Bessel Functions

Michael Taylor

Lecture Notes for Math 524

Contents

1. Introduction

Here we consider solutions to the Bessel equation

$$u''(t) + \frac{1}{t}u'(t) + \Bigl(1 - \frac{\nu^2}{t^2}\Bigr)u(t) = 0, \tag{1.1}$$

for $\nu \in \mathbb{R}$. This equation arises in the study of partial differential equations on regions with special shapes in n-dimensional Euclidean space, and in cases $n = 2$ and $n = 3$ such PDE are very important in mathematical physics. See Appendix A of these notes for material on this.

The equation (1.1) has a regular singular point at $t = 0$. In §2 we convert (1.1) to a first order 2×2 system, of the form

$$t\frac{dx}{dt} = A(t)x, \tag{1.2}$$

with a regular singular point at $t = 0$. This allows us to take advantage of material developed in §11, Chapter 3 of [T1], though our presentation in §2 is actually self contained.

In §3 we study a certain Bessel function $J_\nu(t)$ solving (1.1). Results of §2 guarantee that $\mathcal{J}_\nu(t) = t^{-\nu} J_\nu(t)$ has a convergent power series $\sum_{k=0}^\infty a_k t^{2k}$, and we derive a recursion formula for the coefficients a_k. We produce a solution to this recursion, and hence define $J_\nu(t)$. The solution involves the gamma function $\Gamma(z)$, and we make use of results on $\Gamma(z)$ given in Appendix B. We see that, for $\nu \in \mathbb{R}$, both $J_\nu(t)$ and $J_{-\nu}(t)$ solve (1.1). If ν is not an integer, we show that these solutions are linearly independent, and hence form a basis for solutions to (1.1) on $t \in (0,\infty)$. If $\nu = n$ is an integer, we show $J_{-n}(t) = (-1)^n J_n(t)$. Section 4 produces a solution $Y_\nu(t)$ to (1.1), having the property that the Wronskian

$$W(J_\nu, Y_\nu)(t) = \frac{2}{\pi t}, \tag{1.3}$$

for all $\nu \in \mathbb{R},\ t \in (0,\infty)$, so $\{J_\nu, Y_\nu\}$ forms a basis of solutions to (1.1) for each $\nu \in \mathbb{R}$.

In §5 we give some identities connecting $J_\nu(t)$ with $J_{\nu\pm 1}(t)$. In §6 we discuss $J_{n+1/2}(t)$, which is an elementary function when n is an integer. In §7 we establish an integral formula for $J_\nu(t)$.

In §8 we introduce the Hankel functions $H_\nu^{(1)}$ and $H_\nu^{(2)}$. In §9 we produce results on the asymptotic behavior of $J_\nu(t), H_\nu^{(1)}(t)$, and $Y_\nu(t)$ as $t \to +\infty$. In §10 we apply results of §9 to show that, for each $\nu \in \mathbb{R}$, real valued solutions to (1.1) have infinitely many zeros (say, $\vartheta_{k\nu}$) on $(0,\infty)$, and give their approximate values, for large k. Material in §§8–9 is less self-contained than in other sections. We refer to [W] for a derivation of the integral formula (8.1) for $H_\nu^{(1)}(t)$, and to Chapter 3 of [T3] for the methods of asymptotic analysis that apply to the integral formula (7.1) for $J_\nu(t)$.

As already indicated, Appendix A explains why Bessel functions are so important for certain PDEs, and Appendix B introduces the gamma function, which is a useful tool in the analysis of Bessel functions. Appendix C evaluates the derivative of $1/\Gamma(z)$ for integer z, of use in the formula for $Y_n(t)$ derived in §4.

2. Conversion to first order systems

To begin our treatment of Bessel's equation

$$u''(t) + \frac{1}{t}u'(t) + \Big(1 - \frac{\nu^2}{t^2}\Big)u(t) = 0, \tag{2.1}$$

we convert it to a first-order 2×2 system for

$$x(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad v(t) = tu'(t), \tag{2.2}$$

obtaining

$$\frac{dx}{dt} = \begin{pmatrix} u' \\ u' + tu'' \end{pmatrix}, \tag{2.3}$$

hence

$$t\frac{dx}{dt} = \begin{pmatrix} 0 & -1 \\ \nu^2 - t^2 & 0 \end{pmatrix} x. \tag{2.4}$$

This has the form

$$t\frac{dx}{dt} = A(t)x, \quad A(t) = A_0 + A_2 t^2, \tag{2.5}$$

with

$$A_0 = \begin{pmatrix} 0 & 1 \\ \nu^2 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{2.6}$$

Generally, (2.5) has a solution given as a convergent power series

$$x(t) = x_0 + x_1 t + x_2 t^2 + \cdots, \tag{2.7}$$

as long as $A_0 x_0 = 0$, provided also A_0 has no eigenvalues that are positive integers. This is established in Lemma 11.1, Chapter 3, of [T1]. (A proof that works in the cases we consider here is given in (2.16)–(2.24) below.) In case (2.6), A_0 has a nonzero null space only for $\nu = 0$. To treat the more general cases, we set

$$x(t) = t^\nu y(t). \tag{2.8}$$

This is motivated by the model Euler equation

$$t\frac{dx}{dt} = A_0 x, \tag{2.9}$$

whose solution for $t > 0$ is

$$x(t) = e^{(\log t)A_0} = t^{A_0}, \tag{2.10}$$

upon noting that A_0 in (1.6) has eigenvalues $\pm\nu$.

From (2.8) we have $x' = t^\nu y' + \nu t^{\nu-1} y$, so

$$A(t)x = t\frac{dx}{dt} = t^\nu t\frac{dy}{dt} + \nu t^\nu y, \tag{2.11}$$

so (2.5) is equivalent to

$$t\frac{dy}{dt} = \big(A(t) - \nu I\big)y(t). \tag{2.12}$$

In case (2.5)–(2.6), we have

$$A(t) - \nu I = (A_0 - \nu I) + A_2 t^2, \quad A_0 - \nu I = \begin{pmatrix} -\nu & 1 \\ \nu^2 & -\nu \end{pmatrix}. \tag{2.13}$$

Since

$$(A_0 - \nu I)\begin{pmatrix} 1 \\ \nu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.14}$$

Lemma 11.1 (Chapter 3) of [T1] (which, we recall, is proved in the case needed here below) implies (2.12) has a solution given by a convergent power series

$$y(t) = y_0 + y_1 t + y_2 t^2 + \cdots, \quad y_0 = \begin{pmatrix} 1 \\ \nu \end{pmatrix}, \tag{2.15}$$

provided $A_0 - \nu I$ has no positive integers as eigenvalues. Since $\operatorname{Spec}(A_0 - \nu I) = \{0, -2\nu\}$, this works as long as 2ν is not a negative integer.

Actually, the fact that only even powers of t appear on the right side of (2.13) allows us to improve this a bit. Let us rewrite (2.12)–(2.13) as

$$t\frac{dy}{dt} = B(t^2)y, \tag{2.16}$$

with

$$B(s) = B_0 + B_1 s, \quad B_0 = A_0 - \nu I, \quad B_1 = A_2. \tag{2.17}$$

Set

$$y(t) = z(t^2), \quad t > 0. \tag{2.18}$$

Thus $ty'(t) = 2t^2 z'(t^2) = B(t^2)y(t)$, so

$$s\frac{dz}{ds} = \frac{1}{2}B(s)z(s), \tag{2.19}$$

and since

$$\frac{1}{2}B_0 = \frac{1}{2}\begin{pmatrix} -\nu & 1 \\ \nu^2 & -\nu \end{pmatrix}, \quad \text{Spec } \frac{1}{2}B_0 = \{0, -\nu\}, \tag{2.20}$$

Lemma 11.1 mentioned above applies, to yield a solution to (2.19)

$$z(s) = \sum_{k=0}^{\infty} z_k s^k, \quad z_0 = \begin{pmatrix} 1 \\ \nu \end{pmatrix}, \tag{2.21}$$

as long as ν is not a negative integer. In fact, plugging this power series into (2.19) gives the recursion

$$kz_k = \frac{1}{2}B_0 z_k + \frac{1}{2}B_1 z_{k-1}, \tag{2.22}$$

i.e.,

$$z_k = (2kI - B_0)^{-1} B_1 z_{k-1}, \tag{2.23}$$

starting with $z_0 = (1, \nu)^t$, and then inductively specifying z_k for $k \in \{1, 2, 3, \dots\}$, as long as $2k \notin$ Spec B_0 for each such k, i.e., as long as Spec B_0 contains no positive even integer, hence as long as ν is not a negative integer. Note that (2.23) implies

$$\|z_k\| \le \frac{C}{k} \|B_1\| \cdot \|z_{k-1}\|, \tag{2.24}$$

for k sufficiently large, and this implies convergence of (2.21) for all $s \in \mathbb{R}$. This argument proves the special case of Lemma 11.1 (Chapter 3) of [T1] that we have been citing.

3. The Bessel functions J_ν

Here we construct a solution to (1.1), for $t > 0$, called the Bessel function $J_\nu(t)$. In this case, (2.2) becomes

$$x(t) = \begin{pmatrix} J_\nu(t) \\ tJ'_\nu(t) \end{pmatrix}, \tag{3.1}$$

and (2.8) becomes

$$y(t) = \begin{pmatrix} t^{-\nu} J_\nu(t) \\ t^{1-\nu} J'_\nu(t) \end{pmatrix}. \tag{3.2}$$

Thus we are motivated to set

$$\mathcal{J}_\nu(t) = t^{-\nu} J_\nu(t), \tag{3.3}$$

so $J_\nu(t) = t^\nu \mathcal{J}_\nu(t)$, hence

$$J'_\nu(t) = t^\nu \mathcal{J}'_\nu(t) + \nu t^{\nu-1} \mathcal{J}_\nu(t), \tag{3.4}$$

so

$$y(t) = \begin{pmatrix} \mathcal{J}_\nu(t) \\ t\mathcal{J}'_\nu(t) + \nu \mathcal{J}_\nu(t) \end{pmatrix}. \tag{3.5}$$

The equation (2.12), i.e.,

$$t\frac{dy}{dt} = \big(A(t) - \nu I\big) y(t), \tag{3.6}$$

becomes

$$t\frac{d}{dt} \begin{pmatrix} \mathcal{J}_\nu \\ t\mathcal{J}'_\nu + \nu \mathcal{J}_\nu \end{pmatrix} = \left[\begin{pmatrix} -\nu & 1 \\ \nu^2 & -\nu \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} t^2 \right] \begin{pmatrix} \mathcal{J}_\nu \\ t\mathcal{J}'_\nu + \nu \mathcal{J}_\nu \end{pmatrix}. \tag{3.7}$$

Comparing the first components, we get

$$t\mathcal{J}'_\nu = -\nu \mathcal{J}_\nu + t\mathcal{J}'_\nu + \nu \mathcal{J}_\nu,$$

which is a tautology, and comparing the second components, we get

$$t\big(t\mathcal{J}''_\nu + (\nu+1)\mathcal{J}'_\nu\big) = \nu^2 \mathcal{J}_\nu - \nu(t\mathcal{J}'_\nu + \nu \mathcal{J}_\nu) - t^2 \mathcal{J}_\nu, \tag{3.8}$$

which we rewrite as

$$\mathcal{J}_\nu''(t) + \frac{2\nu+1}{t}\mathcal{J}_\nu'(t) + \mathcal{J}_\nu(t) = 0. \tag{3.9}$$

The material developed in (2.16)–(2.24) applies to (3.7), and guarantees that (3.9) has a solution given as a convergent power series of the form

$$\mathcal{J}_\nu(t) = \sum_{k=0}^\infty a_k t^{2k}, \quad \forall\, t \in \mathbb{R}, \tag{3.10}$$

as long as ν is not a negative integer. Here we give an iterative construction of such coefficients. Given (3.10), the left side of (3.9) is

$$\sum_{k=0}^\infty \Bigl\{(2k+2)(2k+2\nu+2)a_{k+1} + a_k\Bigr\} t^{2k}. \tag{3.11}$$

As long as $\nu \notin \{-1,-2,-3,\dots\}$, one can fix $a_0 = a_0(\nu)$ and solve recursively for a_{k+1}, for each $k \ge 0$:

$$a_{k+1} = -\frac{1}{4}\,\frac{a_k}{(k+1)(k+\nu+1)}. \tag{3.12}$$

We now establish an explicit solution to (3.12), making use of the Euler gamma function $\Gamma(z)$. See Appendix B of these notes for a definition of $\Gamma(z)$ and a derivation of some of its basic properties. For our present purposes, we mention that $\Gamma(z)$ is well defined for $z \notin \{0,-1,-2,-3,\dots\}$, and satisfies

$$\Gamma(z+1) = z\Gamma(z). \tag{3.13}$$

Also, $\Gamma(1) = 1$, so, if k is a positive integer,

$$\Gamma(k+1) = k! \tag{3.14}$$

Moreover,

$$\begin{aligned} &\frac{1}{\Gamma(z)} \ \text{ is well defined for all } z \in \mathbb{R}, \\ &\frac{1}{\Gamma(z)} = 0 \ \text{ for } \ z \in \{0,-1,-2,-3,\dots\}. \end{aligned} \tag{3.15}$$

In light of these results, we see that a solution to (3.12) is given by

$$a_k = \Bigl(-\frac{1}{4}\Bigr)^k \frac{1}{\Gamma(k+1)\Gamma(k+\nu+1)}. \tag{3.16}$$

Furthermore, though (3.12) seems to break down for $\nu \in \{-1, -2, -3, \dots\}$, (3.16) is well defined for all $\nu \in \mathbb{R}$, and all $k \in \mathbb{Z}^+$, and we have

$$4(k+1)(k+\nu+1)a_{k+1} + a_k = 0, \tag{3.17}$$

so (3.11) vanishes. Thus a solution to (3.9) is given by

$$\mathscr{J}_\nu(t) = \frac{1}{2^\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \Big(\frac{t}{2}\Big)^{2k}. \tag{3.18}$$

(We have multiplied the coefficients a_k in (3.16) by $2^{-\nu}$, for consistency with standard conventions.) This is valid for all $\nu, t \in \mathbb{R}$. Convergence of (3.18) follows from the result

$$\frac{1}{\Gamma(k+1)\Gamma(k+\nu+1)} \le \frac{1}{k!}, \quad \text{for } \ k \ge 0, \ k+\nu \text{ sufficiently large},$$

which follows from the fact that $\Gamma(z) \to +\infty$ as $z \to +\infty$, itself a direct consequence of the definition (B.1).

Returning to (3.3), we have the Bessel function

$$J_\nu(t) = t^\nu \mathscr{J}_\nu(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \Big(\frac{t}{2}\Big)^{2k+\nu}, \tag{3.19}$$

solving (1.1), for $\nu \in \mathbb{R}, \ t \in (0, \infty)$.

Note that (1.1) for ν coincides with (1.1) for $-\nu$. Thus, both $J_\nu(t)$ and $J_{-\nu}(t)$ solve (1.1). Now the leading term in (3.19) is

$$J_\nu(t) = \frac{1}{\Gamma(\nu+1)} \Big(\frac{t}{2}\Big)^{\nu} + \cdots. \tag{3.20}$$

As long as ν is not a negative integer, this leading term is nonvanishing, so

$$\nu \notin \mathbb{Z} \Longrightarrow J_\nu(t) \ \text{ and } \ J_{-\nu}(t) \ \text{ are linearly independent solutions of (1.1).} \tag{3.21}$$

However, if ν is a negative integer, $1/\Gamma(\nu+1) = 0$, so (3.20) does not establish such linear independence. Also, clearly $J_\nu = J_{-\nu}$ when $\nu = 0$. Moreover, when $\nu = -n$ is a negative integer, we have

$$\begin{aligned} J_{-n}(t) &= \sum_{k=n}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k-n+1)} \Big(\frac{t}{2}\Big)^{2k-n} \\ &= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell+n}}{\Gamma(\ell+1)\Gamma(\ell+n+1)} \Big(\frac{t}{2}\Big)^{2\ell+n} \\ &= (-1)^n J_n(t). \end{aligned} \tag{3.22}$$

Thus we have not constructed two linearly independent solutions to (1.1) when ν is an integer. We turn to this task in §4.

4. The Bessel functions Y_ν

We have seen that $J_\nu(t)$ and $J_{-\nu}(t)$ are linearly independent solutions to Bessel's equation

$$u''(t) + \frac{1}{t}u'(t) + \Bigl(1 - \frac{\nu^2}{t^2}\Bigr)u(t) = 0, \tag{4.1}$$

as long as ν is not an integer, while if $n \in \mathbb{Z}$, then $J_{-n}(t) = (-1)^n J_n(t)$. We want to construct a basis of solutions to (4.1), uniformly good for all ν. This construction can be motivated by a calculation of the Wronskian.

Generally, for a pair of solutions u_1 and u_2 to a second order ODE

$$a(t)u'' + b(t)u' + c(t)u = 0, \tag{4.2}$$

u_1 and u_2 are linearly independent if and only if the Wronskian

$$W(t) = W(u_1, u_2)(t) = \det \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} \tag{4.3}$$

is nonvanishing. Now $W(t)$ satisfies the first order ODE

$$W'(t) = -\frac{b(t)}{a(t)}W(t). \tag{4.4}$$

In the case of Bessel's equation (4.1), this becomes

$$W'(t) = -\frac{W(t)}{t}, \tag{4.5}$$

so

$$W(t) = \frac{K}{t}, \tag{4.6}$$

for some K, independent of t, but perhaps depending on ν.

To be specific,

$$W(J_\nu, J_{-\nu})(t) = \frac{K(\nu)}{t}. \tag{4.7}$$

We turn to the task of evaluating $K(\nu)$. Using the relations $J_\nu(t) = t^\nu \mathcal{J}_\nu(t)$ and $J_{-\nu}(t) = t^{-\nu}\mathcal{J}_{-\nu}(t)$, we have

$$W(J_\nu, J_{-\nu})(t) = W(\mathcal{J}_\nu, \mathcal{J}_{-\nu})(t) - \frac{2\nu}{t}\mathcal{J}_\nu(t)\mathcal{J}_{-\nu}(t). \tag{4.8}$$

Since $\mathcal{J}_\nu$ and $\mathcal{J}_{-\nu}$ are smooth on $\mathbb{R}$, we get

$$W(J_\nu, J_{-\nu})(t) = -2\nu\, \frac{\mathcal{J}_\nu(0)\mathcal{J}_{-\nu}(0)}{t} + g(t), \tag{4.9}$$

where $g(t)$ is smooth on $[0,\infty)$. Comparison with (4.7) gives

$$W(J_\nu, J_{-\nu})(t) = -2\, \frac{\nu\mathcal{J}_\nu(0)\mathcal{J}_{-\nu}(0)}{t}, \tag{4.10}$$

so, in (4.7), $K(\nu) = -2\nu\mathcal{J}_\nu(0)\mathcal{J}_{-\nu}(0)$. Now (3.18) yields

$$\mathcal{J}_\nu(0) = \frac{1}{2^\nu\Gamma(\nu+1)}, \quad \mathcal{J}_{-\nu}(0) = \frac{1}{2^{-\nu}\Gamma(1-\nu)}, \tag{4.11}$$

so

$$K(\nu) = -\frac{2\nu}{\Gamma(\nu+1)\Gamma(1-\nu)}. \tag{4.10}$$

By (3.13) (cf. (B.3)), $\Gamma(\nu+1) = \nu\Gamma(\nu)$. Furthermore (cf. (B.8)),

$$\Gamma(\nu)\Gamma(1-\nu) = \frac{\pi}{\sin \pi\nu}. \tag{4.13}$$

Hence

$$W(J_\nu, J_{-\nu})(t) = -\frac{2}{\pi}\,\frac{\sin\pi\nu}{t}. \tag{4.14}$$

In particular, this recovers our previous observation that J_ν and $J_{-\nu}$ are linearly independent if and only if $\nu \notin \mathbb{Z}$.

Taking a cue from (4.14), we set

$$Y_\nu(t) = \frac{J_\nu(t)\cos\pi\nu - J_{-\nu}(t)}{\sin\pi\nu}, \tag{4.15}$$

when ν is not an integer. Then, for $n \in \mathbb{Z}$, we define

$$Y_n(t) = \lim_{\nu\to n} Y_\nu(t) = \frac{1}{\pi}\Big[\frac{\partial}{\partial\nu}J_\nu(t) - (-1)^n\frac{\partial}{\partial\nu}J_{-\nu}(t)\Big]\Big|_{\nu=n}. \tag{4.16}$$

Then we have

$$W(J_\nu, Y_\nu)(t) = \frac{2}{\pi t}, \tag{4.17}$$

for all ν.

We now produce a series representation for $Y_n(t)$, when $n \in \mathbb{Z}^+$. Recall that, for $\nu \in \mathbb{R}$,

$$J_\nu(t) = \sum_{k=0}^{\infty} \alpha_k(\nu)\Bigl(\frac{t}{2}\Bigr)^{2k+\nu}, \quad \alpha_k(\nu) = \frac{(-1)^k}{k!\Gamma(k+\nu+1)}. \tag{4.18}$$

Hence

$$\frac{\partial}{\partial\nu} J_\nu(t) = \sum_{k=0}^{\infty} \alpha_k'(\nu)\Bigl(\frac{t}{2}\Bigr)^{2k+\nu} + \Bigl(\log\frac{t}{2}\Bigr)\sum_{k=0}^{\infty} \alpha_k(\nu)\Bigl(\frac{t}{2}\Bigr)^{2k+\nu}, \tag{4.19}$$

Note that the last sum is equal to $J_\nu(t)$. Hence

$$\frac{\partial}{\partial\nu} J_\nu(t)\Big|_{\nu=n} = \sum_{k=0}^{\infty} \alpha_k'(n)\Bigl(\frac{t}{2}\Bigr)^{2k+n} + \Bigl(\log\frac{t}{2}\Bigr) J_n(t). \tag{4.20}$$

Similarly,

$$\frac{\partial}{\partial\nu} J_{-\nu}(t) = -\sum_{k=0}^{\infty} \alpha_k'(-\nu)\Bigl(\frac{t}{2}\Bigr)^{2k-\nu} - \Bigl(\log\frac{t}{2}\Bigr) J_{-\nu}(t). \tag{4.21}$$

Recalling from (3.13) that $\alpha_k(-n) = 0$ for $k < n$, we have

$$\frac{\partial}{\partial\nu} J_{-\nu}(t)\Big|_{\nu=n} = -\sum_{k=0}^{\infty} \alpha_k'(-n)\Bigl(\frac{t}{2}\Bigr)^{2k-n} - \Bigl(\log\frac{t}{2}\Bigr) J_{-n}(t). \tag{4.22}$$

Plugging (4.20) and (4.22) into (4.16) then yields the series for $Y_n(t)$, convergent for $t \in (0,\infty)$.

In detail, if n is a positive integer,

$$\begin{aligned} Y_n(t) = \frac{2}{\pi}\Bigl(\log\frac{t}{2}\Bigr) J_n(t) &+ \frac{(-1)^n}{\pi}\sum_{k=0}^{\infty} \alpha_k'(-n)\Bigl(\frac{t}{2}\Bigr)^{2k-n} \\ &+ \frac{1}{\pi}\sum_{k=0}^{\infty} \alpha_k'(n)\Bigl(\frac{t}{2}\Bigr)^{2k+n}, \end{aligned} \tag{4.23}$$

and

$$Y_0(t) = \frac{2}{\pi}\Bigl(\log\frac{t}{2}\Bigr) J_0(t) + \frac{2}{\pi}\sum_{k=0}^{\infty} \alpha_k'(0)\Bigl(\frac{t}{2}\Bigr)^{2k}. \tag{4.24}$$

The evaluation of $\alpha_k'(\pm n)$ is discussed in Appendix C. We see that

$$Y_0(t) \sim \frac{2}{\pi}\log t, \quad \text{as } t \searrow 0, \tag{4.25}$$

while, if n is a positive integer,

$$Y_n(t) \sim \frac{(-1)^n}{\pi}\alpha_0'(-n)\Bigl(\frac{t}{2}\Bigr)^{-n}, \quad \text{as } t \searrow 0. \tag{4.26}$$

As seen in Appendix C, when n is a positive integer,

$$\alpha_0'(-n) = \beta'(-n+1) = (-1)^{n-1}(n-1)!. \tag{4.27}$$

5. Relations between J_ν and $J_{\nu\pm 1}$

The Bessel functions have the following remarkable relations:

$$J_{\nu+1}(t) = -J_\nu'(t) + \frac{\nu}{t} J_\nu(t), \tag{5.1}$$

$$J_{\nu-1}(t) = J_\nu'(t) + \frac{\nu}{t} J_\nu(t). \tag{5.2}$$

Putting these two identities together yields

$$\Bigl(\frac{d}{dt} - \frac{\nu-1}{t}\Bigr)\Bigl(\frac{d}{dt} + \frac{\nu}{t}\Bigr) J_\nu(t) = J_\nu(t), \tag{5.3}$$

which is equivalent to the Bessel equation (1.1). The identities (5.1) and (5.2) are equivalent, respectively, to

$$\mathcal{J}_{\nu+1}(t) = -\frac{1}{t}\mathcal{J}_\nu'(t), \tag{5.4}$$

and

$$\mathcal{J}_{\nu-1}(t) = t\mathcal{J}_\nu'(t) + 2\nu \mathcal{J}_\nu(t). \tag{5.5}$$

These follow readily from the power series

$$\mathcal{J}_\nu(t) = \frac{1}{2^\nu} \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \Bigl(\frac{t}{2}\Bigr)^{2k}. \tag{5.6}$$

and its corollary

$$\mathcal{J}_\nu'(t) = \frac{1}{2^\nu} \sum_{k=1}^\infty \frac{(-1)^k k}{k!\Gamma(k+\nu+1)} \Bigl(\frac{t}{2}\Bigr)^{2k-1}. \tag{5.7}$$

The reader is invited to verify (5.4)–(5.5), and hence (5.1)–(5.2). We mention that to establish (5.4) one uses $\Gamma(k+1+1) = (k+1)\Gamma(k+1)$, and to establish (5.5) one uses $\Gamma(k+\nu+1) = (k+\nu)\Gamma(k+\nu)$.

We also have analogues of (5.1)–(5.2) for the functions Y_ν:

$$\begin{aligned} Y_{\nu+1}(t) &= -Y_\nu'(t) + \frac{\nu}{t} Y_\nu(t), \\ Y_{\nu-1}(t) &= Y_\nu'(t) + \frac{\nu}{t} Y_\nu(t). \end{aligned} \tag{5.8}$$

We leave it to the reader to establish this, first for $\nu \notin \mathbb{Z}$, via (4.15), and then for $\nu \in \mathbb{Z}$, via a limiting argument.

6. The functions $J_{n+1/2}$

Looking at (3.9), we see that $\mathcal{J}_{-1/2}(t)$ solves

$$u''(t) + u(t) = 0, \tag{6.1}$$

whose solutions are $C_1 \cos t + C_2 \sin t$. Meanwhile, the power series

$$\mathcal{J}_\nu(t) = \frac{1}{2^\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \Big(\frac{t}{2}\Big)^{2k} \tag{6.2}$$

yields

$$\mathcal{J}_{-1/2}(t) = \sqrt{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1/2)} \Big(\frac{t}{2}\Big)^{2k}, \tag{6.3}$$

which has only even powers of t, so $\mathcal{J}_{-1/2}(t) = C_1 \cos t$. Furthermore, (6.3) implies

$$\mathcal{J}_{-1/2}(0) = \frac{\sqrt{2}}{\Gamma(1/2)} = \sqrt{\frac{2}{\pi}}, \tag{6.4}$$

the latter identity by (B.9). Hence

$$\mathcal{J}_{-1/2}(t) = \sqrt{\frac{2}{\pi}} \cos t, \tag{6.5}$$

and consequently

$$J_{-1/2}(t) = \sqrt{\frac{2}{\pi t}} \cos t. \tag{6.6}$$

Applying (5.1) with $\nu = -1/2$ yields

$$J_{1/2}(t) = \sqrt{\frac{2}{\pi t}} \sin t. \tag{6.7}$$

Then repeated applications of (5.1)–(5.2) give, for $n \in \mathbb{Z}^+$,

$$J_{n+1/2}(t) = (-1)^k \Big\{ \prod_{j=1}^{n} \Big(\frac{d}{dt} - \frac{j-1/2}{t} \Big) \Big\} \frac{\sin t}{\sqrt{2\pi t}}, \tag{6.8}$$

and

$$J_{-n-1/2}(t) = \left\{ \prod_{j=1}^{n} \left(\frac{d}{dt} - \frac{j-1/2}{t} \right) \right\} \frac{\cos t}{\sqrt{2\pi t}}. \tag{6.9}$$

Returning to the identity (6.5), we recall that

$$\cos t = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} t^{2k}. \tag{6.10}$$

Comparison with (6.3) gives

$$\sqrt{\pi}(2k)! = 2^{2k} k! \Gamma\left(k + \frac{1}{2}\right), \tag{6.11}$$

which can also be deduced from the identity $\Gamma(1/2) = \sqrt{\pi}$ and, via $\Gamma(z+1) = z\Gamma(z)$,

$$\begin{aligned} \Gamma\left(k + \frac{1}{2}\right) &= \left(k - \frac{1}{2}\right)\Gamma\left(k - \frac{1}{2}\right) \\ &= \cdots \\ &= \frac{2k-1}{2} \frac{2k-3}{2} \cdots \frac{1}{2}\Gamma\left(\frac{1}{2}\right). \end{aligned} \tag{6.12}$$

7. Integral formula for J_ν

Our goal in this section is to establish the following integral formula:

$$J_\nu(t) = \frac{(t/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \int_{-1}^{1} (1-s^2)^{\nu-1/2} e^{ist}\, ds, \tag{7.1}$$

for $\nu > -1/2$. Motivation for this formula can be found in Chapter 3, §6, of [T3].

To verify (7.1), we replace e^{ist} by its power series, integrate term by term, and use an identity from Appendix B. To begin, the integral on the right side of (7.1) is equal to

$$\sum_{k=0}^{\infty} \frac{1}{(2k)!} \int_{-1}^{1} (ist)^{2k} (1-s^2)^{\nu-1/2}\, ds. \tag{7.2}$$

The identity (B.14) implies

$$\int_{-1}^{1} s^{2k} (1-s^2)^{\nu-1/2}\, ds = \frac{\Gamma(k+1/2)\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)}, \tag{7.3}$$

so the right side of (7.1) equals

$$\frac{(t/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \sum_{k=0}^{\infty} \frac{1}{(2k)!} (it)^{2k} \frac{\Gamma(k+1/2)\Gamma(\nu+1/2)}{\Gamma(k+\nu+1)}. \tag{7.4}$$

As seen in (6.11), we have

$$\Gamma\Big(\frac{1}{2}\Big)(2k)! = 2^{2k}\, k!\, \Gamma\Big(k+\frac{1}{2}\Big), \tag{7.5}$$

so the right side of (7.1) is equal to

$$\Big(\frac{t}{2}\Big)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \Big(\frac{t}{2}\Big)^{2k}, \tag{7.6}$$

which agrees with our formula (3.19) for $J_\nu(t)$. This proves (7.1).

Note that the integral in (7.1) is easy to evaluate when $\nu = 1/2$. We have

$$\begin{aligned} J_{1/2}(t) &= \frac{1}{\sqrt{\pi}} \Big(\frac{t}{2}\Big)^{1/2} \int_{-1}^{1} e^{ist}\, ds \\ &= \frac{1}{\sqrt{\pi}} \Big(\frac{t}{2}\Big)^{1/2} \frac{2\sin t}{t} \\ &= \sqrt{\frac{2}{\pi t}}\, \sin t, \end{aligned} \tag{7.7}$$

recovering the formula (6.7).

8. The Hankel functions $H_\nu^{(k)}$

The Hankel functions $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are defined by

$$\begin{aligned} H_\nu^{(1)}(t) &= J_\nu(t) + iY_\nu(t), \\ H_\nu^{(2)}(t) &= J_\nu(t) - iY_\nu(t). \end{aligned} \tag{8.1}$$

The Wronskian relation (4.17) readily yields

$$W(H_\nu^{(1)}, H_\nu^{(2)})(t) = -\frac{4i}{\pi t}. \tag{8.2}$$

Series expansions for $H_\nu^{(1)}(t)$ and $H_\nu^{(2)}(t)$ follow from (3.19) and (4.15) for $\nu \notin \mathbb{Z}$, and (4.23)–(4.24) for $\nu = n \in \mathbb{Z}^+$.

Using (4.15), we have

$$\begin{aligned} H_\nu^{(1)}(t) &= \frac{J_{-\nu}(t) - e^{-\pi i\nu} J_\nu(t)}{i \sin \pi\nu}, \\ H_\nu^{(2)}(t) &= \frac{e^{\pi i\nu} J_\nu(t) - J_{-\nu}(t)}{i \sin \pi\nu}, \end{aligned} \tag{8.3}$$

evaluated as usual by l'Hospital's rule for $\nu \in \mathbb{Z}$. These formulas imply

$$H_{-\nu}^{(1)}(t) = e^{\pi i\nu} H_\nu^{(1)}(t), \quad H_{-\nu}^{(2)}(t) = e^{-\pi i\nu} H_\nu^{(2)}(t). \tag{8.4}$$

We mention the following useful integral formula:

$$H_\nu^{(1)}(t) = \Bigl(\frac{2}{\pi t}\Bigr)^{1/2} \frac{e^{i(t-\pi\nu/2-\pi/4)}}{\Gamma(\nu+1/2)} \int_0^\infty e^{-s} s^{\nu-1/2} \Bigl(1 - \frac{s}{2it}\Bigr)^{\nu-1/2} ds, \tag{8.5}$$

valid for $t > 0,\ \nu > -1/2$. There is a similar formula for $H_\nu^{(2)}(t)$. We will not prove (8.5) here. See [W], p. 168, for a derivation. See also [L], p. 139.

9. Large t behavior

The integral formula (7.1) for $J_\nu(t)$, valid for $\nu > -1/2,\ t > 0$, represents $J_\nu(t)$ as $(t/2)^\nu$ times the Fourier transform of a function supported on the line segment $I = [-1, 1]$, and smooth on the interior of this segment. Classical methods of Fourier analysis (cf. [T3], Chapter 3) read off the asymptotic behavior of such a Fourier integral from the nature of the singularities of $\chi_I(s)(1-s^2)^{\nu-1/2}$ at the endpoints $s = \pm 1$. The following results:

$$J_\nu(t) = \Bigl(\frac{2}{\pi t}\Bigr)^{1/2} \cos\Bigl(t - \frac{\nu\pi}{2} - \frac{\pi}{4}\Bigr) + O(t^{-3/2}), \quad t \to +\infty, \tag{9.1}$$

given $\nu > -1/2$. We also have

$$J_\nu'(t) = -\Bigl(\frac{2}{\pi t}\Bigr)^{1/2} \sin\Bigl(t - \frac{\nu\pi}{2} - \frac{\pi}{4}\Bigr) + O(t^{-3/2}), \quad t \to +\infty. \tag{9.2}$$

In fact, (9.2) follows from the identity (5.1), i.e.,

$$J_\nu'(t) = -J_{\nu+1}(t) + \frac{\nu}{t} J_\nu(t), \tag{9.3}$$

and (9.1), applied both to $J_\nu(t)$ and to $J_{\nu+1}(t)$, and noting that

$$\cos\Bigl(x - \frac{\pi}{2}\Bigr) = \sin x. \tag{9.4}$$

The integral formula (8.5) for $H_\nu^{(1)}(t)$, valid for $\nu > -1/2,\ t > 0$, can be used to obtain an analogous result on the asymptotic behavior of $H_\nu^{(1)}(t)$. In fact, one can show that, if $\nu > -1/2$,

$$\int_0^\infty e^{-s} s^{\nu-1/2} \Bigl(1 - \frac{s}{2it}\Bigr)^{\nu-1/2} ds = \Gamma\Bigl(\nu + \frac{1}{2}\Bigr) + O(t^{-1}), \tag{9.5}$$

as $t \to +\infty$. It follows that

$$H_\nu^{(1)}(t) = \Bigl(\frac{2}{\pi t}\Bigr)^{1/2} e^{i(t-\nu\pi/2-\pi/4)} + O(t^{-3/2}), \quad t \to +\infty, \tag{9.6}$$

given $\nu > -1/2$. Since $H_\nu^{(1)}(t) = J_\nu(t) + iY_\nu(t)$, comparison with (9.1) gives

$$Y_\nu(t) = \Bigl(\frac{2}{\pi t}\Bigr)^{1/2} \sin\Bigl(t - \frac{\nu\pi}{2} - \frac{\pi}{4}\Bigr) + O(t^{-3/2}), \quad t \to +\infty, \tag{9.7}$$

for $\nu > -1/2$. An argument parallel to that yielding (9.2), with (9.3) replaced by (5.8), yields

$$Y_\nu'(t) = \Bigl(\frac{2}{\pi t}\Bigr)^{1/2} \cos\Bigl(t - \frac{\nu\pi}{2} - \frac{\pi}{4}\Bigr) + O(t^{-3/2}), \quad t \to +\infty, \tag{9.8}$$

for $\nu > -1/2$. Note that a comparison with (9.1)–(9.2) gives

$$Y_\nu(t) = -J_\nu'(t) + O(t^{-3/2}), \quad Y_\nu'(t) = J_\nu(t) + O(t^{-3/2}), \tag{9.9}$$

as $t \to +\infty$, for $\nu > -1/2$.

10. Zeros of J_ν and Y_ν

The functions $J_\nu(t)$ and $Y_\nu(t)$ each have an infinite number of zeros on the half line $(0, \infty)$. We show how this follows readily from the asymptotic formulas (9.1) and (9.7), and consider how to approximate these zeros, at least for large t.

More generally, any real-valued solution u_ν to Bessel's equation (1.1) is a linear combination $u_\nu = aJ_\nu + bY_\nu$, with $a, b \in \mathbb{R}$, and hence there exists $\psi \in [0, 2\pi]$ and $A \in \mathbb{R}$ such that

$$u_\nu(t) = A\Big(\frac{2}{\pi t}\Big)^{1/2} \cos\Big(t - \frac{\nu\pi}{2} - \frac{\pi}{4} - \psi\Big) + O(t^{-3/2}), \tag{10.1}$$

as $t \to +\infty$. We assume u_ν is not identically zero, so $A \neq 0$. We can restrict attention to $\nu \in [0, \infty)$, and note that (10.1) applies also to $u_\nu = J_{-\nu}$ and $u_\nu = Y_{-\nu}$. We also have

$$u_\nu'(t) = -A\Big(\frac{2}{\pi t}\Big)^{1/2} \sin\Big(t - \frac{\nu\pi}{2} - \frac{\pi}{4} - \psi\Big) + O(t^{-3/2}). \tag{10.2}$$

Now set

$$\begin{aligned} \alpha_{k\nu} &= k\pi + \frac{\nu\pi}{2} + \frac{\pi}{4} + \psi, \\ \beta_{k\nu} &= \Big(k + \frac{1}{2}\Big)\pi + \frac{\nu\pi}{2} + \frac{\pi}{4} + \psi = \alpha_{k\nu} + \frac{\pi}{2}. \end{aligned} \tag{10.3}$$

We have, with $t = \alpha_{k\nu}$,

$$\begin{aligned} u_\nu(\alpha_{k\nu}) &= (-1)^k A\Big(\frac{2}{\pi t}\Big)^{1/2} + O(t^{-3/2}), \\ u_\nu(\beta_{k\nu}) &= O(t^{-3/2}), \end{aligned} \tag{10.4}$$

and (still with $t = \alpha_{k\nu}$)

$$\begin{aligned} u_\nu'(\alpha_{k\nu}) &= O(t^{-3/2}), \\ u_\nu'(\beta_{k\nu}) &= (-1)^{k+1} A\Big(\frac{2}{\pi t}\Big)^{1/2} + O(t^{-3/2}). \end{aligned} \tag{10.5}$$

The first part of (10.4) guarantees that, for each $\nu \in [0, \infty)$, there exists $T_\nu < \infty$ such that, whenever $\alpha_{k\nu} \geq T_\nu$, $u_\nu(\alpha_{k\nu})$ and $u(\alpha_{(k+1)\nu}) = u_\nu(\alpha_{k\nu} + \pi)$ have opposite signs, hence there exists

$$\vartheta_{k\nu} \in (\alpha_{k\nu}, \alpha_{k\nu} + \pi) \ \text{ such that } \ u_\nu(\vartheta_{k\nu}) = 0. \tag{10.6}$$

The second part of (10.4) suggests that $\vartheta_{k\nu}$ is close to $\beta_{k\nu} = \alpha_{k\nu} + \pi/2$, and indeed the second part of (10.5) yields

$$\vartheta_{k\nu} = \beta_{k\nu} + O(k^{-1}). \tag{10.7}$$

Remark. The Hankel functions have no zeros on the half line $(0, \infty)$. In fact, since $J_\nu(t)$ and $Y_\nu(t)$ are real valued for $t \in (0, \infty)$,

$$t_0 \in (0, \infty),\ H_\nu^{(1)}(t_0) = 0 \Longrightarrow J_\nu(t_0) = Y_\nu(t_0) = 0, \tag{10.8}$$

contradicting the fact that $\{J_\nu, Y_\nu\}$ is a basis of solutions to (1.1).

A. Where Bessel functions come from

Bessel functions arise in the natural generalization of the equation

$$\frac{d^2u}{dx^2} + k^2 u = 0, \tag{A.1}$$

with solutions $\sin kx$ and $\cos kx$, to partial differential equations

$$\Delta u + k^2 u = 0, \tag{A.2}$$

where Δ is the Laplace operator, acting on a function u on a domain $\Omega \subset \mathbb{R}^n$ by

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}. \tag{A.3}$$

We can eliminate k^2 from (A.2) by scaling. Set $u(x) = v(kx)$. Then equation (A.2) becomes

$$(\Delta + 1)v = 0. \tag{A.4}$$

We specialize to the case $n = 2$ and write

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \tag{A.5}$$

For a number of special domains $\Omega \subset \mathbb{R}^2$, such as circular domains, annular domains, angular sectors, and pie-shaped domains, it is convenient to switch to polar coordinates (r, θ), related to (x, y)-coordinates by

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{A.6}$$

In such coordinates,

$$\Delta v = \Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\Big)v. \tag{A.7}$$

A special class of solutions to (A.4) has the form

$$v = w(r)e^{i\nu\theta}. \tag{A.8}$$

By (A.7), for such v,

$$(\Delta + 1)v = \Big[\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \Big(1 - \frac{\nu^2}{r^2}\Big)w\Big]e^{i\nu\theta}, \tag{A.9}$$

so (A.4) holds if and only if

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} + \Big(1 - \frac{\nu^2}{r^2}\Big)w = 0. \tag{A.10}$$

This is Bessel's equation (1.1) (with different variables).

Note that if v solves (A.4) on $\Omega \subset \mathbb{R}^2$ and if Ω is a circular domain or an annular domain, centered at the origin, then ν must be an integer. However, if Ω is an angular sector or a pie-shaped domain, with vertex at the origin, ν need not be an integer.

In n dimensions, the Laplace operator (A.3) can be written

$$\Delta v = \Big(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_S\Big)v, \tag{A.11}$$

where Δ_S is a second-order differential operator acting on functions on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, called the Laplace-Beltrami operator. Generalizing (A.8), one looks for solutions to (A.4) of the form

$$v(x) = w(r)\psi(\omega), \tag{A.12}$$

where $x = r\omega,\ r \in (0,\infty),\ \omega \in S^{n-1}$. Parallel to (A.9), for such v,

$$(\Delta + 1)v = \Big[\frac{d^2w}{dr^2} + \frac{n-1}{r}\frac{dw}{dr} + \Big(1 - \frac{\nu^2}{r^2}\Big)w\Big]\psi(\omega), \tag{A.13}$$

provided

$$\Delta_S\psi = -\nu^2\psi. \tag{A.14}$$

The equation

$$\frac{d^2w}{dr^2} + \frac{n-1}{r}\frac{dw}{dr} + \Big(1 - \frac{\nu^2}{r^2}\Big)w = 0 \tag{A.15}$$

is a variant of Bessel's equation. If we set

$$\varphi(r) = r^{n/2-1}w(r), \tag{A.16}$$

then (A.15) is converted into the Bessel equation

$$\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr} + \Big(1 - \frac{\mu^2}{r^2}\Big)\varphi = 0, \quad \mu^2 = \nu^2 + \Big(\frac{n-2}{2}\Big)^2. \tag{A.17}$$

The study of solutions to (A.14) gives rise to the study of spherical harmonics, and from there to other special functions, such as Legendre functions.

The search for solutions of the form (A.12) is a key example of the method of separation of variables for partial differential equations. It arises in numerous other contexts. Here are a couple of other examples:

$$(\Delta - |x|^2 + k^2)u = 0, \tag{A.18}$$

and

$$\Big(\Delta + \frac{K}{|x|} + k^2\Big)u = 0. \tag{A.19}$$

The first describes the n-dimensional quantum harmonic oscillator. The second (for $n = 3$) describes the quantum mechanical model of a hydrogen atom, according to Schrödinger. Study of these equations leads to other special functions defined by differential equations, such as Hermite functions and Whittaker functions.

Much further material on these topics can be found in books on partial differential equations, such as [T3] (particularly Chapters 3 and 8).

B. The Euler gamma function $\Gamma(z)$

We define the gamma function by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \quad \text{for} \quad z > 0. \tag{B.1}$$

Actually, we can take z to be complex, and (B.1) works for $\operatorname{Re} z > 0$. In such a case, (B.1) is absolutely convergent. There is the crucial identity

$$\begin{aligned} \Gamma(z+1) &= \int_0^\infty e^{-t} t^z\, dt \\ &= -\int_0^\infty \frac{d}{dt}(e^{-t})\, t^z\, dt \\ &= \int_0^\infty e^{-t}\, \frac{d}{dt} t^z\, dt \\ &= z\Gamma(z), \end{aligned} \tag{B.2}$$

for $\operatorname{Re} z > 0$, where we use integration by parts plus the identity $(d/dt)t^z = zt^{z-1}$. In other words,

$$\Gamma(z+1) = z\Gamma(z), \tag{B.3}$$

for $\operatorname{Re} z > 0$. The definition (B.1) readily gives $\Gamma(1) = 1$, so, for each positive integer k, the relation (B.3) gives inductively

$$\Gamma(k) = (k-1)! \tag{B.4}$$

While $\Gamma(z)$ is defined in (B.1) for $\operatorname{Re} z > 0$, note that the left side of (B.3) is well defined for $\operatorname{Re} z > -1$, so this identity extends $\Gamma(z)$ to $\{z \in \mathbb{C} : \operatorname{Re} z > -1\}$, with a pole at $z = 0$. Iterating this argument, we extend $\Gamma(z)$ to $\{z \in \mathbb{C} : z \neq 0, -1, -2, \dots\}$. Similarly,

$$\beta(z) = \frac{1}{\Gamma(z)}, \tag{B.5}$$

initially defined for $\operatorname{Re} z > 0$, satisfies

$$\beta(z) = z\beta(z+1), \tag{B.6}$$

and then β extends to be well defined for all $z \in \mathbb{C}$, with

$$\beta(z) = 0 \quad \text{for} \quad z \in \{0, -1, -2, -3, \dots\}. \tag{B.7}$$

For use in §4 (cf. (4.13)) we record the following identity:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{B.8}$$

A demonstration of this would take us far afield, so we refer to §18 of [T2] for a proof, and for further material on the gamma function.

Note that setting $z = 1/2$ in (B.8) yields

$$\Gamma\Bigl(\frac{1}{2}\Bigr) = \sqrt{\pi}. \tag{B.9}$$

A direct proof of (B.9) can be given as follows. Taking $t = s^2$ in (B.1) gives

$$\Gamma(z) = 2\int_0^\infty e^{-s^2} s^{2z-1}\, ds, \tag{B.10}$$

hence

$$\Gamma\Bigl(\frac{1}{2}\Bigr) = 2\int_0^\infty e^{-s^2}\, ds = \int_{-\infty}^\infty e^{-s^2}\, ds. \tag{B.11}$$

Thus

$$\begin{aligned} \Gamma\Bigl(\frac{1}{2}\Bigr)^2 &= \Bigl(\int_{-\infty}^\infty e^{-x^2}\, dx\Bigr)\Bigl(\int_{-\infty}^\infty e^{-y^2}\, dy\Bigr) = \int_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dx\, dy \\ &= \int_0^{2\pi}\int_0^\infty e^{-r^2} r\, dr\, d\theta = 2\pi \int_0^\infty e^{-r^2} r\, dr \\ &= \pi \int_0^\infty e^{-s}\, ds = \pi. \end{aligned} \tag{B.12}$$

The following identity will be useful in §7. Namely, the beta function, defined for $x, y > 0$ by

$$B(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1}\, ds = \int_0^\infty (1+u)^{-x-y} u^{x-1}\, du \tag{B.13}$$

(with $u = s/(1-s)$), satisfies

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{B.14}$$

To prove this, note that since

$$\Gamma(z) p^{-z} = \int_0^\infty e^{-pt} t^{z-1}\, dt, \tag{B.15}$$

we have

$$(1+u)^{-x-y} = \frac{1}{\Gamma(x+y)} \int_0^\infty e^{-(1+u)t} t^{x+y-1}\, dt, \tag{B.16}$$

so

$$\begin{aligned} B(x,y) &= \frac{1}{\Gamma(x+y)} \int_0^\infty e^{-t} t^{x+y-1} \int_0^\infty e^{-ut} u^{x-1}\, du\, dt \\ &= \frac{\Gamma(x)}{\Gamma(x+y)} \int_0^\infty e^{-t} t^{y-1}\, dt \\ &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \end{aligned} \tag{B.17}$$

as asserted. For closer contact with (7.3), note that setting $s = t^2$ in (B.13) gives

$$B(x,y) = 2 \int_0^1 t^{2x-1} (1-t^2)^{y-1}\, dt, \tag{B.18}$$

so, if $k \in \mathbb{Z}^+$ and $\nu > -1/2$,

$$B\Big(k + \frac{1}{2}, \nu + \frac{1}{2}\Big) = \int_{-1}^1 t^{2k} (1-t^2)^{\nu - 1/2}\, dt. \tag{B.19}$$

C. Derivative of $1/\Gamma(z)$ at integer points

The formula for $Y_n(t)$, given by (4.16), (4.20), and (4.22), involves $\alpha_k'(n)$ and $\alpha_k'(-n)$, where

$$\alpha_k(\nu) = \frac{(-1)^k}{k!\Gamma(k+\nu+1)}. \tag{C.1}$$

Thus

$$\alpha_k'(\nu) = \frac{(-1)^k}{k!}\beta'(k+\nu+1), \tag{C.2}$$

where, as in (B.5),

$$\beta(z) = \frac{1}{\Gamma(z)}. \tag{C.3}$$

Thus we want to evaluate $\beta'(k \pm n+1)$, or equivalently, evaluate $\beta'(\ell)$ for all $\ell \in \mathbb{Z}$.

To start, we have from (B.1) that, for $z > 1$,

$$\Gamma'(z) = \int_0^\infty e^{-t} t^{z-1} (\log t)\, dt. \tag{C.4}$$

In particular,

$$\Gamma'(1) = \int_0^\infty e^{-t} \log t\, dt = -\gamma, \tag{C.5}$$

where γ is Euler's constant,

$$\gamma \approx 0.5772156649\cdots. \tag{C.6}$$

See §18 and Appendix J of [T2] for more on this constant. We deduce that

$$\beta'(1) = \gamma. \tag{C.7}$$

To evaluate $\beta'(\ell)$ for other $\ell \in \mathbb{Z}$, we can use

$$\beta(z) = z\beta(z+1), \tag{C.8}$$

which implies

$$\beta'(z) = \beta(z+1) + z\beta'(z+1). \tag{C.9}$$

Thus

$$\beta'(0) = \beta(1) = 1. \tag{C.10}$$

For $\ell = -m$, a negative integer, we have $\beta(-m+1) = 0$, hence

$$\beta'(-m) = -m\beta'(-m+1), \tag{C.11}$$

hence $\beta'(-1) = -\beta'(0) = -1,\ \beta'(-2) = -2\beta'(-1) = 2$, and, inductively,

$$\beta'(-m) = (-1)^m m!, \tag{C.12}$$

when m is a positive integer. To evaluate $\beta(\ell)$ for an integer $\ell \geq 2$, we can turn (C.9) around:

$$\beta'(z+1) = \frac{\beta'(z) - \beta(z+1)}{z}. \tag{C.13}$$

Hence

$$\beta'(2) = \beta'(1) - \beta(2) = \gamma - 1, \tag{C.14}$$

and generally

$$\beta'(\ell+1) = \frac{\beta'(\ell)}{\ell} - \frac{1}{\ell \cdot \ell!}. \tag{C.15}$$

References

[L] N. Lebedev, Special Functions and Their Applications, Dover, New York, 1972.

[O] F. Olver, Asymptotics and Special Functions, Academic Press, New York, 1974.

[T1] M. Taylor, Introduction to Differential Equations, American Mathematical Society, Providence, RI, 2011.

[T2] M. Taylor, Introduction to Complex Analysis, Lecture Notes, available at http://www.unc.edu/math/Faculty/met/complex.html

[T3] M. Taylor, Partial Differential Equations, Vols. 1–3, Springer-Verlag, New York, 1996 (2nd ed. 2011).

[W] G. Watson, A Treatise on the Theory of Bessel Functions, Cambridge Univ. Press, Cambridge UK, 1944 (Library Ed. 1996).