Hardy Spaces and bmo on Manifolds with Bounded Geometry

Michael Taylor[1]

Abstract. We develop the theory of the "local" Hardy space $\mathfrak{h}^1(M)$ and John-Nirenberg space $\text{bmo}(M)$ when M is a Riemannian manifold with bounded geometry, building on the classic work of Fefferman-Stein and subsequent material, particularly of Goldberg and Ionescu. Results include $\mathfrak{h}^1 - \text{bmo}$ duality, L^p estimates on an appropriate variant of the sharp maximal function, $\mathfrak{h}^1$ and bmo-Sobolev spaces, and action of a natural class of pseudodifferential operators, including a natural class of functions of the Laplace operator, in a setting that unifies these results with results on L^p-Sobolev spaces. We apply results on these topics to some interpolation theorems, motivated in part by the search for dispersive estimates for wave equations.

1. Introduction

The theory of the Hardy space $H^1(\mathbb{R}^n)$ of functions on Euclidean space $\mathbb{R}^n$ and its connection to the John-Nirenberg space $\text{BMO}(\mathbb{R}^n)$ were highly developed in the classic paper [FS]. One characterization of $H^1(\mathbb{R}^n)$ given there goes as follows.

$$H^1(\mathbb{R}^n) = \{f \in L^1_{\text{loc}}(\mathbb{R}^n) : \mathcal{G}f \in L^1(\mathbb{R}^n)\}, \tag{1.1}$$

where

$$\mathcal{G}f(x) = \sup_{0<r<\infty} \sup_{\varphi\in\mathcal{F}} \Bigl| \int \varphi_r(x-y) f(y)\, dy \Bigr|, \tag{1.2}$$

with

$$\varphi_r(x) = r^{-n}\varphi(r^{-1}x), \tag{1.3}$$

and $\mathcal{F}$ a collection of functions that can be rather flexible. For example, one could take

$$\mathcal{F} = \{\varphi \in C^1_0(B_1) : \|\nabla\varphi\|_{L^\infty} \le 1\}, \tag{1.4}$$

or one could take

$$\mathcal{F} = \{\varphi\},$$

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consisting of a *single* function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int \varphi(x)\,dx = 1$. (Cf. Theorem 11 of [FS], and also [Sem].) Such flexibility in specifying $\mathcal{F}$ is itself a useful tool in the study of $H^1(\mathbb{R}^n)$.

One of the major results of [FS] was the proof of the duality

$$H^1(\mathbb{R}^n)' = \mathrm{BMO}(\mathbb{R}^n), \tag{1.5}$$

where the right side is the John-Nirenberg space, defined by

$$\mathrm{BMO}(\mathbb{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : f^{\#} \in L^\infty(\mathbb{R}^n)\} \tag{1.6}$$

(modulo constants), where

$$f^{\#}(x) = \sup_{B \in \mathcal{B}(x)} \frac{1}{V(B)} \int_B |f(y) - f_B|\,dy, \tag{1.7}$$

with

$$\mathcal{B}(x) = \{B_r(x) : 0 < r < \infty\}, \tag{1.8}$$

$B_r(x)$ being the ball centered at x of radius r, and

$$f_B = \frac{1}{V(B)} \int_B f(y)\,dy. \tag{1.9}$$

There are variants giving the same space. For example, one could use cubes containing x instead of balls centered at x (as did [JN] and [FS] in their original works), and one could replace f_B in (1.7) by c_B, chosen to minimize the integral. The flexibility afforded by the equivalence of these different characterizations is again useful (as we will see, in a related context, in §3).

A number of variants of these spaces have been studied. In [G] "local" spaces $\mathfrak{h}^1(\mathbb{R}^n)$ and $\mathrm{bmo}(\mathbb{R}^n)$ were defined, as follows.

$$\mathfrak{h}^1(\mathbb{R}^n) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \mathcal{G}^b f \in L^1(\mathbb{R}^n)\}, \tag{1.10}$$

where

$$\mathcal{G}^b f(x) = \sup_{0<r\le 1} \sup_{\varphi \in \mathcal{F}} \Bigl| \int \varphi_r(x-y) f(y)\,dy \Bigr|, \tag{1.11}$$

with φ_r and $\mathcal{F}$ as described above. It was shown in [G] that (1.5) implies

$$\mathfrak{h}^1(\mathbb{R}^n)' = \mathrm{bmo}(\mathbb{R}^n), \tag{1.12}$$

where $\text{bmo}(\mathbb{R}^n)$ is defined as

$$\text{bmo}(\mathbb{R}^n) = \{f \in L^1_{\text{loc}}(\mathbb{R}^n) : \mathcal{N}f \in L^\infty(\mathbb{R}^n)\}, \tag{1.13}$$

where

$$\mathcal{N}f(x) = \sup_{B\in\mathcal{B}_1(x)} \frac{1}{V(B)} \int\limits_B |f(y) - f_B|\, dy + \frac{1}{V(B_1(x))} \int\limits_{B_1(x)} |f(y)|\, dy, \tag{1.14}$$

with (in place of (1.8))

$$\mathcal{B}_1(x) = \{B_r(x) : 0 < r \le 1\}. \tag{1.15}$$

As shown in [G], the spaces $\mathfrak{h}^1(\mathbb{R}^n)$ and $\text{bmo}(\mathbb{R}^n)$ are invariant under the action of multiplication ($f \mapsto af$) by nice functions, and more generally invariant under the action of pseudodifferential operators $p(x, D) \in OPS^0_{1,0}(\mathbb{R}^n)$, where we recall

$$\begin{gathered} p(x, D)f(x) = (2\pi)^{-n} \iint p(x, \xi) f(y) e^{i(x-y)\cdot\xi}\, dy\, d\xi \\ p(x, \xi) \in S^m_{1,0}(\mathbb{R}^n) \Leftrightarrow |D^\beta_x D^\alpha_\xi p(x, \xi)| \le C_{\alpha\beta}(1 + |\xi|)^{m-|\alpha|}, \end{gathered} \tag{1.16}$$

and then say $p(x, D) \in OPS^m_{1,0}(\mathbb{R}^n)$. Invariance under a class of diffeomorphisms on $\mathbb{R}^n$ is also established, allowing one to define $\mathfrak{h}^1(M)$ and $\text{bmo}(M)$ whenever M is a smooth, compact manifold. An alternative approach to $\mathfrak{h}^1(M)$ for such M had been given in [Str1].

In [CKS], a theory of local Hardy spaces was developed on smoothly bounded domains, and applied to some elliptic boundary problems.

In another direction, [CW1]–[CW2] have studied $H^1(X)$ and $\text{BMO}(X)$ when X is a space of "homogeneous type," a metric space (or more generally a quasi-metric space) with a measure satisfying a doubling condition.

Another class of spaces on which many people do analysis is the class of symmetric spaces of noncompact type, such as n-dimensional hyperbolic space $\mathcal{H}^n$. Despite the fact that these are homogeneous spaces, they are not spaces of "homogeneous type," since balls of large radius grow too rapidly in volume, a fact that influences analysis on these spaces in many ways. In the course of studying some Fourier integral operators on $\mathcal{H}^n$ (and other noncompact symmetric spaces of real rank 1) [I] defines BMO and develops basic properties. Somewhat parallel to (1.6)–(1.8), [I] takes

$$\text{BMO}(\mathcal{H}^n) = \{f \in L^1_{\text{loc}}(\mathcal{H}^n) : f^\# \in L^\infty(\mathcal{H}^n)\}, \tag{1.17}$$

where

$$f^\#(x) = \sup_{B\in\mathcal{B}_1(x)} \frac{1}{V(B)} \int\limits_B |f(y) - f_B|\, dV(y), \tag{1.18}$$

but, in contrast to (1.8), $\mathcal{B}_1(x)$ is as in (1.15). This contrast makes it problematic to produce a unified theory of the spaces $H^1(M)$ and $\mathrm{BMO}(M)$ for a class of manifolds including both $M = \mathbb{R}^n$ and $M = \mathcal{H}^n$.

Our goal in this paper is to produce a unified theory of the "local" spaces $\mathfrak{h}^1(M)$ and $\mathrm{bmo}(M)$, whenever M is a complete Riemannian manifold with bounded geometry. We define "bounded geometry" as follows. First we assume there exists $R_0 \in (0, \infty)$ such that for each $p \in M$, the exponential map

$$\mathrm{Exp}_p : T_pM \longrightarrow M \tag{1.19}$$

has the property

$$\mathrm{Exp}_p : B_{R_0}(0) \longrightarrow B_{R_0}(p) \ \text{ diffeomorphically,} \tag{1.20}$$

where $B_r(p) = \{x \in M : d(x,p) < r\}$, $d(x,p)$ denoting the distance from x to p. Furthermore, the pull-back of the metric tensor from $B_{R_0}(p) \subset M$ to $B_{R_0}(0) \subset T_pM$, identified with $B_{R_0}(0) \subset \mathbb{R}^n$ ($n = \dim M$), uniquely up to an element of $O(n)$, furnishes a collection of $n \times n$ matrices $G_p(x) = (g^p_{jk}(x))$ satisfying

$$\{G_p : p \in M\} \ \text{ is bounded in } \ C^\infty(B_{R_0}(0), \mathrm{End}(\mathbb{R}^n)). \tag{1.21}$$

We also require that

$$\xi \cdot G_p(x)\xi \geq \frac{1}{2}|\xi|^2, \quad \forall\, p \in M, x \in B_{R_0}(0), \xi \in \mathbb{R}^n, \tag{1.22}$$

and that

$$B_{R_0}(p) \ \text{ is geodesically convex, } \forall\, p \in M. \tag{1.23}$$

Such is a complete Riemannian manifold with bounded geometry. Given this, we find it convenient to multiply the metric tensor of M by a constant, if necessary, so we can say the properties above hold with

$$R_0 = 4. \tag{1.24}$$

Having (1.20)–(1.24), we can pick $p_k \in M$, $k \in \mathbb{Z}^+$, such that

$$\{B_{1/2}(p_k) : k \in \mathbb{Z}^+\} \ \text{ covers } \ M, \tag{1.25}$$

while, for some $K = K(M) < \infty$,

$$\forall\, p \in M, \ \text{at most } K \text{ balls } B_2(p_k) \text{ contain } p. \tag{1.26}$$

We can then form a partition of unity $\sum_k \varphi_k = 1$ such that

$$\mathrm{supp}\ \varphi_k \subset B_1(p_k), \quad \varphi_k \circ \mathrm{Exp}_{p_k} \text{ is bounded in } C_0^\infty(B_1(0)). \tag{1.27}$$

We call such $\{\varphi_k : k \in \mathbb{Z}^+\}$ a *tame* partition of unity, and the collection $\{B_1(p_k) : k \in \mathbb{Z}^+\}$ a *tame cover* of M.

The structure of the rest of this paper is as follows. In Sections 2 and 3, respectively, we define $\mathfrak{h}^1(M)$ and $\mathrm{bmo}(M)$, when M has bounded geometry, and establish some basic properties, starting with showing that

$$f \in \mathfrak{h}^1(M),\ g \in \mathrm{bmo}(M) \Longrightarrow af \in \mathfrak{h}^1(M),\ ag \in \mathrm{bmo}(M), \tag{1.28}$$

for a class of functions a containing $L^\infty(M) \cap \mathrm{Lip}(M)$. Having these results, we then show that

$$\begin{aligned} \|f\|_{\mathfrak{h}^1(M)} &\approx \sum_k \|(\varphi_k f) \circ \mathrm{Exp}_{p_k}\|_{\mathfrak{h}^1(\mathbb{R}^n)}, \\ \|g\|_{\mathrm{bmo}(M)} &\approx \sup_k \|(\varphi_k g) \circ \mathrm{Exp}_{p_k}\|_{\mathrm{bmo}(\mathbb{R}^n)}, \end{aligned} \tag{1.29}$$

where $\{\varphi_k : k \in \mathbb{Z}^+\}$ is a tame partition of unity, as in (1.27). The spaces $\mathfrak{h}^1(\mathbb{R}^n)$ and $\mathrm{bmo}(\mathbb{R}^n)$ are as in (1.10)–(1.15). This enables us to make use of the results of [G] (and, by extension, those of [FS]). Doing this, we show in §4 that

$$\mathfrak{h}^1(M)' = \mathrm{bmo}(M) \tag{1.30}$$

whenever M has bounded geometry, extending (1.12).

In §5 we discuss the atomic theory of $\mathfrak{h}^1(M)$. We relate this to the "ionic theory," developed in the context of Lipschitz surfaces in [MT]

In §6 we show that when M has bounded geometry,

$$(\lambda I - \Delta)^{it} : \mathrm{bmo}(M) \longrightarrow \mathrm{bmo}(M), \tag{1.31}$$

with an exponential bound, provided $\lambda \geq \lambda_0(M)$ is sufficiently large. We also get such bounds on $\mathfrak{h}^1(M)$ and on $L^p(M)$, for $p \in (1, \infty)$. We show that the operators in (1.31) belong to a class of pseudodifferential operators on M denoted $\Psi^0_W(M)$, given $W < \sqrt{\lambda}$, where $\Psi^m_W(M)$ consists of operators of the form

$$P = P^\# + P^b, \tag{1.32}$$

of the following nature. The Schwartz kernel of $P^\#$ is supported near the diagonal in $M \times M$ and in local exponential coordinates centered at $p \in M$, $P^\#$, acting on functions supported in $B_4(p)$ (identified with a ball in $\mathbb{R}^n$) belongs to the class $OPS^m_{1,0}(\mathbb{R}^n)$ of classical pseudodifferential operators on $\mathbb{R}^n$, with bounds independent of p. Meanwhile P^b has integral kernel $K^b(x, y)$, satisfying estimates

$$|K^b(x, y)| \leq C_k(1 + d(x, y))^{-k} e^{-W d(x,y)}, \quad \forall\, k \in \mathbb{Z}^+, \tag{1.33}$$

plus such estimates on all its derivatives. Generalizing (1.31), we get

$$P \in \Psi^0_W(M),\ W \geq K_0 \Longrightarrow P : \mathrm{bmo}(M) \to \mathrm{bmo}(M), \tag{1.34}$$

and also bounds on $\mathfrak{h}^1(M)$ and on $L^p(M)$, $p \in (1,\infty)$. Here K_0 is a geometrical constant; cf. (6.16). We also show that

(1.35)
$$P_1 \in \Psi^{m_1}_W(M), P_2 \in \Psi^{m_2}_{W+K_1}(M),\ K_1 > K_0 \Longrightarrow P_1P_2, P_2P_1 \in \Psi^{m_1+m_2}_W(M).$$

These results will be of further use later on.

Another fundamental circle of results in [FS] involved interpolation, showing for example that if $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and also $T : L^\infty(\mathbb{R}^n) \to \mathrm{BMO}(\mathbb{R}^n)$, then $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for each $p \in (2,\infty)$. A key ingredient was an L^p estimate on f in terms of $f^\#$, defined by (1.7). In §§7–9 of this paper we establish analogous results for manifolds with bounded geometry, involving $\mathrm{bmo}(M)$. In §7 we obtain local estimates for $\|f\|_{L^p}$ on a cube Q, in terms of $f^\#|_Q$ and $\|f\|_{L^1(Q)}$. We apply this in §8 to obtain estimates of the form

$$\|f\|_{L^p(M)} \le C_p \|\mathcal{N}f\|_{L^p(M)}, \quad 1 < p < \infty, \tag{1.36}$$

when M is a manifold with bounded geometry and $\mathcal{N}f$ is defined as in (3.4). The interpolation result is then established in §9.

Section 10 introduces L^p, Hardy, and bmo-Sobolev spaces on M and discusses some basic properties. We begin with definitions of

$$H^{k,p}(M), \quad \mathfrak{h}^{k,1}(M), \quad \mathfrak{h}^{k,\infty}(M), \tag{1.37}$$

when $k \in \mathbb{N}$, and then define

$$H^{s,p}(M), \quad \mathfrak{h}^{s,1}(M), \quad \mathfrak{h}^{s,\infty}(M), \tag{1.38}$$

for $s \in \mathbb{R}$. Our first order of business is to show the definitions of (1.38) are equivalent to those of (1.37) when $s = k \in \mathbb{N}$. Results on $\Psi^m_W(M)$ from §6 are useful here, together with the fact that

$$(\lambda I - \Delta)^{-m/2} \in \Psi^{-m}_W(M) \tag{1.39}$$

when $m \in \mathbb{R}$ and $\lambda > \sqrt{W}$. One result of §10 is that

$$P \in \Psi^m_W(M),\ W \ge K_0 \Longrightarrow P : \mathfrak{h}^{s,\infty}(M) \to \mathfrak{h}^{s-m,\infty}(M), \tag{1.40}$$

and corresponding results for the action of such P on $\mathfrak{h}^{s,1}(M)$ and on $H^{s,p}(M)$ for $p \in (1,\infty)$.

In §11 we establish further interpolation results, of the following nature. Given

$$R : L^2(M) \to L^2(M), \quad R : L^1(M) \to \mathfrak{h}^{s,\infty}(M), \tag{1.41}$$

we have, for $\theta \in (0,1)$,

$$R : L^p(M) \longrightarrow H^{(1-\theta)s,p'}(M), \quad p = \frac{2}{2-\theta},\ p' = \frac{2}{\theta}, \tag{1.42}$$

together with associated operator norm bounds. Such results are applicable to estimates on

$$S(t) = \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}, \tag{1.43}$$

for which one has bounds

$$\|S(t)f\|_{H^{1,2}} \le A_1(t)\|f\|_{L^2}, \quad \|S(t)f\|_{\mathfrak{h}^{-(n-1)/2,\infty}} \le A_0(t)\|f\|_{L^1}, \tag{1.44}$$

at least under certain additional geometrical hypotheses on M. We can apply (1.41)–(1.42) to $R(t) = (\lambda I - \Delta)^{1/2}S(t)$, with $\lambda > K_0^2$. The resulting operator estimates on $S(t)$ (known as dispersive estimates) have potential application to nonlinear wave equations on various classes of Riemannian manifolds, such as hyperbolic space. These matters will be taken up elsewhere.

In Appendix A we study the space $\mathrm{vmo}(M)$, which is the closure in $\mathrm{bmo}(M)$ of $C_0^\infty(M)$, in analogy with $\mathrm{VMO}(\mathbb{R}^n)$, introduced in [Sar] as the closure in $\mathrm{BMO}(\mathbb{R}^n)$ of $C_0^\infty(\mathbb{R}^n)$. As shown in [Sar],

$$\mathrm{VMO}(\mathbb{R}^n)' = H^1(\mathbb{R}^n). \tag{1.45}$$

We show that, when M has bounded geometry,

$$\mathrm{vmo}(M)' = \mathfrak{h}^1(M). \tag{1.46}$$

We also show that, under the hypotheses of (1.34),

$$P : \mathrm{vmo}(M) \longrightarrow \mathrm{vmo}(M). \tag{1.47}$$

In Appendix B, we introduce another class of pseudodifferential operators:

$$\widetilde{\Psi}^m_W(M) \subset \Psi^m_W(M). \tag{1.48}$$

It is of some interest that functions of the Laplace operator, such as discussed in §6 and §10, actually belong to this smaller space, particularly in view of the fact that elements of $\widetilde{\Psi}^0_W(M)$ have sharper L^p-operator estimates. These results tie in closely with results of [T3].

In Appendix C, we point out some special geometrical and analytic properties of symmetric spaces of noncompact type. We extend L^p-Sobolev space bounds on operators of the form (1.39) to the result that

$$(-\Delta)^{m/2} : H^{s,p}(M) \longrightarrow H^{s-m,p}(M), \tag{1.49}$$

for $m, s \in \mathbb{R}$, $p \in (1,\infty)$, in the special case that M is a symmetric space of noncompact type. We indicate how this leads to an extension of the dispersive estimates on operators of the form (1.43) described in §11 to dispersive estimates on $e^{it\sqrt{-\Delta}}$, in this setting.

Remark. We call attention to the recent work [CMM], developing a theory of H^1 and BMO for certain nondoubling measured metric measure spaces. This paper takes the approach to H^1 and BMO of [I] and extends its scope considerably. Our paper, emphasizing the "local" spaces $\mathfrak{h}^1$ and bmo, is to some degree complementary to [CMM], though these two papers deal with a number of common themes.

2. The space $\mathfrak{h}^1(M)$

We take M to be a complete Riemannian manifold, of dimension n, with bounded geometry, as defined in the Introduction. As described there, we scale the metric tensor (if necessary) to arrange that the properties (1.20)–(1.24) hold. Given $f \in L^1_{\mathrm{loc}}(M)$, we define the following maximal function:

$$\mathcal{G}^b f(x) = \sup_{0<r\le 1} \mathcal{G}_r f(x), \tag{2.1}$$

where

$$\mathcal{G}_r f(x) = \sup\Big\{\Big|\int \varphi(y) f(y)\,dV(y)\Big| : \varphi \in \mathcal{F}(B_r(x))\Big\}, \tag{2.2}$$

with

$$\mathcal{F}(B_r(x)) = \Big\{\varphi \in C^1_0(B_r(x)) : \|\varphi\|_{\mathrm{Lip}} \le \frac{1}{r^{n+1}}\Big\}. \tag{2.3}$$

We then set

$$\mathfrak{h}^1(M) = \{f \in L^1_{\mathrm{loc}}(M) : \mathcal{G}^b f \in L^1(M)\}, \tag{2.4}$$

with norm

$$\|f\|_{\mathfrak{h}^1} = \|\mathcal{G}^b f\|_{L^1}. \tag{2.5}$$

Remark. One could replace $C^1_0(B_r(x))$ by $\{\varphi \in \mathrm{Lip}(M) : \operatorname{supp}\varphi \subset B_r(x)\}$ and get the same result.

A comparison with (1.10)–(1.11) shows that when $M = \mathbb{R}^n$, the space $\mathfrak{h}^1(M)$ defined above coincides with the space $\mathfrak{h}^1(\mathbb{R}^n)$ defined in the Introduction.

It is convenient to know that $\mathfrak{h}^1(M)$ is a module over $\mathrm{Lip}(M) \cap L^\infty(M)$. In fact, a more precise result holds. Let σ be a modulus of continuity, and say

$$a \in C^\sigma(M) \Longleftrightarrow |a(x) - a(y)| \le L\sigma(d(x,y)), \quad \text{for } \ d(x,y) \le 1, \tag{2.6}$$

for some $L \in [0,\infty)$. Define $\|a\|_{C^\sigma}$ to be the smallest L for which (2.6) holds (this is a seminorm). We then have the following result.

Proposition 2.1. *Let σ be a modulus of continuity satisfying the Dini condition*

$$D(\sigma) = \int_0^1 \frac{\sigma(r)}{r}\, dr < \infty. \tag{2.7}$$

We also assume $\sigma(r)/r$ is monotonically decreasing on $(0,1]$ (or constant). Then

$$a \in L^\infty(M) \cap C^\sigma(M),\ \ f \in \mathfrak{h}^1(M) \Longrightarrow af \in \mathfrak{h}^1(M). \tag{2.8}$$

The proof is quite similar to that for the Euclidean case, but for the sake of completeness we give the details.

Take a and f as in (2.8). To estimate $\mathcal{G}^b(af)(x)$, we compare $\mathcal{G}_r(af)(x)$ with $a(x)\mathcal{G}_r f(x)$. Take $\varphi \in \mathcal{F}(B_r(x))$, and note that

$$\varphi \in \mathcal{F}(B_r(x)) \Longrightarrow \|\varphi\|_{L^\infty} \le \frac{1}{r^n}. \tag{2.9}$$

Hence

$$\begin{aligned} &\Bigl| \int_{B_r(x)} \varphi(y) \bigl[a(y)f(y) - a(x)f(y)\bigr]\, dV(y) \Bigr| \\ &\qquad \le \sup_{d(x,y)\le r} |a(x) - a(y)| \int |\varphi(y)| \cdot |f(y)|\, dV(y) \\ &\qquad \le \|a\|_{C^\sigma} \frac{\sigma(r)}{r^n} \int_{B_r(x)} |f(y)|\, dV(y). \end{aligned} \tag{2.10}$$

That is to say,

$$\bigl|\mathcal{G}_r(af)(x) - a(x)\mathcal{G}_r f(x)\bigr| \le \|a\|_{C^\sigma} \frac{\sigma(r)}{r^n} \int_{B_r(x)} |f|\, dV. \tag{2.11}$$

Hence, by (2.1),

$$\mathcal{G}^b(af)(x) \le |a(x)| \mathcal{G}^b f(x) + \|a\|_{C^\sigma} \mathcal{R}^\sigma f(x), \tag{2.12}$$

where

$$\begin{aligned} \mathcal{R}^\sigma f(x) &= \sup_{0<r\le 1} \frac{\sigma(r)}{r^n} \int_{B_r(x)} |f|\, dV \\ &\le \int_{B_1(x)} \frac{\sigma(d(x,y))}{d(x,y)^n} |f(y)|\, dV(y) \\ &= \int_M K^\sigma(x,y) |f(y)|\, dV(y), \end{aligned} \tag{2.13}$$

where the last identity defines $K^\sigma(x,y)$. Note that $K^\sigma(x,y) \geq 0$ and

$$\int_M K^\sigma(x,y)\,dV(x) = \int_{B_1(y)} \frac{\sigma(d(x,y))}{d(x,y)^n}\,dV(x) \leq C\int_0^1 \frac{\sigma(r)}{r}\,dr, \tag{2.14}$$

with $C < \infty$ depending on the geometrical bounds for M. Consequently

$$\|\mathcal{R}^\sigma f\|_{L^1} \leq CD(\sigma)\|f\|_{L^1}, \tag{2.15}$$

and we have the desired estimate for (2.12), yielding

$$\|\mathcal{G}^b(af)\|_{L^1} \leq \|a\|_{L^\infty}\|\mathcal{G}^b f\|_{L^1} + CD(\sigma)\|a\|_{C^\sigma}\|f\|_{L^1}. \tag{2.16}$$

This proves Proposition 2.1.

Using Proposition 2.1, we can establish the following.

Proposition 2.2. *Let $\{\varphi_k : k \in \mathbb{Z}^+\}$ be a tame partition of unity. Given $f \in L^1_{\text{loc}}(M)$, we have*

$$f \in \mathfrak{h}^1(M) \Longleftrightarrow \sum_k \|\varphi_k f\|_{\mathfrak{h}^1} < \infty, \tag{2.17}$$

and

$$\|f\|_{\mathfrak{h}^1} \approx \sum_k \|\varphi_k f\|_{\mathfrak{h}^1}. \tag{2.18}$$

Proof. The inequality $\|f\|_{\mathfrak{h}^1} \leq \sum_k \|\varphi_k f\|_{\mathfrak{h}^1}$ is elementary. For the converse estimate in (2.18), we argue as follows. We can partition the set $\{p_k : k \in \mathbb{Z}^+\}$ mentioned in (1.25)–(1.27) into $K_1 = K_1(M)$ subsets $\mathcal{S}_1, \dots \mathcal{S}_{K_1}$ such that

$$p_j, p_k \in \mathcal{S}_\nu,\ j \neq k \Longrightarrow d(p_j, p_k) \geq 20. \tag{2.19}$$

For convenience, if $p_k \in \mathcal{S}_\nu$, we also write $k \in \mathcal{S}_\nu$. For each $\nu \in \{1, \dots, K_1\}$, set

$$T_\nu f = \sum_{k \in \mathcal{S}_\nu} \varphi_k f. \tag{2.20}$$

We apply Proposition 2.1 to $a = a_\nu = \sum_{k \in \mathcal{S}_\nu} \varphi_k$ and deduce that

$$T_\nu : \mathfrak{h}^1(M) \to \mathfrak{h}^1(M), \quad \|T_\nu f\|_{\mathfrak{h}^1} \leq C\|f\|_{\mathfrak{h}^1}, \quad 1 \leq \nu \leq K_1. \tag{2.21}$$

On the other hand, by the degree of disjointness of the supports of $\{\varphi_k : k \in \mathcal{S}_\nu\}$, a direct check of the definitions gives

$$\mathcal{G}^b\Big(\sum_{k\in\mathcal{S}_\nu} \varphi_k f\Big) = \sum_{k\in\mathcal{S}_\nu} \mathcal{G}^b(\varphi_k f), \tag{2.22}$$

for each ν, and (again by disjointness)

$$\sum_{k\in\mathcal{S}_\nu} \|\mathcal{G}^b(\varphi_k f)\|_{L^1} = \|\mathcal{G}^b T_\nu f\|_{L^1}, \tag{2.23}$$

i.e.,

$$\sum_{k\in\mathcal{S}_\nu} \|\varphi_k f\|_{\mathfrak{h}^1} = \|T_\nu f\|_{\mathfrak{h}^1}. \tag{2.24}$$

Summing over $\nu \in \{1, \dots, K_1\}$ then gives

$$\sum_k \|\varphi_k f\|_{\mathfrak{h}^1} \le C K_1 \|f\|_{\mathfrak{h}^1}, \tag{2.25}$$

proving (2.18).

Proposition 2.2 combines nicely with the following elementary result.

Proposition 2.3. *We have, uniformly in* $k \in \mathbb{Z}^+$,

$$\|\varphi_k f\|_{\mathfrak{h}^1(M)} \approx \|(\varphi_k f)\circ \operatorname{Exp}_{p_k}\|_{\mathfrak{h}^1(\mathbb{R}^n)}. \tag{2.26}$$

We recall that there is an isometric isomorphism of the n-dimensional inner product space T_pM with $\mathbb{R}^n$, determined uniquely up to the action of $O(n)$.

Corollary 2.4. *In the setting of Proposition 2.2,*

$$\|f\|_{\mathfrak{h}^1(M)} \approx \sum_k \|(\varphi_k f)\circ \operatorname{Exp}_{p_k}\|_{\mathfrak{h}^1(\mathbb{R}^n)}. \tag{2.27}$$

The following result is occasionally useful.

Proposition 2.5. *The space* $C_0^\infty(M)$ *is dense in* $\mathfrak{h}^1(M)$.

Proof. Take $f \in \mathfrak{h}^1(M)$. Via Proposition 2.2, it suffices to approximate each term $\varphi_k f$ in $\mathfrak{h}^1$-norm by an element of $C_0^\infty(B_2(p_k)) \approx C_0^\infty(B_2(0))$. This reduces matters to treating the case $M = \mathbb{R}^n$. In this case, the result is proven in [G], p. 35.

We record some other useful facts. First, the Lebesgue density theorem immediately gives

$$|f(x)| \le C\mathcal{G}^b f(x), \tag{2.28}$$

a.e. on M, for each $f \in L^1_{\text{loc}}(M)$, with $C = C(M) < \infty$. We also have the following:

Proposition 2.6. *Assume f is a locally finite Borel measure on M. Define $\mathcal{G}^b f(x)$ as the natural variant of (2.1)–(2.1). If $\mathcal{G}^b f \in L^1(M)$, then $f \in L^1_{\mathrm{loc}}(M)$, hence $f \in \mathfrak{h}^1(M)$.*

Proof. Let J_ε be a mollifier, with integral kernel supported in $\{(x,y) : d(x,y) \le \varepsilon\}$. Then $f_\varepsilon = J_\varepsilon f \in C^\infty(M)$ and $f_\varepsilon \to f$ weak* in $\mathcal{M}_{\mathrm{loc}}(M)$. Since $\langle \varphi, f_\varepsilon\rangle = \langle J_\varepsilon^* \varphi, f\rangle$, we have, for $\varepsilon \in (0,1]$,

$$\begin{aligned} \mathcal{G}^b f_\varepsilon(x) &\le C \sup_{0<r\le 1+\varepsilon} \mathcal{G}_r f(x) \\ &\le C\mathcal{G}^b f(x) + C_1(x), \end{aligned} \tag{2.29}$$

where $C_1(x)$ is the total variation of f on $B_2(x)$, a locally bounded function. We can apply (2.28) to f_ε and get

$$|f_\varepsilon(x)| \le C\mathcal{G}^b f(x) + C_1(x), \tag{2.30}$$

again with $C < \infty$, $C_1(x)$ locally bounded, independent of $\varepsilon \in (0,1]$. The uniform estimate (2.30) implies $f \in L^1_{\mathrm{loc}}(M)$, so we are in the setting of the definition of $\mathfrak{h}^1(M)$ given at the beginning of this section.

3. The space $\mathrm{bmo}(M)$

As usual, M is a complete Riemannian manifold, of dimension n, with bounded geometry, and with the properties (1.20)–(1.24). We set up the following maximal functions. Given $f \in L^1_{\mathrm{loc}}(M)$, let

$$f^\#(x) = \sup_{B\in\mathcal{B}(x)} \frac{1}{V(B)} \int_B |f - f_B|\, dV, \tag{3.1}$$

where

$$f_B = \frac{1}{V(B)} \int_B f\, dV, \tag{3.2}$$

and

$$\mathcal{B}(x) = \{B_r(x) : 0 < r \le 1\}. \tag{3.3}$$

Then define

$$\mathcal{N} f(x) = f^\#(x) + \mathcal{N}_0 f(x), \quad \mathcal{N}_0 f(x) = \frac{1}{V(B_1(x))} \int_{B_1(x)} |f|\, dV. \tag{3.4}$$

We set

$$\text{bmo}(M) = \{f \in L^1_{\text{loc}}(M) : \mathcal{N}f \in L^\infty(M)\}, \tag{3.5}$$

with norm

$$\|f\|_{\text{bmo}} = \|\mathcal{N}f\|_{L^\infty}. \tag{3.6}$$

In case $M = \mathbb{R}^n$, the definition of $\text{bmo}(M)$ given here is clearly equivalent to that of $\text{bmo}(\mathbb{R}^n)$ given in the Introduction.

It is useful to make note of some equivalent norms. For example, in place of $f^\#$, consider

$$f^s(x) = \sup_{B \in \mathcal{B}(x)} \inf_{c_B \in \mathbb{C}} \frac{1}{V(B)} \int\limits_B |f - c_B| \, dV. \tag{3.7}$$

Given $B \in \mathcal{B}(x)$ and taking c_B to realize this infimum, we have

$$\begin{aligned} |f_B - c_B| &= \Bigl| \frac{1}{V(B)} \int\limits_B (f - c_B) \, dV \Bigr| \\ &\le \frac{1}{V(B)} \int\limits_B |f - c_B| \, dV \\ &\le f^s(x), \end{aligned} \tag{3.8}$$

and hence

$$\frac{1}{V(B)} \int\limits_B |f - f_B| \, dV \le \frac{1}{V(B)} \int\limits_B \bigl(|f - c_B| + |f_B - c_B|\bigr) \, dV \le 2 f^s(x). \tag{3.9}$$

Consequently,

$$f^s(x) \le f^\#(x) \le 2 f^s(x). \tag{3.10}$$

It is also useful to note that one can fix $a, b, c \in (0, \infty)$, with $a < b$, and replace $\mathcal{B}(x)$ by

$$\widetilde{\mathcal{B}}(x) = \{Q^\alpha_r(x) : 0 < r \le 1, \ \alpha \in \mathcal{A}\}, \tag{3.11}$$

where $Q^\alpha_r(x)$ is a family of measurable sets with the property that for each $r \in (0, 1]$,

$$\begin{aligned} V(Q^\alpha_r(x)) \ge cV(B_r(x)), \quad Q^\alpha_r(x) &\subset B_{br}(x), \quad \text{for all } \alpha, \text{ and} \\ B_{ar}(x) &\subset Q^\alpha_r(x), \quad \text{for some } \alpha. \end{aligned} \tag{3.12}$$

One gets functions comparable in size in (3.7) and hence also in (3.1). In connection with this, we recall that the original treatments in [JN] and [FS] used cubes containing x in place of balls centered at x. One consequence of this observation is that the John-Nirenberg estimate, proven in [JN] for functions defined on a cube in $\mathbb{R}^n$, is applicable in our current situation. We have, for each ball $B \subset M$ of radius ≤ 1,

$$\frac{1}{V(B)} \int\limits_B e^{\alpha|f-f_B|}\, dV \le \gamma, \tag{3.13}$$

with

$$\alpha = \frac{\beta}{\|f\|_{\text{bmo}}}, \quad \beta, \gamma \text{ constants.} \tag{3.14}$$

Cf. (3′) of [JN].

We next aim to show that

$$a \in \text{Lip}(M) \cap L^\infty(M),\ f \in \text{bmo}(M) \Longrightarrow af \in \text{bmo}(M). \tag{3.15}$$

In fact, we will obtain a much more precise result, which can be compared with Proposition 2.1. To begin, note that

$$\mathcal{N}_0(af)(x) \le \|a\|_{L^\infty} \mathcal{N}_0(f)(x), \tag{3.16}$$

so it suffices to estimate $(af)^s(x)$. For $B \in \mathcal{B}(x)$, we take $c_B = a(x) f_B$. Then we have

$$\begin{aligned} &\frac{1}{V(B)} \int\limits_B |a(y)f(y) - a(x)f_B|\, dV(y) \\ &\le \frac{1}{V(B)} \int\limits_B |a| \cdot |f - f_B|\, dV + \frac{1}{V(B)} \int\limits_B |a(y) - a(x)| \cdot |f_B|\, dV(y) \\ &\le \|a\|_{L^\infty} f^{\#}(x) + \Bigl(\sup_{y\in B} |a(x) - a(y)|\Bigr) |f_B|. \end{aligned} \tag{3.17}$$

Let us assume that $a \in C^\sigma(M)$, defined in (2.6). The last term in (3.17) is bounded by

$$\|a\|_{C^\sigma} \sigma(r) |f_B|, \quad \text{if } \ B = B_r(x). \tag{3.18}$$

We use the John-Nirenberg estimate (3.13)–(3.14) to estimate $|f_B|$. With α as in (3.14), $B = B_r(x),\ 0 < r \le 1$,

$$\begin{aligned} e^{\alpha|f_B|/2} &= \frac{1}{V(B)} \int\limits_B e^{\alpha|f_B|/2}\, dV \\ &= \frac{1}{V(B)} \int\limits_B e^{\alpha|f-(f-f_B)|/2}\, dV \\ &\le \frac{1}{V(B)} \int\limits_{B_1} e^{\alpha|f|}\, dV + \frac{1}{V(B)} \int\limits_B e^{\alpha|f-f_B|}\, dV, \end{aligned} \tag{3.19}$$

where $B_1 = B_1(x)$. The estimate (3.13) applies to the second term on the last line of (3.19). To estimate the first term on that line, we have

$$\begin{aligned}\int\limits_{B_1} e^{\alpha|f|}\,dV &= \int\limits_{B_1} e^{\alpha|f-f_{B_1}+f_{B_1}|}\,dV \\ &\le e^{\alpha|f_{B_1}|}\int\limits_{B_1} e^{\alpha|f-f_{B_1}|}\,dV \\ &\le \gamma V(B_1)e^{\alpha|f_{B_1}|} \\ &\le \gamma V(B_1)e^{\beta},\end{aligned} \tag{3.20}$$

the penultimate inequality by (3.13) and the last inequality because

$$|f_{B_1}| \le V(B_1)^{-1}\int_{B_1} |f|\,dV \le \|f\|_{\mathrm{bmo}}.$$

Using this in (3.19) we get (for some constant γ_1)

$$e^{\alpha|f_B|/2} \le \frac{\gamma_1}{V(B)}, \quad \text{hence} \quad |f_B| \le \frac{2}{\alpha}\log\frac{\gamma_1}{V(B)}. \tag{3.21}$$

Recalling that our goal is to estimate (3.18), we have

$$\begin{aligned}\sigma(r)|f_B| &\le \frac{2}{\alpha}\sigma(r)\,\log\frac{\gamma_2}{r^n} \\ &= \frac{2n}{\beta}\,\|f\|_{\mathrm{bmo}}\,\sigma(r)\,\log\frac{\gamma_3}{r}.\end{aligned} \tag{3.22}$$

We have the following sharpening of (3.15).

Proposition 3.1. *If $a \in L^\infty(M)\cap C^\sigma(M)$ with*

$$\sigma(r) = \Bigl(\log\frac{1}{r}\Bigr)^{-1}, \quad 0 < r \le \frac{1}{2}, \tag{3.23}$$

then

$$f \in \mathrm{bmo}(M) \Longrightarrow af \in \mathrm{bmo}(M). \tag{3.24}$$

Remark. Note that the Dini condition (2.7) just barely fails for $\sigma(r)$ given by (3.23). We discuss this further after establishing $\mathfrak{h}^1$-bmo duality in §4.

We can use Proposition 3.1 to establish the following counterpart to Proposition 2.2.

Proposition 3.2. *Let $\{\varphi_k : k \in \mathbb{Z}^+\}$ be a tame partition of unity. Given $f \in L^1_{\mathrm{loc}}(M)$, we have*

$$f \in \mathrm{bmo}(M) \Longleftrightarrow \sup_k \|\varphi_k f\|_{\mathrm{bmo}} < \infty, \tag{3.25}$$

and

$$\|f\|_{\mathrm{bmo}} \approx \sup_k \|\varphi_k f\|_{\mathrm{bmo}}. \tag{3.26}$$

Proof. The Lipschitz bounds on φ_k yield

$$\|\varphi_k f\|_{\mathrm{bmo}} \le C\|f\|_{\mathrm{bmo}}, \tag{3.27}$$

with C independent of k, by Proposition 3.1. The converse inequality

$$\|f\|_{\mathrm{bmo}} \le C \sup_k \|\varphi_k f\|_{\mathrm{bmo}} \tag{3.28}$$

follows directly from the identity

$$f = \sum_k \varphi_k f = \sum_{\nu=1}^{K_1} \sum_{k\in\mathcal{S}_\nu} \varphi_k f, \tag{3.29}$$

with $\mathcal{S}_\nu$ as in (2.19), the support conditions on $\{\varphi_k f\}$, and the definition of the bmo-norm.

From here we easily have the following counterparts to Proposition 2.3 and Corollary 2.4.

Proposition 3.3. *We have, uniformly in $k \in \mathbb{Z}^+$,*

$$\|\varphi_k f\|_{\mathrm{bmo}(M)} \approx \|(\varphi_k f) \circ \mathrm{Exp}_{p_k}\|_{\mathrm{bmo}(\mathbb{R}^n)}. \tag{3.30}$$

Proof. This follows via the equivalence between the use of (3.1) and (3.7), and the equivalence between the use of (3.3) and (3.11)–(3.12).

Corollary 3.4. *In the setting of Proposition 3.2,*

$$\|f\|_{\mathrm{bmo}(M)} \approx \sup_k \|(\varphi_k f) \circ \mathrm{Exp}_{p_k}\|_{\mathrm{bmo}(\mathbb{R}^n)}. \tag{3.31}$$

4. $\mathfrak{h}^1$ – bmo **duality**

As before, we take M to be a complete Riemannian manifold with bounded geometry, of dimension n. Here our aim is to prove the following extension of (1.12).

Proposition 4.1. *We have*

$$\mathfrak{h}^1(M)' = \mathrm{bmo}(M). \tag{4.1}$$

Proof. To begin, we take $f \in \mathfrak{h}^1(M)$ and $g \in \mathrm{bmo}(M)$ and show the pairing $\langle f, g\rangle$ is well defined. Let $\{\varphi_k : k \in \mathbb{Z}^+\}$ be a tame partition of unity, as in (1.25)–(1.27). Also take a bounded family $\psi_k \in C_0^\infty(B_2(p_k)) \approx C_0^\infty(B_2(0))$ such that $\psi_k \equiv 1$ on $\operatorname{supp}\varphi_k$. We attempt to define $\langle f, g\rangle$ as

$$\langle f, g\rangle = \sum_k \langle f_k, \psi_k g\rangle, \quad f_k = \varphi_k f. \tag{4.2}$$

By Proposition 2.2 we have

$$\tilde{f}_k = f_k \circ \mathrm{Exp}_{p_k} \in \mathfrak{h}^1(\mathbb{R}^n), \quad \mathbb{R}^n \approx T_{p_k} M, \tag{4.3}$$

and as in Proposition 3.2 we have

$$\tilde{g}_k = (\psi_k g) \circ \mathrm{Exp}_{p_k} \in \mathrm{bmo}(\mathbb{R}^n). \tag{4.4}$$

The volume element on $B_4(p_k) \subset M$ pulls back to

$$A_k \in C^\infty(B_4(0)), \ \text{ bounded uniformly in } \ k. \tag{4.5}$$

Hence we can set

$$\langle f_k, \psi_k g\rangle = \langle A_k \tilde{f}_k, \tilde{g}_k\rangle, \tag{4.6}$$

the pairing on the right side defined by the duality $\mathfrak{h}^1(\mathbb{R}^n)' = \mathrm{bmo}(\mathbb{R}^n)$, proven in Corollary 1 of [G] (making essential use of the result $H^1(\mathbb{R}^n)' = \mathrm{BMO}(\mathbb{R}^n)$ from [FS]). We have

$$\begin{aligned} |\langle f_k, \psi_k g\rangle| &\le \|A_k \tilde{f}_k\|_{\mathfrak{h}^1(\mathbb{R}^n)} \|\tilde{g}_k\|_{\mathrm{bmo}(\mathbb{R}^n)} \\ &\le C \|f_k\|_{\mathfrak{h}^1(M)} \|g\|_{\mathrm{bmo}(M)}, \end{aligned} \tag{4.7}$$

the last inequality by Proposition 2.1 (with $M = \mathbb{R}^n$, applied to $\tilde{f}_k \mapsto A_k \tilde{f}_k$), together with Propositions 2.3, 3.1, and 3.3. Then we apply Proposition 2.2 to see that the series (4.2) converges and

$$|\langle f, g\rangle| \le C \|f\|_{\mathfrak{h}^1} \|g\|_{\mathrm{bmo}}. \tag{4.8}$$

This shows that $\mathrm{bmo}(M) \subset \mathfrak{h}^1(M)'$.

For the converse, we let ω be a continuous linear functional on $\mathfrak{h}^1(M)$, and take up the task of associating an element $g_\omega \in \text{bmo}(M)$. To start, we apply Proposition 2.2 and for $f \in \mathfrak{h}^1(M)$ write

$$\omega(f) = \sum_k \omega(\varphi_k f) = \sum_k \omega_k(f), \tag{4.9}$$

where $\omega_k(f) = \omega(\varphi_k f)$. Another appeal to Corollary 1 of [G] and arguments similar to those done above give

$$\omega_k(f) = \langle f, g_k \rangle \tag{4.10}$$

with

$$\text{supp}\, g_k \subset B_1(p_k), \quad \|g_k\|_{\text{bmo}(M)} \le C\|\omega_k\|_{\mathfrak{h}^1(M)'} \le C'\|\omega\|_{\mathfrak{h}^1(M)'}. \tag{4.11}$$

The properties listed in (4.11) (in concert with (1.26)) in turn give

$$g_\omega = \sum_k g_k \in \text{bmo}(M), \quad \|g_\omega\|_{\text{bmo}} \le C\|\omega\|_{\mathfrak{h}^1(M)'}, \tag{4.12}$$

satisfying

$$\omega(f) = \langle f, g_\omega \rangle. \tag{4.13}$$

This completes the proof of Proposition 4.1.

Remark. From Proposition 4.1 we deduce that the multipliers $f \mapsto af$ on $\mathfrak{h}^1(M)$ given by Proposition 2.1 are also multipliers on $\text{bmo}(M)$, and the multipliers on $\text{bmo}(M)$ given by Proposition 3.1 are also multipliers on $\mathfrak{h}^1(M)$. This is of some interest, since the classes of multipliers treated in these two propositions are slightly different.

5. Atomic theory (ionic theory) of $\mathfrak{h}^1(M)$

In the theory of $H^1(\mathbb{R}^n)$, an atom is a function a satisfying, for some $p \in \mathbb{R}^n,\ r \in (0, \infty)$,

$$\text{supp}\, a \subset B_r(p), \quad \|a\|_{L^\infty} \le r^{-n}, \tag{5.1}$$

and

$$\int a\, dx = 0. \tag{5.2}$$

It is not hard to show that for such a

$$\|\mathcal{G}a\|_{L^1} \le C, \quad \text{independent of} \ \ p, r, \tag{5.3}$$

and hence

$$f = \sum \lambda_j a_j, \ a_j \text{ atoms} \implies \|f\|_{H^1} \le C \sum |\lambda_j|. \tag{5.4}$$

The converse result is that each $f \in H^1(\mathbb{R}^n)$ has such an atomic decomposition; cf. [St] for a treatment, due originally to R. Coifman for $n = 1$ and R. Latter for $n > 1$. In [G] the atomic decomposition of $\mathfrak{h}^1(\mathbb{R}^n)$ was given in terms of atoms which, this time, satisfy (5.1)–(5.2) for $r \le 1$ but only (5.1) for $r > 1$.

Note that if a is an atom satisfying (5.1)–(5.2) and φ is Lipschitz, with $\|\varphi\|_{L^\infty} \le 1$ and $\|\varphi\|_{\mathrm{Lip}} \le L$, then $b = \varphi a$ satisfies

$$\operatorname{supp} b \subset B_r(p), \quad \|b\|_{L^\infty} \le r^{-n}, \tag{5.5}$$

and

$$\Bigl| \int b \, dx \Bigr| \le V(B_1(p)) \, L(r \wedge 1), \tag{5.6}$$

Adapting material from Appendix A of [MT], if $r \in (0, 1]$, we call such b an *ion*. More generally, if b satisfies (5.5)–(5.6), we can write

$$b = a + h, \tag{5.7}$$

with

$$h = b_{B_r(p)} \chi_{B_r(p)}, \quad |b_{B_r(p)}| \le A_n L r^{1-n}, \tag{5.8}$$

with a satisfying (5.1), up to a factor of $1 + A_n L$. For such a function it is not hard to show that

$$\|\mathcal{G}^b h\|_{L^1} \le CL, \quad C \ \text{ independent of } \ p, r \in (0, 1]. \tag{5.9}$$

In concert with (5.3) this gives

$$\|\mathcal{G}^b b\|_{L^1} \le C(L + 1). \tag{5.10}$$

Hence, for ions satisfying (5.5)–(5.6) (with fixed $L < \infty$ and with $r \in (0, 1]$), we have

$$f = \sum \lambda_j b_j, \ b_j \ \text{ ions } \implies \|f\|_{\mathfrak{h}^1} \le C(L + 1) \sum |\lambda_j|. \tag{5.11}$$

The existence of an ionic decomposition of a general $f \in \mathfrak{h}^1(\mathbb{R}^n)$ follows from the atomic decomposition of [G] mentioned above.

We move now to the setting of a complete n-dimensional Riemannian manifold M with bounded geometry (and with metric tensor satisfying (1.19)–(1.24)).

Definition. An ion (of ionic norm ≤ 2) is a function b on M satisfying the following properties:

$$\begin{aligned} &\operatorname{supp} b \subset B_r(p) \ \text{ for some } \ p \in M,\ r \in (0,1], \\ &\|b\|_{L^\infty} \le r^{-n}, \\ &\Bigl|\int b\, dV\Bigr| \le r. \end{aligned} \tag{5.12}$$

The results discussed above in concert with the results of §2 yield the following.

Proposition 5.1. *If $\{b_j : j \in \mathbb{Z}^+\}$ are ions, then*

$$\begin{aligned} &f = \sum \lambda_j b_j,\ \sum |\lambda_j| < \infty \Longrightarrow \\ &f \in \mathfrak{h}^1(M) \quad \text{and} \quad \|f\|_{\mathfrak{h}^1} \le C \sum |\lambda_j|. \end{aligned} \tag{5.13}$$

Conversely, if $f \in \mathfrak{h}^1(M)$, then there exist ions b_j and $\lambda_j \in \mathbb{C}$ such that

$$f = \sum \lambda_j b_j, \quad \sum |\lambda_j| \le C \|f\|_{\mathfrak{h}^1}. \tag{5.14}$$

6. Action of $(\lambda I - \Delta)^{it}$ and other pseudodifferential operators on bmo, $\mathfrak{h}^1$, and L^p

We take M as in §§1–5, particularly enforcing (1.19)–(1.24). Our first goal is to prove the following.

Proposition 6.1. *With M as above, let Δ be the Laplace-Beltrami operator on M. There exist $\lambda_0 = \lambda_0(M) \in (0,\infty)$ such that whenever $\lambda \ge \lambda_0$,*

$$(\lambda I - \Delta)^{it} : \operatorname{bmo}(M) \longrightarrow \operatorname{bmo}(M), \tag{6.1}$$

with uniformly bounded operator norms for $|t| \le 1$, hence, with $C = C(M,\lambda)$,

$$\|(\lambda I - \Delta)^{it} f\|_{\operatorname{bmo}} \le C e^{C|t|} \|f\|_{\operatorname{bmo}}. \tag{6.2}$$

To begin the analysis, we write

$$(\lambda I - \Delta)^{it} = \Phi_{-it,\lambda}(\sqrt{-\Delta}), \tag{6.3}$$

where

$$\Phi_{-it,\lambda}(\zeta) = (\zeta^2 + \lambda)^{it}. \tag{6.4}$$

Results of [CGT] apply to analyze the integral kernel $K_{t,\lambda}(x, y)$ in

$$(\lambda I - \Delta)^{it} f(x) = \int\limits_M K_{t,\lambda}(x, y) f(y)\, dV(y). \tag{6.5}$$

They are described as follows. Given $W > 0,\ m \in \mathbb{R}$, we say

$$\Phi \in \mathcal{S}^m_W \tag{6.6}$$

provided Φ is holomorphic and even on the strip

$$\Omega_W = \{\zeta \in \mathbb{C} : |\mathrm{Im}\, \zeta| < W\} \tag{6.7}$$

and satisfies $S^m_{1,0}$ estimates on Ω_W:

$$|\Phi^{(j)}(\zeta)| \le C_j (1 + |\zeta|)^{m-j}, \quad \zeta \in \Omega_W. \tag{6.8}$$

Compare [CGT], Definition 3.1. As shown in §3 of [CGT] (cf. (3.45)), with an improvement given by (1.12) of [T3], if K_Φ is the integral kernel of $\Phi(\sqrt{-\Delta})$,

$$\Phi \in \mathcal{S}^m_W \Longrightarrow |K_\Phi(x, y)| \le C_k d(x, y)^{-k} e^{-W d(x,y)}, \quad \text{for} \ \ d(x, y) \ge 1,\ k \in \mathbb{Z}^+, \tag{6.9}$$

when M has bounded geometry. Here C depends on M and finitely many of the constants in (6.8). Now $\Phi_{-it,\lambda}$, given by (6.4), satisfies, for each $\delta \in (0, \lambda)$,

$$\Phi_{-it,\lambda} \in \mathcal{S}^0_{\sqrt{\lambda-\delta}}, \tag{6.10}$$

uniformly in $t \in [-1, 1]$, so we deduce that the kernel $K_{t,\lambda}$ in (6.5) satisfies the estimate

$$|K_{t,\lambda}(x, y)| \le C e^{-\sqrt{\lambda-\delta}\, d(x,y)}, \quad \text{for} \ \ d(x, y) \ge 1, \tag{6.11}$$

with $C = C(M, \lambda, \delta)$, independent of $t \in [-1, 1]$.

The near diagonal behavior is covered by the following result. To state it, first for each $p \in M$, use $\mathrm{Exp}_p : T_pM \to M$, satisfying (1.19)–(1.24), to identify $B_4(p) \subset M$ with $B_4(0) \subset T_pM$, further identified with $B_4(0) \subset \mathbb{R}^n$, uniquely up to the action of $O(n)$. Thus functions supported on $B_4(p) \subset M$ are identified with functions supported on $B_4(0) \subset \mathbb{R}^n$. With this convention in place, we state the result, which, by (6.10), is a special case of Theorem 3.3 of [CGT].

Proposition 6.2. *Given $p \in M$, take $\varphi_j \in C_0^\infty(B_4(0)) \approx C_0^\infty(B_4(p))$, and set $M_{\varphi_j} f = \varphi_j f$. Then, for each $\lambda > 0$,*

$$M_{\varphi_1}(\lambda I - \Delta)^{it} M_{\varphi_2} \in OPS^0_{1,0}(\mathbb{R}^n), \tag{6.12}$$

with uniform bounds for $t \in [-1,1]$, $p \in M$.

Having these results, we now take $f \in \mathrm{bmo}(M)$ and estimate $g_t = (\lambda I - \Delta)^{it} f$. It suffices to show that for each $p \in M$, with $B_1 = B_1(p)$,

$$\|g_t^{\#}\|_{L^\infty(B_1)} \le C\|f\|_{\mathrm{bmo}}, \quad \|g_t\|_{L^1(B_1)} \le C\|f\|_{\mathrm{bmo}}, \tag{6.13}$$

with C independent of p and of $t \in [-1,1]$. We also set $B_r = B_r(p)$. Take $\varphi \in C_0^\infty(M)$ such that

$$\varphi = 1 \ \text{ on } \ B_2, \quad \operatorname{supp} \varphi \subset B_3, \tag{6.14}$$

and write

$$f = \varphi f + (1-\varphi) f. \tag{6.15}$$

We first estimate $(\lambda I - \Delta)^{it}(1-\varphi) f$ on B_1. To do this, we use (6.11) together with the following well known volume estimate. (See [CGT], Proposition 4.1, for a stronger result.)

Lemma 6.3. *Given a Riemannian manifold M with bounded geometry, there exists $C_0 = C_0(M)$, $\mu_0 = \mu_0(M)$, and $K_0 = K_0(M)$ such that for each $p \in M$, $r \in (0,\infty)$,*

$$Vol\, B_r(p) \le C_0 (1+r)^{\mu_0} e^{K_0 r}. \tag{6.16}$$

Making use of this, we have

$$\begin{aligned} x \in B_1 \Longrightarrow \Big| \int K_{t,\lambda}(x,y)(1-\varphi(y)) f(y)\, dV(y) \Big| & \\ \le C \int\limits_{M \setminus B_1} e^{-\sqrt{\lambda - \delta}\, d(p,y)}\, dV(y)\ \|f\|_{\mathrm{bmo}} & \\ \le C \int_1^\infty r^{\mu_0} e^{(K_0 - \sqrt{\lambda-\delta}) r}\, dr\ \|f\|_{\mathrm{bmo}}. & \end{aligned} \tag{6.17}$$

The first inequality in (6.17) results from (6.11) plus the fact that $\int_B |f|\, dV \le C\|f\|_{\mathrm{bmo}}$, uniformly for all unit balls $B \subset M$. The last line in (6.17) is $\le C\|f\|_{\mathrm{bmo}}$ provided $\lambda > K_0^2 + \delta$. Thus, to make Proposition 6.1 work, we require

$$\lambda_0 > K_0^2. \tag{6.18}$$

In such a case, we have an L^∞ estimate on B_1 for $(\lambda I-\Delta)^{it}(1-\varphi)f$.

It remains to estimate $(\lambda I-\Delta)^{it}\varphi f$ on B_1. Let us fix $\varphi_2\in C_0^\infty(B_3(0))\approx C_0^\infty(B_3(p))$ such that $\varphi_2=1$ on $\operatorname{supp}\varphi$. Then we can apply Proposition 6.2 to get

$$M_\varphi(\lambda I-\Delta)^{it}M_{\varphi_2}\in OPS^0_{1,0}(\mathbb{R}^n), \tag{6.19}$$

with uniform bounds for $p\in M,\ t\in[-1,1]$. Theorem 4 of [G] implies this family of operators is uniformly bounded on $\mathfrak{h}^1(\mathbb{R}^n)$, and by duality we get

$$\|\varphi_2(\lambda I-\Delta)^{it}\varphi f\|_{\mathrm{bmo}}\le C\|\varphi_2 f\|_{\mathrm{bmo}}\le C'\|f\|_{\mathrm{bmo}}, \tag{6.20}$$

where we also use Proposition 3.3. This completes the proof of Proposition 6.1.

A similar argument also proves:

Proposition 6.4. *In the setting of Proposition 6.1,*

$$(\lambda I-\Delta)^{it}:\mathfrak{h}^1(M)\longrightarrow\mathfrak{h}^1(M). \tag{6.21}$$

Furthermore, we have

$$(\lambda I-\Delta)^{it}:L^p(M)\longrightarrow L^p(M),\quad 1<p<\infty. \tag{6.22}$$

Remark. In fact, (6.22) is known in a much more general context; cf. [St0].

To put the results just established in context, and for use in later sections, it is useful to identify some distinguished classes of "pseudodifferential operators" on M and record some of their mapping properties. To make the following definition, we retain the identification made in the statement of Proposition 6.2 of functions in $C_0^\infty(B_4(0))$, where $B_4(0)\subset\mathbb{R}^n$, and functions in $C_0^\infty(B_4(p))$, where $B_4(p)\subset M$, via the exponential map. Let $\varphi_j\in C^\infty(B_4(0))\approx C_0^\infty(B_4(p))$ be as in that proposition, and assume $\varphi_j=1$ on $B_2(0)$. Given an operator $P:C_0^\infty(M)\to\mathcal{D}'(M)$, we will say

$$P\in\Psi^m_\#(M) \tag{6.23}$$

provided the following conditions hold. First, we assume its Schwartz kernel $K_P\in\mathcal{D}'(M\times M)$ satisfies

$$\begin{aligned}\operatorname{supp}K_P&\subset\{(x,y)\in M\times M:d(x,y)\le 1\},\\ \operatorname{sing\,supp}K_P&\subset\operatorname{diag}(M\times M)=\{(x,x):x\in M\}.\end{aligned} \tag{6.24}$$

Next, we assume that for each $p\in M$,

$$M_{\varphi_1}PM_{\varphi_2}\in OPS^m_{1,0}(\mathbb{R}^n), \tag{6.25}$$

with uniform bounds, independent of $p\in M$. These conditions define (6.23).

In case $m=0$, the condition (6.25), with uniform bounds, implies uniform operator bounds for $M_{\varphi_1}PM_{\varphi_2}$ on $\mathfrak{h}^1(\mathbb{R}^n)$ and $\mathrm{bmo}(\mathbb{R}^n)$ (by Theorem 4 of [G] and duality), and on $L^p(\mathbb{R}^n)$ for $1<p<\infty$ (by Calderon-Zygmund theory). From here we get as before the following.

Proposition 6.5. *We have*

$$
\begin{aligned}
P \in \Psi^0_{\#}(M) \Rightarrow\ & P : \mathfrak{h}^1(M) \to \mathfrak{h}^1(M), \\
& P : \mathrm{bmo}(M) \to \mathrm{bmo}(M), \\
& P : L^p(M) \to L^p(M), \quad 1 < p < \infty.
\end{aligned} \tag{6.26}
$$

To proceed, given $W \in (0,\infty)$, we say

$$
P \in \Psi^m_W(M), \tag{6.27}
$$

provided we can write

$$
P = P^{\#} + P^b, \tag{6.28}
$$

where $P^{\#} \in \Psi^m_{\#}(M)$ and P^b has the form

$$
P^b f(x) = \int\limits_M K^b(x,y) f(y)\, dV(y), \tag{6.29}
$$

where $K^b(x,y) \in C^\infty(M \times M)$ satisfies

$$
|K^b(x,y)| \le C_k (1 + d(x,y))^{-k} e^{-W d(x,y)}, \quad \forall\, k \in \mathbb{Z}^+, \tag{6.30}
$$

and also such estimates hold for all x and y-derivatives of $K^b(x,y)$ (say in local exponential coordinate systems). When (6.29)–(6.30) hold, we say

$$
P^b \in \Psi^{-\infty}_W(M). \tag{6.31}
$$

The main result in §3 of [CGT] can be summarized as follows:

Proposition 6.6. *For $W \in (0,\infty),\ m \in \mathbb{R}$,*

$$
\Phi \in \mathcal{S}^m_W \Longrightarrow \Phi(\sqrt{-\Delta}) \in \Psi^m_W(M). \tag{6.32}
$$

We recall that the proof of this result in [CGT] began with the representation

$$
\begin{aligned}
&\Phi(\sqrt{-\Delta}) f \\
&= \int_{-\infty}^{\infty} \widehat{\Phi}(t) \cos t\sqrt{-\Delta} f\, dt \\
&= \int_{-\infty}^{\infty} \widehat{\Phi}(t)\psi(t) \cos t\sqrt{-\Delta} f\, dt + \int_{-\infty}^{\infty} \widehat{\Phi}(t)(1-\psi(t)) \cos t\sqrt{-\Delta} f\, dt \\
&= \Phi^{\#}(\sqrt{-\Delta}) f + \Phi^b(\sqrt{-\Delta}) f,
\end{aligned}
$$

where in the second identity we take $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi(t) = 1$ for $|t| \le 1/4$, 0 for $|t| \ge 1/2$. As shown in [CGT], we have $\Phi^\#(\sqrt{-\Delta}) \in \Psi^m_\#(M)$, defined as in (6.23)–(6.25), and $\Phi^b(\sqrt{-\Delta}) \in \Psi^{-\infty}_W(M)$. The result $\Phi^\#(\sqrt{-\Delta}) \in \Psi^m_\#(M)$ is established via finite propagation speed and a parametrix construction for the solution operator $\cos t\sqrt{-\Delta}$ to the wave equation. (The slightly greater precision of having the factors $C_k(1 + d(x, y))^{-k}$ arises as in (1.12) of [T3].) Related results are also discussed in [T1] and in Chapter 5 of [T2].

As for boundedness on function spaces, given the estimate (6.16), we claim that if P^b satisfies (6.29)–(6.30), i.e., $P^b \in \Psi^{-\infty}_W(M)$, then

$$\begin{aligned} W \ge K_0 \Longrightarrow P^b &: \mathfrak{h}^1(M) \to \mathfrak{h}^1(M), \\ P^b &: \mathrm{bmo}(M) \to \mathrm{bmo}(M), \\ P^b &: L^p(M) \to L^p(M), \quad 1 \le p \le \infty. \end{aligned} \tag{6.33}$$

In fact, one has the following. Let $\{\varphi_k : k \in \mathbb{Z}^+\}$ be a tame partition of unity, as in (1.27). Given $p, q \in [1, \infty]$, say

$$f \in L^{(p)}_{(q)}(M) \Longleftrightarrow (\|\varphi_k f\|_{L^q}) \in \ell^p, \tag{6.34}$$

Note that $L^{(p)}_{(p)}(M) = L^p(M)$ for $p \in [1, \infty]$, and

$$\mathrm{bmo}(M) \subset L^{(\infty)}_{(1)}(M), \quad L^{(1)}_{(\infty)}(M) \subset \mathfrak{h}^1(M). \tag{6.35}$$

The following results are straightforward, and imply (6.33):

$$\begin{aligned} P^b \in \Psi^{-\infty}_W(M),\ W \ge K_0 \Longrightarrow P^b &: L^{(p)}_{(1)}(M) \to L^{(p)}_{(\infty)}(M),\ \forall\, p \in [1, \infty] \\ \Longrightarrow P^b &: L^1(M) \to \mathfrak{h}^1(M) \ \text{ and} \\ P^b &: \mathrm{bmo}(M) \to L^\infty(M). \end{aligned} \tag{6.36}$$

Together with Proposition 6.5, (6.33) gives:

Proposition 6.7. *Given K_0 as in (6.16),*

$$\begin{aligned} W \ge K_0,\ P \in \Psi^0_W(M) \Rightarrow P &: \mathfrak{h}^1(M) \to \mathfrak{h}^1(M), \\ P &: \mathrm{bmo}(M) \to \mathrm{bmo}(M), \\ P &: L^p(M) \to L^p(M), \quad 1 < p < \infty. \end{aligned} \tag{6.37}$$

In particular, these mapping properties hold for $\Phi(\sqrt{-\Delta})$, given $\Phi \in \mathcal{S}^0_W,\ W > K_0$.

Remark. A sharper L^p-boundedness result on $\Phi(\sqrt{-\Delta})$ was demonstrated in [T3], namely, for $p \in (1, \infty)$,

$$\Phi \in \mathcal{S}^0_W,\ W \ge \left|\frac{1}{p} - \frac{1}{2}\right| \cdot K_0 \Longrightarrow \Phi(\sqrt{-\Delta}) : L^p(M) \to L^p(M). \tag{6.38}$$

In fact, the following more general result is proven in [T3]. Suppose $A \geq 0$ and

$$\text{(6.38A)} \qquad \operatorname{Spec}(-\Delta) \subset [A, \infty) \quad \text{on} \quad L^2(M).$$

Then, for $p \in (1, \infty)$,

$$\text{(6.38B)} \quad \Phi \in \mathcal{S}_W^0, \ W \geq \left| \frac{1}{p} - \frac{1}{2} \right| \cdot K_0, \ L = \Delta + A \Longrightarrow \Phi(\sqrt{-L}) : L^p(M) \to L^p(M).$$

The condition (6.38A) holds with $A > 0$ for hyperbolic space and other symmetric spaces of noncompact type. See Appendices B and C for more on this.

For later use, we establish the following result on composition.

Proposition 6.8. *Given $W \geq K_0$, we have*

$$\text{(6.39)} \qquad P_j \in \Psi_W^{m_j}(M) \Longrightarrow P_1 P_2 \in \Psi_{W - K_0/2}^{m_1 + m_2}(M).$$

Proof. Write $P_j = P_j^\# + P_j^b$ with $P_j^\# \in \Psi_\#^{m_j}(M)$ and $P_j^b \in \Psi_W^{-\infty}(M)$. Furthermore, arrange that the Schwartz kernels of $P_j^\#$ are supported in $\{(x, y) \in M \times M : d(x, y) \leq 1/2\}$. We claim that

$$\text{(6.40)} \qquad P_1^\# P_2^\# \in \Psi_\#^{m_1 + m_2}(M),$$

$$\text{(6.41)} \qquad P_1^\# P_2^b, \ P_1^b P_2^\# \in \Psi_W^{-\infty}(M),$$

$$\text{(6.42)} \qquad P_1^b P_2^b \in \Psi_{W - K_0/2}^{-\infty}(M).$$

These results imply (6.39).

Of these results, (6.40) follows from standard Euclidean space pseudodifferential operator calculus, especially the composition results

$$\text{(6.43)} \qquad OPS_{1,0}^{m_1}(\mathbb{R}^n) \times OPS_{1,0}^{m_2}(\mathbb{R}^n) \longrightarrow OPS_{1,0}^{m_1 + m_2}(\mathbb{R}^n).$$

The first result in (6.41) follows from the fact that

$$\text{(6.44)} \qquad K_{P_1^\# P_2^b}(\cdot, y) = P_1^\# K_{P_2^b}(\cdot, y),$$

plus standard pseudodifferential operator estimates, and the second part from the first, by passing to the adjoint.

This leaves (6.42). Note that

$$\text{(6.45)} \qquad |K_{P_1^b P_2^b}(x, y)| \leq C_k \int\limits_M \langle d(x, z) \rangle^{-k} \langle d(z, y) \rangle^{-k} e^{-W[d(x,z) + d(z,y)]} \, dV(z).$$

Write

$$
\begin{aligned}
M &= A \cup B \cup C, \quad C = M \setminus (A \cup B), \\
A &= \{z \in M : d(x,z) \le \tfrac{1}{2} d(x,y)\}, \\
B &= \{z \in M : d(y,z) \le \tfrac{1}{2} d(x,y)\}.
\end{aligned} \tag{6.46}
$$

Since $d(x,z)+d(y,z) \ge d(x,y)$ and $\langle d(x,z)\rangle\langle d(z,y)\rangle \ge C\langle d(x,y)\rangle$, we have

$$
\begin{aligned}
&\int_A \langle d(x,z)\rangle^{-k} \langle d(z,y)\rangle^{-k} e^{-W[d(x,z)+d(z,y)]}\, dV(z) \\
&\le C_k \langle d(x,y)\rangle^{-k} e^{-W d(x,y)} \operatorname{Vol}(A) \\
&\le C_k \langle d(x,y)\rangle^{-k+\mu_0} e^{-(W-K_0/2)d(x,y)},
\end{aligned} \tag{6.47}
$$

the latter estimate by (6.16). There is a similar estimate on $\int_B$. Finally, since $d(z,y) \ge d(x,y)/2$ on C,

$$
\begin{aligned}
&\int_C \langle d(x,z)\rangle^{-k} \langle d(z,y)\rangle^{-k} e^{-W[d(x,z)+d(z,y)]}\, dV(z) \\
&\le C_k \langle d(x,y)\rangle^{-k/2} e^{-W d(x,y)/2} \int_{M\setminus A} \langle d(x,z)\rangle^{-k/2} e^{-W d(x,z)}\, dV(z) \\
&\le C_k \langle d(x,y)\rangle^{-k/2} e^{-W d(x,y)/2} \int_{d(x,y)/2}^{\infty} \langle r\rangle^{\mu_0-k/2} e^{-Wr} e^{K_0 r}\, dr \\
&= C_k \langle d(x,y)\rangle^{-k/2} e^{-(W-K_0/2)d(x,y)},
\end{aligned} \tag{6.48}
$$

provided $W \ge K_0$ (taking k large enough). Similar estimates hold for derivatives of $K_{P_1^b P_2^b}(x,y)$. This completes the proof.

The following variant of Proposition 6.8 will also prove useful.

Proposition 6.9. *Given $W > 0$, we have*

$$
P_1 \in \Psi_W^{m_1}(M),\ P_2 \in \Psi_{W+K_0}^{m_2}(M) \Longrightarrow P_1P_2,\ P_2P_1 \in \Psi_W^{m_1+m_2}(M). \tag{6.49}
$$

Proof. With $P_j = P_j^{\#} + P_j^b$ as before, the results (6.40)–(6.41) are readily verified, together with their analogues with the subscripts 1 and 2 interchanged. In place of (6.42), this time we claim

$$
P_1^b P_2^b,\ P_2^b P_1^b \in \Psi_W^{-\infty}(M). \tag{6.50}
$$

We treat $P_1^b P_2^b$; a similar argument will handle the other product. In place of (6.45), we have

$$
|K_{P_1^b P_2^b}(x,y)| \le C_k \int_M \langle d(x,z)\rangle^{-k} \langle d(z,y)\rangle^{-k} e^{-W[d(x,z)+d(z,y)]} e^{-K_0 d(z,y)}\, dV(z). \tag{6.51}
$$

Using again the observations below (6.46), we see that the right side of (6.51) is

$$
\begin{aligned}
&\le C_k \langle d(x,y)\rangle^{-k/2} e^{-W d(x,y)} \int_M \langle d(z,y)\rangle^{-k/2} e^{-K_0 d(z,y)}\, dV(z) \\
&\le C_k \langle d(x,y)\rangle^{-k/2} e^{-W d(x,y)},
\end{aligned}
\tag{6.52}
$$

the latter estimate by (6.16). There are similar estimates on derivatives, giving (6.50).

7. Local L^p estimates

The purpose of this section is to obtain a local version of the L^p estimates in Theorem 5 of [FS], which will allow us to obtain global L^p estimates in §8.

Here, let

$$
Q_1 = \{x \in \mathbb{R}^n : 0 \le x_j \le 1,\ 1 \le j \le n\} \subset \mathbb{R}^n. \tag{7.1}
$$

For this section, we set

$$
f^{\#}(x) = \sup_{Q \in \mathcal{Q}(x)} \frac{1}{V(Q)} \int_Q |f(x) - f_Q|\, dx, \tag{7.2}
$$

where V is Lebesgue measure, as usual, f_Q is the mean of f over f_Q, and

$$
\mathcal{Q}(x) = \{Q \subset Q_1 : Q \text{ cube},\ Q \ni x\}. \tag{7.3}
$$

Our goal is to prove the following.

Proposition 7.1. *For $p \in (1,\infty)$, there exists $C_{n,p} < \infty$ with the following property. Given $f \in L^1(Q_1)$ such that*

$$
\|f\|_{L^1(Q_1)} \le 1, \quad \|f^{\#}\|_{L^p(Q_1)} \le 1, \tag{7.4}
$$

it follows that

$$
\|f\|_{L^p(Q_1)} \le C_{n,p}. \tag{7.5}
$$

To begin the proof, for $\alpha \in [1,\infty)$, subdivide Q_1 dyadically, into 2^n cubes of edge $1/2$, and denote by Q_j^α any such cubes for which

$$
\alpha < \frac{1}{V(Q_j^\alpha)} \int_{Q_j^\alpha} |f|\, dx \le 2^n \alpha. \tag{7.6}
$$

For those dyadic cubes for which (7.6) fails, subdivide these dyadically, retaining those for which (7.6) holds, and continue this process, obtaining a family $\{Q_j^\alpha\}$ for which (7.6) holds. Note that

$$|f(x)| \le \alpha \quad \text{a.e. on} \quad Q_1 \setminus \bigcup_j Q_j^\alpha. \tag{7.7}$$

Note that if $\beta > \alpha$ the cubes in $\{Q_j^\beta\}$ are sub-cubes of cubes in $\{Q_j^\alpha\}$. Set

$$\mu(\alpha) = \sum_j V(Q_j^\alpha). \tag{7.8}$$

Parallel to (4.4) of [FS], we aim to show that

$$\mu(\alpha) \le V\Bigl(\Bigl\{x \in Q_1 : f^\#(x) > \frac{\alpha}{A}\Bigr\}\Bigr) + \frac{2}{A}\mu(2^{-n-1}\alpha), \tag{7.9}$$

whenever

$$\alpha \ge 2^{n+1}, \quad A \ge 1. \tag{7.10}$$

The proof is similar to that of [FS]. Fix a cube $Q_0 = Q_{j_0}^{\alpha/2^{n+1}}$ and look at all the cubes $Q_j^\alpha \subset Q_0$. Consider two cases:

Case I: $Q_0 \subset \{x : f^\#(x) > \alpha/A\}$. In this case,

$$\sum_{Q_j^\alpha \subset Q_0} V(Q_j^\alpha) \le V\Bigl(Q_0 \cap \Bigl\{x : f^\#(x) \ge \frac{\alpha}{A}\Bigr\}\Bigr). \tag{7.11}$$

Case II: $Q_0 \not\subset \{x : f^\#(x) > \alpha/A\}$. In this case,

$$\frac{1}{V(Q_0)} \int_{Q_0} |f(x) - f_{Q_0}|\, dx \le \frac{\alpha}{A}. \tag{7.12}$$

Now (7.6) implies both $|f_{Q_0}| \le \alpha/2$ and $|f|_{Q_j^\alpha} > \alpha$. Hence

$$\int_{Q_j^\alpha} |f(x) - f_{Q_0}|\, dx \ge \frac{\alpha}{2} V(Q_j^\alpha), \tag{7.13}$$

for each $Q_j^\alpha \subset Q_0$ of the form described above. Now sum (7.13) over all such cubes and compare the result to (7.12). This yields, in Case II,

$$\sum_{Q_j^\alpha \subset Q_0} V(Q_j^\alpha) \le \frac{2}{A} V(Q_0). \tag{7.14}$$

Now sum over all the cubes Q_0, i.e., over all the cubes in $\{Q_j^{\alpha/2^{n+1}}\}$, taking into account the estimates (7.11) in Case I and (7.14) in Case II, to obtain the asserted estimate (7.9).

Next, parallel to (4.8) of [FS], we bring in

$$\lambda(\alpha) = V(\{x \in Q_1 : Mf(x) > \alpha\}), \tag{7.15}$$

where

$$Mf(x) = \sup_{Q \in \mathcal{Q}(x)} \frac{1}{V(Q)} \int\limits_Q |f|\,dx. \tag{7.16}$$

Clearly $Mf(x) > \alpha$ whenever $x \in Q_j^\alpha$, so

$$\mu(\alpha) \le \lambda(\alpha), \quad \forall\, \alpha \in [1,\infty). \tag{7.17}$$

We next aim to prove

$$\lambda\big((1+8^n)\alpha\big) \le 2^n \mu(\alpha), \quad \forall\, \alpha \in [1,\infty). \tag{7.18}$$

To get this, we bring in the following notation. Given a cube Q, let $\widetilde{Q}$ and $\widehat{Q}$ denote the concentric cubes dilated by factors of 2 and 4, respectively. Now, take $\{Q_j^\alpha\}$ as above, and consider $x \in Q_1 \setminus \cup_j \widetilde{Q}_j^\alpha$. Let $Q \in \mathcal{Q}(x)$, defined by (7.3). Consider

$$\int\limits_Q |f(y)|\,dy = \int\limits_{Q \cap (\cup_j Q_j^\alpha)} |f(y)|\,dy + \int\limits_{Q \setminus (\cup_j Q_j^\alpha)} |f(y)|\,dy. \tag{7.19}$$

Since $|f| \le \alpha$ on $Q \setminus (\cup_j Q_j^\alpha)$, the last integral in (7.19) is $\le \alpha V(Q)$. For the first integral on the right side of (7.19), we use the fact that

$$\begin{aligned} \text{Given } x \in Q \setminus \widetilde{Q}_j^\alpha, \quad Q \cap Q_j^\alpha \neq \emptyset &\Rightarrow Q \not\subset \widetilde{Q}_j^\alpha \\ &\Rightarrow Q_j^\alpha \subset \widehat{Q}. \end{aligned} \tag{7.20}$$

Hence

$$\begin{aligned} \int\limits_{Q \cap (\cup_j Q_j^\alpha)} |f(y)|\,dy &\le \sum_{Q_j^\alpha \subset \widehat{Q}} \int\limits_{Q_j^\alpha} |f(y)|\,dy \\ &\le \sum_{Q_j^\alpha \subset \widehat{Q}} 2^n \alpha V(Q_j^\alpha) \\ &\le 8^n \alpha V(Q). \end{aligned} \tag{7.21}$$

Thus, in (7.19), we have

$$\int_Q |f(y)|\,dy \le (1+8^n)\alpha V(Q). \tag{7.22}$$

Since this is true for all $Q \in \mathcal{Q}(x)$, we deduce that

$$x \in Q_1 \setminus \bigcup_j \widetilde{Q}_j^\alpha \Rightarrow Mf(x) \le (1+8^n)\alpha, \quad \forall\, \alpha \ge 1. \tag{7.23}$$

Hence

$$\{x \in Q_1 : Mf(x) > (1+8^n)\alpha\} \subset \bigcup_j \widetilde{Q}_j^\alpha, \quad \forall\, \alpha \ge 1, \tag{7.24}$$

and (7.18) is established.

Moving on towards the proof of (7.5), we will actually estimate $\|Mf\|_{L^p(Q_1)}$. Note that

$$\|Mf\|^p_{L^p(Q_1)} = p\int_0^\infty \alpha^{p-1}\lambda(\alpha)\,d\alpha = \lim_{N\to\infty} p\int_0^N \alpha^{p-1}\lambda(\alpha)\,d\alpha. \tag{7.25}$$

Furthermore, for $N \ge 16^{n+1}$,

$$p\int_0^N \alpha^{p-1}\lambda(\alpha)\,d\alpha = p\int_0^{16^{n+1}} \alpha^{p-1}\lambda(\alpha)\,d\alpha + p\int_{16^{n+1}}^N \alpha^{p-1}\lambda(\alpha)\,d\alpha. \tag{7.26}$$

To estimate the first integral on the right, we use

$$\lambda(\alpha) \le \frac{C}{\alpha}\|f\|_{L^1}, \tag{7.27}$$

which follows by weak (1,1) boundedness of $f \mapsto Mf$, to obtain

$$\begin{aligned} p\int_0^{16^{n+1}} \alpha^{p-1}\lambda(\alpha)\,d\alpha &\le Cp\int_0^{16^{n+1}} \alpha^{p-2}\,d\alpha\ \|f\|_{L^1} \\ &= \frac{Cp}{p-1}\,16^{(n+1)(p-1)}\|f\|_{L^1}. \end{aligned} \tag{7.28}$$

To treat the second integral on the right side of (7.26), we bring in (7.18). We have

$$\begin{aligned} p\int_{16^{n+1}}^N \alpha^{p-1}\lambda(\alpha)\,d\alpha &= p\,8^{(n+1)p}\int_{2^{n+1}}^{N/8^{n+1}} \beta^{p-1}\lambda(8^{n+1}\beta)\,d\beta \\ &\le p\,2^n 8^{(n+1)p}\int_{2^{n+1}}^{N/8^{n+1}} \beta^{p-1}\mu(\beta)\,d\beta. \end{aligned} \tag{7.29}$$

Having this, we follow [FS] and study, for $M \geq 2^{n+1}$,

$$I_M = p \int_{2^{n+1}}^{M} \alpha^{p-1} \mu(\alpha)\, d\alpha. \tag{7.30}$$

Using (7.9), we have

$$\begin{aligned} I_M &\leq p \int_{2^{n+1}}^{M} \alpha^{p-1} V\Big(\Big\{x \in Q_1 : f^{\#}(x) > \frac{\alpha}{A}\Big\}\Big)\, d\alpha \\ &\quad + \frac{2}{A} p \int_{2^{n+1}}^{M} \alpha^{p-1} \mu(2^{-n-1}\alpha)\, d\alpha \\ &\leq A^p \|f^{\#}\|_{L^p(Q_1)}^p + \frac{2}{A} 2^{(n+1)p} p \int_1^{M/2^{n+1}} \beta^{p-1} \mu(\beta)\, d\beta. \end{aligned} \tag{7.31}$$

Provided $M > 4^{n+1}$, we write the last integral as

$$\int_1^{2^{n+1}} \beta^{p-1} \mu(\beta)\, d\beta + \int_{2^{n+1}}^{M/2^{n+1}} \beta^{p-1} \mu(\beta)\, d\beta. \tag{7.32}$$

Since $\mu(\beta) \leq \lambda(\beta)$, we have as in (7.28) the estimate

$$\int_1^{2^{n+1}} \beta^{p-1} \mu(\beta)\, d\beta \leq \frac{C}{p-1} 2^{(n+1)(p-1)} \|f\|_{L^1}, \tag{7.33}$$

while the last integral in (7.32), multiplied by p, is $\leq I_M$. Hence (7.31) yields

$$I_M \leq A^p \|f^{\#}\|_{L^p}^p + \frac{2C}{A} \frac{p}{p-1} 2^{(n+1)(2p-1)} \|f\|_{L^1} + \frac{2 \cdot 2^{(n+1)p}}{A} I_M. \tag{7.34}$$

This holds for each $A \geq 1$. Take $A = 4 \cdot 2^{(n+1)p}$ to obtain

$$\frac{1}{2} I_M \leq A^p \|f^{\#}\|_{L^p}^p + C_{n,p} \|f\|_{L^1}, \quad \forall\, M \geq 4^{n+1}. \tag{7.35}$$

From here on, $C_{n,p}$ denotes various (unevaluated) constants. Bringing in (7.25)–(7.30), we deduce

$$\begin{aligned} \|Mf\|_{L^p}^p &\leq C_{n,p} \|f\|_{L^1} + \lim_{M \to \infty} C_{n,p} I_M \\ &\leq C_{n,p} (\|f^{\#}\|_{L^p}^p + \|f\|_{L^1}), \end{aligned} \tag{7.36}$$

the last inequality by (7.35). This holds under the hypothesis (7.4). Hence the conclusion (7.5) is established, and Proposition 7.1 is proven.

The following is a straightforward corollary.

Corollary 7.2. *Given $p \in (1,\infty)$, if $f \in L^1(Q_1)$ and $f^\# \in L^p(Q_1)$, then $f \in L^p(Q_1)$, and*

$$\|f\|_{L^p(Q_1)} \le C_{n,p}\big(\|f^\#\|_{L^p(Q_1)} + \|f\|_{L^1(Q_1)}\big). \tag{7.37}$$

8. Global L^p estimates

We return to the setting of a complete Riemannian manifold M with bounded geometry, and as in (3.1)–(3.4) define the following operators:

(8.1)
$$\mathcal{N}f(x) = f^\#(x) + \mathcal{N}_0 f(x),$$
(8.2)
$$f^\#(x) = \sup_{B\in\mathcal{B}(x)} \frac{1}{V(B)} \int\limits_B |f - f_B|\, dV, \quad \mathcal{B}(x) = \{B_r(x) : 0 < r \le 1\},$$
(8.3)
$$\mathcal{N}_0 f(x) = \frac{1}{V(B_1(x))} \int\limits_{B_1(x)} |f|\, dV,$$

which in turn define the bmo-norm:

$$\|f\|_{\mathrm{bmo}} = \|\mathcal{N}f\|_{L^\infty}. \tag{8.4}$$

It is clear that

$$\mathcal{N}f(x) \le 3\mathcal{M}_1 f(x) = \sup_{0<r\le 1} \frac{3}{V(B_r(x))} \int\limits_{B_r(x)} |f|\, dV, \tag{8.5}$$

and the Hardy-Littlewood estimates

$$\|\mathcal{M}_1 f\|_{L^p(M)} \le A_p \|f\|_{L^p(M)}, \quad 1 < p < \infty, \tag{8.6}$$

are readily established when M has bounded geometry. Thus

$$\|f\|_{L^p(M)} \le C_p \|\mathcal{N}f\|_{L^p(M)} \tag{8.7}$$

for $p \in (1,\infty)$.

Our next goal is to establish the converse:

Proposition 8.1. *Assume* $1 < p < \infty$, $f \in L^1_{\mathrm{loc}}(M)$, *and* $\mathcal{N}f \in L^p(M)$. *Then* $f \in L^p(M)$, *and*

$$\|f\|_{L^p(M)} \le B_p \|\mathcal{N}f\|_{L^p(M)}. \tag{8.8}$$

In connection with this, we recall the following results. First, Theorem 5 of [FS] gives

$$\|f\|_{L^p(M)} \le B_p \|f^\#\|_{L^p(M)}, \quad 1 < p < \infty, \tag{8.9}$$

for $M = \mathbb{R}^n$, but when $f^\#$ is given, not by (8.2), but by (1.7)–(1.8). Second, Proposition 1 of [I] gives (8.9) for $M = \mathcal{H}^n$, hyperbolic space, and more generally when M is a rank one symmetric space of noncompact type. It is noted in [I] that (8.9) fails for $M = \mathbb{R}^n$ when $f^\#$ is defined by (8.2). For a counterexample, [I] mentions the family of characteristic functions of large balls in $\mathbb{R}^n$. Of course, such a family of functions does not furnish a counterexample to (8.8) in case $M = \mathbb{R}^n$.

In order to interface with the results of §7, we next obtain some estimates on $\mathcal{N}_0 f$. Note that $f \in L^1_{\mathrm{loc}}(M) \Rightarrow \mathcal{N}_0 f \in C(M)$.

Lemma 8.2. *There exist* $C = C(M) < \infty$ *and* $K = K(M) < \infty$ *with the following properties. For each* $x \in M$ *there are points* $y_1, \dots, y_K \in B_1(x)$ *such that*

$$\{B_{1/2}(y_j) : 1 \le j \le K\} \quad \text{covers} \quad B_1(x), \tag{8.10}$$

and for each $f \in L^1_{\mathrm{loc}}(M)$, *there exist* $y_j' \in B_{1/2}(y_j)$ *such that*

$$\mathcal{N}_0 f(y_j') \le C \|\mathcal{N}_0 f\|_{L^1(B_1(x))}. \tag{8.11}$$

Proof. The existence of K and y_j satisfying (8.10) follows from the bounded geometry conditions on M. One can furthermore arrange that

$$V\big(B_{1/2}(y_j) \cap B_1(x)\big) \ge c_0, \tag{8.12}$$

for some $c_0 = c_0(M) > 0$. Then Chebycheff's inequality guarantees the existence of $C < \infty$ (independent of f) and $y_j' \in B_{1/2}(y_j)$ (depending on f) such that (8.11) holds.

The definition (8.3) implies

$$\mathcal{N}_0 f(x) \le \sum_{j=1}^{K} \mathcal{N}_0 f(y_j'), \tag{8.13}$$

so we have:

Corollary 8.3. *In the setting of Lemma 8.2,*

$$V(B_1(x))^{-1}\|f\|_{L^1(B_1(x))} = \mathcal{N}_0 f(x) \le CK\|\mathcal{N}_0 f\|_{L^1(B_1(x))}. \tag{8.14}$$

For notational simplicity, let us replace CK by C, and furthermore denote by C various constants $C(M)$ in the estimates below.

Lemma 8.4. *Let $\{B_1(q_k) : k \in \mathbb{Z}^+\}$ be a tame cover of M, i.e., assume (1.25)–(1.26) hold. Then there exists $C = C(M) < \infty$ such that*

$$\sum_k \|f\|^p_{L^1(B_1(q_k))} \le C\|\mathcal{N}_0 f\|^p_{L^p(M)}. \tag{8.15}$$

Proof. Starting with (8.14), we have

$$\begin{aligned}\sum_k \|f\|^p_{L^1(B_1(q_k))} &\le C_1 \sum_k \|\mathcal{N}_0 f\|^p_{L^1(B_1(q_k))}\\ &\le C_2 \sum_k \|\mathcal{N}_0 f\|^p_{L^p(B_1(q_k))}\\ &\le C\|\mathcal{N}_0 f\|^p_{L^p(M)},\end{aligned} \tag{8.16}$$

the last inequality via (1.26).

In order to interface with §7, it also helps to recall from (3.7)–(3.10) that we can replace $f^{\#}$ in (8.2) by f^s, defined by

$$f^s(x) = \sup_{B\in\mathcal{B}(x)} \inf_{c_B\in\mathbb{C}} \frac{1}{V(B)} \int_B |f - c_B|\, dV, \tag{8.17}$$

since

$$f^s(x) \le f^{\#}(x) \le 2f^s(x). \tag{8.18}$$

Furthermore, as in (3.11)–(3.12), we can fix $a, b, c \in (0,\infty)$, with $a < b$, and replace $\mathcal{B}(x)$ by

$$\widetilde{\mathcal{B}}(x) = \{Q^\alpha_r(x) : 0 < r \le 1,\ \alpha \in \mathcal{A}\}, \tag{8.19}$$

where $Q^\alpha_r(x)$ is a family of measurable sets with the property that for each $r \in (0,1]$,

$$V(Q^\alpha_r(x)) \ge cV(B_r(x)),\quad Q^\alpha_r(x) \subset B_{br}(x),\quad \text{for all}\ \ \alpha, \tag{8.20}$$

and

$$B_{ar}(x) \subset Q^\alpha_r(x),\quad \text{for some}\ \ \alpha. \tag{8.21}$$

We can make the replacement first in (8.17), and then, by another application of (8.18), we can also make this replacement in (8.2). Furthermore, if we denote

$$f^\sigma(x) = \sup_{\widehat{\mathcal{B}}(x)} \frac{1}{V(B)} \int_B |f - f_B| \, dV, \tag{8.22}$$

where $\widehat{\mathcal{B}}(x)$ is a family of the form (8.15), satisfying (8.20), but not (8.21), we still have

$$f^\sigma(x) \le C f^\#(x). \tag{8.23}$$

This allows us to bring in the result of §7 as follows. Assume $f \in L^1_{\text{loc}}(M)$, $\mathcal{N} f \in L^p(M)$. Given $x \in M$, let Q be the cube in $T_x M$, of edge $1/\sqrt{n}$, centered at $0 \in T_x M$, identified with a subset of M via $\text{Exp}_x : T_x M \to M$. Then the function $(f|_Q)^\#$, with $^\#$ defined as in §7, has the form $f^\sigma|_Q$, for a certain class $\widehat{\mathcal{B}}(x)$ for which (8.23) holds, with $f^\#$ given by (8.2). Hence (dilating the cubes as needed) Corollary 7.2 (plus (8.23)) yields

$$\big\|f\big|_Q\big\|^p_{L^p(Q)} \le C\Big(\|f^\#\|^p_{L^p(Q)} + \|f\|^p_{L^1(Q)}\Big). \tag{8.24}$$

Summing over a family of such "cubes," tamely covering M, and taking (8.15) into account, we have the proof of (8.8).

9. An interpolation result

In this section we establish the following variant of Corollary 2 in [FS]. As usual, M is a complete Riemannian manifold with bounded geometry.

Proposition 9.1. *Given $p \in (1, \infty)$, assume*

$$T : L^p(M) \longrightarrow L^p(M), \quad \|Tf\|_{L^p} \le M_1 \|f\|_{L^p}. \tag{9.1}$$

Assume also that

$$T : L^\infty(M) \longrightarrow \text{bmo}(M), \quad \|Tf\|_{\text{bmo}} \le M_0 \|f\|_{L^\infty}. \tag{9.2}$$

Then, for $q \in (p, \infty)$, i.e., $q = p/\theta$, $\theta \in (0,1)$, we have

$$T : L^q(M) \longrightarrow L^q(M), \quad \|Tf\|_{L^q} \le C M_1^\theta M_0^{1-\theta} \|f\|_{L^q}. \tag{9.3}$$

Proof. Take $f \in L^q(M)$ and produce a holomorphic family f_z, for z in

$$\overline{\Omega} = \{z \in \mathbb{C} : 0 \le \text{Re}\, z \le 1\},$$

with values in $L^p(M) + L^\infty(M)$, such that

$$f_\theta = f, \quad \|f_z\|_{L^{q(z)}} \le C\|f\|_{L^q}, \quad q(z) = \frac{p}{\operatorname{Re} z}, \ q = \frac{p}{\theta}. \tag{9.4}$$

Thus we have L^∞ bounds on f_{it} and L^p bounds on f_{1+it}, $t \in \mathbb{R}$. For example, we can take $f_z = (f/|f|)|f|^{z/\theta}$. Now set

$$F_z = e^{z^2} T f_z. \tag{9.5}$$

To proceed, it is convenient to mollify F_z as follows. Pick $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi(t) = 1$ for $|t| \le 1/2$, 0 for $|t| \ge 1$, and set $\psi_\varepsilon(x) = \varphi(\varepsilon \operatorname{dist}^2(x, x_0))$ for some fixed $x_0 \in M$. Then set

$$G_z(x) = G_z^\varepsilon(x) = \psi_\varepsilon(x) e^{\varepsilon\Delta} F_z(x). \tag{9.6}$$

We drop the ε and denote the family of functions on M by G_z for notational simplicity. We will obtain estimates for G_z^ε that are independent of ε.

Now, taking a cue from [FS], we let $x \mapsto B(x)$ be a measurable assignment to each $x \in M$ of a ball $B(x) \in \mathcal{B}(x)$ (defined by (3.3)), we take

$$\eta \in L^\infty(M \times M), \quad |\eta(x,y)| \equiv 1, \tag{9.7}$$

and we set

$$G_z^{B,\eta}(x) = \frac{1}{V(B(x))} \int\limits_{B(x)} \big[G_z(y) - (G_z)_{B(x)}\big] \eta(x,y)\, dV(y). \tag{9.8}$$

Then

$$G_z^{\#}(x) = \sup_{B,\eta} |G_z^{B,\eta}(x)|, \tag{9.9}$$

the sup over B, η as described above, when $G_z^{\#}$ is defined as in (8.2). In addition, for η as in (9.7), set

$$N_0^\eta G_z(x) = \frac{1}{V(B_1(x))} \int\limits_{B_1(x)} G_z(y) \eta(x,y)\, dV(y). \tag{9.10}$$

We have

$$\mathcal{N}_0 G_z(x) = \sup_\eta |N_0^\eta G_z(x)|, \tag{9.11}$$

the sup being over η as in (9.7), with $\mathcal{N}_0$ defined as in (8.3).

Now we have the following estimates on $G_z^{B,\eta}$:

$$\|G_{it}^{B,\eta}\|_{L^\infty} \le \|G_{it}^{\#}\|_{L^\infty} \le \|G_{it}\|_{\mathrm{bmo}} \le CM_0\|f\|_{L^q}, \tag{9.12}$$

$$\|G_{1+it}^{B,\eta}\|_{L^p} \le \|G_{1+it}^{\#}\|_{L^p} \le C_p\|G_{1+it}\|_{L^p} \le CM_1\|f\|_{L^q}. \tag{9.13}$$

The second inequality in (9.12) follows from the definition of bmo, and the second inequality in (9.13) holds because of (8.5)–(8.6). From here, the standard interpolation inequalities for the L^q-interpolation scale yield

$$\|G_\theta^{B,\eta}\|_{L^q} \le CM_1^\theta M_0^{1-\theta}\|f\|_{L^q}, \tag{9.14}$$

with C independent of $x \mapsto B(x)$ and of η. Hence we have

$$\|G_\theta^{\#}\|_{L^q} \le CM_1^\theta M_0^{1-\theta}\|f\|_{L^q}. \tag{9.15}$$

Similarly we have the following estimates for $N_0^\eta G_z$:

$$\|N_0^\eta G_{it}\|_{L^\infty} \le \|\mathcal{N}_0 G_{it}\|_{L^\infty} \le \|G_{it}\|_{\mathrm{bmo}} \le CM_0\|f\|_{L^q}, \tag{9.16}$$

(9.17)

$$\|N_0^\eta G_{1+it}\|_{L^p} \le \|\mathcal{N}_0 G_{1+it}\|_{L^p} \le C\|G_{1+it}\|_{L^p} \le CM_1\|f\|_{L^q}.$$

Again standard interpolation gives

$$\|N_0^\eta G_\theta\|_{L^q} \le CM_1^\theta M_0^{1-\theta}\|f\|_{L^q}, \tag{9.18}$$

with C independent of the choice of η, hence

$$\|\mathcal{N}_0 G_\theta\|_{L^q} \le CM_1^\theta M_0^{1-\theta}\|f\|_{L^q}. \tag{9.19}$$

We are almost done with the proof of Proposition 9.1. Combining (9.15) and (9.19) yields

$$\|\mathcal{N} G_\theta\|_{L^q} \le CM_1^\theta M_0^{1-\theta}\|f\|_{L^q}, \tag{9.20}$$

and then Proposition 8.1 gives an estimate on $\|G_\theta\|_{L^q}$, yielding

$$\|\psi_\varepsilon e^{\varepsilon\Delta} Tf\|_{L^q} \le CM_1^\theta M_0^{1-\theta}\|f\|_{L^q}, \tag{9.21}$$

with C independent of $\varepsilon \in (0,1]$. Taking $\varepsilon \searrow 0$ then proves (9.3).

10. L^p, $\mathfrak{h}^1$, and bmo-Sobolev spaces

As usual, M is a Riemannian manifold with bounded geometry, satisfying (1.19)–(1.24). We want to define and study the spaces $H^{s,p}(M)$, $\mathfrak{h}^{s,1}(M)$, and $\mathfrak{h}^{s,\infty}(M)$ of functions (or distributions) with s derivatives in $L^p(M)$, $\mathfrak{h}^1(M)$, and $\mathrm{bmo}(M)$, respectively. Related results can be found in Chapter 7 of [Tri].

Here is one natural definition of these spaces when $s = k$ is a positive integer. Let $\mathcal{V}^1(M)$ denote the space of smooth vector fields X on M with the property that, in each exponential coordinate system $\mathrm{Exp}_q : T_qM \supset B_1(0) \to B_1(q)$, there is a uniform bound (independent of q) on the coefficients of X and, for each k, a uniform bound on all the derivatives of these coefficients of order $\le k$. Let $\mathcal{V}^k(M)$ denote the set of linear combinations of operators of the form $L = X_1 \cdots X_j$, with $X_\nu \in \mathcal{V}^1(M)$ and $j \le k$. Then we can define

$$H^{k,p}(M) = \{u \in L^p(M) : Lu \in L^p(M),\ \forall\, L \in \mathcal{V}^k(M)\}, \tag{10.1}$$

$$\mathfrak{h}^{k,1}(M) = \{u \in \mathfrak{h}^1(M) : Lu \in \mathfrak{h}^1(M),\ \forall\, L \in \mathcal{V}^k(M)\}, \tag{10.2}$$

$$\mathfrak{h}^{k,\infty}(M) = \{u \in \mathrm{bmo}(M) : Lu \in \mathrm{bmo}(M),\ \forall\, L \in \mathcal{V}^k(M)\}. \tag{10.3}$$

There are alternative characterizations of these spaces. For one, let $\{B_1(p_\ell) : \ell \in \mathbb{Z}^+\}$ be a tame cover of M and $\{\varphi_\ell : \ell \in \mathbb{Z}^+\}$ a tame partition of unity, as defined in (1.25)–(1.27). Given a function u on M, set

$$u_\ell = (\varphi_\ell u) \circ \mathrm{Exp}_{p_\ell}, \tag{10.4}$$

a function supported on $B_1(0) \subset T_{p_\ell}M$, which we can identify with $B_1(0) \subset \mathbb{R}^n$, uniquely up to the action of an element of $O(n)$. Then (given $p < \infty$)

$$u \in H^{k,p}(M) \Leftrightarrow \sum_\ell \sum_{|\alpha|\le k} \|D^\alpha u_\ell\|^p_{L^p(B_1(0))} < \infty, \tag{10.5}$$

$$u \in \mathfrak{h}^{k,1}(M) \Leftrightarrow \sum_\ell \sum_{|\alpha|\le k} \|D^\alpha u_\ell\|_{\mathfrak{h}^1(M)} < \infty, \tag{10.6}$$

$$u \in \mathfrak{h}^{k,\infty}(M) \Leftrightarrow \sup_\ell \sum_{|\alpha|\le k} \|D^\alpha u_\ell\|_{\mathrm{bmo}(M)} < \infty \tag{10.7}$$

Of the results just stated, given the definitions (10.1)–(10.3), the result (10.5) is straightforward, and (10.6)–(10.7) follow readily from the results of §§2–3, particularly Corollary 2.4 and Corollary 3.4.

We next define these Sobolev spaces for arbitrary index of regularity $s \in \mathbb{R}$, as

$$H^{s,p}(M) = (\lambda I - \Delta)^{-s/2} L^p(M), \tag{10.8}$$

$$\mathfrak{h}^{s,1}(M) = (\lambda I - \Delta)^{-s/2} \mathfrak{h}^1(M), \tag{10.9}$$

$$\mathfrak{h}^{s,\infty}(M) = (\lambda I - \Delta)^{-s/2}\, \mathrm{bmo}(M), \tag{10.10}$$

where we take λ as in Proposition 6.1, i.e., a sufficiently large positive number. More precisely, as in (6.18), take $\lambda > K_0^2$, where K_0 is as in (6.16). From here on, we work under the condition

$$1 < p < \infty. \tag{10.11}$$

Of course, we need to show that when $s = k$ is a positive integer, (10.1)–(10.3) are equivalent to (10.8)–(10.10). Before tackling this, we first need to show that the right sides of (10.8)–(10.10) are well defined. This will follow from results obtained in §6. To begin, we write

$$(\lambda I - \Delta)^{-s/2} = \Phi_{s,\lambda}(\sqrt{-\Delta}), \tag{10.12}$$

where

$$\Phi_{s,\lambda}(\zeta) = (\zeta^2 + \lambda)^{-s/2}. \tag{10.13}$$

With $\mathcal{S}_W^m$ defined as in (6.6)–(6.8), we have

$$\Phi_{s,\lambda} \in \mathcal{S}_W^{-s}, \quad \forall\, W < \sqrt{\lambda}. \tag{10.14}$$

Hence, by Proposition 6.6, given $\lambda > 0$,

$$(\lambda I - \Delta)^{-s/2} \in \Psi_W^{-s}(M), \quad \forall\, W < \sqrt{\lambda}. \tag{10.15}$$

We can now establish the following.

Proposition 10.1. *Given $\lambda > K_0^2$,*

$$(\lambda I - \Delta)^{-k/2} : L^p(M) \longrightarrow H^{k,p}(M), \tag{10.16}$$

$$(\lambda I - \Delta)^{-k/2} : \mathfrak{h}^1(M) \longrightarrow \mathfrak{h}^{k,1}(M), \tag{10.17}$$

$$(\lambda I - \Delta)^{-k/2} : \mathrm{bmo}(M) \longrightarrow \mathfrak{h}^{k,\infty}(M), \tag{10.18}$$

where the spaces on the right are defined by (10.1)–(10.3).

Proof. Note that $\mathcal{V}^k(M) \subset \Psi_{\#}^k(M)$. Hence, by (6.40)–(6.41),

$$L \in \mathcal{V}^k(M) \Longrightarrow L(\lambda I - \Delta)^{-k/2} \in \Psi_W^0(M), \quad \forall\, W < \sqrt{\lambda}. \tag{10.19}$$

As long as we can take $W \geq K_0$, we can apply Proposition 6.1 to conclude that such $L(\lambda I - \Delta)^{-k/2}$ is bounded on $L^p(M)$, $p \in (1, \infty)$, on $\mathfrak{h}^1(M)$, and on $\mathrm{bmo}(M)$, establishing (10.16)–(10.18).

At this point, we have the spaces defined on the right sides of (10.8)–(10.10) contained in the spaces defined in (10.1)–(10.3), when $s = k$ is a positive integer.

To proceed, it will be convenient to know that

$$(\lambda I-\Delta)^{-r/2}(\lambda I-\Delta)^{-s/2}f=(\lambda I-\Delta)^{-(r+s)/2}f,\quad \forall\, r,s\in\mathbb{R}, \tag{10.20}$$

whenever $f\in L^p(M),\ 1<p<\infty$, or $f\in\mathfrak{h}^1(M)$, or $f\in\mathrm{bmo}(M)$. The result (10.20) for $f\in L^2(M)$ is a well known consequence of Hilbert space spectral theory. In that case, the self-duality of $L^2(M)$ extends to produce the duality

$$\Big((\lambda I-\Delta)^{-s/2}L^2(M)\Big)'=(\lambda I-\Delta)^{s/2}L^2(M),\quad \forall\, s\in\mathbb{R}. \tag{10.21}$$

Now, given that $(\lambda I-\Delta)^{-k/2}L^2(M)$ is contained in $H^{k,2}(M)$ as defined by (10.1), or by (10.5), we have

$$(\lambda I-\Delta)^{-k/2}L^2(M)\subset L^\infty(M),\quad \forall\, k>\frac{n}{2}, \tag{10.22}$$

and hence, by duality,

$$L^1(M)\subset(\lambda I-\Delta)^{k/2}L^2(M),\quad \forall\, k>\frac{n}{2}, \tag{10.23}$$

from which it follows that whenever $k>n/2$,

$$\begin{aligned} &L^p(M)\subset(\lambda I-\Delta)^{k/2}L^2(M),\quad \forall\, p\in(1,2],\\ &\mathfrak{h}^1(M)\subset(\lambda I-\Delta)^{k/2}L^2(M). \end{aligned} \tag{10.24}$$

We can now prove:

Lemma 10.2. *The identity (10.20) holds for all $f\in L^p(M),\ 1<p<\infty$, for all $f\in\mathfrak{h}^1(M)$, and for all $f\in\mathrm{bmo}(M)$.*

Proof. We have seen that (10.20) holds for all $f\in L^2(M)$. The result (10.24) implies (10.20) holds on $\mathfrak{h}^1(M)$ and on $L^p(M)$ for $p\in(1,2]$. The facts that (10.20) holds on $\mathrm{bmo}(M)$ and on $L^p(M)$ for $p\in(2,\infty)$ follow by duality.

We are now prepared to prove:

Proposition 10.3. *If $s=k$ is a positive integer, the spaces defined by (10.1)–(10.3) coincide with those defined by (10.8)–(10.10) (assuming $p\in(1,\infty)$).*

Proof. We have one set of inclusions. For the converse, assume u has the property

$$u,\ X_1\cdots X_j u\in\mathfrak{X},\quad \forall\, j\le k,\ X_\nu\in\mathcal{V}^1(M), \tag{10.25}$$

where either $\mathfrak{X}=L^p(M),\ 1<p<\infty$, or $\mathfrak{X}=\mathfrak{h}^1(M)$, or $\mathfrak{X}=\mathrm{bmo}(M)$. We claim

$$f=(\lambda I-\Delta)^{k/2}u\in\mathfrak{X}. \tag{10.26}$$

If so, then, by (10.20),

$$u = (\lambda I - \Delta)^{-k/2} f, \tag{10.27}$$

and we are done.

The result (10.26) is elementary if $k = 2j$ is an *even* integer. Then $(\lambda I - \Delta)^j$ is a differential operator, and it is a finite linear combination of operators of the form appearing in (10.25). Now suppose $k = 2j + 1$. The same argument shows that

$$v = (\lambda I - \Delta)^j u \tag{10.28}$$

has the property

$$v,\ Xv \in \mathfrak{X}, \quad \forall\, X \in \mathcal{V}^1(M). \tag{10.29}$$

If we can show that for such v,

$$(\lambda I - \Delta)^{1/2} v \in \mathfrak{X}, \tag{10.30}$$

we will be done. To get this, write

$$(\lambda I - \Delta)^{1/2} v = \Phi^{\#}_{-1,\lambda}(\sqrt{-\Delta}) v + \Phi^{b}_{-1,\lambda}(\sqrt{-\Delta}) v, \tag{10.31}$$

with $\Phi^{\#}_{-1,\lambda}(\sqrt{-\Delta}) \in \Psi^1_{\#}(M)$ and $\Phi^{b}_{-1,\lambda}(\sqrt{-\Delta}) \in \Psi^{-\infty}_W(M)$, for all $W < \sqrt{\lambda}$. Estimates in (6.33) give

$$\Phi^{b}_{-1,\lambda}(\sqrt{-\Delta}) : \mathfrak{X} \longrightarrow \mathfrak{X}, \tag{10.32}$$

for such spaces $\mathfrak{X}$. It remains to show that

$$\Phi^{\#}_{-1,\lambda}(\sqrt{-\Delta}) v \in \mathfrak{X}, \tag{10.33}$$

whenever (10.29) holds. Indeed, since $P = \Phi^{\#}_{-1,\lambda}(\sqrt{-\Delta}) \in \Psi^1_{\#}(M)$, as defined in (6.23), pseudodifferential operator calculus allows us to write

$$P = Q_0 + \sum_{j=1}^{N} Q_j X_j, \tag{10.34}$$

with

$$X_1, \dots, X_N \in \mathcal{V}^1(M), \quad Q_j \in \Psi^0_{\#}(M). \tag{10.35}$$

Hence, if v satisfies (10.29),

$$Pv = Q_0 v + \sum_{j=1}^{N} Q_j (X_j v) \in \mathfrak{X}, \tag{10.36}$$

for $\mathfrak{X} = \mathfrak{h}^1(M),\ \text{bmo}(M)$, or $L^p(M),\ 1 < p < \infty$, by Proposition 6.5.

Having identified the spaces (10.1)–(10.3) with their counterparts in (10.8)–(10.10) when $s = k \in \mathbb{N}$, we next show that for general $s \in \mathbb{R}$, the spaces (10.8)–(10.10) are independent of the choice of λ, as long as $\lambda > K_0^2$.

Proposition 10.4. *Let* $\mathfrak{X} = L^p(M),\ p \in (1,\infty)$*, or* $\mathfrak{X} = \mathfrak{h}^1(M)$*, or* $\mathfrak{X} = \mathrm{bmo}(M)$*. Then, for each* $s \in \mathbb{R}$*,*

$$\mu, \lambda > K_0^2 \Longrightarrow (\lambda I - \Delta)^{-s/2}\mathfrak{X} = (\mu I - \Delta)^{-s/2}\mathfrak{X}. \tag{10.37}$$

Proof. Note that

$$(\mu I - \Delta)^{s/2}(\lambda I - \Delta)^{-s/2} = \psi_{s,\mu,\lambda}(\sqrt{-\Delta}), \quad \psi_{s,\mu\lambda}(\zeta) = \Big(\frac{\mu + \zeta^2}{\lambda + \zeta^2}\Big)^{s/2}, \tag{10.38}$$

and $\psi_{s,\mu,\lambda} \in \mathcal{S}^0_W$ for all $W < \min(\mu, \lambda)$. Hence

$$(\mu I - \Delta)^{s/2}(\lambda I - \Delta)^{-s/2} : \mathfrak{X} \longrightarrow \mathfrak{X}, \tag{10.39}$$

by Propositions 6.6–6.7, with inverse $\psi_{s,\lambda,\mu}(\sqrt{-\Delta})$, so (10.39) is an isomorphism for each such $\mathfrak{X}$. This gives (10.37).

We next record how elements of $\Psi^m_W(M)$ act on these Sobolev spaces.

Proposition 10.5. *Take* $m, s \in \mathbb{R}$ *and assume* $W \ge K_0$*. Then*

$$\begin{aligned} P \in \Psi^m_W(M) \Rightarrow \ \ & P : H^{s,p}(M) \to H^{s-m,p}(M), \quad \forall\, p \in (1,\infty), \\ & P : \mathfrak{h}^{s,1}(M) \to \mathfrak{h}^{s-m,1}(M), \\ & P : \mathfrak{h}^{s,\infty}(M) \to \mathfrak{h}^{s-m,\infty}(M). \end{aligned} \tag{10.40}$$

Proof. The results in (10.40) are equivalent to the existence of $\lambda > K_0^2$ such that

$$Q = (\lambda I - \Delta)^{(s-m)/2} P (\lambda I - \Delta)^{-s/2} \tag{10.41}$$

has the mapping properties

$$\begin{aligned} Q &: L^p(M) \to L^p(M), \quad p \in (1,\infty), \\ Q &: \mathfrak{h}^1(M) \to \mathfrak{h}^1(M), \\ Q &: \mathrm{bmo}(M) \to \mathrm{bmo}(M). \end{aligned} \tag{10.42}$$

To get this, take $\lambda > (W + K_0)^2$, so $(\lambda I - \Delta)^{\sigma/2} \in \Psi^\sigma_{W+K_0}(M)$. An application of Proposition 6.9 gives $P(\lambda I - \Delta)^{-s/2} \in \Psi^{m-s}_W(M)$, and a second application gives $Q \in \Psi^0_W(M)$. Then the mapping properties in (10.42) follow from Proposition 6.7.

11. Interpolations of type (p, p')

Our goal here is to establish the following interpolation result, of potential use for dispersive estimates in PDE.

Proposition 11.1. *Take $s \in \mathbb{R}$. Assume we have a bounded operator*

$$R : L^2(M) \longrightarrow L^2(M), \quad R : L^1(M) \longrightarrow \mathfrak{h}^{s,\infty}(M), \tag{11.1}$$

satisfying

$$\|Rf\|_{L^2} \le M_1 \|f\|_{L^2}, \quad \|Rf\|_{\mathfrak{h}^{s,\infty}} \le M_0 \|f\|_{L^1}. \tag{11.2}$$

Then, for $\theta \in (0,1)$,

$$R : L^{p(\theta)}(M) \longrightarrow H^{(1-\theta)s,p(\theta)'}(M), \quad p(\theta) = \frac{2}{2-\theta}, \ p(\theta)' = \frac{2}{\theta}, \tag{11.3}$$

and (with $C_\theta \in (0,\infty)$ independent of R and of f),

$$\|Rf\|_{H^{(1-\theta)s,p(\theta)'}} \le C_\theta M_1^\theta M_0^{1-\theta} \|f\|_{L^{p(\theta)}}. \tag{11.4}$$

The proof combines methods of §9 and Sobolev results of §10. To start, set

$$\Omega = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}. \tag{11.5}$$

Given $\theta \in (0,1)$, $f \in L^{2/(2-\theta)}(M)$, take $f_z(x)$ holomorphic in $z \in \Omega$, bounded and continuous on $\overline{\Omega}$ with vaules in $L^1(M) + L^2(M)$, satisfying

$$f_\theta = f, \quad \|f_z\|_{L^{p(z)}} \le C\|f\|_{L^{p(\theta)}}, \quad p(z) = \frac{2}{2 - \operatorname{Re} z}. \tag{11.6}$$

Note that $f_{it} \in L^1(M)$ and $f_{1+it} \in L^2(M)$, for $t \in \mathbb{R}$. Now take $\lambda > K_0^2$ and set

$$F_z(x) = e^{z^2} (\lambda I - \Delta)^{s(1-z)/2} R f_z(x). \tag{11.7}$$

The hypotheses (11.1)–(11.2), plus results of §6 and §10, give

$$F_{it} \in \operatorname{bmo}(M), \quad F_{1+it} \in L^2(M), \tag{11.8}$$

and

$$\|F_{it}\|_{\operatorname{bmo}} \le C M_0 \|f\|_{L^{p(\theta)}}, \quad \|F_{1+it}\|_{L^2} \le C M_1 \|f\|_{L^{p(\theta)}}. \tag{11.9}$$

To proceed, it is convenient to mollify F_z as follows (parallel to the construction leading to (9.6)). Pick $\psi_\varepsilon(x) = \varphi(\varepsilon\, d(x,x_0)^2)$ as in (9.6), and set

$$G_z(x) = G_z^\varepsilon(x) = \psi_\varepsilon(x) e^{\varepsilon \Delta} F_z(x), \tag{11.10}$$

with $F_z(x)$ as in (11.7). We often drop the ε and denote the family of functions on M by G_z for notational simplicity; we will obtain estimates for G_z^ε that will be independent of ε.

With G_z defined as above, take η as in (9.7), let $x \mapsto B(x)$ be a measurable assignment, with $B(x) \in \mathcal{B}(x)$, and define $G_z^{B,\eta}$ as in (9.8). As in (9.9), we have

$$G_z^{\#}(x) = \sup_{B,\eta} |G_z^{B,\eta}(x)|. \tag{11.11}$$

Next define $N_0^\eta G_z$ as in (9.10), so, as in (9.11),

$$\mathcal{N}_0 G_z(x) = \sup_{\eta} |N_0^\eta G_z(x)|. \tag{11.12}$$

In the current setting, we have the following variants of the estimates (9.12)–(9.13),

$$\begin{aligned} \|G_{it}^{B,\eta}\|_{L^\infty} &\le \|G_{it}^{\#}\|_{L^\infty} \le \|G_{it}\|_{\mathrm{bmo}} \le CM_0 \|f\|_{L^{p(\theta)}}, \\ \|G_{1+it}^{B,\eta}\|_{L^2} &\le \|G_{1+it}^{\#}\|_{L^2} \le C\|G_{1+it}\|_{L^2} \le CM_1 \|f\|_{L^{p(\theta)}}, \end{aligned} \tag{11.13}$$

for the same reasons used to justify (9.12)–(9.13). Then standard interpolation gives

$$\|G_\theta^{B,\eta}\|_{L^{p(\theta)'}} \le CM_1^\theta M_0^{1-\theta} \|f\|_{L^{p(\theta)}}, \tag{11.14}$$

parallel to (9.14), with C independent of $x \mapsto B(x)$ and of η. Hence

$$\|G_\theta^{\#}\|_{L^{p(\theta)'}} \le CM_1^\theta M_0^{1-\theta} \|f\|_{L^{p(\theta)}}. \tag{11.15}$$

We also have the following variants of (9.16)–(9.17):

$$\begin{aligned} \|N_0^\eta G_{it}\|_{L^\infty} &\le \|\mathcal{N}_0 G_{it}\|_{L^\infty} \le \|G_{it}\|_{\mathrm{bmo}} \le CM_0 \|f\|_{L^{p(\theta)}}, \\ \|N_0^\eta G_{1+it}\|_{L^2} &\le \|\mathcal{N}_0 G_{1+it}\|_{L^2} \le C\|G_{1+it}\|_{L^2} \le CM_1 \|f\|_{L^{p(\theta)}}. \end{aligned} \tag{11.16}$$

Again standard interpolation gives

$$\|N_0^\eta G_\theta\|_{L^{p(\theta)'}} \le CM_1^\theta M_0^{1-\theta} \|f\|_{L^{p(\theta)}}, \tag{11.17}$$

with C independent of the choice of η, hence

$$\|\mathcal{N}_0 G_\theta\|_{L^{p(\theta)'}} \le CM_1^\theta M_0^{1-\theta} \|f\|_{L^{p(\theta)}}. \tag{11.18}$$

Combining (11.15) and (11.18) yields

$$\|\mathcal{N} G_\theta\|_{L^{p(\theta)'}} \le CM_1^\theta M_0^{1-\theta} \|f\|_{L^{p(\theta)}}. \tag{11.19}$$

Then Proposition 8.1 gives an estimate on $\|G_\theta\|_{L^{p(\theta)}}$, yielding

$$\|\psi_\varepsilon e^{\varepsilon\Delta}(\lambda I-\Delta)^{s(1-\theta)/2}Rf\|_{L^{p(\theta)'}} \le CM_1^\theta M_0^{1-\theta}\|f\|_{L^{p(\theta)}}, \tag{11.20}$$

with C independent of $\varepsilon \in (0,1]$. Taking $\varepsilon \searrow 0$ gives

$$(\lambda I-\Delta)^{s(1-\theta)/2}Rf \in L^{p(\theta)'}(M), \tag{11.21}$$

with norm estimates, implying (11.3) and (11.4).

We illustrate Proposition 11.1 by considering estimates on

$$S(t)=\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}. \tag{11.22}$$

In general, if there are no pairs of conjugate points in M (and under some additional technical hypotheses),

$$\begin{aligned} S(t) &: L^2(M)\longrightarrow H^{1,2}(M),\\ S(t) &: L^1(M)\longrightarrow \mathfrak{h}^{-(n-1)/2,\infty}(M), \end{aligned} \tag{11.23}$$

where $N=\dim M$. The first mapping property holds by spectral theory and the second by the parametrix construction for solutions to the wave equation; it is this second part that requires the absence of conjugate points. Let us assume we have estimates of the form

$$\begin{aligned} \|S(t)f\|_{H^{1,2}} &\le A_1(t)\|f\|_{L^2},\\ \|S(t)f\|_{\mathfrak{h}^{-(n-1)/2,\infty}} &\le A_0(t)\|f\|_{L^1}. \end{aligned} \tag{11.24}$$

It is well known that

$$\begin{aligned} M=\mathbb{R}^n \Longrightarrow A_1(t) &= a(1+|t|),\\ A_0(t) &= b_n|t|^{-(n-1)/2}. \end{aligned} \tag{11.25}$$

This estimate for $A_1(t)$ is universally valid, by the spectral theorem. On the other hand, for some manifolds with bounded geometry, one can do better. Namely, the following might apply:

$$\operatorname{Spec}(-\Delta)\subset[B^2,\infty),\ B>0 \Longrightarrow A_1(t)=a_B. \tag{11.26}$$

While the estimate on $A_0(t)$ in (11.25) is typically sharp for $|t|\le 1$, for general M with bounded geometry (and, say, with sectional curvature ≤ 0), sometimes there can be faster decay as $|t|\to\infty$.

To apply Proposition 11.1, it is convenient to pick $\lambda > K_0^2$ and consider

$$R(t)=(\lambda I-\Delta)^{1/2}S(t). \tag{11.27}$$

Then (11.23)–(11.24) yield

$$R(t):L^2(M)\longrightarrow L^2(M),\quad R(t):L^1(M)\longrightarrow \mathfrak{h}^{-(n+1)/2,\infty}(M), \tag{11.28}$$

and

$$\|R(t)f\|_{L^2}\le CA_1(t)\|f\|_{L^2},\quad \|R(t)f\|_{\mathfrak{h}^{-(n+1)/2,\infty}}\le CA_0(t)\|f\|_{L^1}. \tag{11.29}$$

Then we can apply Proposition 11.1 to get, for $\theta\in(0,1)$,

$$R(t):L^p(M)\longrightarrow H^{-(1-\theta)(n+1)/2,p'}(M),\quad p=\frac{2}{2-\theta},\ p'=\frac{2}{\theta}, \tag{11.30}$$

with operator norm bounded by $C_\theta A_1(t)^\theta A_0(t)^{1-\theta}$. Returning to $S(t)$, we have

$$S(t):L^p(M)\longrightarrow H^{1-(1-\theta)(n+1)/2,p'}(M),\quad p=\frac{2}{2-\theta},\ p'=\frac{2}{\theta}, \tag{11.31}$$

with

$$\|S(t)f\|_{H^{1-(1-\theta)(n+1)/2,p'}}\le C_\theta A_1(t)^\theta A_0(t)^{1-\theta}\|f\|_{L^p}. \tag{11.32}$$

For example, we can pick θ so that

$$1-(1-\theta)\frac{n+1}{2}=0,\quad \text{i.e.,}\ \ \theta=\frac{n-1}{n+1}, \tag{11.33}$$

and obtain

$$\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}:L^p(M)\longrightarrow L^{p'}(M),\quad p=2\frac{n+1}{n+3},\ p'=2\frac{n+1}{n-1}, \tag{11.34}$$

with

$$\Big\|\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}f\Big\|_{L^{p'}}\le CA_1(t)^{(n-1)/(n+1)}A_0(t)^{2/(n+1)}\|f\|_{L^p}. \tag{11.35}$$

In case $M=\mathbb{R}^n$, we obtain (with p,p' as in (11.34)) the well known result

$$\Big\|\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}f\Big\|_{L^{p'}}\le Ct^{-(n-1)/(n+1)}\|f\|_{L^p}, \tag{11.36}$$

at least for $|t|\le 1$, but then a simple scaling argument gives (11.36) for all t. For non-euclidean manifolds M, scaling is not available. In many cases one gets an estimate of the form (11.36) valid for $|t| \le 1$, and in some cases one might get a *better* estimate for $|t| \ge 1$.

In case M is hyperbolic space $\mathcal{H}^n$, results stronger than those obtained in (11.35) via (11.23) are possible, along the following lines. In such a case, (11.26) holds with $B = (n-1)/2$. One can set

$$L = \Delta - \Bigl(\frac{n-1}{2}\Bigr)^2, \tag{11.37}$$

write

$$\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} = S_0(t) + S_1(t), \quad S_0(t) = \frac{\sin t\sqrt{-L}}{\sqrt{-L}}, \tag{11.38}$$

and estimate $\|S_0(t)f\|_{H^{1-s,p'}}$, $s = (1-\theta)(n+1)/2$, via arguments leading to (11.32) (obtaining stronger estimates), while applying other techniques to estimate $\|S_1(t)f\|_{H^{1-s,p'}}$. Work on this will be taken up elsewhere.

A. The space vmo(M)

Given a Riemannian manifold M with bounded geometry, we define vmo(M) to be the closure in bmo(M) of the space $C_*(M)$ of continuous functions vanishing at infinity. This is parallel to the characterization of VMO($\mathbb{R}^n$), introduced in [Sar], as the closure in BMO($\mathbb{R}^n$) of $C_*(\mathbb{R}^n)$. There are equivalent characterizations of vmo(M), e.g., the closure in bmo(M) of $C_0^\infty(M)$. We set

$$\begin{aligned} \|f\|_{\mathrm{vmo}} = \|f\|_{\mathrm{bmo}} \quad &\text{if} \quad f \in \mathrm{vmo}(M), \\ \infty \quad &\text{if} \quad f \notin \mathrm{vmo}(M). \end{aligned} \tag{A.1}$$

It readily follows from Proposition 3,1 that

$$f \in \mathrm{vmo}(M) \Longrightarrow af \in \mathrm{vmo}(M) \tag{A.2}$$

as long as $a \in L^\infty(M) \cap \mathrm{Lip}(M)$, or more generally $a \in L^\infty(M) \cap C^\sigma(M)$, with σ given by (3.23). From here, an argument parallel to the proof of Proposition 3.2 gives:

Proposition A.1. *Let $\{\varphi_k : k \in \mathbb{Z}^+\}$ be a tame partition of unity. Given $f \in L^1_{\mathrm{loc}}(M)$, we have*

$$f \in \mathrm{vmo}(M) \Longleftrightarrow \sup_k \|\varphi_k f\|_{\mathrm{vmo}} < \infty, \quad \textit{and} \quad \lim_{k\to\infty} \|\varphi_k f\|_{\mathrm{vmo}} = 0. \tag{A.3}$$

In [Sar] it was proven that

$$\mathrm{VMO}(\mathbb{R}^n)' = H^1(\mathbb{R}^n). \tag{A.4}$$

Our next goal is to prove the following analogue:

Proposition A.2. *If M is a Riemannian manifold with bounded geometry,*

$$\text{vmo}(M)' = \mathfrak{h}^1(M). \tag{A.5}$$

The proof of Proposition 4.1 yields a natural map $\mathcal{I} : \mathfrak{h}^1(M) \to \text{vmo}(M)'$, which is clearly one-to-one. To show that $\mathcal{I}$ is surjective, we will construct the inverse $\mathcal{J} : \text{vmo}(M)' \to \mathfrak{h}^1(M)$.

Before tackling this, we note that Proposition 4.1 implies there exists $C_0 = C_0(M)$ such that

$$C_0^{-1} \|f\|_{\mathfrak{h}^1} \le \sup\{\langle f, g\rangle : g \in \text{bmo}(M),\ \|g\|_{\text{bmo}} \le 1\} \le C_0 \|f\|_{\mathfrak{h}^1}. \tag{A.6}$$

Part of the content of (A.5) is that

$$C_0^{-1} \|f\|_{\mathfrak{h}^1} \le \sup\{\langle f, g\rangle : g \in \text{vmo}(M),\ \|g\|_{\text{vmo}} \le 1\} \le C_0 \|f\|_{\mathfrak{h}^1}. \tag{A.7}$$

Of course the second inequality in (A.7) follows from its counterpart in (A.6). Our task is to prove the first inequality in (A.7). We start with the case

$$\mathbb{T}^n = \mathbb{R}^n / 2\pi \mathbb{Z}^n. \tag{A.8}$$

Lemma A.3. *The result (A.7) holds when $M = \mathbb{T}^n$.*

Proof. As mentioned, we need only establish the first inequality in (A.7). Fix $f \in \mathfrak{h}^1(M)$. Given $\delta > 0$, pick $g_\delta \in \text{bmo}(\mathbb{T}^n)$ such that

$$\|g_\delta\|_{\text{bmo}} \le 1, \quad \|f\|_{\mathfrak{h}^1} \le C_0(1+\delta)\langle f, g_\delta\rangle. \tag{A.9}$$

Now

$$\begin{aligned} \langle f, e^{\varepsilon\Delta} g_\delta\rangle &= \langle e^{\varepsilon\Delta} f, g_\delta\rangle \\ &= \langle f, g_\delta\rangle - \langle f - e^{\varepsilon\Delta} f, g_\delta\rangle, \end{aligned} \tag{A.10}$$

and, since $C^\infty(\mathbb{T}^n)$ is dense in $\mathfrak{h}^1(\mathbb{T}^n)$,

$$|\langle f - e^{\varepsilon\Delta} f, g_\delta\rangle| \le C \|f - e^{\varepsilon\Delta} f\|_{\mathfrak{h}^1} \|g_\delta\|_{\text{bmo}} \le \eta(f, \varepsilon), \tag{A.11}$$

where $\eta(f,\varepsilon) \to 0$ as $\varepsilon \searrow 0$. Hence

$$\langle f, e^{\varepsilon\Delta} g_\delta\rangle \ge \frac{1}{C_0(1+\delta)} \|f\|_{\mathfrak{h}^1} - \eta(f, \varepsilon). \tag{A.12}$$

Now $e^{\varepsilon\Delta} g_\delta \in C^\infty(\mathbb{T}^n) \subset \text{vmo}(\mathbb{T}^n)$ for each $\varepsilon > 0$, and $\|e^{\varepsilon\Delta} g_\delta\|_{\text{vmo}} \le 1$. Taking positive ε and δ arbitrarily small yields (A.7), for $M = \mathbb{T}^n$.

We now prove Proposition A.2 in case $M = \mathbb{T}^n$. Thus, let

$$\omega : \mathrm{vmo}(\mathbb{T}^n) \longrightarrow \mathbb{R} \tag{A.13}$$

be a continuous linear functional. Also set, for $\varepsilon > 0$,

$$\omega_\varepsilon(f) = \omega(e^{\varepsilon\Delta} f). \tag{A.14}$$

Clearly we have a unique $g_\varepsilon \in C^\infty(\mathbb{T}^n)$ such that

$$\omega_\varepsilon(f) = \langle f, g_\varepsilon \rangle = \int\limits_{\mathbb{T}^n} f g_\varepsilon \, dx. \tag{A.15}$$

By Lemma A.3, we have

$$\|g_\varepsilon\|_{\mathfrak{h}^1} \le C_0 \|\omega_\varepsilon\| \le C_0 \|\omega\| < \infty. \tag{A.16}$$

Now we have (perhaps passing to a subsequence $\varepsilon = \varepsilon_j \searrow 0$)

$$g_\varepsilon \longrightarrow g \ \text{ weak}^* \text{ in } \ \mathcal{M}(\mathbb{T}^n), \tag{A.17}$$

where $\mathcal{M}(\mathbb{T}^n)$ denotes the space of finite Borel measures on $\mathbb{T}^n$, and

$$\omega(f) = \langle f, g \rangle, \quad \forall\, f \in C(\mathbb{T}^n). \tag{A.18}$$

To conclude the proof of Proposition A.2 for $M = \mathbb{T}^n$, it remains to show that

$$g \in \mathfrak{h}^1(\mathbb{T}^n). \tag{A.19}$$

To see this, note that

$$\mathcal{G}^\rho g(x) \nearrow \mathcal{G}^b g(x) \ \text{ as } \ \rho \searrow 0, \tag{A.20}$$

where, for $\rho \in (0,1)$,

$$\begin{aligned} \mathcal{G}^\rho g(x) &= \sup_{r \in [\rho,1]} \mathcal{G}^r g(x) \\ &= \sup \Big\{ |\langle \varphi, g \rangle| : \varphi \in \bigcup_{r \in [\rho,1]} \mathcal{F}(B_r(x)) \Big\}, \end{aligned} \tag{A.21}$$

with $\mathcal{F}(B_r(x))$ as in (2.3). Since $\cup_{x \in \mathbb{T}^n} \cup_{r \in [\rho,1]} \mathcal{F}(B_r(x))$ is a relatively compact subset of $C(\mathbb{T}^n)$ for each $\rho \in (0,1)$, we have

$$\mathcal{G}^\rho g_\varepsilon(x) \to \mathcal{G}^\rho g(x) \ \text{ as } \ \varepsilon \to 0, \tag{A.22}$$

uniformly in x, for each $\rho > 0$. Hence, by (A.16),

$$\|\mathcal{G}^\rho g\|_{L^1} \le C_0 \|\omega\|. \tag{A.23}$$

We then deduce from (A.20) that

$$\mathcal{G}^b g \in L^1(\mathbb{T}^n). \tag{A.24}$$

This implies the desired result (A.19).

We proceed with the proof of Proposition A.2 for general M with bounded geometry. To set things up, bring in a tame partition of unity $\{\varphi_k : k \in \mathbb{Z}^+\}$, $\varphi_k \in C_0^\infty(B_1(p_k))$, as defined in (1.25)–(1.27). As in the proof of Proposition 2.2, partition $\mathbb{Z}^+$ into $K_1 = K_1(M)$ sets $\mathcal{S}_1, \dots, \mathcal{S}_{K_1}$ such that $j, k \in \mathcal{S}_\nu, j \neq k \Rightarrow d(p_j, p_k) \ge 20$, and for each $\nu \in \{1, \dots, K_1\}$, set

$$T_\nu f = \sum_{k \in \mathcal{S}_\nu} \varphi_k f. \tag{A.25}$$

We have

$$T_\nu : \mathrm{vmo}(M) \longrightarrow \mathrm{vmo}(M), \quad \|T_\nu f\|_{\mathrm{vmo}} \le C \|f\|_{\mathrm{vmo}}, \ 1 \le \nu \le K_1. \tag{A.26}$$

Of course, $f = \sum_\nu T_\nu f$.

Now let $\omega : \mathrm{vmo}(M) \to \mathbb{R}$ be a continuous linear functional. We want to define $\mathcal{J}\omega \in \mathfrak{h}^1(M)$. To do this it suffices to define $\mathcal{J}\omega_\nu$ for each $\nu \in \{1, \dots, K_1\}$, where $\omega_\nu = \omega \circ T_\nu$. Also define $\omega_k : \mathrm{vmo}(M) \to \mathbb{R}$ by

$$\omega_k(f) = \omega(\varphi_k f). \tag{A.27}$$

Using the identification of $B_2(p_k)$ with $B_2(0) \subset \mathbb{R}^n$ via Exp_{p_k}, and then identifying $B_2(0) \subset \mathbb{R}^n$ with $B_2(0) \subset \mathbb{T}^n$, we can use the special case just proven to write

$$\omega_k(f) = \langle f, g_k \rangle, \tag{A.28}$$

where $g_k \in \mathfrak{h}^1(\mathbb{T}^n)$ has support in $B_2(0)$. Multiplying by the volume form, we can identify g_k with an element of $\mathfrak{h}^1(M)$, supported in $B_2(p_k)$:

$$g_k \in \mathfrak{h}^1(M), \quad \operatorname{supp} g_k \subset B_2(p_k). \tag{A.29}$$

We next claim that

$$\omega_\nu(f) = \langle f, g_\nu \rangle, \tag{A.30}$$

with

$$g_\nu = \sum_{k \in \mathcal{S}_\nu} g_k \in \mathfrak{h}^1(M). \tag{A.31}$$

Note that the terms in this sum have widely disjoint supports, and if $\widetilde{\mathcal{S}}_\nu \subset \mathcal{S}_\nu$ is any finite subset, then

$$\tilde{g}_\nu = \sum_{\widetilde{\mathcal{S}}_\nu} g_k \Rightarrow \sum_{k\in\widetilde{\mathcal{S}}_\nu} \omega(\varphi_k f) = \langle f, \tilde{g}_\nu\rangle, \quad \|\tilde{g}_\nu\|_{\mathfrak{h}^1} = \sum_{k\in\widetilde{\mathcal{S}}_\nu} \|g_k\|_{\mathfrak{h}^1}. \tag{A.32}$$

Using Lemma A.3, we can produce $f_k \in \mathrm{vmo}(M)$ such that

$$\operatorname{supp} f_k \subset B_2(p_k), \quad \|f_k\|_{\mathrm{vmo}} \le C_1, \quad \langle f_k, g_k\rangle \ge \|g_k\|_{\mathfrak{h}^1}, \tag{A.33}$$

with $C_1 = C_1(M) < \infty$. Then, given any finite subset $\widetilde{\mathcal{S}}_\nu \subset \mathcal{S}_\nu$,

$$\tilde{f}_\nu = \sum_{k\in\widetilde{\mathcal{S}}_\nu} f_k \Longrightarrow \|\tilde{f}_\nu\|_{\mathrm{vmo}} \le C_1 \tag{A.34}$$

and

$$\begin{aligned} \sum_{k\in\widetilde{\mathcal{S}}_\nu} \|g_k\|_{\mathfrak{h}^1} &\le \sum_{k\in\widetilde{\mathcal{S}}_\nu} \langle f_k, g_k\rangle \\ &= \langle \tilde{f}_\nu, \tilde{g}_\nu\rangle \\ &= \omega_\nu(\tilde{f}_\nu) \le C_2 < \infty, \end{aligned} \tag{A.35}$$

the last inequality by (A.34). This implies

$$g_\nu = \sum_{k\in\mathcal{S}_\nu} g_k \in \mathfrak{h}^1(M), \tag{A.36}$$

and we have $g_\nu = \mathcal{J}\omega_\nu$, finishing the proof of Proposition A.2.

We next examine the action of pseudodifferential operators on $\mathrm{vmo}(M)$. The following result complements Proposition 6.7.

Proposition A.4. *Given K_0 as in (6.16),*

$$W > K_0, \ P \in \Psi^0_W(M) \Longrightarrow P : \mathrm{vmo}(M) \to \mathrm{vmo}(M). \tag{A.37}$$

Proof. As in §6, write $P = P^\# + P^b$, with $P^\# \in \Psi^0_\#(M)$, $P^b \in \Psi^{-\infty}_W(M)$. One readily verifies the following:

$$P^\# : C^\infty_0(M) \longrightarrow C^\infty_0(M), \quad P^b : C^\infty_0(M) \longrightarrow C_*(M), \tag{A.38}$$

so

$$P : C^\infty_0(M) \longrightarrow C_*(M). \tag{A.39}$$

The result (A.37) follows from this together with the boundedness on bmo(M) given in (6.37).

B. The operator class $\widetilde{\Psi}^m_W(M)$

Given $m \in \mathbb{R},\ W > 0$, we define an operator class $\widetilde{\Psi}^m_W(M)$, smaller than $\Psi^m_W(M)$, which was defined in §6, and discuss some properties. Parallel to (6.27)–(6.31), we set

$$\widetilde{\Psi}^m_W(M) = \{P^\# + P^b : P^\# \in \Psi^m_\#(M),\ P^b \in \widetilde{\Psi}^{-\infty}_W(M)\}, \tag{B.1}$$

where $\Psi^m_\#(M)$ is as in (6.23)–(6.25), and we say $P^b \in \widetilde{\Psi}^{-\infty}_W(M)$ provided it has the form

$$P^b f(x) = \int\limits_M k^b(x,y) f(y)\, dV(y), \tag{B.2}$$

where $k^b \in C^\infty(M \times M)$ satisfies, for each $x, y \in M,\ r \in (0,\infty),\ \ell \in \mathbb{Z}^+$,

$$\begin{aligned} \|k^b(\cdot,y)\|_{L^2(M\setminus B_r(y))} &\le C_\ell \langle r\rangle^{-\ell} e^{-Wr}, \\ \|k^b(x,\cdot)\|_{L^2(M\setminus B_r(x))} &\le C_\ell \langle r\rangle^{-\ell} e^{-Wr}, \end{aligned} \tag{B.3}$$

with similar estimates on all x and y-derivatives of $k^b(x,y)$.

It is easy to check that $\widetilde{\Psi}^{-\infty}_W(M) \subset \Psi^{-\infty}_W(M)$, and hence $\widetilde{\Psi}^m_W(M) \subset \Psi^m_W(M)$. The following result improves Proposition 6.6.

Proposition B.1. *For $W > 0,\ m \in \mathbb{R}$,*

$$\Phi \in \mathcal{S}^m_W \Longrightarrow \Phi(\sqrt{-\Delta}) \in \widetilde{\Psi}^m_W(M). \tag{B.4}$$

More generally, if

$$\operatorname{Spec}(-\Delta) \subset [B^2, \infty) \tag{B.5}$$

and $L = \Delta + B^2$, then

$$\Phi \in \mathcal{S}^m_W \Longrightarrow \Phi(\sqrt{-L}) \in \widetilde{\Psi}^m_W(M). \tag{B.6}$$

The proof of this is given in (1.8)–(1.13) of [T3].

In light of this, the following result has stronger consequences for L^p estimates on $\Phi(\sqrt{-\Delta})$ than Proposition 6.7 does. Recall the volume estimate (6.16):

$$\operatorname{Vol}(B_r(p)) \le C_0 (1+r)^{\mu_0} e^{K_0 r}, \quad \forall\, p \in M,\ r \in (0,\infty). \tag{B.7}$$

Proposition B.2. *If $W \geq K_0/2$, then*

$$P^b \in \widetilde{\Psi}_W^{-\infty}(M) \Longrightarrow P : L^p(M) \to L^p(M), \quad \forall\, p \in [1,\infty]. \tag{B.8}$$

Hence

$$\begin{aligned} P \in \widetilde{\Psi}^0_W(M) \Longrightarrow P : L^p(M) \to L^p(M), \quad \forall\, p \in (1,\infty), \\ P : \mathfrak{h}^1(M) \to L^1(M), \\ P : L^\infty(M) \to \mathrm{bmo}(M). \end{aligned} \tag{B.9}$$

Proof. It suffices to prove (B.8), since Proposition 6.5 then gives (B.9). If $k^b(x,y)$ is the integral kernel of P^b, to prove (B.8) it suffices to show that

$$\begin{aligned} \sup_y \int\limits_M |k^b(x,y)|\, dV(x) < \infty, \quad \text{and} \\ \sup_x \int\limits_M |k^b(x,y)|\, dV(y) < \infty. \end{aligned} \tag{B.10}$$

We estimate the first integral in (B.10) by dividing M into shells

$$A_j(y) = \{x \in M : j \leq d(x,y) \leq j+1\}. \tag{B.11}$$

We have the following estimate:

$$\begin{aligned} \int\limits_M |k^b(x,y)|\, dV(x) &= \sum_{j\geq 0} \int\limits_{A_j(y)} |k^b(x,y)|\, dV(x) \\ &\leq \sum_{j\geq 0} \big(\mathrm{Vol}\, A_j(y)\big)^{1/2} \|k^b(\cdot,y)\|_{L^2(A_j(y))} \\ &\leq C \sum_{j\geq 0} \langle j\rangle^{\mu_0/2} e^{jK_0/2} \|k^b(\cdot,y)\|_{L^2(A_j(y))}. \end{aligned} \tag{B.12}$$

Bringing in (B.3), we have

$$\|k^b(\cdot,y)\|_{L^2(A_j(y))} \leq C_\ell \langle j\rangle^{-\ell} e^{-jW}, \tag{B.13}$$

and taking $\ell > \mu_0/2 + 1$ yields the first bound in (B.10), as long as $W \geq K_0/2$. The second bound in (B.10) is proven similarly.

Propositions B.1–B.2 yield the following improvement over Proposition 6.7. This result can be compared with Theorem 10.2 of [CMM].

Corollary B.3. *If the volume estimate (B.7) holds, and if (B.5) holds and $L = \Delta + B^2$, then*

$$\begin{aligned}\Phi \in \mathcal{S}^0_W,\ W \geq \frac{K_0}{2} \Longrightarrow \Phi(\sqrt{-L}) &: L^p(M) \to L^p(M), \quad p \in (1, \infty),\\ \Phi(\sqrt{-L}) &: \mathfrak{h}^1(M) \to L^1(M),\\ \Phi(\sqrt{-L}) &: L^\infty(M) \to \mathrm{bmo}(M).\end{aligned} \tag{B.14}$$

Regarding L^p-estimates, (B.14) plus an application of the Stein interpolation theorem yields the following (Theorem 1.6 of [T3]):

Proposition B.4. *If $\Phi \in \mathcal{S}^0_W$, then*

$$\Phi(\sqrt{-L}) : L^p(M) \longrightarrow L^p(M), \tag{B.15}$$

provided

$$p \in (1, \infty), \quad \textit{and} \ \ W \geq \left|\frac{1}{p} - \frac{1}{2}\right| \cdot K_0.$$

We finish with the following improvement of Proposition B.2 and Corollary B.3.

Proposition B.5. *If $W \geq K_0/2$, then*

$$\begin{aligned}P^b : \widetilde{\Psi}^{-\infty}_W(M) \Longrightarrow P^b &: \mathrm{bmo}(M) \to L^\infty(M),\\ P^b &: L^1(M) \to \mathfrak{h}^1(M).\end{aligned} \tag{B.16}$$

Hence

$$\begin{aligned}P : \widetilde{\Psi}^0_W(M) \Longrightarrow P &: \mathrm{bmo}(M) \to \mathrm{bmo}(M),\\ P &: \mathfrak{h}^1(M) \to \mathfrak{h}^1(M).\end{aligned} \tag{B.17}$$

Consequently, in the setting of Corollary B.3,

$$\begin{aligned}\Phi \in \mathcal{S}^0_W \Longrightarrow \Phi(\sqrt{-L}) &: \mathrm{bmo}(M) \to \mathrm{bmo}(M),\\ \Phi(\sqrt{-L}) &: \mathfrak{h}^1(M) \to \mathfrak{h}^1(M).\end{aligned} \tag{B.18}$$

Proof. We prove the first part of (B.16). This readily yields

$$P^b : \mathrm{vmo}(M) \longrightarrow C_*(M), \tag{B.19}$$

the latter space consisting of continuous functions on M vanishing at infinity, and the second part follows by duality. From here, (B.17) and (B.18) follow by the same arguments as used above.

To proceed, take $f \in \mathrm{bmo}(M)$. Pick $\lambda > K_0^2$, and write

$$P^b f = P^b(\lambda I - \Delta)\big((\lambda I - \Delta)^{-1} f\big). \tag{B.20}$$

By Proposition 10.1 and the characterization (10.7) of $\mathfrak{h}^{2,\infty}(M)$, we have

$$f \in \mathrm{bmo}(M) \Longrightarrow (\lambda I - \Delta)^{-1} f \in L^\infty(M). \tag{B.21}$$

On the other hand, since the integral kernel of $P^b(\lambda I - \Delta)$ is $(\lambda I - \Delta_y)k^b(x,y)$, it is clear from the definition that

$$P^b \in \widetilde{\Psi}_W^{-\infty}(M) \Longrightarrow P^b(\lambda I - \Delta) \in \widetilde{\Psi}_W^{-\infty}(M). \tag{B.22}$$

Thus Proposition B.2 gives

$$P^b(\lambda I - \Delta) : L^\infty(M) \longrightarrow L^\infty(M), \tag{B.23}$$

and the proof is done.

C. Further results for symmetric spaces of noncompact type

A symmetric space of noncompact type is a Riemannian manifold $M = G/K$, where G is a semisimple Lie group of noncompact type and K a maximal compact subgroup. Examples include hyperbolic space $\mathcal{H}^n$, with constant sectional curvature -1, amongst others. (However, this definition excludes Euclidean space.) We refer to [Hel] for basic material; basic results are also summarized in §2 of [T3]. Without going into details, we mention the following key fact: there exists a positive quantity, denoted $|\rho|^2$, with the property that

$$\mathrm{Spec}\,(-\Delta) = [|\rho|^2, \infty) \quad \text{on} \quad L^2(M) \tag{C.1}$$

and

$$\mathrm{Vol}\, B_r(p) \sim C r^\beta e^{2|\rho| r}, \quad r \to \infty, \tag{C.2}$$

for some $\beta \in (0,\infty)$. Cf. [T3], (2.2) and (2.9). When $M = \mathcal{H}^n$, $|\rho| = (n-1)/2$. Now, if we set

$$L = \Delta + |\rho|^2, \tag{C.3}$$

so $\mathrm{Spec}\,(-L) = [0,\infty)$ on $L^2(M)$, we can apply Proposition B.4 to deduce that, for $p \in (1,\infty)$,

$$\Phi \in \mathcal{S}_W^0,\ W > \Big|\frac{2}{p} - 1\Big| \cdot |\rho| \Longrightarrow \Phi(\sqrt{-L}) : L^p(M) \to L^p(M). \tag{C.4}$$

Using this, we can establish the following variant of the fact that

$$(\lambda I - \Delta)^{m/2} : H^{s,p}(M) \longrightarrow H^{s-m,p}(M), \tag{C.5}$$

for $s, m \in \mathbb{R},\ p \in (1,\infty)$, given $\lambda > 0$ sufficiently large, which was proven in §10, in the setting of general manifolds with bounded geometry.

Proposition C.1. *If M is a symmetric space of noncompact type, then for $s,m\in\mathbb{R},\ p\in(1,\infty)$,*

$$(-\Delta)^{m/2}:H^{s,p}(M)\longrightarrow H^{s-m,p}(M). \tag{C.6}$$

Remark. This fails when $M=\mathbb{R}^n$.

Proof. In light of the results of §10, (C.6) is equivalent to the assertion that, for $\lambda>0$ sufficiently large,

$$(\lambda I-\Delta)^{(s-m)/2}(-\Delta)^{m/2}(\lambda I-\Delta)^{-s/2}:L^p(M)\longrightarrow L^p(M). \tag{C.7}$$

We can write this operator as

$$(\lambda I+|\rho|^2-L)^{(s-m)/2}(|\rho|^2-L)^{m/2}(\lambda I+|\rho|^2-L)^{-s/2}=\Phi(\sqrt{-L}), \tag{C.8}$$

where $\Phi(\zeta)=(\lambda+|\rho|^2+\zeta^2)^{-m/2}(|\rho|^2+\zeta^2)^{m/2}$, and we see that

$$\Phi\in\mathcal{S}^0_W,\quad \forall\, W<|\rho|. \tag{C.9}$$

Now for each $p\in(1,\infty),\ |2/p-1|<1$, so (C.7) follows from (C.4).

Proposition C.1 interfaces with results of §11 as follows. As stated there, for

$$\theta\in(0,1),\quad p=\frac{2}{2-\theta},\quad p'=\frac{2}{\theta},\quad s=(1-\theta)\frac{n+1}{2}, \tag{C.10}$$

we obtain in various circumstances estimates of the form

$$\Big\|\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}\Big\|_{H^{1-s,p'}}\le\psi_\theta(t)\|f\|_{L^p}, \tag{C.11}$$

known as dispersive estimates. Similar estimates yield

$$\|\cos t\sqrt{-\Delta}\,f\|_{H^{-s,p'}}\le A_1\psi_\theta(t)\|f\|_{L^p}. \tag{C.12}$$

Obtaining such estimates, e.g., for $M=\mathcal{H}^n$ involves, amongst other things, rather explicit formulas for (distributional) integral kernels of these operators. Such explicit formulas are lacking for the opeators $\sin t\sqrt{-\Delta}$; however applying Proposition C.1 to (C.11) gives

$$\|\sin t\sqrt{-\Delta}\,f\|_{H^{-s,p'}}\le A_2\psi_\theta(t)\|f\|_{L^p}, \tag{C.13}$$

and putting (C.12)–(C.13) together gives for $e^{it\sqrt{-\Delta}} = \cos t\sqrt{-\Delta} + i \sin t\sqrt{-\Delta}$ the estimate

$$\|e^{it\sqrt{-\Delta}} f\|_{H^{-s,p'}} \le (A_1 + A_2)\psi_\theta(t)\|f\|_{L^p}, \tag{C.14}$$

in case M is a symmetric space of noncompact type. Such an estimate is a convenient variant of (C.11) for the purpose of passing from dispersive estimates to Strichartz estimates. This matter will be pursued elsewhere.

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