it for $x > 0$. Note that

$$\begin{aligned} xS_0(x) &= -\int_2^\infty \frac{1}{\log \xi} \frac{d}{d\xi} \cos x\xi \, d\xi \\ &= -\int_2^\infty \frac{1}{\xi(\log \xi)^2} \cos x\xi \, d\xi + \frac{\cos 2x}{\log 2}, \end{aligned} \tag{A.3}$$

the latter identity by integration by parts. Compare (7.5), with $a = 0$. Now, for $u > 1$,

$$\int_u^\infty \frac{d\xi}{\xi(\log \xi)^2} = -\int_u^\infty \frac{d}{d\xi} \frac{1}{\log \xi} \, d\xi = \frac{1}{\log u}, \tag{A.4}$$

so (A.3) yields

$$xS_0(x) = \int_2^\infty \frac{1}{\xi(\log \xi)^2}(1 - \cos x\xi) \, d\xi - \frac{1 - \cos 2x}{\log 2}. \tag{A.5}$$

Assuming $0 < x < 1/2$, we break this integral into an integral over $[2, 1/x]$ and an integral over $[1/x, \infty)$, and we separate out the terms in the integrand of the latter integral, obtaining

$$\begin{aligned} xS_0(x) &+ \frac{1 - \cos 2x}{\log 2} \\ &= \int_2^{1/x} \frac{1}{\xi(\log \xi)^2}(1 - \cos x\xi) \, d\xi \\ &\quad + \int_{1/x}^\infty \frac{1}{\xi(\log \xi)^2} \, d\xi \\ &\quad - \int_{1/x}^\infty \frac{1}{\xi(\log \xi)^2} \cos x\xi \, d\xi \\ &= r_1(x) + v(x) - r_2(x). \end{aligned} \tag{A.6}$$

By (A.4), for $0 < x < 1$,

$$v(x) = \frac{1}{\log 1/x}. \tag{A.7}$$

Next, since

$$|1 - \cos x\xi| \leq x^2\xi^2 \quad \text{for} \quad |x\xi| \leq 1, \tag{A.8}$$

we have

$$\begin{aligned} |r_1(x)| &\leq x^2 \int_2^{1/x} \frac{\xi}{(\log \xi)^2}\, d\xi \\ &\leq Cx^2 \cdot \frac{1}{x} \cdot \frac{1/x}{(\log 1/x)^2} \\ &= \frac{C}{(\log 1/x)^2}, \end{aligned} \tag{A.9}$$

the second inequality because the integrand is monotonically increasing for large ξ. It remains to treat

$$\begin{aligned} r_2(x) &= \frac{1}{x} \int_{1/x}^{\infty} \frac{1}{\xi(\log \xi)^2} \frac{d}{d\xi} \sin x\xi \, d\xi \\ &= -\frac{1}{x} \int_{1/x}^{\infty} \frac{d}{d\xi} \frac{1}{\xi(\log \xi)^2} \sin x\xi \, d\xi + \frac{\sin 1}{(\log 1/x)^2}, \end{aligned} \tag{A.10}$$

the latter identity by integration by parts. A computation gives

$$\Big| \frac{d}{d\xi} \frac{1}{\xi(\log \xi)^2} \Big| \leq \frac{C}{\xi^2(\log \xi)^2}, \tag{A.11}$$

which readily yields

$$|r_2(x)| \leq \frac{C}{(\log 1/x)^2}. \tag{A.12}$$

This proves (A.1), so we have (1.3).

Similar analysis can be done on

$$\begin{aligned} S_1(x) &= \int_2^{\infty} \frac{1}{\xi \log \xi} \sin x\xi \, d\xi \\ &= \frac{1}{x} \int_2^{\infty} \frac{1}{\xi \log \xi} \frac{d}{d\xi}(1 - \cos x\xi)\, d\xi \\ &= \frac{1}{x} \int_2^{\infty} \Big[\frac{1}{\xi^2 \log \xi} + \frac{1}{\xi^2 (\log \xi)^2} \Big] (1 - \cos x\xi)\, d\xi \\ &\quad - \frac{1}{x} \frac{1 - \cos 2x}{2 \log 2}. \end{aligned} \tag{A.13}$$

If we split the last integral into an integral over $[2, 1/|x|]$ and an integral over $[1/|x|, \infty)$, and apply (A.8) to the first integral, we get

$$|S_1(x)| \le \frac{C}{\log 1/|x|}. \tag{A.14}$$

From this and (1.21) one readily deduces (1.20), hence (1.10). (This argument is different from the one in §2, using (2.9)–(2.15).) One can go further, splitting the integral over $[1/|x|, \infty)$ into two terms by splitting $1 - \cos x\xi$, obtaining for $S_1(x)$ the first term in the asymptotic expansion in powers of $(\log |x|)^{-1}$ arising from (1.38). We leave the pursuit of this to the reader. Of course, results obtained in this way are less precise than those given in (1.27) and (1.38).

B Asymptotics for (1.5)

By similar reasoning as used in (1.12)–(1.19), the distribution v_L on $\mathbb{T}^1$ given by (1.5) belongs to $C^\infty(\mathbb{T}^1 \setminus 0)$ and has the same singular behavior at $\theta = 0$ as does

$$\int_0^\infty (\log \xi) \sin x\xi \, d\xi \tag{B.1}$$

at $x = 0$. This is minus the imaginary part of

$$W(x) = \int_0^\infty (\log \xi) e^{-ix\xi} \, d\xi, \tag{B.2}$$

which is seen to satisfy

$$\begin{aligned} \frac{d}{dx}(xW) &= \lim_{\varepsilon \searrow 0} \int_0^\infty e^{-(\varepsilon + ix)\xi} \, d\xi \\ &= \lim_{\varepsilon \searrow 0} \frac{1}{i} \frac{1}{x - i\varepsilon}, \end{aligned} \tag{B.3}$$

hence

$$xW = \lim_{\varepsilon \searrow 0} \frac{1}{i} \log(x - i\varepsilon) + \text{const.} \tag{B.4}$$

Note that $\operatorname{Re} W$ is even and $\operatorname{Im} W$ is odd, so $\operatorname{Re} xW$ is odd and $\operatorname{Im} xW$ is even. Consequently, up to a purely imaginary additive constant, xW is given by

$$\frac{1}{i} \log |x| + \frac{\pi}{2} \operatorname{sgn} x. \tag{B.5}$$

To evaluate the constant, we use

$$
\begin{aligned}
\operatorname{Im} W(1) &= -\int_0^\infty (\log \xi) \sin \xi \, d\xi \\
&= -\int_0^1 (\log \xi) \frac{d}{d\xi}(1-\cos\xi)\, d\xi + \int_1^\infty (\log \xi) \frac{d}{d\xi} \cos \xi \, d\xi \\
&= \int_0^1 \frac{1-\cos\xi}{\xi}\, d\xi - \int_1^\infty \frac{\cos\xi}{\xi}\, d\xi \\
&= \gamma,
\end{aligned}
\tag{B.6}
$$

where γ is Euler's constant. See p. 41 of [2] for the last identity here. It follows that

$$
xW = \frac{1}{i}(\log|x| - \gamma) + \frac{\pi}{2} \operatorname{sgn} x, \tag{B.7}
$$

so

$$
\int_0^\infty (\log \xi) \cos x\xi \, d\xi = \frac{\pi}{2} \operatorname{PF} \frac{1}{|x|}, \tag{B.8}
$$

and

$$
\int_0^\infty (\log \xi) \sin x\xi \, d\xi = \operatorname{PV} \frac{1}{x} \log|x| - \gamma \operatorname{PV} \frac{1}{x}. \tag{B.9}
$$

C The family of operators $A_n(s)$

In §3 we produced the family of operators $A_n(s)$, acting on $C^\infty(S^{n-1})$ as follows:

$$
\begin{aligned}
A_n(s) h_\ell(\omega) = (-i)^\ell 2^{n/2-s} \Gamma\Big(\frac{n-s+\ell}{2}\Big) \Gamma\Big(\frac{s+\ell}{2}\Big)^{-1} h_\ell(\omega), \\
\text{for} \quad s \notin \{n+\ell, n+\ell+2, n+\ell+4, \dots\},
\end{aligned}
\tag{C.1}
$$

when h_ℓ is a harmonic polynomial on $\mathbb{R}^n$, homogeneous of degree ℓ. Equivalently, with $\Lambda \in OPS^1(S^{n-1})$ as in (3.44), an elliptic pseudodifferential operator satisfying $\Lambda h_\ell = \ell h_\ell$,

$$
A_n(s) = 2^{n/2-s} e^{-\pi i \Lambda/2} \Phi_{n,s}(\Lambda). \tag{C.2}
$$

See (3.46)–(3.47). From the asymptotic relation (3.35), i.e.,

$$
\Gamma\Big(\frac{n-s+\ell}{2}\Big) \Gamma\Big(\frac{s+\ell}{2}\Big)^{-1} \sim \Big(\frac{\ell}{2}\Big)^{n/2-s}, \tag{C.3}
$$

as $\ell \to +\infty$, given $s \in \mathbb{C}$, we have

$$
\begin{aligned}
A_n(s) : C^\infty(S^{n-1}) \longrightarrow C^\infty(S^{n-1}), \\
\text{holomorphic for } s \notin \{n, n+1, n+2, \dots\}.
\end{aligned}
\tag{C.4}
$$

Our goal here is to prove (C.3), and also to derive a more precise result, needed to estabish (3.48), i.e.,

$$\Phi_{n,s}(\Lambda) \in OPS^{-s+n/2}(S^{n-1}), \quad \text{for} \ \ s \notin \{n, n+1, n+2, \dots\}. \tag{C.5}$$

Before getting to this, we mention some refinements of (C.4), which follow from (C.1) and (C.3). Namely, if $C^\infty_{\text{even}}(S^{n-1})$ and $C^\infty_{\text{odd}}(S^{n-1})$ denote the spaces of smooth functions $p(\omega)$ satisfying, respectively $p(-\omega) = p(\omega)$ and $p(-\omega) = -p(\omega)$, we have

$$\begin{aligned} A_n(s) : C^\infty_{\text{even}}(S^{n-1}) &\longrightarrow C^\infty_{\text{even}}(S^{n-1}), \\ &\text{holomorphic for } s \notin \{n, n+2, n+4, \dots\}, \end{aligned} \tag{C.6}$$

and

$$\begin{aligned} A_n(s) : C^\infty_{\text{odd}}(S^{n-1}) &\longrightarrow C^\infty_{\text{odd}}(S^{n-1}), \\ &\text{holomorphic for } s \notin \{n+1, n+3, n+5, \dots\}. \end{aligned} \tag{C.7}$$

Also (for $k = 0, 1, 2, \dots$), if

$$C^\infty_{[k]}(S^{n-1}) = \{p \in C^\infty(S^{n-1}) : p \perp E_{n,\ell}, \ \forall \, \ell \le k\}, \tag{C.8}$$

with $E_{n,\ell}$ as in (3.22), then

$$\begin{aligned} A_n(s) : C^\infty_{[k]}(S^{n-1}) &\longrightarrow C^\infty_{[k]}(S^{n-1}), \\ &\text{holomorphic for } s \notin \{n+k+1, n+k+2, \dots\}. \end{aligned} \tag{C.9}$$

To establish (C.3) and refinements, we use Stirling's formula, in the following form:

$$\log \Gamma(z) = \Bigl(z - \frac{1}{2}\Bigr) \log z - z + \frac{1}{2} \log 2\pi + \omega(z), \tag{C.10}$$

for $\operatorname{Re} z \ge 0$, where

$$\omega(z) = \int_0^\infty \Bigl(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\Bigr) \frac{1}{t} e^{-tz} \, dt, \tag{C.11}$$

the Laplace transform of a smooth, bounded function on $[0, \infty)$, which has an asymptotic expansion

$$\omega(z) \sim \sum_{k \ge 0} \beta_k z^{-2k-1}, \quad z \to \infty, \ \operatorname{Re} z \ge 0. \tag{C.12}$$

See [2], §1.4 for a derivation, and [11], §12.33 for a related derivation, with a different formula for (C.11). See also [10]. Applying (C.10) to the left side of (C.3) yields, for $\ell \geq \operatorname{Re} s - n$ and $\ell \geq \operatorname{Re}(-s)$,

$$\begin{aligned}
&\log\Big(\Gamma\Big(\frac{n-s+\ell}{2}\Big)\Gamma\Big(\frac{s+\ell}{2}\Big)^{-1}\Big) \\
&= \frac{n-s+\ell-1}{2}\log\Big(\frac{n-s+\ell}{2}\Big) - \frac{s+\ell-1}{2}\log\Big(\frac{s+\ell}{2}\Big) \\
&\quad - \frac{n-s+\ell}{2} + \frac{s+\ell}{2} + \omega\Big(\frac{n-s+\ell}{2}\Big) - \omega\Big(\frac{s+\ell}{2}\Big) \\
&= \frac{n-2\ell}{2}\log\Big(\frac{n-s+\ell}{2}\Big) + \frac{s+\ell-1}{2}\log\Big(1+\frac{n-2s}{\ell+s}\Big) \\
&\quad + \frac{2s-n}{2} + \omega\Big(\frac{n-s+\ell}{2}\Big) - \omega\Big(\frac{s+\ell}{2}\Big).
\end{aligned} \tag{C.13}$$

Now, for ℓ such that $|\ell+s| > |n-2s|$,

$$\begin{aligned}
\frac{s+\ell-1}{2}\log\Big(1+\frac{n-2s}{\ell+s}\Big) &= \frac{\ell+s-1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\Big(\frac{n-2s}{\ell+s}\Big)^k \\
&= \sum_{k=0}^{\infty}\alpha_k(s)(\ell+s)^{-k}, \quad \alpha_0(s) = \frac{n}{2} - s.
\end{aligned} \tag{C.14}$$

Hence, for ℓ sufficiently large,

$$\begin{aligned}
&\Gamma\Big(\frac{n-s+\ell}{2}\Big)\Gamma\Big(\frac{s+\ell}{2}\Big)^{-1} \\
&= \Big(\frac{\ell+n-s}{2}\Big)^{n/2-s}\exp\Big(\sum_{k=1}^{\infty}\alpha_k(s)(\ell+s)^{-k} + \omega\Big(\frac{n-s+\ell}{2}\Big) - \omega\Big(\frac{s+\ell}{2}\Big)\Big).
\end{aligned} \tag{C.15}$$

By (C.12),

$$\begin{aligned}
&\omega\Big(\frac{n-s+\ell}{2}\Big) - \omega\Big(\frac{s+\ell}{2}\Big) \\
&\quad\sim \sum_{k\geq 0}\beta_k\Big[\Big(\frac{2}{\ell+n-s}\Big)^{2k+1} - \Big(\frac{2}{\ell+s}\Big)^{2k+1}\Big].
\end{aligned} \tag{C.16}$$

We deduce that, as $\ell \to +\infty$,

$$\Gamma\Big(\frac{n-s+\ell}{2}\Big)\Gamma\Big(\frac{s+\ell}{2}\Big)^{-1} \sim \Big(\frac{\ell}{2}\Big)^{n/2-s}\Big(1+\sum_{k\geq 0}\gamma_{k,n}(s)\ell^{-k}\Big), \tag{C.17}$$

for certain $\gamma_{k,n}(s) \in \mathbb{C}$. This is a more precise version of (C.3). It implies that the left side of (C.17) is a classical symbol (in ℓ) of order $n/2 - s$, as long as either $s \in \mathbb{C} \setminus [n, \infty)$ or $\ell > s - n$. The expansion (C.17) yields (C.5).

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