A Critical Besov Space Embedding into BMO

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Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be supported on $\{\xi \in \mathbb{R}^n : 1/4 \le |\xi| \le 4\}$. Assume ψ is radial, $\psi \ge 0$, and $\psi(\xi) > 0$ for $1/2 \le |\xi| \le 2$. Take $f \in L^p(\mathbb{R}^n)$. As is well known, given $s > 0,\ p \in (1,\infty)$, the Besov space $B^s_{p,\infty}(\mathbb{R}^n)$ has the following characterization.

$$f \in B^s_{p,\infty}(\mathbb{R}^n) \Leftrightarrow \|\psi(tD)f\|_{L^p(\mathbb{R}^n)} \le Ct^s, \quad t \in (0,1]. \tag{1}$$

Our goal in this note is to show that $B^{n/2}_{2,\infty}(\mathbb{R}^n)$ is contained in the localized John-Nirenberg space $\mathrm{bmo}(\mathbb{R}^n)$, and discuss how this allows one to obtain an n-dimensional version of a result of J. Cima and K. Petersen [CP].

We begin with some background from two books of E. Stein. The following is proved in [S1], p. 151.

Proposition 1. *Given* $0 < s < 1,\ 1 < p < \infty,\ f \in L^p(\mathbb{R}^n)$,

$$f \in B^s_{p,\infty}(\mathbb{R}^n) \Leftrightarrow \|\tau_y f - f\|_{L^p} \le C|y|^s, \quad |y| \le 1. \tag{2}$$

Here $\tau_y f(x) = f(x+y)$.

The following is proved in [S2], p. 159.

Proposition 2. *Given* $f \in L^2(\mathbb{R}^n),\ f \in \mathrm{bmo}(\mathbb{R}^n)$ *if and only if there exists* $C < \infty$ *such that*

$$\int_{T(B)} |\psi(tD)f(x)|^2\, t^{-1}\, dx\, dt \le C\ \mathrm{Vol}(B), \tag{3}$$

for each ball $B = B_r(x_0) \subset \mathbb{R}^n$, *where* $T(B) = \{(t,x) : x \in B_r(x_0), 0 < t \le r\}$, *and we let* $r \in (0,1]$.

We now aim to prove the following.

Proposition 3. *For* $n \ge 1$,

$$B^{n/2}_{2,\infty}(\mathbb{R}^n) \subset \mathrm{bmo}(\mathbb{R}^n). \tag{4}$$

Proof. Assume $f \in B^{n/2}_{2,\infty}(\mathbb{R}^n)$. Then, by (1),

$$\|\psi(tD)f\|^2_{L^2(\mathbb{R}^n)} \le Ct^n, \quad t \in (0,1]. \tag{5}$$

Hence, with $B = B_r(x_0)$ as above,

$$\begin{aligned} \int_{T(B)} |\psi(tD)f(x)|^2\, t^{-1}\, dx\, dt &\le C \int_0^r t^n\, t^{-1}\, dt \\ &= C' r^n \\ &= C'' \operatorname{Vol}(B). \end{aligned} \tag{6}$$

This completes the proof.

Note that (4) is a bit sharper than the well known L^2-Sobolev space inclusion

$$H^{n/2,2}(\mathbb{R}^n) \subset \mathrm{bmo}(\mathbb{R}^n). \tag{7}$$

Propositions 1 and 3 together give a result similar to that in [CP], when $n = 1$. Note that, for $s > 0$, $f \in L^2(\mathbb{R}^n)$,

$$f \in B^s_{2,\infty}(\mathbb{R}^n) \Leftrightarrow \int_{K \le |\xi| \le 2K} |\hat{f}(\xi)|^2\, d\xi \le C K^{-2s}, \quad \forall\, K \ge 1. \tag{8}$$

Thus

$$\begin{aligned} |\hat{f}(\xi)| \le C(1 + |\xi|)^{-n} &\Rightarrow f \in B^{n/2}_{2,\infty}(\mathbb{R}^n) \\ &\Rightarrow f \in \mathrm{bmo}(\mathbb{R}^n). \end{aligned} \tag{9}$$

A variant of the case $n = 1$ of (9) is also noted in [CP].

References

[CP] J. Cima and K. Petersen, Some analytic functions whose boundary values have bounded mean oscillation, Math. Zeit. 147 (1976), 237–247.

[S1] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton NJ, 1970.

[S2] E. Stein, Harmonic Analysis, Princeton Univ. Press, Princeton NJ, 1993.