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Function Space and Operator Theory for Nonlinear Analysis

Introduction

This chapter examines a number of analytical techiques, which will be applied to diverse nonlinear problems in the remaining chapters. For example, we study Sobolev spaces based on L^p, rather than just L^2. Sections 1 and 2 discuss the definition of Sobolev spaces $H^{k,p}$, for $k \in \mathbb{Z}^+$, and inclusions of the form $H^{k,p} \subset L^q$. Estimates based on such inclusions have refined forms, due to E. Gagliardo and L. Nirenberg. We discuss these in §3, together with results of J. Moser on estimates on nonlinear functions of an element of a Sobolev space, and on commutators of differential operators and multiplication operators. In §4 we establish some integral estimates of N. Trudinger, on functions in Sobolev spaces for which L^∞-bounds just fail. In these sections we use such basic tools as Hölder's inequality and integration by parts.

The Fourier transform is not as effective for analysis on L^p as on L^2. One result that does often serve when, in the L^2-theory, one could appeal to the Plancherel theorem, is Mikhlin's Fourier multiplier theorem, established in §5. This enables interpolation theory to be applied to the study of the spaces $H^{s,p}$, for noninteger s, in §6. In §7 we apply some of this material to the study of L^p-spectral theory of the Laplace operator, on compact manifolds, possibly with boundary.

In §8 we study spaces C^r of Hölder continuous functions, and their relation with Zygmund spaces C^r_*. We derive estimates in these spaces for solutions to elliptic boundary problems.

The next two sections extend results on pseudodifferential operators, introduced in Chapter 7. Section 9 considers symbols $p(x, \xi)$ with minimal regularity in x. We derive both L^p- and Hölder estimates. Section 10 considers paradifferential operators, a variant of pseudodifferential operator calculus particularly well suited to nonlinear analysis. Sections 9 and 10 are largely taken from [T2].

In §11 we consider "fuzzy functions," consisting of a pair (f, λ), where f is a function on a space Ω and λ is a measure on $\Omega \times \mathbb{R}$, with the property that $\iint y\varphi(x)\, d\lambda(x, y) = \int \varphi(x)f(x)\, dx$. The measure λ is known as a Young measure. It incorporates information on how f may have arisen as a weak limit of smooth ("sharply defined") functions, and it is useful for analyses of nonlinear maps that do not generally preserve weak convergence.

In §12 there is a brief discussion of Hardy spaces, subspaces of $L^1(\mathbb{R}^n)$ with many desirable properties, only a few of which are discussed here. Much more on this topic can be found in [S3], but material covered here will be useful for some elliptic regularity results in §12B of Chapter 14.

We end this chapter with Appendix A, discussing variants of the complex interpolation method introduced in Chapter 4 and used a lot in the early sections of this chapter. It turns out that slightly different complex interpolation functors are better suited to the scale of Zygmund spaces.

1. L^p-Sobolev spaces

Let $p \in [1, \infty)$. In analogy with the definition of the Sobolev spaces in Chapter 4, we set, for $k = 0, 1, 2, \dots,$

$$H^{k,p}(\mathbb{R}^n) = \big\{u \in L^p(\mathbb{R}^n) : D^\alpha u \in L^p(\mathbb{R}^n) \text{ for } |\alpha| \le k\big\}. \tag{1.1}$$

It is easy to see that $\mathcal{S}(\mathbb{R}^n)$ is dense in each space $H^{k,p}(\mathbb{R}^n)$, with its natural norm

$$\|u\|_{H^{k,p}} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p}. \tag{1.2}$$

For $p \neq 2$, we cannot characterize the spaces $H^{k,p}(\mathbb{R}^n)$ conveniently in terms of the Fourier transform. It is still possible to define spaces $H^{s,p}(\mathbb{R}^n)$ by interpolation; we will examine this in §6. Here we will consider only the spaces $H^{k,p}(\mathbb{R}^n)$ with k a nonnegative integer.

The chain rule allows us to say that if $\chi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism that is linear outside a compact set, then $\chi^* : H^{k,p}(\mathbb{R}^n) \to H^{k,p}(\mathbb{R}^n)$. Also multiplication by an element $\varphi \in C_0^\infty(\mathbb{R}^n)$ maps $H^{k,p}(\mathbb{R}^n)$ to itself. This allows us to define $H^{k,p}(M)$ for a compact manifold M via a partition of unity subordinate to a coordinate chart. Also, for compact M, if we define $\mathrm{Diff}^k(M)$ to be the set of differential operators of order $\le k$ on M, with smooth coefficients, then

$$H^{k,p}(M) = \{u \in L^p(M) : Pu \in L^p(M) \text{ for all } P \in \mathrm{Diff}^k(M)\}. \tag{1.3}$$

We can define $H^{k,p}(\mathbb{R}^n_+)$ as in (1.1), with $\mathbb{R}^n$ replaced by $\mathbb{R}^n_+$. The extension operator defined by (4.2)–(4.4) of Chapter 4 also works to produce extension maps $E : H^{k,p}(\mathbb{R}^n_+) \to H^{k,p}(\mathbb{R}^n)$. Similarly, if M is a compact manifold with smooth boundary, with double N, we can define $H^{k,p}(M)$

via coordinate charts and the notion of $H^{k,p}(\mathbb{R}^n_+)$, or by (1.3), and we have extension operators $E : H^{k,p}(M) \to H^{k,p}(N)$.

We also note the obvious fact that

$$D^\alpha : H^{k,p}(\mathbb{R}^n) \longrightarrow H^{k-|\alpha|,p}(\mathbb{R}^n), \tag{1.4}$$

for $|\alpha| \le k$, and

$$P : H^{k,p}(M) \longrightarrow H^{k-\ell,p}(M) \text{ if } P \in \text{Diff}^\ell(M), \tag{1.5}$$

provided $\ell \le k$.

Exercises

1. A Friedrichs mollifier on $\mathbb{R}^n$ is a family of smoothing operators $J_\varepsilon u(x) = j_\varepsilon * u(x)$ where

$$j_\varepsilon(x) = \varepsilon^{-n} j(\varepsilon^{-1}x), \quad \int j(x)dx = 1, \quad j \in \mathcal{S}(\mathbb{R}^n).$$

Equivalently, $J_\varepsilon u(x) = \varphi(\varepsilon D)u(x),\ \varphi \in \mathcal{S}(\mathbb{R}^n),\ \varphi(0) = 1$. Show that, for each $p \in [1, \infty),\ k \in \mathbb{Z}^+$,

$$J_\varepsilon : H^{k,p}(\mathbb{R}^n) \longrightarrow \bigcap_{\ell<\infty} H^{\ell,p}(\mathbb{R}^n),$$

for each $\varepsilon > 0$, and

$$J_\varepsilon u \to u \quad \text{in } H^{k,p}(\mathbb{R}^n)$$

as $\varepsilon \to 0$ if $u \in H^{k,p}(\mathbb{R}^n)$.

2. Suppose $A \in C^1(\mathbb{R}^n)$, with $\|A\|_{C^1} = \sup_{|\alpha|\le 1} \|D^\alpha A\|_{L^\infty}$. Show that when J_ε is a Friedrichs mollifier as above, then

$$\|[A, J_\varepsilon]v\|_{H^{1,p}} \le C\|A\|_{C^1}\|v\|_{L^p},$$

with C independent of $\varepsilon \in (0, 1]$. (*Hint*: Write $A(x) - A(y) = \sum B_k(x, y)(x_k - y_k),\ |B_k(x, y)| \le K$, and, with $q_\ell(x) = \partial j/\partial x_\ell$,

$$\partial_\ell[A, J_\varepsilon]v(x) = \sum \int B_k(x, y)\Big[\varepsilon^{-n} q_\ell\Big(\frac{x-y}{\varepsilon}\Big) \cdot \frac{x_k - y_k}{\varepsilon}\Big] v(y)\ dy,$$

with absolute value bounded by

$$K\ \varepsilon^{-n} \sum \int \Big|\varphi_{k\ell}\Big(\varepsilon^{-1}(x - y)\Big)\Big| \cdot |v(y)|\ dy,$$

where $\varphi_{k\ell}(x) = x_k q_\ell(x)$.)

3. Using Exercise 2, show that

$$\|[A, J_\varepsilon]\partial_j v\|_{L^p} \le C\|A\|_{C^1}\|v\|_{L^p}.$$

2. Sobolev imbedding theorems

We will derive various inclusions of the type $H^{k,p}(M) \subset H^{\ell,q}(M)$. We will concentrate on the case $M = \mathbb{R}^n$. The discussion of §1 will give associated results when M is a compact manifold, possibly with (smooth) boundary.

One technical tool useful for our estimates is the following generalized Hölder inequality:

Lemma 2.1. *If $p_j \in [1, \infty]$, $\sum p_j^{-1} = 1$, then*

$$\int_M |u_1 \cdots u_m| \, dx \le \|u_1\|_{L^{p_1}(M)} \cdots \|u_m\|_{L^{p_m}(M)}. \tag{2.1}$$

The proof follows by induction from the case $m = 2$, which is the usual Hölder inequality.

Our first Sobolev imbedding theorem is the following:

Proposition 2.2. *For $p \in [1, n)$,*

$$H^{1,p}(\mathbb{R}^n) \subset L^{np/(n-p)}(\mathbb{R}^n). \tag{2.2}$$

In fact, there is an estimate

$$\|u\|_{L^{np/(n-p)}} \le C \|\nabla u\|_{L^p}, \tag{2.3}$$

for $u \in H^{1,p}(\mathbb{R}^n)$, with $C = C(p, n)$.

Proof. It suffices to establish (2.3) for $u \in C_0^\infty(\mathbb{R}^n)$. Clearly,

$$|u(x)| \le \int_{-\infty}^{\infty} |D_j u| \, dx_j, \tag{2.4}$$

so

$$|u(x)|^{n/(n-1)} \le \Big\{ \prod_{j=1}^n \int_{-\infty}^{\infty} |D_j u| \, dx_j \Big\}^{1/(n-1)}. \tag{2.5}$$

We can integrate (2.5) successively over each variable x_j, $j = 1, \dots, n$, and apply the generalized Hölder inequality (2.1) with $m = p_1 = \cdots = p_m = n - 1$ after each integration. We get

$$\|u\|_{L^{n/(n-1)}} \le \Big\{ \prod_{j=1}^n \int_{\mathbb{R}^n} |D_j u| \, dx \Big\}^{1/n} \le C \|\nabla u\|_{L^1}. \tag{2.6}$$

This establishes (2.3) in the case $p = 1$. We can apply this to $v = |u|^\gamma$, $\gamma > 1$, obtaining

$$\big\| |u|^\gamma \big\|_{L^{n/(n-1)}} \le C \big\| |u|^{\gamma-1} |\nabla u| \big\|_{L^1} \le C \big\| |u|^{\gamma-1} \big\|_{L^{p'}} \big\| \nabla u \big\|_{L^p}. \tag{2.7}$$

For $p < n$, pick $\gamma = (n-1)p/(n-p)$. Then (2.7) gives (2.3) and the proposition is proved.

Given $u \in H^{k,p}(\mathbb{R}^n)$, we can apply Proposition 2.2 to estimate the $L^{np/(n-p)}$-norm of $D^{k-1}u$ in terms of $\|D^k u\|_{L^p}$, where we use the notation

$$D^k u = \{D^\alpha u : |\alpha| = k\}, \quad \|D^k u\|_{L^p} = \sum_{|\alpha|=k} \|D^\alpha u\|_{L^p}, \tag{2.8}$$

and proceed inductively, obtaining the following corollary.

Proposition 2.3. *For $kp < n$,*

$$H^{k,p}(\mathbb{R}^n) \subset L^{np/(n-kp)}(\mathbb{R}^n). \tag{2.9}$$

The same result holds with $\mathbb{R}^n$ replaced by a compact manifold of dimension n. If we take $p = 2$, then for the Sobolev spaces $H^k(\mathbb{R}^n) = H^{k,2}(\mathbb{R}^n)$, we have

$$H^k(\mathbb{R}^n) \subset L^{2n/(n-2k)}(\mathbb{R}^n), \quad k < \frac{n}{2}. \tag{2.10}$$

Consequently, the interpolation theory developed in Chapter 4 implies

$$H^s(\mathbb{R}^n) \subset L^{2n/(n-2s)}(\mathbb{R}^n), \tag{2.11}$$

for any real $s \in [0, k]$, $k < n/2$ an integer. Actually, (2.11) holds for any real $s \in [0, n/2)$, as will be shown in §6. We write down some particular examples, for $n = 2, 3, 4$, which will play a role later in various nonlinear evolution equations, such as the Navier-Stokes equations. The cases $n = 3, 4$ follow from the results proved above, while the case $n = 2$ follows from the general case of (2.11) established in §6.

(2.12)

$$\begin{array}{ccc} & H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3) & H^1(\mathbb{R}^4) \subset L^4(\mathbb{R}^4) \\ & H^{3/4}(\mathbb{R}^3) \subset L^4(\mathbb{R}^3) & \\ H^{1/2}(\mathbb{R}^2) \subset L^4(\mathbb{R}^2) & H^{1/2}(\mathbb{R}^3) \subset L^3(\mathbb{R}^3) & \end{array}$$

Note that interpolation of the $\mathbb{R}^2$-result with $L^2(\mathbb{R}^2) = L^2(\mathbb{R}^2)$ yields

$$H^{1/3}(\mathbb{R}^2) \subset L^3(\mathbb{R}^2).$$

The next result provides a partial generalization of the Sobolev imbedding theorem,

$$H^s(\mathbb{R}^n) \subset C(\mathbb{R}^n), \quad s > \frac{n}{2},$$

proved in Chapter 4. A more complete generalization is given in §6.

Proposition 2.4. *We have*

$$H^{k,p}(\mathbb{R}^n) \subset C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \quad \textit{for } kp > n. \tag{2.13}$$

Proof. It suffices to obtain a bound on $\|u\|_{L^\infty(\mathbb{R}^n)}$ for $u \in H^{k,p}(\mathbb{R}^n)$, if $kp > n$. In turn, it suffices to bound $u(0)$ appropriately, for $u \in C_0^\infty(\mathbb{R}^n)$. Use polar coordinates, $x = r\omega$, $\omega \in S^{n-1}$. Let $g \in C^\infty(\mathbb{R})$ have the property that $g(r) = 1$ for $r < 1/2$ and $g(r) = 0$ for $r > 3/4$. Then, for each ω, we have

$$\begin{aligned} u(0) &= -\int_0^1 \frac{\partial}{\partial r}\big[g(r)u(r,\omega)\big]\,dr \\ &= \frac{(-1)^k}{(k-1)!}\int_0^1 r^{k-n}\Big\{\Big(\frac{\partial}{\partial r}\Big)^k \big[g(r)u(r,\omega)\big]\Big\}\, r^{n-1}dr, \end{aligned}$$

upon integrating by parts $k-1$ times. Integrating over $\omega \in S^{n-1}$ gives

$$|u(0)| \le C \int_B r^{k-n}\Big|\Big(\frac{\partial}{\partial r}\Big)^k \big[g(r)u(x)\big]\Big|\,dx,$$

where B is the unit ball centered at 0. Hölder's inequality gives

$$|u(0)| \le C\|r^{k-n}\|_{L^{p'}(B)}\big\|\partial_r^k[g(r)u(x)]\big\|_{L^p(B)}, \tag{2.14}$$

with $1/p + 1/p' = 1$. We claim that $(\partial/\partial r)^k$ is a linear combination of D^α, $|\alpha| = k$, with L^∞-coefficients. To see this, note that ∂_r^k annihilates x^α for $|\alpha| < k$, so we get

$$\Big(\frac{\partial}{\partial r}\Big)^k = \sum_{|\alpha|=k} a_\alpha(x)\partial^\alpha, \tag{2.15}$$

with $a_\alpha(x) = (1/\alpha!)\partial_r^k x^\alpha$, for $|\alpha| = k$, or

$$a_\alpha(r\omega) = \frac{k!}{\alpha!}\,\omega^\alpha,$$

so $a_\alpha(x)$ is homogeneous of degree 0 in x and smooth on $\mathbb{R}^n \setminus 0$.

Returning to the estimate of (2.14), our information on $(\partial/\partial r)^k$ implies that the last factor on the right side is bounded by the $H^{k,p}$-norm of u. The factor $\|r^{k-n}\|_{L^{p'}(B)}$ is finite provided $kp > n$, so the proposition is proved.

To close this section, we note the following simple consequence of Proposition 2.2, of occasional use in analysis. Let $\mathcal{M}(\mathbb{R}^n)$ denote the space of locally finite Borel measures (not necessarily positive) on $\mathbb{R}^n$. Let us assume that $n \ge 2$.

Proposition 2.5. *If we have $u \in \mathcal{M}(\mathbb{R}^n)$ and $\nabla u \in \mathcal{M}(\mathbb{R}^n)$, then it follows that $u \in L^{n/(n-1)}_{\mathrm{loc}}(\mathbb{R}^n)$.*

Proof. Using a cut-off in C_0^∞, we can assume u has compact support. Applying a mollifier, we get $u_j = \chi_j * u \in C_0^\infty(\mathbb{R}^n)$ such that $u_j \to u$ and

$\nabla u_j \to \nabla u$ in $\mathcal{M}(\mathbb{R}^n)$. In particular, we have a uniform L^1-norm estimate on ∇u_j. By (2.3) we have a uniform $L^{n/(n-1)}$-norm estimate on u_j, which gives the result, since $L^{n/(n-1)}(\mathbb{R}^n)$ is reflexive.

Exercises

1. If $p_j \in [1,\infty]$ and $u_j \in L^{p_j}$, show that $u_1 u_2 \in L^r$ provided $r^{-1} = {p_1}^{-1} + {p_2}^{-1} \in [0,1]$. Show that this implies Lemma 2.1.
2. Use the containment (which follows from Proposition 2.2)
$$H^{k,p}(\mathbb{R}^n) \subset H^{1,np/(n-(k-1)p)}(\mathbb{R}^n) \quad \text{if } (k-1)p < n$$
to show that if Proposition 2.4 is proved in the case $k=1$, then it follows in general. Note that the proof in the text of Proposition 2.4 is slightly simpler in the case $k=1$ than for $k \geq 2$.
3. Suppose $k = 2\ell$ is even. Suppose $u \in \mathcal{S}'(\mathbb{R}^n)$ and
$$(-\Delta + 1)^\ell u = f \in L^p(\mathbb{R}^n).$$
Show that
$$u = \mathcal{J}_k * f, \quad \widehat{\mathcal{J}}_k(\xi) = \langle \xi \rangle^{-k}.$$
Using estimates on $\mathcal{J}_k(x)$ established in Chapter 3, §8, show that
$$kp > n \Longrightarrow u \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$
Show that this gives an alternative proof of Proposition 2.4 in case k is even.
4. Suppose $k = 2\ell + 1$ is odd, $kp > 1$. Use the containment
$$H^{k,p}(\mathbb{R}^n) \subset H^{k-1,np/(n-p)}(\mathbb{R}^n) \quad \text{if } p < n,$$
which follows from Proposition 2.2, to deduce from Exercise 3 that Proposition 2.4 holds for all integers $k \geq 2$.
5. Establish the following variant of the $k=1$ case of (2.14):
$$|u(0) - u(x)| \leq C\|\nabla u\|_{L^p(B)}, \quad p > n, \ x \in \partial B. \tag{2.16}$$
(*Hint*: Suppose $x = e_1$. If γ_z is the line segment from 0 to z, followed by the line segment from z to e_1, write
$$u(e_1) - u(0) = \int_\Sigma \Bigl(\int_{\gamma_z} du \Bigr) \, dS(z), \quad \Sigma = \Bigl\{ x \in B : x_1 = \frac{1}{2} \Bigr\}.$$
Show that this gives $u(e_1) - u(0) = \int_B \nabla u(z) \cdot \varphi(z) \, dz$, with $\varphi \in L^q(B)$, $\forall \, q < n/(n-1)$.)
6. Show that $H^{n,1}(\mathbb{R}^n) \subset C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.
(*Hint*: $u(x) = \int_{-\infty}^0 \cdots \int_{-\infty}^0 D_1 \cdots D_n u(x+y) \, dy_1 \cdots dy_n$.)

3. Gagliardo-Nirenberg-Moser estimates

In this section we establish further estimates on various L^p-norms of derivatives of functions, which are very useful in nonlinear PDE. Estimates of this sort arose in work of Gagliardo [Gag], Nirenberg [Ni], and Moser [Mos]. Our first such estimate is the following. We keep the convention (2.8).

Proposition 3.1. *For real* $k \ge 1,\ 1 \le p \le k$, *we have*

$$\|D_j u\|^2_{L^{2k/p}} \le C\|u\|_{L^{2k/(p-1)}} \cdot \|D_j^2 u\|_{L^{2k/(p+1)}}, \tag{3.1}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$, *hence for all* $u \in L^{q_2}(\mathbb{R}^n) \cap H^{2,q_1}$, *where*

$$q_1 = \frac{2k}{p+1}, \quad q_2 = \frac{2k}{p-1}. \tag{3.2}$$

Proof. Given $v \in C_0^\infty(\mathbb{R}^n),\ q \ge 2$, we have $v|v|^{q-2} \in C_0^1(\mathbb{R}^n)$ and

$$D_j(v|v|^{q-2}) = (q-1)(D_j v)|v|^{q-2}.$$

Letting $v = D_j u$, we have

$$|D_j u|^q = D_j(u\ D_j u|D_j u|^{q-2}) - (q-1)u\ D_j^2 u|D_j u|^{q-2}.$$

Integrating this, we have, by the generalized Hölder inequality (2.1),

$$\|D_j u\|^q_{L^q} \le |q-1| \cdot \|u\|_{L^{q_2}}\|D_j^2 u\|_{L^{q_1}}\|D_j u\|^{q-2}_{L^q}, \tag{3.3}$$

where $q = 2k/p$ and q_1 and q_2 are given by (3.2). Dividing by $\|D_j u\|^{q-2}_{L^q}$ gives the estimate (3.1) for $u \in C_0^\infty(\mathbb{R}^n)$, and the proposition follows.

If we apply (3.1) to $D^{\ell-1}u$, we get

$$\|D^\ell u\|^2_{L^{2k/p}} \le C\|D^{\ell-1}u\|_{L^{2k/(p-1)}}\|D^{\ell+1}u\|_{L^{2k/(p+1)}}, \tag{3.4}$$

for real $k \ge 1,\ p \in [1,k],\ \ell \ge 1$. Consequently, for any $\varepsilon > 0$,

$$\|D^\ell u\|_{L^{2k/p}} \le C\varepsilon\|D^{\ell-1}\|_{L^{2k/(p-1)}} + C(\varepsilon)\|D^{\ell+1}u\|_{L^{2k/(p+1)}}. \tag{3.5}$$

If $p \in [2,k]$ and $\ell \ge 2$, we can apply (3.5) with p replaced by $p-1$ and $D^{\ell-1}u$ replaced by $D^{\ell-2}u$, to get, for any $\varepsilon_1 > 0$,

$$\|D^{\ell-1}u\|_{L^{2k/(p-1)}} \le C\varepsilon_1\|D^{\ell-2}u\|_{L^{2k/(p-2)}} + C(\varepsilon_1)\|D^\ell u\|_{L^{2k/p}}. \tag{3.6}$$

Now we can plug (3.6) into (3.5); fix ε_1 (e.g., $\varepsilon_1 = 1$), and pick ε so small that $C\varepsilon C(\varepsilon_1) \le 1/2$, so the term $C\varepsilon C(\varepsilon_1)\|D^\ell u\|_{L^{2k/p}}$ can be absorbed on the left, to yield

$$\|D^\ell u\|_{L^{2k/p}} \le C\varepsilon\|D^{\ell-2}u\|_{L^{2k/(p-2)}} + C(\varepsilon)\|D^{\ell+1}u\|_{L^{2k/(p+1)}}, \tag{3.7}$$

for real $k \ge 2,\ p \in [2,k],\ \ell \ge 2$. Continuing in this fashion, we get

$$\|D^\ell u\|_{L^{2k/p}} \le C\varepsilon\|D^{\ell-j}u\|_{L^{2k/(p-j)}} + C(\varepsilon)\|D^{\ell+1}u\|_{L^{2k/(p+1)}}, \tag{3.8}$$

for $j \le p \le k,\ \ell \ge j$. Similarly working on the last term in (3.8), we have the following:

Proposition 3.2. *If $j \le p \le k+1-m,\ \ell \ge j$, then (for sufficiently small $\varepsilon > 0$)*

$$\|D^\ell u\|_{L^{2k/p}} \le C\varepsilon \|D^{\ell-j} u\|_{L^{2k/(p-j)}} + C(\varepsilon)\|D^{\ell+m} u\|_{L^{2k/(p+m)}}. \tag{3.9}$$

Here, j, ℓ, and m must be positive integers, but p and k are real. Of course, the full content of (3.9) is represented by the case $\ell = j$, which reads

$$\|D^\ell u\|_{L^{2k/p}} \le C\varepsilon \|u\|_{L^{2k/(p-\ell)}} + C(\varepsilon)\|D^{\ell+m} u\|_{L^{2k/(p+m)}}, \tag{3.10}$$

for $\ell \le p \le k+1-m$. Taking $p+m = k$, we note the following important special case.

Corollary 3.3. *If ℓ, p, and k are positive integers satisfying $\ell \le p \le k-1$, then*

$$\|D^\ell u\|_{L^{2k/p}} \le C\varepsilon \|u\|_{L^{2k/(p-\ell)}} + C(\varepsilon)\|D^{k+\ell-p} u\|_{L^2}. \tag{3.11}$$

In particular, taking $p = \ell$, if $\ell < k$, then

$$\|D^\ell u\|_{L^{2k/\ell}} \le C\varepsilon \|u\|_{L^\infty} + C(\varepsilon)\|D^k u\|_{L^2}, \tag{3.12}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

We want estimates for the left sides of (3.11) and (3.12) which involve products, as in (3.1), rather than sums. The following simple general result produces such estimates.

Proposition 3.4. *Let ℓ, μ, and m be nonnegative integers satisfying $\ell \le \max(\mu, m)$, and let q, r, and ρ belong to $[1,\infty]$. Suppose the estimate*

$$\|D^\ell u\|_{L^q} \le C_1 \|D^\mu u\|_{L^r} + C_2 \|D^m u\|_{L^\rho} \tag{3.13}$$

is valid for all $u \in C_0^\infty(\mathbb{R}^n)$. Then

$$\|D^\ell u\|_{L^q} \le (C_1 + C_2)\|D^\mu u\|_{L^r}^{\beta/(\alpha+\beta)} \cdot \|D^m u\|_{L^\rho}^{\alpha/(\alpha+\beta)}, \tag{3.14}$$

with

$$\alpha = \frac{n}{q} - \frac{n}{r} + \mu - \ell, \quad \beta = -\frac{n}{q} + \frac{n}{\rho} - m + \ell, \tag{3.15}$$

provided these quantities are not both zero. If (3.13) is valid and the quantities (3.15) are both nonzero, then they have the same sign.

Proof. Replacing $u(x)$ in (3.13) by $u(sx)$ produces from (3.13), which we write schematically as $Q \le C_1 R + C_2 P$, the estimate

$$s^{\ell-n/q} Q \le C_1 s^{\mu-n/r} R + C_2 s^{m-n/\rho} P, \quad \text{for all } s > 0,$$

or equivalently,

$$Q \le C_1 s^\alpha R + C_2 s^{-\beta} P, \quad \text{for all } s > 0,$$

with α and β given by (3.15). If α and β have opposite signs, one can take $s \to 0$ or $s \to \infty$ to produce the absurd conclusion $Q = 0$. If they have the same sign, one can take s so that $s^\alpha R = s^{-\beta} P = P^a R^b$, which can be done with $a = \alpha/(\alpha+\beta),\ b = \beta/(\alpha+\beta)$, and the estimate (3.14) results.

Applying Proposition 3.4 to the estimate (3.11), we find $\alpha = (n-2k)\ell/2k,\ \beta = (n-2k)(k-p)/2k$, which gives the following:

Proposition 3.5. *If ℓ, p, and k are positive integers satisfying $\ell \le p \le k-1$, then*

$$\|D^\ell u\|_{L^{2k/p}} \le C\|u\|_{L^{2k/(p-\ell)}}^{(k-p)/(k+\ell-p)} \cdot \|D^{k+\ell-p}u\|_{L^2}^{\ell/(k+\ell-p)}. \tag{3.16}$$

In particular, taking $p = \ell$, if $\ell < k$, then

$$\|D^\ell u\|_{L^{2k/\ell}} \le C\|u\|_{L^\infty}^{1-\ell/k} \cdot \|D^k u\|_{L^2}^{\ell/k}. \tag{3.17}$$

One of the principal applications of such an inequality as (3.17) is to bilinear estimates, such as the following.

Proposition 3.6. *If $|\beta| + |\gamma| = k$, then*

$$\|(D^\beta f)(D^\gamma g)\|_{L^2} \le C\|f\|_{L^\infty}\|g\|_{H^k} + C\|f\|_{H^k}\|g\|_{L^\infty}, \tag{3.18}$$

for all $f, g \in C_o(\mathbb{R}^n) \cap H^k(\mathbb{R}^n)$.

Proof. With $|\beta| = \ell, |\gamma| = m$, and $\ell + m = k$, we have

$$\begin{aligned} \|(D^\beta f)(D^\gamma g)\|_{L^2} &\le \|D^\beta f\|_{L^{2k/\ell}} \cdot \|D^\gamma g\|_{L^{2k/m}} \\ &\le C\|f\|_{L^\infty}^{1-\ell/k} \cdot \|f\|_{H^k}^{\ell/k} \cdot \|g\|_{L^\infty}^{1-m/k} \cdot \|g\|_{H^k}^{m/k}, \end{aligned} \tag{3.19}$$

using Hölder's inequality and (3.17). We can write the right side of (3.19) as

$$C\big(\|f\|_{L^\infty}\|g\|_{H^k}\big)^{m/k}\big(\|f\|_{H^k}\|g\|_{L^\infty}\big)^{\ell/k}, \tag{3.20}$$

and this is readily dominated by the right side of (3.18).

The two estimates of the next proposition are major implications of (3.18).

Proposition 3.7. *We have the estimates*

$$\|f \cdot g\|_{H^k} \le C\|f\|_{L^\infty}\|g\|_{H^k} + C\|f\|_{H^k}\|g\|_{L^\infty} \tag{3.21}$$

and, for $|\alpha| \le k$,

$$\|D^\alpha(f \cdot g) - fD^\alpha g\|_{L^2} \le C\|f\|_{H^k}\|g\|_{L^\infty} + C\|\nabla f\|_{L^\infty}\|g\|_{H^{k-1}}. \tag{3.22}$$

Proof. The estimate (3.21) is an immediate consequence of (3.18). To prove (3.22), write

$$D^\alpha(f\cdot g)=\sum_{\beta+\gamma=\alpha}\binom{\alpha}{\beta}\,(D^\beta f)\,(D^\gamma g), \tag{3.23}$$

so, if $|\alpha|=k$,

$$\begin{aligned} D^\alpha(f\cdot g)-fD^\alpha g&=\sum_{\beta+\gamma=\alpha,\beta>0}\binom{\alpha}{\beta}\,(D^\beta f)\,(D^\gamma g)\\ &=\sum_{|\beta|+|\gamma|=k-1}C_{j\beta\gamma}(D^\beta D_j f)(D^\gamma g). \end{aligned} \tag{3.24}$$

Hence, with $u_j=D_jf$,

$$\|D^\alpha(fg)-fD^\alpha g\|_{L^2}\le C\sum_{|\beta|+|\gamma|=k-1}\|(D^\beta u_j)(D^\gamma g)\|_{L^2}. \tag{3.25}$$

From here, the estimate (3.22) follows immediately from (3.18), and Proposition 3.7 is proved. Note that on the right side of (3.22), we can replace $\|f\|_{H^k}$ by $\|\nabla f\|_{H^{k-1}}$.

From Proposition 3.4 there follow further estimates involving products of norms, which can be quite useful. We record a few here.

Proposition 3.8. *We have the estimates*

$$\|u\|_{L^\infty}\le C\|D^{m+1}u\|_{L^2}^{1/2}\cdot\|D^{m-1}u\|_{L^2}^{1/2},\quad \textit{for } u\in C_0^\infty(\mathbb{R}^{2m}), \tag{3.26}$$

and

$$\|u\|_{L^\infty}\le C\|D^{m+1}u\|_{L^2}^{1/2}\cdot\|D^{m}u\|_{L^2}^{1/2},\quad \textit{for } u\in C_0^\infty(\mathbb{R}^{2m+1}). \tag{3.27}$$

Proof. It is easy to see that

$$\|u\|_{L^\infty}^2\le C\|D^{m+1}u\|_{L^2}^{2}+C\|D^{m-1}u\|_{L^2}^{2},\quad \text{for } u\in C_0^\infty(\mathbb{R}^{2m}), \tag{3.28}$$

and

$$\|u\|_{L^\infty}^2\le C\|D^{m+1}u\|_{L^2}^{2}+C\|D^{m}u\|_{L^2}^{2},\quad \text{for } u\in C_0^\infty(\mathbb{R}^{2m+1}). \tag{3.29}$$

Proposition 3.4 then yields $\alpha=\beta=1$ in case (3.28) and $\alpha=\beta=1/2$ in case (3.29), proving (3.26) and (3.27).

A more delicate L^∞-estimate will be proved in §8.

It is also useful to have the following estimates on compositions.

Proposition 3.9. *Let F be smooth, and assume $F(0)=0$. Then, for $u\in H^k\cap L^\infty$,*

$$\|F(u)\|_{H^k}\le C_k(\|u\|_{L^\infty})\,(1+\|u\|_{H^k}). \tag{3.30}$$

Proof. The chain rule gives

$$D^\alpha F(u) = \sum_{\beta_1+\cdots\beta_\mu=\alpha} C_\beta u^{(\beta_1)} \cdots u^{(\beta_\mu)} F^{(\mu)}(u),$$

hence

$$\|D^k F(u)\|_{L^2} \le C_k(\|u\|_{L^\infty}) \sum \big\| u^{(\beta_1)} \cdots u^{(\beta_\mu)} \big\|_{L^2}. \tag{3.31}$$

From here, (3.30) is obtained via the following simple generalization of Proposition 3.6:

Lemma 3.10. *If* $|\beta_1| + \cdots + |\beta_\mu| = k$, *then*

$$\|f_1^{(\beta_1)} \cdots f_\mu^{(\beta_\mu)}\|_{L^2} \le C \sum_\nu \Big[\|f_1\|_{L^\infty} \cdots \widehat{\|f_\nu\|_{L^\infty}} \cdots \|f_\mu\|_{L^\infty} \Big] \|f\|_{H^k}. \tag{3.32}$$

Proof. The generalized Hölder inequality dominates the left side of (3.32) by

$$\|f_1^{(\beta_1)}\|_{L^{2k/|\beta_1|}} \cdots \|f_\mu^{(\beta_\mu)}\|_{L^{2k/|\beta_\mu|}}. \tag{3.33}$$

Then applying (3.17) dominates this by

$$C\|f_1\|_{L^\infty}^{1-|\beta_1|/k} \cdot \|f_1\|_{H^k}^{|\beta_1|/k} \cdots \|f_\mu\|_{L^\infty}^{1-|\beta_\mu|/k} \cdot \|f_\mu\|_{H^k}^{|\beta_\mu|/k}, \tag{3.34}$$

which in turn is easily bounded by the right side of (3.32) (with $f = (f_1, \dots, f_\mu)$).

We remark that Proposition 3.9 also works if u takes values in $\mathbb{R}^L$. The estimates in Propositions 3.7 and 3.8 are called *Moser estimates*, and are very useful in nonlinear PDE. Some extensions will be given in (10.20) and (10.52).

Exercises

1. Show that the proof of Proposition 3.1 yields

$$\|D_j u\|_{L^q}^2 \le C\|u\|_{L^{q_1}} \cdot \|D^2 u\|_{L^{q_2}} \tag{3.35}$$

   whenever $2 \le q < \infty$, $1 \le q_j \le \infty$, and $1/q_1 + 1/q_2 = 2/q$. Show that if $q_2 < q < q_1$, then (3.35) and (3.1) are equivalent. Is (3.35) valid if the hypothesis $q \ge 2$ is relaxed to $q \ge 1$?
2. Show directly that (3.35) holds with $q_1 = q_2 = q \in [1, \infty]$. (*Hint*: Do the next exercise.)
3. Let A generate a contraction semigroup on a Banach space B. Show that

$$\|Au\|^2 \le 8\|u\| \cdot \|A^2 u\|, \quad \text{for } u \in \mathcal{D}(A^2). \tag{3.36}$$

   (*Hint*: Use the identity $-tAu = t(t-A)^{-1}A^2u + t^2u - t^2t(t-A)^{-1}u$ together with the estimate $\|t(t-A)^{-1}\| \le 1$, for $t > 0$, to obtain the estimate $t\|Au\| \le$

$\|A^2u\| + 2t^2\|u\|$, for $t > 0$.) Try to improve the 8 to a 4 in (3.36), in case B is a *Hilbert* space.

4. Show that (3.10) implies

$$\|D^\ell u\|_{L^q} \le C_1\|u\|_{L^r} + C_2\|D^{\ell+m}u\|_{L^\rho} \tag{3.37}$$

when $\rho < q < r$ are related by

$$\frac{1}{q} = \frac{m}{m+\ell}\frac{1}{r} + \frac{\ell}{m+\ell}\frac{1}{\rho}, \tag{3.38}$$

as long as we require furthermore that $q > 2$, in order to satisfy the hypothesis $p/k \le 1 - (m-1)/k$ used for (3.10). In how much greater generality can you establish (3.37)? Note that if Proposition 3.4 is applied to (3.37), one gets

$$\|D^\ell u\|_{L^q} \le C\|u\|_{L^r}^{m/(m+\ell)} \cdot \|D^{\ell+m}u\|_{L^\rho}^{\ell/(m+\ell)}, \tag{3.39}$$

provided (3.38) holds.

5. Generalize Propositions 3.6 and 3.7, replacing L^2 and H^k by L^p and $H^{k,p}$. Use (3.10) to do this for $p \ge 2$. Can you also treat the case $1 \le p < 2$?
6. Show that in (3.30) you can use $C_k(\|u\|_{L^\infty})$ with

$$C_k(\lambda) = \sup_{|x|\le\lambda, \mu\le k} |F^{(\mu)}(x)|. \tag{3.40}$$

7. Extend the Moser estimates in Propositions 3.7 and 3.9 to estimates in $H^{k,p}$-norms.

4. Trudinger's inequalities

The space $H^{n/2}(\mathbb{R}^n)$ does not quite belong to $L^\infty(\mathbb{R}^n)$, although $H^{n/2}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for all $p \in [2,\infty)$. In fact, quite a bit more is true; exponential functions of $u \in H^{n/2}(\mathbb{R}^n)$ are locally integrable. The proof of this starts with the following estimate of $\|u\|_{L^p(\mathbb{R}^n)}$ as $p \to \infty$.

Proposition 4.1. *If $u \in H^{n/2}(\mathbb{R}^n)$, then, for $p \in [2,\infty)$,*

$$\|u\|_{L^p(\mathbb{R}^n)} \le C_n p^{1/2}\|u\|_{H^{n/2}(\mathbb{R}^n)}. \tag{4.1}$$

Proof. We have $u = \Lambda^{-n/2}v$ for $v \in L^2(\mathbb{R}^n)$, where, recall,

$$\big(\Lambda^{-s}v\big)\hat{}\,(\xi) = \langle\xi\rangle^{-s}\hat{v}(\xi). \tag{4.2}$$

Hence, with $v \in L^2(\mathbb{R}^n)$,

$$u = \mathcal{J}_{n/2} * v, \tag{4.3}$$

where

$$\widehat{\mathcal{J}}_{n/2}(\xi) = \langle\xi\rangle^{-n/2}. \tag{4.4}$$

The behavior of $\mathcal{J}_{n/2}(x)$ follows results of Chapter 3. By Proposition 8.2 of Chapter 3, $\mathcal{J}_{n/2}(x)$ is C^∞ on $\mathbb{R}^n \setminus 0$ and vanishes rapidly as $|x| \to \infty$.

By Proposition 9.2 of Chapter 3, we have

$$\mathcal{J}_{n/2}(x) \le C|x|^{-n/2}, \quad \text{for } |x| \le 1. \tag{4.5}$$

Consequently, $\mathcal{J}_{n/2}$ just misses being in $L^2(\mathbb{R}^n)$; we have, for $\delta \in (0,1]$,

$$\|\mathcal{J}_{n/2}\|^{2-\delta}_{L^{2-\delta}(\mathbb{R}^n)} \le C + C\int_0^1 r^{n\delta/2-1}\,dr \le \frac{C_n}{\delta}. \tag{4.6}$$

Now the map K_v defined by $K_v f = v * f$, with v given in $L^2(\mathbb{R}^n)$, satisfies

$$K_v : L^2 \to L^\infty, \quad K_v : L^1 \to L^2, \tag{4.7}$$

both maps having operator norm $\|v\|_{L^2}$. By interpolation,

$$\|K_v f\|_{L^p(\mathbb{R}^n)} \le \|f\|_{L^q(\mathbb{R}^n)} \cdot \|v\|_{L^2(\mathbb{R}^n)}, \quad \text{for } q \in [1,2], \tag{4.8}$$

where p is defined by $1/q - 1/p = 1/2$. Taking $f = \mathcal{J}_{n/2},\ q = 2-\delta$, we have, for $v \in L^2(\mathbb{R}^n)$,

$$\|\mathcal{J}_{n/2} * v\|_{L^p} \le \Bigl(\frac{C_n}{\delta}\Bigr)^{1/(2-\delta)} \|v\|_{L^2}, \quad p = \frac{2(2-\delta)}{\delta}, \tag{4.9}$$

which gives (4.1).

The following result, known as Trudinger's inequality, is a direct consequence of (4.1):

Proposition 4.2. *If $u \in H^{n/2}(\mathbb{R}^n)$, there is a constant $\gamma = \gamma(u) > 0$, of the form*

$$\gamma(u) = \frac{\gamma_n}{\|u\|^2_{H^{n/2}}}, \tag{4.10}$$

such that

$$\int_{\mathbb{R}^n} \Bigl(e^{\gamma|u(x)|^2} - 1\Bigr)\,dx < \infty. \tag{4.11}$$

If M is a compact manifold, possibly with boundary, of dimension n, and if $u \in H^{n/2}(M)$, then there exists $\gamma = \gamma(M)/\|u\|^2_{H^{n/2}(M)}$ such that

$$\int_M e^{\gamma|u(x)|^2}\,dV(x) < \infty. \tag{4.12}$$

Proof. We have

$$e^{\gamma|u(x)|^2} - 1 = \gamma|u(x)|^2 + \frac{\gamma^2}{2}\,|u(x)|^4 + \cdots + \frac{\gamma^m}{m!}\,|u(x)|^{2m} + \cdots.$$

By (4.1),

$$\frac{\gamma^m}{m!}\int |u(x)|^{2m}\,dV(x) \le C_n^{2m}\frac{\gamma^m}{m!}\,(2m)^m\|u\|^{2m}_{H^{n/2}}, \tag{4.13}$$

which is bounded by $C'\kappa^m$, for some $\kappa < 1$, if γ has the form (4.10), with $\gamma_n < 1/(2eC_n^2)$, as can be seen via Stirling's formula for $m!$. This proves the proposition.

We note that the same argument involving (4.2)–(4.8) also shows that, for any $p \in [2, \infty)$, there is an $\varepsilon > 0$ such that

$$H^{n/2-\varepsilon}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n). \tag{4.14}$$

Similarly, we have $H^{n/2-\varepsilon}(M) \subset L^p(M)$, when M is a compact manifold, perhaps with boundary, of dimension n. By virtue of Rellich's theorem, we have for such M that the natural inclusion

$$\iota : H^{n/2}(M) \hookrightarrow L^p(M) \text{ is compact, for all } p < \infty. \tag{4.15}$$

Using this, we obtain the following result:

Proposition 4.3. *If M is a compact manifold (with boundary) of dimension n, $\alpha \in \mathbb{R}$, then*

$$u_j \to u \text{ weakly in } H^{n/2}(M) \Longrightarrow e^{\alpha u_j} \to e^{\alpha u} \text{ in } L^1(M)\text{-norm.} \tag{4.16}$$

Proof. We have

$$\left|e^{\alpha u_j} - e^{\alpha u}\right| \le \sum_{m\ge 1} \frac{|\alpha|^m}{m!} \left||u_j(x)|^m - |u(x)|^m\right|.$$

If $\|u_j\|_{H^{n/2}(M)} \le A$, we obtain

$$\begin{aligned} \|e^{\alpha u_j} - e^{\alpha u}\|_{L^1} \le & \sum_{m\le k} \frac{|\alpha|^m}{m!} \|u_j - u\|_{L^m} \cdot m\Big[\|u_j\|_{L^m}^{m-1} + \|u\|_{L^m}^{m-1}\Big] \\ & + C\sum_{m>k} \frac{m^{m/2}}{m!} |AC_n\alpha|^m, \end{aligned} \tag{4.17}$$

where we use

$$\big||u_j|^m - |u|^m\big| \le m|u_j - u|\big(|u_j|^{m-1} + |u|^{m-1}\big)$$

to estimate the sum over $m \le k$, and we use (4.1) to estimate the sum over $m > k$. By (4.15), for any k, the first sum on the right side of (4.17) goes to 0 as $j \to \infty$. Meanwhile the second sum vanishes as $k \to \infty$, so (4.16) follows.

Exercises

1. Partially generalizing (4.10), let $p \in (1,\infty)$, and let $u \in H^{k,p}(\mathbb{R}^n)$, with $kp = n,\ k \in \mathbb{Z}^+$. Show that there exists $\gamma = \gamma_p(u)$ such that

$$\int_{|x|\le R} e^{\gamma |u(x)|^{p/(p-1)}}\, dx \le C_{pR}. \tag{4.18}$$

For a more complete generalization, see Exercise 5 of §6.
Note: Finding the best constant γ in (4.18) is subtle and has some important uses; see [Mo2], [Au], particularly for the case $k = 1,\ p = n$.

5. Singular integral operators on L^p

One way the Fourier transform makes analysis on $L^2(\mathbb{R}^n)$ easier than analysis on other L^p-spaces is by the definitive result the Plancherel theorem gives as a condition that a convolution operator $k * u = P(D)u$ be L^2-bounded, namely that $\hat{k}(\xi) = P(\xi)$ be a bounded function of ξ. A replacement for this that advances our ability to pursue analysis on L^p is the next result, established by S. Mikhlin, following related work for $L^p(\mathbb{T}^n)$ by J. Marcinkiewicz.

Theorem 5.1. *Suppose $P(\xi)$ satisfies*

$$|D^\alpha P(\xi)| \le C_\alpha \langle \xi \rangle^{-|\alpha|}, \tag{5.1}$$

for $|\alpha| \le n+1$. Then

$$P(D) : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \quad \text{for } 1 < p < \infty. \tag{5.2}$$

Stronger results have been proved; one needs (5.1) only for $|\alpha| \le [n/2]+1$, and one can use certain L^2-estimates on the derivatives of $P(\xi)$. These sharper results can be found in [H1] and [S1]. Note that the characterization of $P(\xi) \in S^0_1(\mathbb{R}^n)$ is that (5.1) hold for all α.

The theorem stated above is a special case of a result that applies to pseudodifferential operators with symbols in $S^0_{1,\delta}(\mathbb{R}^n)$. As shown in §2 of Chapter 7, if $p(x,\xi)$ satisfies the estimates

$$|D^\beta_x D^\alpha_\xi p(x,\xi)| \le C_{\alpha\beta} \langle \xi \rangle^{-|\alpha|+|\beta|}, \tag{5.3}$$

for

$$|\beta| \le 1, \quad |\alpha| \le n+1+|\beta|, \tag{5.4}$$

then the Schwartz kernel $K(x,y)$ of $P = p(x,D)$ satisfies the estimates

$$|K(x,y)| \le C|x-y|^{-n} \tag{5.5}$$

and

$$|\nabla_{x,y}K(x,y)| \le C|x-y|^{-n-1}. \tag{5.6}$$

Furthermore, at least when $\delta < 1$, we have an L^2-bound:

$$\|Pu\|_{L^2} \le K\|u\|_{L^2}, \tag{5.7}$$

and smoothings of such an operator have smooth Schwartz kernels satisfying (5.5)–(5.7) for fixed C, K. (Results in §9 of this chapter will contain another proof of this L^2-estimate. Note that when $p(x,\xi) = p(\xi)$ the estimate (5.7) follows from the Plancherel theorem.) Our main goal here is to give a proof of the following fundamental result of A. P. Calderon and A. Zygmund:

Theorem 5.2. *Suppose $P : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a weak limit of operators with smooth Schwartz kernels satisfying (5.5)–(5.7) uniformly. Then*

$$P : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \quad 1 < p < \infty. \tag{5.8}$$

In particular, this holds when $P \in OPS^0_{1,\delta}(\mathbb{R}^n),\ \delta \in [0,1)$.

The hypotheses do not imply boundedness on $L^1(\mathbb{R}^n)$ or on $L^\infty(\mathbb{R}^n)$. They will imply that P is of weak type $(1,1)$. By definition, an operator P is of weak type (q,q) provided that, for any $\lambda > 0$,

$$\text{meas}\ \{x : |Pu(x)| > \lambda\} \le C\lambda^{-q}\|u\|^q_{L^q}. \tag{5.9}$$

Any bounded operator on L^q is a fortiori of weak type (q,q), in view of the simple inequality

$$\text{meas}\ \{x : |u(x)| > \lambda\} \le \lambda^{-1}\|u\|_{L^1}. \tag{5.10}$$

A key ingredient in proving Theorem 5.2 is the following result:

Proposition 5.3. *Under the hypotheses of Theorem 5.2, P is of weak type* $(1,1)$.

Once this is established, Theorem 5.2 will then follow from the next result, known as the *Marcinkiewicz interpolation theorem.*

Proposition 5.4. *If $r < p < q$ and if T is both of weak type (r,r) and of weak type (q,q), then $T : L^p \to L^p$.*

Proof. Write $u = u_1 + u_2$, with $u_1(x) = u(x)$ for $|u(x)| > \lambda$ and $u_2(x) = u(x)$ for $|u(x)| \le \lambda$. With the notation

$$\mu_f(\lambda) = \ \text{meas}\ \{x : |f(x)| \ge \lambda\}, \tag{5.11}$$

we have

$$\begin{aligned}\mu_{Tu}(2\lambda) &\le \mu_{Tu_1}(\lambda) + \mu_{Tu_2}(\lambda) \\ &\le C_1\lambda^{-r}\|u_1\|_{L^r}^r + C_2\lambda^{-q}\|u_2\|_{L^q}^q.\end{aligned} \tag{5.12}$$

Also, there is the formula

$$\int |f(x)|^p\,dx = p\int_0^\infty \mu_f(\lambda)\lambda^{p-1}\,d\lambda.$$

Hence

$$\begin{aligned}\int |Tu(x)|^p\,dx &= p\int_0^\infty \mu_{Tu}(\lambda)\lambda^{p-1}\,d\lambda \\ &\le C_1 p\int_0^\infty \lambda^{p-1-r}\Big(\int\limits_{|u|>\lambda} |u(x)|^r\,dx\Big)\,d\lambda \\ &\quad + C_2 p\int_0^\infty \lambda^{p-1-q}\Big(\int\limits_{|u|\le\lambda} |u(x)|^q\,dx\Big)\,d\lambda.\end{aligned} \tag{5.13}$$

Now

$$\int_0^\infty \lambda^{p-1-r}\Big(\int\limits_{|u|>\lambda} |u(x)|^r\,dx\Big)\,d\lambda = \frac{1}{p-r}\int |u(x)|^p\,dx \tag{5.14}$$

and, similarly,

$$\int_0^\infty \lambda^{p-1-q}\Big(\int\limits_{|u|\le\lambda} |u(x)|^q\,dx\Big)\,d\lambda = \frac{1}{q-p}\int |u(x)|^p\,dx. \tag{5.15}$$

Combining these gives the desired estimate on $\|Tu\|_{L^p}^p$.

We will apply Proposition 5.4 in conjunction with the following covering lemma of Calderon and Zygmund:

Lemma 5.5. *Let $u \in L^1(\mathbb{R}^n)$ and $\lambda > 0$ be given. Then there exist $v, w_k \in L^1(\mathbb{R}^n)$ and disjoint cubes Q_k, $1 \le k < \infty$, with centers x_k, such that*

$$u = v + \sum_k w_k, \quad \|v\|_{L^1} + \sum_k \|w_k\|_{L^1} \le 3\|u\|_{L^1}, \tag{5.16}$$

$$|v(x)| \le 2^n\lambda, \tag{5.17}$$

$$\int\limits_{Q_k} w_k(x)\,dx = 0 \ \text{ and supp } w_k \subset Q_k, \tag{5.18}$$

$$\sum_k \text{meas}(Q_k) \le \lambda^{-1}\|u\|_{L^1}. \tag{5.19}$$

Proof. Tile $\mathbb{R}^n$ with cubes of volume greater than $\lambda^{-1}\|u\|_{L^1}$. The mean value of $|u(x)|$ over each such cube is $< \lambda$. Divide each of these cubes into 2^n equal cubes, and let $I_{11}, I_{12}, I_{13}, \dots$ be those so obtained over which the mean value of $|u(x)|$ is $\geq \lambda$. Note that

$$\lambda \ \mathrm{meas}(I_{1k}) \leq \int_{I_{1k}} |u(x)| \, dx \leq 2^n \lambda \ \mathrm{meas}(I_{1k}). \tag{5.20}$$

Now set

$$v(x) = \frac{1}{\mathrm{meas}(I_{1k})} \int_{I_{1k}} u(y) \, dy, \quad \text{for } x \in I_{1k}, \tag{5.21}$$

and

$$\begin{aligned} w_{1k}(x) = u(x) - v(x), &\quad \text{for } x \in I_{1k}, \\ 0, &\quad \text{for } x \notin I_{1k}. \end{aligned} \tag{5.22}$$

Next take all the cubes that are not among the I_{1k}, subdivide each into 2^n equal parts, select those new cubes $I_{21}, I_{22}, \dots,$ over which the mean value of $|u(x)|$ is $\geq \lambda$, and extend the definitions (5.21)–(5.22) to these cubes, in the natural fashion. Continue in this way, obtaining disjoint cubes I_{jk} and functions w_{jk}. Then reorder these cubes and functions as $Q_1, Q_2, \dots,$ and $w_1, w_2, \dots$. Complete the definition of v by setting $v(x) = u(x)$, for $x \notin \cup Q_k$. Then we have the first part of (5.16). Since

$$\int_{Q_k} \big(|v(x)| + |w_k(x)|\big) \, dx \leq 3 \int_{Q_k} |u(x)| \, dx, \tag{5.23}$$

and since the cubes are disjoint, w_k is supported in Q_k, and $v = u$ on $\mathbb{R}^n \setminus \cup Q_k$, we obtain the rest of (5.16).

Next, (5.17) follows from (5.20) if $x \in \cup Q_k$. But if $x \notin \cup Q_k$, there are arbitrarily small cubes containing x over which the mean value of $|u(x)|$ is $< \lambda$, so (5.17) holds almost everywhere on $\mathbb{R}^n \setminus \cup Q_k$ as well. The assertion (5.18) is obvious from the construction, and (5.19) follows by summing (5.20). The lemma is proved.

One thinks of v as the "good" piece and $w = \sum w_k$ as the "bad" piece. What is "good" about v is that $\|v\|_{L^2}^2 \leq 2^n \lambda \|u\|_{L^1}$, so

$$\|Pv\|_{L^2}^2 \leq K^2 \|v\|_{L^2}^2 \leq 4^n K^2 \lambda \|u\|_{L^1}. \tag{5.24}$$

Hence

$$\Big(\frac{\lambda}{2}\Big)^2 \ \mathrm{meas}\Big\{x : |Pv(x)| > \frac{\lambda}{2}\Big\} \leq C\lambda \|u\|_{L^1}. \tag{5.25}$$

To treat the action of P on the "bad" term w, we make use of the following essentially elementary estimate on the Schwartz kernel K. The proof is an exercise.

Lemma 5.6. *There is a* $C_0 < \infty$ *such that, for any* $t > 0$, *if* $|y| \le t, x_0 \in \mathbb{R}^n$,

$$\int_{|x-x_0|\ge 2t} \big|K(x, x_0 + y) - K(x, x_0)\big|\ dx \le C_0. \tag{5.26}$$

To estimate Pw, we have

$$\begin{aligned} Pw_k(x) &= \int K(x,y) w_k(y)\ dy \\ &= \int_{Q_k} \big[K(x,y) - K(x,x_k)\big] w_k(y)\ dy. \end{aligned} \tag{5.27}$$

Before we make further use of this, a little notation: Let Q_k^* be the cube concentric with Q_k, enlarged by a linear factor of $2n^{1/2}$, so meas $Q_k^* = (4n)^{n/2}$ meas Q_k. For some $t_k > 0$, we can arrange that

$$\begin{aligned} Q_k &\subset \{x : |x - x_k| \le t_k\}, \\ Y_k &= \mathbb{R}^n \setminus Q_k^* \subset \{x : |x - x_k| > 2t_k\}. \end{aligned} \tag{5.28}$$

Furthermore, set $\mathcal{O} = \cup Q_k^*$, and note that

$$\text{meas}\ \mathcal{O} \le L\lambda^{-1} \|u\|_{L^1}, \tag{5.29}$$

with $L = (4n)^{n/2}$. Now, from (5.27), we have

$$\begin{aligned} &\int_{Y_k} |Pw_k(x)|\ dx \\ &\le \int_{|y|\le t_k,} \int_{|x|\ge 2t_k} \big|K(x + x_k, x_k + y) - K(x + x_k, x_k)\big| \\ &\qquad\qquad \cdot |w_k(y + x_k)|\ dx\ dy \\ &\le C_0 \|w_k\|_{L^1}, \end{aligned} \tag{5.30}$$

the last estimate using Lemma 5.6. Thus

$$\int_{\mathbb{R}^n \setminus \mathcal{O}} |Pw(x)|\ dx \le 3C_0 \|u\|_{L^1}. \tag{5.31}$$

Together with (5.29), this gives

$$\text{meas}\Big\{x : |Pw(x)| > \frac{\lambda}{2}\Big\} \le \frac{C_1}{\lambda} \|u\|_{L^1}, \tag{5.32}$$

and this estimate together with (5.25) yields the desired weak (1,1)-estimate:

$$\text{meas}\big\{x : |Pu(x)| > \lambda\big\} \le \frac{C_2}{\lambda} \|u\|_{L^1}. \tag{5.33}$$

This proves Proposition 5.3.

To complete the proof of Theorem 5.2, we apply Marcinkiewicz interpolation to obtain (5.8) for $p \in (1,2]$. Note that the Schwartz kernel of P^* also satisfies the hypotheses of Theorem 5.2, so we have $P^* : L^p \to L^p$, for $1 < p \le 2$. Thus the result (5.8) for $p \in [2,\infty)$ follows by duality.

We remark that if (5.6) is weakened to $|\nabla_y K(x,y)| \le C|x-y|^{-n-1}$, while the hypotheses (5.5) and (5.7) are retained, then Lemma 5.6 still holds, and hence so does Proposition 5.3. Thus, we still have $P : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $1 < p \le 2$, but the duality argument gives only $P^* : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $2 \le p < \infty$.

We next describe an important generalization to operators acting on Hilbert space-valued functions. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and suppose

$$P : L^2(\mathbb{R}^n, \mathcal{H}_1) \longrightarrow L^2(\mathbb{R}^n, \mathcal{H}_2). \tag{5.34}$$

Then P has an $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$-operator-valued Schwartz kernel K. Let us impose on K the hypotheses of Theorem 5.2, where now $|K(x,y)|$ stands for the $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$-norm of $K(x,y)$. Then all the steps in the proof of Theorem 5.2 extend to this case. Rather than formally state this general result, we will concentrate on an important special case.

Proposition 5.7. *Let $P(\xi) \in C^\infty(\mathbb{R}^n, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ satisfy*

$$\|D_\xi^\alpha P(\xi)\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \le C_\alpha \langle \xi \rangle^{-|\alpha|}, \tag{5.35}$$

for all $\alpha \ge 0$. Then

$$P(D) : L^p(\mathbb{R}^n, \mathcal{H}_1) \longrightarrow L^p(\mathbb{R}^n, \mathcal{H}_2), \quad \text{for } 1 < p < \infty. \tag{5.36}$$

This leads to an important circle of results known as *Littlewood-Paley theory*. To obtain this, start with a partition of unity

$$1 = \sum_{j=0}^{\infty} \varphi_j(\xi)^2, \tag{5.37}$$

where $\varphi_j \in C^\infty$, $\varphi_0(\xi)$ is supported on $|\xi| \le 1$, $\varphi_1(\xi)$ is supported on $1/2 \le |\xi| \le 2$, and $\varphi_j(\xi) = \varphi_1(2^{1-j}\xi)$ for $j \ge 2$. We take $\mathcal{H}_1 = \mathbb{C}$, $\mathcal{H}_2 = \ell^2$, and look at

$$\Phi : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n, \ell^2) \tag{5.38}$$

given by

$$\Phi(f) = (\varphi_0(D)f, \varphi_1(D)f, \varphi_2(D)f, \dots). \tag{5.39}$$

This is clearly an isometry, though of course it is not surjective. The adjoint

$$\Phi^* : L^2(\mathbb{R}^n, \ell^2) \longrightarrow L^2(\mathbb{R}^n), \tag{5.40}$$

given by

$$\Phi^*(g_0, g_1, g_2, \dots) = \sum \varphi_j(D) g_j, \tag{5.41}$$

satisfies

$$\Phi^*\Phi = I \tag{5.42}$$

on $L^2(\mathbb{R}^n)$. Note that $\Phi = \Phi(D)$, where

$$\Phi(\xi) = (\varphi_0(\xi), \varphi_1(\xi), \varphi_2(\xi), \dots). \tag{5.43}$$

It is easy to see that the hypothesis (5.35) is satisfied by both $\Phi(\xi)$ and $\Phi^*(\xi)$. Hence, for $1 < p < \infty$,

$$\begin{aligned} \Phi : L^p(\mathbb{R}^n) &\longrightarrow L^p(\mathbb{R}^n, \ell^2), \\ \Phi^* : L^p(\mathbb{R}^n, \ell^2) &\longrightarrow L^p(\mathbb{R}^n). \end{aligned} \tag{5.44}$$

In particular, Φ maps $L^p(\mathbb{R}^n)$ isomorphically onto a closed subspace of $L^p(\mathbb{R}^n, \ell^2)$, and we have compatibility of norms:

$$\|u\|_{L^p} \approx \|\Phi u\|_{L^p(\mathbb{R}^n, \ell^2)}. \tag{5.45}$$

In other words,

$$C_p' \|u\|_{L^p} \le \Big\| \Big\{ \sum_{j=0}^{\infty} |\varphi_j(D) u|^2 \Big\}^{1/2} \Big\|_{L^p} \le C_p \|u\|_{L^p}, \tag{5.46}$$

for $1 < p < \infty$.

Exercises

1. Estimate the family of symbols $a_y(\xi) = \langle \xi \rangle^{iy}$, $y \in \mathbb{R}$. Show that if $\Lambda^{iy} = a_y(D)$, then

$$\|\Lambda^{iy} u\|_{L^p(\mathbb{R}^n)} \le C_p \langle y \rangle^{n+1} \|u\|_{L^p(\mathbb{R}^n)}. \tag{5.47}$$

   This estimate will be useful for the development of the Sobolev spaces $H^{s,p}$ in the next section.
2. Let $\tilde{\psi}_1(\xi)$ be supported on $1/4 \le |\xi| \le 4$, $\tilde{\psi}_1(\xi) = 1$ for $1/2 \le |\xi| \le 2$, and $\tilde{\psi}_j(\xi) = \tilde{\psi}_1(2^{1-j}\xi)$ for $j \ge 2$. Let $s \in \mathbb{R}$. Show that

$$A(D), B(D) : L^p(\mathbb{R}^n, \ell^2) \longrightarrow L^p(\mathbb{R}^n, \ell^2), \quad 1 < p < \infty,$$

   for

$$\begin{aligned} A_{jk}(\xi) &= 2^{ks} \langle \xi \rangle^{-s} \tilde{\psi}_j(\xi) \delta_{jk}, \\ B_{jk}(\xi) &= 2^{-ks} \langle \xi \rangle^{s} \tilde{\psi}_j(\xi) \delta_{jk}, \end{aligned}$$

   by applying Proposition 5.7.
3. Give a proof that

$$\int |f(x)|^p \, dx = p \int_0^\infty \mu_f(\lambda) \, \lambda^{p-1} \, d\lambda, \tag{5.48}$$

used in (5.13). Also, demonstrate (5.14) and (5.15). (*Hint*: After doing (5.48), get an analogous identity for the integral of $|f(x)|^p$ over the set $\{x : |f(x)| > \lambda\}$, resp., $\leq \lambda$.)

4. Give a detailed proof of Lemma 5.6.
5. Let $A \in OPS^1_{1,0}(\mathbb{R}^n)$, and suppose $A(x,\xi) = 0$ for $x_n = 0$. Define $Tf = Af\Big|_{\mathbb{R}^n_-}$, where $\mathbb{R}^n_\pm = \{x \in \mathbb{R}^n : \pm x_n \geq 0\}$. Show that, for $1 \leq p \leq \infty$,

$$f \in L^p(\mathbb{R}^n), \quad \text{supp } f \subset \mathbb{R}^n_+ \Longrightarrow Tf \in L^p(\mathbb{R}^n_-). \tag{5.49}$$

(*Hint*: Apply Proposition 5.1 of Appendix A. Compare with Exercise 3 in §5 of Appendix A.)

6. The spaces $H^{s,p}$

Here we define and study $H^{s,p}$ for any $s \in \mathbb{R},\ p \in (1,\infty)$. In analogy with the characterization of $H^s(\mathbb{R}^n) = H^{s,2}(\mathbb{R}^n)$ given in §1 of Chapter 4, we set

$$H^{s,p}(\mathbb{R}^n) = \Lambda^{-s} L^p(\mathbb{R}^n). \tag{6.1}$$

Given the results of §5, we can establish the following.

Proposition 6.1. *When $s = k$ is a positive integer, $p \in (1,\infty)$, the spaces $H^{k,p}(\mathbb{R}^n)$ of §1 coincide with (6.1).*

Proof. For $|\alpha| \leq k,\ \xi^\alpha \langle\xi\rangle^{-k}$ belongs to $S^0_1(\mathbb{R}^n)$. Thus, by Theorem 5.1, $D^\alpha \Lambda^{-k}$ maps $L^p(\mathbb{R}^n)$ to itself. Thus any $u \in \Lambda^{-k} L^p(\mathbb{R}^n)$ satisfies the definition of $H^{k,p}(\mathbb{R}^n)$ given in §1. For the converse, note that one can write

$$\langle\xi\rangle^k = \sum_{|\alpha|\leq k} q_\alpha(\xi)\xi^\alpha, \tag{6.2}$$

with coefficients $q_\alpha \in S^0_1(\mathbb{R}^n)$. Thus if $D^\alpha u \in L^p(\mathbb{R}^n)$ for all $|\alpha| \leq k$, it follows that $\Lambda^k u \in L^p(\mathbb{R}^n)$.

We next prove an interpolation theorem generalizing the identity

$$\big[L^2(\mathbb{R}^n), H^s(\mathbb{R}^n)\big]_\theta = H^{\theta s}(\mathbb{R}^n), \quad \text{for } \theta \in [0,1],$$

proven in §2 of Chapter 4.

Proposition 6.2. *For $s \in \mathbb{R},\ \theta \in (0,1)$, and $p \in (1,\infty)$,*

$$\big[L^p(\mathbb{R}^n), H^{s,p}(\mathbb{R}^n)\big]_\theta = H^{\theta s,p}(\mathbb{R}^n). \tag{6.3}$$

Proof. The proof is parallel to that of Proposition 2.2 of Chapter 4, except that we use the estimate (5.47) of the last section in place of the obvious identity $\|A^{iy}\| = 1$ for a unitary operator A^{iy} on a Hilbert space. Thus, if $v \in H^{\theta s,p}(\mathbb{R}^n)$, let

$$u(z) = e^{z^2} \Lambda^{(\theta - z)s} v. \tag{6.4}$$

Then $u(\theta) = e^{\theta^2} v$, $u(iy) = e^{-y^2} \Lambda^{-iys} \big(\Lambda^{s\theta} v\big)$ is bounded in $L^p(\mathbb{R}^n)$, by (5.47), and also $u(1 + iy) = e^{-(y-i)^2} \Lambda^{-s} \Lambda^{-iys} \big(\Lambda^{s\theta} v\big)$ is bounded in the space $H^{s,p}(\mathbb{R}^n)$. Therefore, such a function v belongs to the left side of (6.3). The reverse containment is similarly established as in the proof of Proposition 2.2 of Chapter 4.

This sort of argument yields more generally that, for $\sigma, s \in \mathbb{R}$, $\theta \in (0, 1)$, and $p \in (1, \infty)$,

$$[H^{\sigma,p}(\mathbb{R}^n), H^{s,p}(\mathbb{R}^n)]_\theta = H^{\theta s + (1-\theta)\sigma,p}(\mathbb{R}^n). \tag{6.5}$$

With Proposition 6.2 established, we can define and analyze spaces $H^{s,p}$ on compact manifolds in the same way as we did for $p = 2$ in Chapter 4. If M is a compact manifold without boundary, one defines $H^{s,p}(M)$ in analogy with $H^s(M)$, via coordinate charts, and proves

$$[H^{\sigma,p}(M), H^{s,p}(M)]_\theta = H^{\theta s + (1-\theta)\sigma,p}(M), \tag{6.6}$$

for $p \in (1, \infty)$, $\theta \in (0, 1)$. If Ω is a compact subdomain of M with smooth boundary, we define $H^{k,p}(\Omega)$ as in §1, and recall the extension operator $E : H^{k,p}(\Omega) \to H^{k,p}(M)$. If we define $H^{s,p}(\Omega)$ for $s > 0$ by

$$H^{s,p}(\Omega) = [L^p(\Omega), H^{k,p}(\Omega)]_\theta, \quad \theta \in (0, 1), \quad s = k\theta, \tag{6.7}$$

it follows that $E : H^{s,p}(\Omega) \to H^{s,p}(M)$ and hence

$$H^{s,p}(\Omega) \approx H^{s,p}(M)/\{u : u = 0 \text{ on } \Omega\}. \tag{6.8}$$

Also, of course, $H^{s,p}(\Omega)$ agrees with the characterization of §1 when $s = k$ is a positive integer. Generalizing the theorem of Rellich, Proposition 4.4 of Chapter 4, one has, for $s \ge 0$, $1 < p < \infty$,

$$\iota : H^{s+\sigma,p}(\Omega) \hookrightarrow H^{s,p}(\Omega) \text{ is compact for } \sigma > 0. \tag{6.9}$$

By the arguments used in Chapter 4, we easily reduce this to showing that, for $\sigma > 0, 1 < p < \infty$,

$$\Lambda^{-\sigma} : L^p(\mathbb{T}^n) \longrightarrow L^p(\mathbb{T}^n) \text{ is compact.} \tag{6.10}$$

Indeed, the operator (6.10) is of the form $\Lambda^{-\sigma} u = k_\sigma * u$, with $k_\sigma \in L^1(\mathbb{T}^n)$ for any $\sigma > 0$. Thus k_σ is an L^1-norm limit of $k_{\sigma,j} \in C^\infty(\mathbb{T}^n)$, so $\Lambda^{-\sigma}$ is an operator norm limit of convolution maps $L^p(\mathbb{T}^n) \to C^\infty(\mathbb{T}^n)$, which are clearly compact on $L^p(\mathbb{T}^n)$.

We now extend some of the Sobolev imbedding theorems of §2. Once they are obtained on $\mathbb{R}^n$, they easily yield similar results for functions on compact manifolds, perhaps with boundary.

Proposition 6.3. *If $s > n/p$, then $H^{s,p}(\mathbb{R}^n) \subset C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.*

Proof. $\Lambda^{-s}u = \mathcal{J}_s * u$, where $\widehat{\mathcal{J}}_s(\xi) = \langle\xi\rangle^{-s}$. It suffices to show that

$$\mathcal{J}_s \in L^{p'}(\mathbb{R}^n), \quad \text{for } s > \frac{n}{p}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{6.11}$$

Indeed, estimates established in §8 of Chapter 3 imply that $\mathcal{J}_s(x)$ is smooth on $\mathbb{R}^n \setminus 0$, rapidly decreasing as $|x| \to \infty$, and

$$|\mathcal{J}_s(x)| \le C|x|^{-n+s}, \quad |x| \le 1, \quad s < n, \tag{6.12}$$

which is sufficient. Compare estimates for $s = n/2$ in (4.4)–(4.9).

Next we generalize (2.9).

Proposition 6.4. *For $sp < n,\ p \in (1,\infty)$, we have*

$$H^{s,p}(\mathbb{R}^n) \subset L^{np/(n-sp)}(\mathbb{R}^n). \tag{6.13}$$

Proof. Suppose $s = k + \sigma,\ k \in \mathbb{Z}^+,\ \sigma \in [0,1)$. Then $u \in H^{s,p} \Rightarrow \Lambda^\sigma u \in H^{k,p}$, and by (2.9) this gives $\Lambda^\sigma u \in L^q(\mathbb{R}^n)$, with $q = np/(n-kp)$. Note that $q \in (1,\infty)$ and $np/(n-sp) = nq/(n-\sigma q)$, so also $\sigma q < n$. Hence it suffices to show that

$$\Lambda^{-\sigma} : L^q(\mathbb{R}^n) \longrightarrow L^{nq/(n-\sigma q)}(\mathbb{R}^n), \tag{6.14}$$

when $\sigma \in (0,1),\ q \in (1,\infty)$, and $\sigma q < n$. We divide the analysis into cases.

Case I. $1 < q < n$. In this case, we have, by (2.2),

$$H^{1,q}(\mathbb{R}^n) \subset L^{nq/(n-q)}(\mathbb{R}^n). \tag{6.15}$$

Fixing $v \in L^q(\mathbb{R}^n)$, consider $\Lambda^{-z}v$ for $z \in \overline{\Omega} = \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}$. Note that Proposition 5.7 implies

$$\|\Lambda^{iy}v\|_{L^q} \le Ae^{B|y|}\|v\|_{L^q}, \tag{6.16}$$

for $y \in \mathbb{R}$. Making use also of (6.15), we have

$$\|\Lambda^{-(1+iy)}v\|_{L^{nq/(n-q)}} \le Ae^{B|y|}\|v\|_{L^q}. \tag{6.17}$$

From here a complex interpolation argument gives (6.14) in this case.

Case II. $2 \le n \le q < \infty$. In this case, set $r = nq/(n-\sigma q)$. Note that

$$\frac{1}{r} = \frac{1}{q} - \frac{\sigma}{n} \quad \text{and} \quad \frac{1}{r'} = \frac{1}{q'} + \frac{\sigma}{n}, \tag{6.18}$$

where r' is the dual exponent to r. We have $r > q \ge n \ge 2$, so $r' < 2 \le n$, and Case I gives

$$\Lambda^{-\sigma} : L^{r'}(\mathbb{R}^n) \longrightarrow L^{q'}(\mathbb{R}^n). \tag{6.19}$$

Then (6.14) follows by duality.

CASE III. $n = 1$. Here one needs a different approach. Since this case is not so crucial for PDE, we omit it. Various proofs that include this case can be found in [S1], [S3], and [BL].

The following result is an immediate consequence of the definition (6.1), the pseudodifferential operator calculus, and the L^p-boundedness result of Theorem 5.2.

Proposition 6.5. *If* $P \in OPS^m_{1,\delta}(\mathbb{R}^n)$, $0 \le \delta < 1$, *and* $1 < p < \infty$, *then*

$$P : H^{s,p}(\mathbb{R}^n) \longrightarrow H^{s-m,p}(\mathbb{R}^n). \tag{6.20}$$

In view of the construction of parametrices for elliptic operators, we deduce various $H^{s,p}$-regularity results for solutions to linear elliptic equations. A sequence of exercises on generalized div-curl lemmas given below will make use of this.

Exercises

1. Let $\varphi_j(\xi)^2 = \psi_j(\xi)$ be the partition of unity (5.37). Using the Littlewood-Paley estimates, show that, for $p \in (1,\infty)$, $s \in \mathbb{R}$,

$$\|u\|_{H^{s,p}(\mathbb{R}^n)} \approx \Bigl\| \Bigl\{ \sum_{k=0}^{\infty} 4^{ks} |\varphi_j(D)u|^2 \Bigr\}^{1/2} \Bigr\|_{L^p(\mathbb{R}^n)}. \tag{6.21}$$

(*Hint*: From (5.37), we have the left side of (6.21)

$$\approx \Bigl\| \Bigl\{ \sum_{k=0}^{\infty} \Bigl| \Lambda^s \varphi_j(D)u \Bigr|^2 \Bigr\}^{\frac{1}{2}} \Bigr\|_{L^p(\mathbb{R}^n)}. \tag{6.22}$$

Now apply Exercise 2 of §5.)

Exercises 2–4 lead up to a demonstration that if

$$\Psi_k(\xi) = \sum_{\ell \le k} \varphi_\ell(\xi)^2, \tag{6.23}$$

then, for $s > 0$, $p \in (1,\infty)$,

$$\Bigl\| \sum_{k=0}^{\infty} \Psi_k(D) f_k \Bigr\|_{H^{s,p}} \le C_{sp} \Bigl\| \Bigl\{ \sum_{k=0}^{\infty} 4^{ks} |f_k|^2 \Bigr\}^{1/2} \Bigr\|_{L^p}. \tag{6.24}$$

2. Show that the left side of (6.24) is

$$\approx \Bigl\| \Bigl\{ \sum_{\ell=0}^{\infty} \Bigl| \psi_\ell(D) \sum_{k=\ell}^{\infty} u_k \Bigr|^2 \Bigr\}^{1/2} \Bigr\|_{L^p} \approx \Bigl\| \Bigl\{ \sum_{\ell=0}^{\infty} 4^{\ell s} \Bigl| \psi_\ell(D) \sum_{k=\ell}^{\infty} f_k \Bigr|^2 \Bigr\}^{1/2} \Bigr\|_{L^p},$$

where $f_k = \Lambda^{-s} u_k$. (*Hint*: Use arguments similar to those needed for Exercise 1.)

3. Taking $w_k = 2^{ks} f_k$, argue that (6.24) follows given continuity of

$$\Gamma(D) : L^p(\mathbb{R}^n, \ell^2) \longrightarrow L^p(\mathbb{R}^n, \ell^2), \tag{6.25}$$

where

$$\begin{aligned} \Gamma_{k\ell}(\xi) = \psi_k(\xi) 2^{-(\ell-k)s}, &\quad \text{for } \ell \geq k, \\ 0, &\quad \text{for } \ell < k. \end{aligned} \tag{6.26}$$

4. Demonstrate the continuity (6.25), for $p \in (1, \infty)$, $s > 0$.
(*Hint*: To apply Proposition 5.7, you need

$$\|D_\xi^\alpha \Gamma(\xi)\|_{\mathcal{L}(\ell^2)} \leq C_s \langle \xi \rangle^{-|\alpha|}, \quad s > 0.$$

Obtain this by establishing

$$\sum_k |D_\xi^\alpha \Gamma_{k\ell}(\xi)| \leq C \langle \xi \rangle^{-|\alpha|}, \quad s \geq 0,$$

and

$$\sum_\ell |D_\xi^\alpha \Gamma_{k\ell}(\xi)| \leq C_s \langle \xi \rangle^{-|\alpha|}, \quad s > 0.)$$

5. If $u \in H^{n/p,p}(\mathbb{R}^n)$, $p \in (1, \infty)$, show that, for $q \in [p, \infty)$,

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C_n q^{(p-1)/p} \|u\|_{H^{n/p,p}(\mathbb{R}^n)}.$$

Deduce that, for some constant $\gamma = \gamma(u) > 0$,

$$\int_{\mathbb{R}^n} \left(e^{\gamma |u(x)|^{p/(p-1)}} - 1 \right) dx < \infty, \tag{6.27}$$

thus extending Trudinger's estimate (4.10). See [Str].

The purpose of the next exercise is to extend the Gagliardo-Nirenberg estimates (3.10) to nonintegral cases, namely

$$\|u\|_{H^{\lambda, s/p}} \leq C_1 \|u\|_{L^{s/(p-\lambda)}} + C_2 \|u\|_{H^{\lambda+\mu, s/(p+\mu)}}, \tag{6.28}$$

given real p, s, λ, and μ satisfying

$$1 < p < \infty,\ 0 < \mu < s - p, \quad \text{and} \quad \lambda \in (0, p). \tag{6.29}$$

6. Establish the interpolation result

$$\Big[L^{s/(p-\lambda)}(\mathbb{R}^n), H^{\lambda+\mu, s/(p+\mu)}(\mathbb{R}^n) \Big]_\theta \subset H^{\lambda, s/p}(\mathbb{R}^n), \quad \theta = \frac{\lambda}{\lambda + \mu}, \tag{6.30}$$

under the hypotheses (6.29). Show that this implies (6.28).
(*Hint*: If $f = u(\theta)$ belongs to the left side of (6.30), with $u(z)$ holomorphic, $u(iy)$ and $u(1 + iy)$ appropriately bounded, consider $v(z) = \Lambda^{-(\lambda+\mu)z} u(z)$. Use the interpolation result

$$\Big[L^{s/(p-\lambda)}, L^{s/(p+\mu)} \Big]_\theta = L^{s/p}, \quad \theta = \frac{\lambda}{\lambda + \mu}.)$$

Can you treat the $p = \lambda$ case, where $L^{s/(p-\lambda)} = L^\infty$?
7. Extend (6.30) to Sobolev inclusions for $[H^{s,p}, H^{\sigma,q}]_\theta$.

Exercises on generalized div-curl lemmas.

Let M be a compact, oriented Riemannian manifold, and assume that, for $j = 1, \dots, k$, $\nu \in \mathbb{Z}^+$, $\sigma_{j\nu}$ are ℓ_j-forms on M, such that

$$\sigma_{j\nu} \longrightarrow \sigma_j \quad \text{weakly in} \quad L^{p_j}(M), \qquad \text{as } \nu \to \infty, \tag{6.31}$$

and

$$\{d\sigma_{j\nu} : \nu \geq 0\} \quad \text{compact in} \quad H^{-1,p_j}(M). \tag{6.32}$$

Assume that

$$p_j \in (1, \infty), \qquad \frac{1}{p_1} + \cdots + \frac{1}{p_k} \leq 1. \tag{6.33}$$

The goal is to deduce that

$$\sigma_{1\nu} \wedge \cdots \wedge \sigma_{k\nu} \longrightarrow \sigma_1 \wedge \cdots \wedge \sigma_k \quad \text{in} \quad \mathcal{D}'(M), \tag{6.34}$$

as $\nu \to \infty$. An exercise set in §8 of Chapter 5 deals with the case $k = 2$, $p_1 = p_2 = 2$, which includes the div-curl lemma of F. Murat [Mur]. As in that exercise set, we follow [RRT].

1. Show that you can write $\sigma_{j\nu} = d\alpha_{j\nu} + \beta_{j\nu}$, where $\alpha_{j\nu} \to \alpha_j$ weakly in $H^{1,p_j}(M)$ and $\{\beta_{j\nu}\}$ is compact in $L^{p_j}(M)$. (*Hint*: Use the Hodge decomposition $\sigma = d\delta G\sigma + \delta dG\sigma + P\sigma$. Set $\alpha_{j\nu} = \delta G\sigma_{j\nu}$.)
2. Show that, for $j \leq k$,
$$d\alpha_{1\nu} \wedge \cdots \wedge d\alpha_{j\nu} \longrightarrow d\alpha_1 \wedge \cdots \wedge d\alpha_j$$
in $\mathcal{D}'(M)$. If ${p_1}^{-1} + \cdots + {p_j}^{-1} = {q_j}^{-1} < 1$, show that this convergence holds weakly in $L^{q_j}(M)$.
(*Hint*: Use induction on j, via
$$\int d\alpha_{1\nu} \wedge \cdots \wedge d\alpha_{j+1,\nu} \wedge \varphi = \pm \int d\alpha_{1\nu} \wedge \cdots \wedge d\alpha_{j\nu} \wedge \alpha_{j+1,\nu} \wedge d\varphi.)$$
3. Now prove (6.34). (*Hint*: Expand $(d\alpha_{1\nu} + \beta_{1\nu}) \wedge \cdots \wedge (d\alpha_{k\nu} + \beta_{k\nu})$. For a term
$$\pm\Bigl(d\alpha_{\ell_1\nu} \wedge \cdots \wedge d\alpha_{\ell_i\nu}\Bigr) \wedge \Bigl(\beta_{\ell_{i+1}\nu} \wedge \cdots \wedge \beta_{\ell_k\nu}\Bigr),$$
establish and exploit weak L^q-convergence of the first factor (if $i < k$) plus strong L^r convergence of the second factor, with $q^{-1} + r^{-1} \leq 1$.)
4. Localize the result (6.31)–(6.33) $\Rightarrow$ (6.34), replacing M by an open set $\Omega \subset \mathbb{R}^n$. (*Hint*: Apply a cutoff $\chi \in C_0^\infty(\Omega)$.)
5. (The div-curl lemma.) Let $\dim M = 3$, and let X_ν and Y_ν be two sequences of vector fields such that
$$X_\nu \to X \text{ weakly in } L^{p_1}, \quad Y_\nu \to Y \text{ weakly in } L^{p_2},$$

$$\text{div } X_\nu \text{ compact in } H^{-1,p_1}, \quad \text{curl } Y_\nu \text{ compact in } H^{-1,p_2},$$

where $1 < p_j < \infty$, ${p_1}^{-1} + {p_2}^{-1} \le 1$. Show that $X_\nu \cdot Y_\nu \to X \cdot Y$ in $\mathcal{D}'$. Formulate the analogue for dim $M = 2$.

6. Let $F_\nu : \mathbb{R}^n \to \mathbb{R}^n$ be a sequence of maps. Assume

$$F_\nu \to F \text{ weakly in } H^{1,n}(\mathbb{R}^n). \tag{6.35}$$

Show that

$$\det DF_\nu \to \det DF \text{ in } \mathcal{D}'(\mathbb{R}^n). \tag{6.36}$$

(*Hint*: Set $\sigma_{j\nu} = d\alpha_{j\nu} = F_\nu^* dx_j$.)
More generally, if $2 \le k \le n$ and

$$F_\nu \to F \text{ weakly in } H^{1,k}(\mathbb{R}^n), \tag{6.37}$$

then

$$\Lambda^k DF_\nu \to \Lambda^k DF \text{ in } \mathcal{D}'(\mathbb{R}^n), \tag{6.38}$$

and hence

$$\text{Tr } \Lambda^k DF_\nu \to \text{Tr } \Lambda^k DF \text{ in } \mathcal{D}'(\mathbb{R}^n). \tag{6.39}$$

7. L^p-spectral theory of the Laplace operator

We will apply material developed in §§5 and 6 to study spectral properties of the Laplace operator Δ on L^p-spaces. We first consider Δ on $L^p(M)$, where M is a compact Riemannian manifold, without boundary. For any $\lambda > 0$, $(\lambda - \Delta)^{-1}$ is bijective on $\mathcal{D}'(M)$, and results of §6 imply $(\lambda - \Delta)^{-1} : L^p(M) \to H^{2,p}(M)$, provided $1 < p < \infty$. Thus if we define the unbounded operator Δ_p on $L^p(M)$ to be Δ acting on $H^{2,p}(M)$, it follows that Δ_p is a closed operator with nonempty resolvent set, and compact resolvent, hence a discrete spectrum, with finite-dimensional generalized eigenspaces. Elliptic regularity implies that each of these generalized eigenspaces consists of functions in $C^\infty(M)$, and then these functions are easily seen to be actual eigenfunctions. Thus, in such a case, the L^p-spectrum of Δ coincides with its L^2-spectrum.

It is desirable to mention properties of Δ_p, related to spectral properties. In particular, the heat semigroup $e^{t\Delta}$ defines a strongly continuous semigroup $H_p(t)$ on $L^p(M)$, for each $p \in [1, \infty)$. For $p \in [2, \infty)$, this can be seen by applying the L^2-theory, the maximum principle (for data in L^∞), and interpolating, to get $H_p(t) : L^p(M) \to L^p(M)$, for $p \in [2, \infty]$. Strong continuity for $p < \infty$ follows from denseness of $C^\infty(M)$ in $L^p(M)$. Then the action of $H_p(t)$ as a semigroup on $L^p(M)$ for $p \in (1, 2)$ follows by duality. One can also take the adjoint of the action of $e^{t\Delta}$ on $C(M)$ to get $e^{t\Delta}$ acting on $\mathfrak{M}(M)$, the space of finite Borel measures on M, and $e^{t\Delta}$ then preserves $L^1(M)$, the closure of $C^\infty(M)$ in $\mathfrak{M}(M)$.

Alternatively, the strongly continuous action of the heat semigroup on $L^p(M)$ for $p \in [1,\infty)$ can be perceived directly from the parametrix for $e^{t\Delta}$ constructed in Chapter 7, §13.

Let $\mathcal{K}$ be a closed cone in the right half-plane of $\mathbb{C}$, with vertex at 0. Assume $\mathcal{K}$ is symmetric about the positive real axis and has angle $\alpha \in (0,\pi)$. If $P(z) : X \to X$ is a family of bounded operators on a Banach space X, for $z \in \mathcal{K}$, we say it is a holomorphic semigroup if it satisfies $P(z_1)P(z_2) = P(z_1 + z_2)$ for $z_j \in \mathcal{K}$, is strongly continuous in $z \in \mathcal{K}$, and is holomorphic in the interior, $z \in \overset{\circ}{\mathcal{K}}$. The strong continuity implies that $\|P(z)\|$ is locally uniformly bounded on $\mathcal{K}$.

Clearly, $e^{t\Delta}$ gives a holomorphic semigroup on $L^2(M)$. Also, $e^{z\Delta}f$ is defined in $\mathcal{D}'(M)$ whenever $f \in \mathcal{D}'(M)$ and $\operatorname{Re} z \ge 0$, and $e^{z\Delta}f \in C^\infty(M)$ when $\operatorname{Re} z > 0$. Also $u(z,x) = e^{z\Delta}f(x)$ is holomorphic in z in $\{\operatorname{Re} z > 0\}$. This establishes all but one "small" point in the following.

Proposition 7.1. *$e^{z\Delta}$ defines a holomorphic semigroup $H_p(z)$ on $L^p(M)$, for each $p \in [1,\infty)$.*

Proof. Here, $\mathcal{K}$ can be any cone of the sort described above. It remains to establish strong continuity, $H_p(z)f \to f$ in $L^p(M)$ as $z \to 0$ in $\mathcal{K}$, for any $f \in L^p(M)$. Since $C^\infty(M)$ is dense in $L^p(M)$, it suffices to prove that $\{H_p(z) : z \in \mathcal{K}, |z| \le 1\}$ has uniformly bounded operator norm on $L^p(M)$. This can be done by checking that the parametrix construction for $e^{t\Delta}$ extends from $t \in \mathbb{R}^+$ to $z \in \mathcal{K}$, yielding integral operators whose norms on $L^p(M)$ are readily bounded. The reader can check this.

Since the heat semigroup on $L^p(\Omega)$ for a compact manifold with boundary has a parametrix of a form more complicated than it does on $L^p(M)$, this "small" point gets bigger when we extend Proposition 7.1 to the case of compact manifolds with boundary.

Here is a useful property of holomorphic semigroups.

Proposition 7.2. *Let $P(z)$ be a holomorphic semigroup on a Banach space X, with generator A. Then*

$$t > 0,\ f \in X \Longrightarrow P(t)f \in \mathcal{D}(A) \tag{7.1}$$

and

$$\|AP(t)f\|_X \le \frac{C}{t}\|f\|_X, \quad \text{for } 0 < t \le 1. \tag{7.2}$$

Proof. For some $a > 0$, there is a circle $\gamma(t)$, centered at t, of radius $a|t|$, such that $\gamma(t) \in \mathcal{K}$, for all $t \in (0, \infty)$. Thus

$$AP(t)f = P'(t)f = -\frac{1}{2\pi i} \int_{\gamma(t)} (t-\zeta)^{-2} P(\zeta) f \, d\zeta. \tag{7.3}$$

Since $\|P(\zeta)f\| \le C_2 \|f\|$ for $\zeta \in \mathcal{K},\ |\zeta| \le 1 + a$, we have (7.2).

In particular, we have that, for $p \in (1, \infty),\ 0 < t \le 1$,

$$f \in L^p(M) \Longrightarrow \|e^{t\Delta} f\|_{H^{2,p}(M)} \le \frac{C}{t} \|f\|_{L^p(M)}, \tag{7.4}$$

where $C = C_p$. This result could also be verified using the parametrix for $e^{t\Delta}$. Note that applying interpolation to (7.4) yields

$$\|e^{t\Delta} f\|_{H^{s,p}(M)} \le C t^{-s/2} \ \|f\|_{L^p(M)}, \quad \text{for } 0 \le s \le 2,\ 0 < t \le 1, \tag{7.5}$$

when $p \in (1, \infty),\ C = C_p$. We will find it very useful to extend such an estimate to the case of $e^{t\Delta}$ acting on $L^p(\Omega)$ when Ω has a boundary.

We now look at Δ on a compact Riemannian manifold with (smooth) boundary $\overline{\Omega}$, with Dirichlet boundary condition. Assume $\overline{\Omega}$ is connected and $\partial\Omega \ne \emptyset$. We know that, for $\lambda \ge 0$,

$$R_\lambda = (\lambda - \Delta)^{-1} : L^2(\Omega) \to L^2(\Omega), \tag{7.6}$$

with range $H^2(\Omega) \cap H_0^1(\Omega)$. We can analyze $R_\lambda f$ for $f \in L^\infty(\Omega)$ by noting that R_λ is positivity preserving:

$$\lambda \ge 0,\ g \ge 0 \text{ on } \Omega \Longrightarrow R_\lambda g \ge 0 \text{ on } \Omega, \tag{7.7}$$

a result that follows from the positivity property of $e^{t\Delta}$ and the resolvent formula. From this and regularity estimates on $R_\lambda 1$, it easily follows that, for $\lambda \ge 0$,

$$R_\lambda : C(\overline{\Omega}) \to C(\overline{\Omega}) \quad \text{and} \quad R_\lambda : L^\infty(\Omega) \to L^\infty(\Omega). \tag{7.8}$$

Taking the adjoint of R_λ acting on $C(\overline{\Omega})$, we have R_λ acting on $\mathfrak{M}(\overline{\Omega})$, the space of finite Borel measures on $\overline{\Omega}$. Since the closure of $L^2(\Omega)$ in $\mathfrak{M}(\overline{\Omega})$ is $L^1(\Omega)$, we have

$$R_\lambda : L^1(\Omega) \to L^1(\Omega). \tag{7.9}$$

Interpolation yields

$$R_\lambda : L^p(\Omega) \longrightarrow L^p(\Omega), \quad 1 \le p \le \infty. \tag{7.10}$$

We next want to prove that

$$R_\lambda : L^p(\Omega) \longrightarrow H^{2,p}(\Omega), \quad p \in (1, \infty), \tag{7.11}$$

when $\lambda \ge 0$. To do this, it is convenient to assume that $\overline{\Omega} \subset M$, where M is a compact Riemannian manifold without boundary, diffeomorphic to the double of $\overline{\Omega}$. Let $R : M \to M$ be an involution that fixes $\partial\Omega$ and that, near $\partial\Omega$, is the reflection of each geodesic normal to $\partial\Omega$ about the point of intersection of the geodesic with $\partial\Omega$. Then extend f to be 0 on $M \setminus \Omega$, defining $\widetilde{f}$, and define v by

$$(\Delta - \lambda)v = \widetilde{f} \quad \text{on } M, \tag{7.12}$$

so $v \in H^{2,p}(M)$. Set $u = R_\lambda f$. Take

$$u_1(x) = v(x) - v\big(R(x)\big), \quad x \in \Omega. \tag{7.13}$$

With $v^r(x) = v\big(R(x)\big)$, we have $(L - \lambda)v^r(x) = \widetilde{f}\big(R(x)\big)$, where L is the Laplace operator for R^*g, the metric on M pulled back via R. Thus $L = \Delta + L^b$, where L^b is a differential operator of order 2, whose principal symbol vanishes on $\partial\Omega$. Thus $u_1 \in H^{2,p}(\Omega)$, $u_1 = 0$ on $\partial\Omega$, and $w_1 = u - u_1$ satisfies

$$(\Delta - \lambda)w_1 = r_1 \text{ on } \Omega, \quad w_1\big|_{\partial\Omega} = 0, \tag{7.14}$$

with

$$r_1 = (\Delta - \lambda)v^r\big|_\Omega = -L^b v^r\big|_\Omega. \tag{7.15}$$

It follows from (5.49) that

$$L^b v^r\big|_\Omega \in H^{1,p}(\Omega) \subset L^{p_2}(\Omega), \tag{7.16}$$

for some $p_2 > p$. If $p_2 < \infty$, repeat the construction above, applying it to (7.14), to obtain

$$w_1 = u_2 + w_2, \quad u_2 \in H^{2,p_2}(\Omega), \quad u_2\big|_{\partial\Omega} = 0, \tag{7.17}$$

and

$$(\Delta - \lambda)w_2 = r_2 \text{ on } \Omega, \quad w_2\big|_{\partial\Omega} = 0, \quad r_2 \in H^{1,p_2}(\Omega) \subset L^{p_3}(\Omega). \tag{7.18}$$

Continue, obtaining

$$u = u_1 + \cdots + u_k + w_k, \quad u_j \in H^{2,p_j}(\Omega), \quad u_j\big|_{\partial\Omega} = 0, \tag{7.19}$$

such that

$$(\Delta - \lambda)w_j = r_j \text{ on } \Omega, \quad w_j\big|_{\partial\Omega} = 0, \quad r_j \in H^{1,p_j}(\Omega) \subset L^{p_{j+1}}(\Omega). \tag{7.20}$$

We continue until $p_k > n = \dim \Omega$. At this point, we use a couple of results that will be established in the next section. Given $s \in (0,1)$, let $C^s(\overline{\Omega})$ denote the space of Hölder-continuous functions on $\overline{\Omega}$, with Hölder exponent s. We have

$$r_k \in H^{1,p_k}(\Omega) \subset C^s(\overline{\Omega}), \tag{7.21}$$

for some $s \in (0,1)$, appealing to Proposition 8.5 for the last inclusion in (7.21). Then the estimates in Theorem 8.9 imply

$$w_k \in C^{2+s}(\overline{\Omega}) \subset H^{2,p}(\Omega). \tag{7.22}$$

This proves (7.11).

Arguments parallel to those used for M show that the heat semigroup $e^{t\Delta}$, defined a priori on $L^2(\Omega)$, yields also a well-defined, strongly continuous semigroup $H_p(t)$ on $L^p(\Omega)$, for each $p \in [1,\infty)$. If Δ_p denotes the generator of the heat semigroup on $L^p(\Omega)$, with Dirichlet boundary condition, then (7.11) implies

$$\mathcal{D}(\Delta_p) \subset H^{2,p}(\Omega), \quad p \in (1,\infty). \tag{7.23}$$

We see that Δ_p has compact resolvent. Furthermore, arguments such as used above for M show that the spectrum of Δ_p coincides with the L^2-spectrum of Δ.

We now extend Proposition 7.1.

Proposition 7.3. *For $p \in (1,\infty)$, $e^{z\Delta}$ defines a holomorphic semigroup on $L^p(\Omega)$, on any symmetric cone $\mathcal{K}$ about $\mathbb{R}^+$ of angle $< \pi$.*

Proof. As in the proof of Proposition 7.1, the point we need to establish is the local uniform boundedness of the $L^p(\Omega)$-operator norm of $e^{z\Delta}$, for $z \in \mathcal{K}$. In other words, we need estimates for the solution u to

$$\frac{\partial u}{\partial t} = \Delta u \text{ on } \mathcal{K} \times \Omega, \quad u(0) = f, \quad u\big|_{\mathcal{K}\times\partial\Omega} = 0, \tag{7.24}$$

of the form

$$\|u(t)\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}, \quad t \in \mathcal{K}, \quad \text{Re}\, t \le 1. \tag{7.25}$$

By duality, it suffices to do this for $p \in (1,2]$. The case $p = 2$ is obvious, so for the rest of the proof we will assume $p \in (1,2)$. We will also assume $n = \dim \Omega > 1$, since the reflection principle works easily when $n = 1$.

To begin, define v by

$$\frac{\partial v}{\partial t} = \Delta v \text{ on } \mathcal{K} \times M, \quad v(0) = \tilde{f} \in L^p(M), \tag{7.26}$$

where $\tilde{f}$ is f on Ω, zero on $M \setminus \Omega$. Making use of Proposition 7.2, which we know applies to $e^{t\Delta}$ on $L^p(M)$, we have

$$\|v(t)\|_{H^{1,p}(M)} \le C|t|^{-1/2} \; \|f\|_{L^p(\Omega)}. \tag{7.27}$$

Now, if $R : M \to M$ is the involution on M used above, for $x \in \Omega$ we set

$$u_1(t,x) = v(t,x) - v\big(t,R(x)\big); \quad u_1 \in C\big(\mathcal{K}, L^p(\Omega)\big). \tag{7.28}$$

We have

$$\frac{\partial u_1}{\partial t} = \Delta u_1 + g \text{ on } \mathcal{K} \times \Omega, \quad u_1(0) = f, \quad u_1\big|_{\mathcal{K}\times\partial\Omega} = 0, \tag{7.29}$$

and, by an argument parallel to (7.16), we derive from (7.27) an estimate

$$\|g(t)\|_{L^p(\Omega)} \le C|t|^{-1/2} \; \|f\|_{L^p(\Omega)}. \tag{7.30}$$

In this case, we replace appeal to (5.49) by the parametrix construction for $e^{t\Delta}$ on $\mathcal{D}'(M)$ made in Chapter 7, §13.

We regard u_1 as a first approximation to u, but we seek a more accurate approximation rather than rely on an estimate at this point of the error. So now we define v_2 by

$$\frac{\partial v_2}{\partial t} = \Delta v_2 - \widetilde{g} \text{ on } \mathcal{K} \times M, \quad v_2(0) = 0, \tag{7.31}$$

where $\widetilde{g}$ is g on $\mathcal{K} \times \Omega$ and zero on $\mathcal{K} \times (M \setminus \Omega)$. We have

$$v_2(t) = -\int_0^t e^{(t-s)\Delta} \widetilde{g}(s)\, ds, \tag{7.32}$$

and the estimate $\|\widetilde{g}(s)\|_{L^p(M)} \le C|s|^{-1/2}$ from (7.30), together with the operator norm estimate of $e^{(t-s)\Delta}$ on $L^p(M)$, from Proposition 7.2, yields

$$v_2 \in C\big(\mathcal{K}, H^{1,p}(M)\big). \tag{7.33}$$

Now, for $x \in \Omega$, set

$$u_2(t,x) = v_2(t,x) - v_2\big(t, R(x)\big); \quad u_2 \in C\big(\mathcal{K}, H^{1,p}(\Omega)\big). \tag{7.34}$$

Thus

$$\frac{\partial u_2}{\partial t} = \Delta u_2 - g + g_2 \text{ on } \mathcal{K} \times \Omega, \quad u_2(0) = 0, \quad u_2\big|_{\mathcal{K} \times \partial\Omega} = 0, \tag{7.35}$$

and we have, parallel to but better than (7.30),

$$\|g_2(t)\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}. \tag{7.36}$$

Next, solve

$$\frac{\partial v_3}{\partial t} = \Delta v_3 - \widetilde{g}_2 \text{ on } \mathcal{K} \times M, \quad v_3(0) = 0, \tag{7.37}$$

where $\widetilde{g}_2$ is g_2 on $\mathcal{K} \times \Omega$ and zero on $\mathcal{K} \times (M \setminus \Omega)$. The argument involving (7.32) and (7.33) this time yields the better estimate

$$v_3 \in C\big(\mathcal{K}, H^{2-\varepsilon,p}(M)\big), \quad \forall\, \varepsilon > 0, \tag{7.38}$$

hence, by the Sobolev imbedding result of Proposition 6.4, with $s = 1 - \varepsilon$,

$$v_3 \in C\big(\mathcal{K}, H^{1,p_3}(M)\big), \quad p_3 = \frac{np}{n - (1-\varepsilon)p} > p, \tag{7.39}$$

provided $p < n$. Now we set

$$u_3(t,x) = v_3(t,x) - v_3\big(t, R(x)\big); \quad u_3 \in C\big(\mathcal{K}, H^{1,p_3}(\Omega)\big), \tag{7.40}$$

and we get

$$\frac{\partial u_3}{\partial t} = \Delta u_3 - g_2 + g_3 \text{ on } \mathcal{K} \times \Omega, \quad u_3(0) = 0, \quad u_3\big|_{\mathcal{K} \times \partial\Omega} = 0, \tag{7.41}$$

with the following improvement on (7.36):

$$\|g_3(t)\|_{L^{p_3}(\Omega)} \le C\|f\|_{L^p(\Omega)}. \tag{7.42}$$

Continuing in this fashion, we get

$$u_j \in C\big(\mathcal{K}, H^{2-\varepsilon,p_{j-1}}(\Omega)\big) \subset C\big(\mathcal{K}, H^{1,p_j}(\Omega)\big), \tag{7.43}$$

with $p = p_2 < p_3 < \cdots \nearrow$. Given $p \in (1,2)$, some p_k is ≥ 2. Then $u_k \in C\big(\mathcal{K}, H^1(\Omega)\big)$ satisfies

$$\frac{\partial u_k}{\partial t} = \Delta u_k - g_{k-1} + g_k \text{ on } \mathcal{K} \times \Omega, \quad u_k(0) = 0, \quad u_k\big|_{\mathcal{K}\times\partial\Omega} = 0, \tag{7.44}$$

with

$$g_k \in C\big(\mathcal{K}, L^2(\Omega)\big). \tag{7.45}$$

Now we solve for w the equation

$$\frac{\partial w}{\partial t} = \Delta w - g_k \text{ on } \mathcal{K} \setminus \Omega, \quad w(0) = 0, \quad w\big|_{\mathcal{K}\times\partial\Omega} = 0. \tag{7.46}$$

The easy L^2-estimates yield

$$w \in C\big(\mathcal{K}, H^{2-\varepsilon}(\Omega)\big), \tag{7.47}$$

and the solution to (7.24) is

$$u = u_1 + \cdots + u_k + w. \tag{7.48}$$

This proves the desired estimate (7.25), for $p \in (1,2)$, which is enough to prove Proposition 7.3.

We mention that an interpolation argument yields that $e^{z\Delta}$ is a holomorphic semigroup on $L^p(\Omega)$ on a cone $\mathcal{K}$ that is symmetric about $\mathbb{R}^+$ and has angle $\pi\big(1 - |2/p - 1|\big)$. (See [RS], Vol. 2, p. 255.) This result is valid even if Ω has nasty boundary, as well as in other settings. On the other hand, ingredients of the argument used above will also be useful for other results, presented below.

Note that once we have the holomorphy of $e^{t\Delta}$ on $L^p(\Omega)$, for all $p \in (1,\infty)$, we can apply Proposition 7.2. In particular, suppose we carry out the construction of the u_k above, not stopping as soon as $p_k \geq 2$, but letting p_k become arbitrarily large. Then (7.44) is replaced by $g_k \in C\big(\mathcal{K}, L^{p_k}(\Omega)\big)$, and we can now apply Proposition 7.2 to improve (7.47) to

$$w \in C\big(\mathcal{K}, H^{2-\varepsilon,p_k}(\Omega)\big), \tag{7.49}$$

making use of (7.2), (7.11), and interpolation to estimate the norm of $e^{t\Delta} : L^p(\Omega) \to H^{2-\varepsilon,p}(\Omega)$.

We now consider the construction (7.24)–(7.44) when $u(0) = f \in L^\infty(\Omega)$. We will restrict attention to $t \in \mathbb{R}^+$. A direct inspection of the parametrix for the heat kernel, constructed in Chapter 7, §13, shows that $e^{t\Delta} : L^\infty(M) \to C^1(M)$, with norm $\leq Ct^{-1/2}$, for $t \in (0,1]$, so v in (7.26) satisfies the estimate $\|v(t)\|_{C^1(M)} \leq Ct^{-1/2}\|f\|_{L^\infty(\Omega)}$, and $\|u_1(t)\|_{C^1(\overline{\Omega})}$ satisfies a similar estimate. Thus g in (7.29) satisfies the estimate (7.30), with

$p = \infty$, and consequently v_2 in (7.32) satisfies $\|v_2(t)\|_{C^1(M)} \le C$. Hence $\|u_2(t)\|_{C^1(\overline{\Omega})} \le C$, and g_2 in (7.35) satisfies (7.36) with $p = \infty$. Thus $u = u_1 + u_2 + w$, where w satisfies

$$\frac{\partial w}{\partial t} = \Delta w - g_2 \text{ on } \mathbb{R}^+ \times \Omega, \quad w(0) = 0, \quad w\big|_{\mathbb{R}^+ \times \partial\Omega} = 0. \tag{7.50}$$

By the holomorphy of $e^{t\Delta}$ on $L^p(\Omega)$ for $p \in (1,\infty)$, we have

$$w \in C\big([0,\infty), H^{2-\varepsilon,p}(\Omega)\big), \tag{7.51}$$

for any $\varepsilon > 0$ and arbitrarily large $p < \infty$, hence $w \in C\big(\mathbb{R}^+, C^{2-\delta}(\overline{\Omega})\big)$, for any $\delta > 0$. We deduce that

$$\|e^{t\Delta} f\|_{C^1(\overline{\Omega})} \le C t^{-1/2} \, \|f\|_{L^\infty(\Omega)}, \quad 0 < t \le 1. \tag{7.52}$$

The estimate (7.52), together with the following result, will be useful for the study of semilinear parabolic equations on domains with boundary, in §3 of Chapter 15.

Proposition 7.4. *If $\overline{\Omega}$ is a compact Riemannian manifold with boundary, on which the Dirichlet condition is placed, then $e^{t\Delta}$ defines a strongly continuous semigroup on the Banach space*

$$C_b^1(\overline{\Omega}) = \big\{ f \in C^1(\overline{\Omega}) : f|_{\partial\Omega} = 0 \big\}. \tag{7.53}$$

Proof. It is easy to verify that, for $N > 1 + (\dim M)/4$,

$$\mathcal{D}(\Delta^N) \subset C_b^1(\overline{\Omega}), \quad \text{densely.}$$

Since $e^{t\Delta}$ is a strongly continuous semigroup on $\mathcal{D}(\Delta^N)$, it suffices to show that for each $f \in C_b^1(\overline{\Omega})$, $\{e^{t\Delta} f : 0 \le t \le 1\}$ is uniformly bounded in $\mathrm{Lip}(\overline{\Omega})$. To see this, we analyze solutions to

$$\frac{\partial u}{\partial t} = \Delta u, \text{ for } x \in \Omega, \quad u(0,x) = f(x), \quad u(t,x) = 0, \text{ for } x \in \partial\Omega,$$

when

$$f \in C^1(\overline{\Omega}), \quad f\big|_{\partial\Omega} = 0. \tag{7.54}$$

We will to some extent follow the proof of Proposition 7.3, and also use that result. In this case, for $\widetilde{f}$ equal to f on $\overline{\Omega}$ and to zero on $M \setminus \overline{\Omega}$, we have $\widetilde{f} \in \mathrm{Lip}(M)$. Thus, for v defined by

$$\frac{\partial v}{\partial t} = \Delta v \text{ on } \mathbb{R}^+ \times M, \quad v(0) = \widetilde{f},$$

we have

$$v \in \mathcal{C}\big(\mathbb{R}^+, \mathrm{Lip}(M)\big), \tag{7.55}$$

where the "$\mathcal{C}$" stands for "weak" continuity in t, (i.e., $v(t)$ is bounded in $\mathrm{Lip}(M)$ and continuous in t, with values in $H^{1,p}(M)$, for each $p < \infty$).

Hence

$$u_1(t,x) = v(t,x) - v\big(t, R(x)\big)\big|_{\mathbb{R}^+\times\overline{\Omega}}$$

satisfies

$$u_1 \in \mathcal{C}\big(\mathbb{R}^+, \mathrm{Lip}(\overline{\Omega})\big). \tag{7.56}$$

We have

$$\frac{\partial u_1}{\partial t} = \Delta u_1 + g, \quad u_1(0) = f, \quad u_1\big|_{\mathbb{R}^+\times\partial\Omega} = 0,$$

where

$$g = L^b v^r\big|_{\mathbb{R}^+\times\Omega}.$$

Here, as in (7.15), L^b is a second-order differential operator whose principal symbol vanishes on $\partial\Omega$, and $v^r(x) = v\big(R(x)\big)$. Consequently, again an analogue of (5.49) gives

$$g \in \mathcal{C}\big(\mathbb{R}^+, L^\infty(\Omega)\big). \tag{7.57}$$

Now, we have $u = u_1 + w$, where w satisfies

$$\frac{\partial w}{\partial t} = \Delta w - g, \quad w(0) = 0, \quad w\big|_{\mathbb{R}^+\times\partial\Omega} = 0, \tag{7.58}$$

and, by (7.57), $g \in C\big(\mathbb{R}^+, L^p(\Omega)\big)$, for all $p < \infty$. This implies

$$w \in C\big(\mathbb{R}^+, H^{2-\varepsilon,p}(\Omega)\big), \quad \forall p < \infty,\ \varepsilon > 0, \tag{7.59}$$

since $e^{t\Delta}$ is a holomorphic semigroup on $L^p(\Omega)$. This proves Proposition 7.4.

Exercises

1. Extend results of this section to the Neumann boundary condition.

In Exercises 2 and 3, let Ω be an open subset, with smooth boundary, of a compact Riemannian manifold M. Assume there is an isometry $\tau : M \to M$ that is an involution, fixing $\partial\Omega$, so M is the isometric double of $\overline{\Omega}$.

2. Suppose X_j are smooth vector fields on $\overline{\Omega}$, $f_j \in L^p(\Omega)$ for some $p \in [2,\infty)$, and u is the unique solution in $H_0^{1,2}(\Omega)$ to
$$\Delta u = \sum X_j f_j.$$
Show that $u \in H^{1,p}(\Omega)$. (*Hint*: Reduce to the case where each X_j is a smooth vector field on M, such that $\tau_\# X_j = \pm X_j$. Extend f_j to $f_j \in L^p(M)$, so that $\tau^* f_j = \mp f_j$. Thus $\sum X_j f_j \in H^{-1,p}(M)$ is odd under τ.)
3. Extend the result of Exercise 2 to the case $f_j \in L^p(\Omega)$ when $1 < p < 2$, appropriately weakening the a priori hypothesis on u.

4. Try to extend the results of Exercises 2 and 3 to general, compact, smooth $\overline{\Omega}$, not necessarily having an isometric double.
5. Show that (7.5) can be improved to

$$R_\lambda : L^\infty(\Omega) \longrightarrow C(\overline{\Omega}),$$

for $\lambda \geq 0$. (*Hint*: Use (7.11). Show that, in fact, for $\lambda \geq 0$,

$$R_\lambda : L^\infty(\Omega) \longrightarrow C^r(\overline{\Omega}), \quad \forall\, r < 2.)$$

A sharper result will be contained in (8.54)–(8.55).

8. Hölder spaces and Zygmund spaces

If $0 < s < 1$, we define the space $C^s(\mathbb{R}^n)$ of Hölder-continuous functions on $\mathbb{R}^n$ to consist of bounded functions u such that

$$|u(x+y) - u(x)| \leq C|y|^s. \tag{8.1}$$

For $k = 0, 1, 2, \dots,$ we take $C^k(\mathbb{R}^n)$ to consist of bounded, continuous functions u such that $D^\beta u$ is bounded and continuous, for $|\beta| \leq k$. If $s = k + r,\ 0 < r < 1$, we define $C^s(\mathbb{R}^n)$ to consist of functions $u \in C^k(\mathbb{R}^n)$ such that, for $|\beta| = k,\ D^\beta u$ belongs to $C^r(\mathbb{R}^n)$.

For nonintegral s, the Hölder spaces $C^s(\mathbb{R}^n)$ have a characterization similar to that for L^p and more generally $H^{s,p}$, in (5.46) and (6.23), via the Littlewood-Paley partition of unity used in (5.37),

$$1 = \sum_{j=0}^{\infty} \varphi_j(\xi)^2,$$

with φ_j supported on $\langle \xi \rangle \sim 2^j$, and $\varphi_j(\xi) = \varphi_1(2^{1-j}\xi)$ for $j \geq 1$. Let $\psi_j(\xi) = \varphi_j(\xi)^2$.

Proposition 8.1. *If $u \in C^s(\mathbb{R}^n)$, then*

$$\sup_k\ 2^{ks} \|\psi_k(D)u\|_{L^\infty} < \infty. \tag{8.2}$$

Proof. To see this, first note that it is obvious for $s = 0$. For $s = \ell \in \mathbb{Z}^+$, it then follows from the elementary estimate

$$\begin{aligned} C_1 2^{k\ell} \|\psi_k(D)u(x)\|_{L^\infty} &\leq \sum_{|\alpha| \leq \ell} \|\psi_k(D) D^\alpha u(x)\|_{L^\infty} \\ &\leq C_2 2^{k\ell} \|\psi_k(D)u(x)\|_{L^\infty}. \end{aligned} \tag{8.3}$$

Thus it suffices to establish that $u \in C^s$ implies (8.2) for $0 < s < 1$. Since $\hat{\psi}_1(x)$ has zero integral, we have, for $k \geq 1$,

$$
\begin{aligned}
|\psi_k(D)u(x)| &= \Big| \int \hat{\psi}_k(y) \big[u(x-y) - u(x) \big] \, dy \Big| \\
&\leq C \int |\hat{\psi}_k(y)| \cdot |y|^s \, dy,
\end{aligned}
\tag{8.4}
$$

which is readily bounded by $C \cdot 2^{-ks}$.

This result has a partial converse.

Proposition 8.2. *If s is not an integer, finiteness in* (8.2) *implies* $u \in C^s(\mathbb{R}^n)$.

Proof. It suffices to demonstrate this for $0 < s < 1$. With $\Psi_k(\xi) = \sum_{j \leq k} \psi_j(\xi)$, if $|y| \sim 2^{-k}$, write

$$
\begin{aligned}
u(x+y) - u(x) = &\int_0^1 y \cdot \nabla \Psi_k(D) u(x+ty) \, dt \\
&+ \big(I - \Psi_k(D) \big) \big(u(x+y) - u(x) \big)
\end{aligned}
\tag{8.5}
$$

and use (8.2) and (8.3) to dominate the L^∞-norm of both terms on the right by $C \cdot 2^{-sk}$, since $\|\nabla \Psi_k(D)u\|_{L^\infty} \leq C \cdot 2^{(1-s)k}$.

This converse breaks down if $s \in \mathbb{Z}^+$. We define the *Zygmund space* $C^s_*(\mathbb{R}^n)$ to consist of u such that (8.2) is finite, using that to define the C^s_*-norm, namely,

$$
\|u\|_{C^s_*} = \sup_k \; 2^{ks} \|\psi_k(D)u\|_{L^\infty}. \tag{8.6}
$$

Thus

$$
C^s = C^s_* \;\text{ if } s \in \mathbb{R}^+ \setminus \mathbb{Z}^+, \quad C^k \subset C^k_*, \quad k \in \mathbb{Z}^+. \tag{8.7}
$$

The class $C^s_*(\mathbb{R}^n)$ can be defined for any $s \in \mathbb{R}$, as the set of elements $u \in \mathcal{S}'(\mathbb{R}^n)$ such that (8.6) is finite.

The following complements previous boundedness results for Fourier multipliers $P(D)$ on $L^p(\mathbb{R}^n)$ and on $H^{s,p}(\mathbb{R}^n)$.

Proposition 8.3. *If $P(\xi) \in S^m_1(\mathbb{R}^n)$, then, for all $s \in \mathbb{R}$,*

$$
P(D) : C^s_* \longrightarrow C^{s-m}_*. \tag{8.8}
$$

Proof. Consider first the case $m = 0$. Pick $\tilde{\psi}_j(\xi) \in C_0^\infty(\mathbb{R}^n)$ such that $\tilde{\psi}_j(\xi) = 1$ on supp ψ_j and $\tilde{\psi}_j(\xi) = \tilde{\psi}_1(2^{1-j}\xi)$, for $j \geq 2$. It follows readily from the analysis of the Schwartz kernel of $P(D)$ made in §2 of Chapter 7, particularly in the proof of Proposition 2.2 there, that

$$P(\xi) \in S^0_1(\mathbb{R}^n) \Longrightarrow \sup_j \|\tilde{\psi}_j(\xi)P(\xi)\|_{\mathcal{F}L^1} < \infty, \tag{8.9}$$

where $\|Q\|_{\mathcal{F}L^1} = \|\widehat{Q}\|_{L^1}$. Also, it is clear that

$$\|\psi_k(D)P(D)u\|_{L^\infty} \le C\|\tilde{\psi}_k P\|_{\mathcal{F}L^1} \cdot \|\psi_k(D)u\|_{L^\infty}, \tag{8.10}$$

which implies (8.8) for $m = 0$. The extension to general $m \in \mathbb{R}$ is straightforward.

In particular, with $\Lambda = (1 - \Delta)^{1/2}$,

$$\Lambda^m : C^s_* \longrightarrow C^{s-m}_* \quad \text{is an isomorphism.} \tag{8.11}$$

Note that in light of (8.9) and (8.10), we have

$$\|P(D)u\|_{C^s_*} \le C \sup_{\xi\in\mathbb{R}^n, |\alpha|\le[n/2]+1} \|P^{(\alpha)}(\xi)\langle\xi\rangle^{|\alpha|}\|_{L^\infty} \cdot \|u\|_{C^s_*}. \tag{8.12}$$

In particular, for $y \in \mathbb{R}$,

$$\|\Lambda^{iy}u\|_{C^s_*} \le C\langle y\rangle^{n/2+1}\|u\|_{C^s_*}. \tag{8.13}$$

Compare with (5.47).

The Sobolev imbedding theorem, Proposition 6.3, can be sharpened and extended to the following:

Proposition 8.4. *For all $s \in \mathbb{R},\ p \in (1, \infty)$,*

$$H^{s,p}(\mathbb{R}^n) \subset C^r_*(\mathbb{R}^n), \quad r = s - \frac{n}{p}. \tag{8.14}$$

Proof. In light of (8.11), it suffices to consider the case $s = n/p$. Let $L_m(\xi) \in S^m_1(\mathbb{R}^n)$ be nowhere vanishing and satisfy $L_m(\xi) = |\xi|^m$, for $|\xi| \ge 1/100$. It suffices to show that, for $p \in (1, \infty)$,

$$\|\psi_k(D)L_{-n/p}(D)u\|_{L^\infty} \le C\|u\|_{L^p(\mathbb{R}^n)}, \tag{8.15}$$

with C independent of k. We can restrict attention to $k \ge 2$. Then $A_k(\xi) = \psi_k(\xi)L_{-n/p}(\xi)$ satisfies

$$A_{k+1}(\xi) = 2^{-nk/p}\ A_1(2^{-k}\xi).$$

Hence $\widehat{A}_{k+1}(x) \in \mathcal{S}(\mathbb{R}^n)$ and

$$\|\widehat{A}_{k+1}\|_{L^{p'}(\mathbb{R}^n)} = C, \quad \text{independent of } k \ge 2. \tag{8.16}$$

Thus the left side of (8.15) is dominated by $\|\widehat{A}_k\|_{L^{p'}} \cdot \|u\|_{L^p}$, which in turn is dominated by the right side of (8.15). This completes the proof.

It is useful to extend Proposition 8.3 to the following.

Proposition 8.5. *If* $p(x,\xi) \in S^m_{1,0}(\mathbb{R}^n)$*, then, for* $s \in \mathbb{R}$*,*

$$p(x,D) : C^s_*(\mathbb{R}^n) \longrightarrow C^{s-m}_*(\mathbb{R}^n). \tag{8.17}$$

Proof. In light of (8.11), it suffices to consider the case $m = 0$. Also, it suffices to consider one fixed s, which we can take to be positive. First we prove (8.18) in the special case where $p(x,\xi)$ has compact support in x. Then we can write

$$p(x,D)u = \int e^{ix\cdot\eta}\, q_\eta(D)u\, d\eta, \tag{8.18}$$

with

$$q_\eta(\xi) = (2\pi)^{-n} \int e^{-ix\cdot\eta}\, p(x,\xi)\, dx. \tag{8.19}$$

Via the estimates used to prove Proposition 8.3, it follows that, for any given $s \in \mathbb{R}$, $q_\eta(D) \in \mathcal{L}\big(C^s_*(\mathbb{R}^n)\big)$ has an operator norm that is a rapidly decreasing function of η. It is easy to establish the estimate

$$\|e^{ix\cdot\eta}\, u\|_{C^s_*} \le C(s)\, \langle\eta\rangle^s\, \|u\|_{C^s_*} \quad (s > 0), \tag{8.20}$$

first for $s \notin \mathbb{Z}^+$, by using the characterization (8.1) of $C^s = C^s_*$, then for general $s > 0$ by interpolation. The desired operator bound on (8.18) follows easily.

To do the general case, one can use a partition of unity in the x-variables, of the form

$$1 = \sum_{j\in\mathbb{Z}^n} \varphi_j(x), \quad \varphi_j(x) = \varphi_0(x+j), \quad \varphi_0 \in C^\infty_0(\mathbb{R}^n),$$

and exploit the estimates on $p_j(x,D)u = \varphi_j(x)p(x,D)u$ obtained by the argument above, in concert with the rapid decrease of the Schwartz kernel of the operator $p(x,D)$ away from the diagonal. Details are left to the reader.

In §9 we will establish a result that is somewhat stronger than Proposition 8.5, but this relatively simple result is already useful for Hölder estimates on solutions to linear, elliptic PDE.

It is useful to note that we can define Zygmund spaces $C^s_*(\mathbb{T}^n)$ on the torus just as in (8.6), but using Fourier series. We again have (8.7) and Propositions 8.3–8.5.

The issue of how Zygmund spaces form a complex interpolation scale is more subtle than the analogous situation for L^p-Sobolev spaces, treated in §6. A different type of complex interpolation functor, $[X,Y]^b_\theta$, defined in Appendix A at the end of this chapter, does a better job than $[X,Y]_\theta$. We have the following result established in Appendix A.

Proposition 8.6. *For* $r, s \in \mathbb{R},\ \theta \in (0,1)$,

$$[C_*^r(\mathbb{T}^n), C_*^s(\mathbb{T}^n)]_\theta^b = C_*^{\theta s+(1-\theta)r}(\mathbb{T}^n). \tag{8.21}$$

It is straightforward to extend the notions of Hölder and Zygmund spaces to spaces $C^s(M)$ and $C_*^s(M)$ when M is a compact manifold without boundary. Furthermore, the analogue of (8.21) is readily established, and we have

$$P : C_*^s(M) \longrightarrow C_*^{s-m}(M) \quad \text{if } P \in OPS_{1,0}^m(M). \tag{8.22}$$

If $\overline{\Omega}$ is a compact manifold with boundary, there is an obvious notion of $C^s(\overline{\Omega})$, for $s \geq 0$. We will define $C_*^s(\overline{\Omega})$ below, for $s \geq 0$. For now we look further at $C^s(\overline{\Omega})$. The following simple observation is useful. Give $\overline{\Omega}$ a Riemannian metric and let $\delta(x) = \operatorname{dist}(x, \partial\Omega)$.

Proposition 8.7. *Let* $r \in (0,1)$. *Assume* $f \in C^1(\Omega)$ *satisfies*

$$|\nabla f(x)| \leq C\ \delta(x)^{r-1}, \quad x \in \Omega. \tag{8.23}$$

Then f *extends continuously to* $\overline{\Omega}$, *as an element of* $C^r(\overline{\Omega})$.

Proof. There is no loss of generality in assuming that Ω is the unit ball in $\mathbb{R}^n$. When estimating $f(x_2) - f(x_1)$, we may as well assume that x_1 and x_2 are a distance $\leq 1/4$ from $\partial\Omega$ and $|x_1 - x_2| \leq 1/4$. Write

$$f(x_2) - f(x_1) = \int_\gamma df(x),$$

where γ is a path from x_1 to x_2 of the following sort. Let y_j lie on the ray segment from 0 to x_j, a distance $d = |x_1 - x_2|$ from x_j. Then γ goes from x_1 to y_1 on a line, from y_1 to y_2 on a line, and from y_2 to x_2 on a line, as illustrated in Fig. 8.1. Then

$$|f(x_j) - f(y_j)| \leq C \int_{1-d}^1 (1-\rho)^{r-1}\ d\rho = C \int_0^d \tau^{r-1}\ d\tau \leq C' d^r, \tag{8.24}$$

while

$$|f(y_1) - f(y_2)| \leq C|y_1 - y_2| d^{r-1} \leq C' d^r, \tag{8.25}$$

so

$$|f(x_2) - f(x_1)| \leq C|x_1 - x_2|^r, \tag{8.26}$$

as asserted.

Now consider $\overline{\Omega}$ of the form $\overline{\Omega} = [0,1] \times M$, where M is a compact Riemannian manifold without boundary. We want to consider the action on $f \in C^r(M)$ of a family of operators of Poisson integral type, such

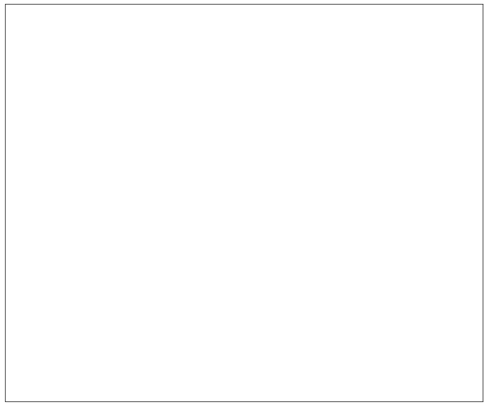

FIGURE 8.1

as were studied in Chapter 7, §12, to construct parametrices for regular elliptic boundary problems. We recall from (12.35) of Chapter 7 the class $OP\mathcal{P}^{-j}$ consisting of families $A(y)$ of pseudodifferential operators on M, parameterized by $y \in [0,1]$:

$$A(y) \in OP\mathcal{P}^{-j} \Longleftrightarrow y^k D_y^\ell A(y) \text{ bounded in } OPS_{1,0}^{-j-k+\ell}(M). \tag{8.27}$$

Furthermore, if $L \in OPS^1(M)$ is a positive, self-adjoint, elliptic operator, then operators of the form $A(y)e^{-yL}$, with $A(y) \in OP\mathcal{P}^{-j}$, belong to $OP\mathcal{P}_e^{-j}$. In addition (see (12.50)), any $A(y) \in OP\mathcal{P}_e^{-j}$ can be written in the form $e^{-yL}B(y)$ for some such elliptic L and some $B(y) \in OP\mathcal{P}^{-j}$. The following result is useful for Hölder estimates on solutions to elliptic boundary problems.

Proposition 8.8. *If $A(y) \in OP\mathcal{P}_e^{-j}$ and $f \in C_*^r(M)$, then*

$$u(y,x) = A(y)f(x) \Longrightarrow u \in C^{j+r}(I \times M), \tag{8.28}$$

provided $j + r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$.

Note that we allow $r < 0$ if $j > 0$.

Proof. First consider the case $j = 0,\ 0 < r < 1$, and write

$$A(y)f = e^{-\kappa y \Lambda}\, B(y)f, \quad B(y) \in OP\mathcal{P}^0. \tag{8.29}$$

We can assume without loss of generality that $\Lambda = (1-\Delta)^{1/2}$, and we can replace M by $\mathbb{R}^n$. In such a case, we will show that

$$|\nabla_{y,x} u(y,x)| \le C y^{r-1}\, \|u\|_{C^r} \tag{8.30}$$

if $0 < r < 1$, which by Proposition 8.7 will yield $u \in C^r(I \times M)$. Now if we set $\partial_j = \partial/\partial x_j$ for $1 \le j \le n,\ \partial_0 = \partial/\partial y$, then we can write

$$y\partial_j u(y,x) = y\Lambda e^{-\kappa y \Lambda} B_j(y)f, \quad B_j(y) \in OP\mathcal{P}^0. \tag{8.31}$$

Now, given $f \in C^r(M),\ 0 < r < 1$, we have $B_j(y)f$ bounded in $C^r(M)$, for $y \in [0,1]$. Then the estimate (8.30) follows from

$$\|\varphi(y\Lambda)g\|_{L^\infty} \le Cy^r\|g\|_{C^r_*}, \tag{8.32}$$

for $0 < r < 1$, where $\varphi(\lambda) = \lambda e^{-\kappa\lambda}$, which vanishes at $\lambda = 0$ and is rapidly decreasing as $\lambda \to +\infty$. In turn, this follows easily from the characterization (8.6) of the C^r_*-norm.

If $f \in C^{k+r}(M),\ k \in \mathbb{Z}^+,\ 0 < r < 1$, and $j = 0$ then given $|\alpha| \le k$,

$$D^\alpha_{y,x}u = e^{-\kappa y\Lambda}B_\alpha(y)\Lambda^k f, \quad B_\alpha(y) \in OP\mathcal{P}^0, \tag{8.33}$$

so the analysis of (8.29), with f replaced by $\Lambda^k f$, applies to yield $D^\alpha_{y,x}u \in C^r(I \times M)$, for $|\alpha| \le k$.

Similarly, the extension from $j = 0$ to general $j \in \mathbb{Z}^+$ is straightforward, so Proposition 8.8 is proved.

As we have said above, Proposition 8.8 is important because it yields Hölder estimates on solutions to elliptic boundary problems, as defined in Chapter 5, §11. The principal consequence is the following:

Theorem 8.9. *Let $(P, B_j, 1 \le j \le \ell)$ be a regular elliptic boundary problem. Suppose P has order m and each B_j has order m_j. If u solves*

$$Pu = 0 \text{ on } \Omega, \quad B_j u = g_j \text{ on } \partial\Omega, \tag{8.34}$$

then, for $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$,

$$g_j \in C^{r-m_j}_*(\partial\Omega) \Longrightarrow u \in C^r(\overline{\Omega}). \tag{8.35}$$

Proof. Of course, $u \in C^\infty(\Omega)$. On a collar neighborhood of $\partial\Omega$, diffeomorphic to $[0,1] \times \partial\Omega$, we can write, modulo $C^\infty\big([0,1] \times \partial\Omega\big)$,

$$u = \sum Q_j(y)g_j, \quad Q_j(y) \in OP\mathcal{P}^{-m_j}_e, \tag{8.36}$$

by Theorem 12.6 of Chapter 7, so the implication (8.35) follows directly from (8.28).

We next want to define Zygmund spaces on domains with boundary. Let Ω be an open set with smooth boundary (and closure $\overline{\Omega}$) in a compact manifold M. We want to consider Zygmund spaces $C^r_*(\overline{\Omega}),\ r > 0$. The approach we will take is to define $C^r_*(\overline{\Omega})$ by interpolation:

$$C^r_*(\overline{\Omega}) = \big[C^{s_1}(\overline{\Omega}), C^{s_2}(\overline{\Omega})\big]^b_\theta, \tag{8.37}$$

where $0 < s_1 < r < s_2,\ 0 < \theta < 1,\ r = (1-\theta)s_1 + \theta s_2$ (and $s_j \notin \mathbb{Z}$). As in (8.21), we are using the complex interpolation functor defined in Appendix A. We need to show that this is independent of choices of such s_j. Using an argument parallel to one in §6, for any $N \in \mathbb{Z}^+$, we have an extension operator

$$E : C^s(\overline{\Omega}) \longrightarrow C^s(M), \quad s \in (0, N) \setminus \mathbb{Z}, \tag{8.38}$$

providing a right inverse for the surjective restriction operator

$$\rho : C^s(M) \longrightarrow C^s(\overline{\Omega}). \tag{8.39}$$

From Proposition 8.6, we can deduce that whenever $r > 0$ and s_j and θ are as above, $C^r_*(M) = \big[C^{s_1}(M), C^{s_2}(M)\big]^b_\theta$. Thus, by interpolation, we have, for $r > 0$,

$$E : C^r_*(\overline{\Omega}) \longrightarrow C^r_*(M), \quad \rho : C^r_*(M) \longrightarrow C^r_*(\overline{\Omega}), \tag{8.40}$$

and $\rho E = I$ on $C^r_*(\overline{\Omega})$. Hence

$$C^r_*(\overline{\Omega}) \approx C^r_*(M) \big/ \{u \in C^r_*(M) : u\big|_\Omega = 0\}. \tag{8.41}$$

This characterization is manifestly independent of the choices made in (8.37). Note that the right side of (8.41) is meaningful even for $r \le 0$.

By Propositions 8.1 and 8.2, we know that $C^r_*(M) = C^r(M)$, for $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, so

$$C^r_*(\overline{\Omega}) = C^r(\overline{\Omega}), \quad \text{for } r \in \mathbb{R}^+ \setminus \mathbb{Z}^+. \tag{8.42}$$

Using the spaces $C^r_*(\overline{\Omega})$, we can fill in the gaps (at $r \in \mathbb{Z}^+$) in the estimates of Theorem 8.9.

Proposition 8.10. *If $(P, B_j, 1 \le j \le \ell)$ is a regular elliptic boundary problem as in Theorem 8.9 and u solves (8.34), then, for all $r \in (0, \infty)$,*

$$g_j \in C^{r-m_j}_*(\partial\Omega) \Longrightarrow u \in C^r_*(\overline{\Omega}). \tag{8.43}$$

Proof. For $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, this is equivalent to (8.35). Since the solution u is given, mod $C^\infty(\overline{\Omega})$, by the operator (8.36), the rest follows by interpolation.

In a sense, the C^0_*-norm is only a tad weaker than the C^0-norm. The following is a quantitative version of this statement, which will prove very useful for the study of nonlinear evolution equations, particularly in Chapter 17.

Proposition 8.11. *If $s > n/2 + \delta$, then there is $C < \infty$ such that, for all $\varepsilon \in (0, 1]$,*

$$\|u\|_{L^\infty} \le C\varepsilon^\delta \|u\|_{H^s} + C\Big(\log \frac{1}{\varepsilon}\Big) \|u\|_{C^0_*}. \tag{8.44}$$

Proof. By (8.6), $\|u\|_{C^0_*} = \sup \|\psi_k(D)u\|_{L^\infty}$. Now, with $\Psi_j = \sum_{\ell \le j} \psi_\ell$, make the decomposition $u = \Psi_j(D)u + \big(1 - \Psi_j(D)\big)u$; let $\varepsilon = 2^{-j}$. Clearly,

$$\|\Psi_j(D)u\|_{L^\infty} \le j\|u\|_{C^0_*}. \tag{8.45}$$

Meanwhile, using the Sobolev imbedding theorem, since $n/2 < s - \delta$,

$$\begin{aligned}\|(1-\Psi_j(D))u\|_{L^\infty} &\le C\|(1-\Psi_j(D))u\|_{H^{s-\delta}} \\ &\le C\, 2^{-j\delta}\|(1-\Psi_j(D))u\|_{H^s},\end{aligned} \tag{8.46}$$

the last identity holding since $\{2^{j\delta}\langle\xi\rangle^{-\delta}\big(1-\Psi_j(\xi)\big) : j \in \mathbb{Z}^+\}$ is uniformly bounded. This proves (8.44).

Suppose the norms satisfy $\|u\|_{C^0_*} \le C\|u\|_{H^s}$. If we substitute $\varepsilon^\delta = C^{-1}\|u\|_{C^0_*}/\|u\|_{H^s}$ into (8.44), we obtain the estimate (for a new $C = C(\delta)$)

$$\|u\|_{L^\infty} \le C\|u\|_{C^0_*}\Big[1 + \log\Big(\frac{\|u\|_{H^s}}{\|u\|_{C^0_*}}\Big)\Big]. \tag{8.47}$$

We note that a number of variants of (8.44) and (8.47) hold. For some of them, it is useful to strengthen the last observation in the proof above to

$$\big\{2^{j\delta}\langle\xi\rangle^{-\delta}\big(1-\Psi_j(\xi)\big) : j \in \mathbb{Z}^+\big\} \text{ is bounded in } S^0_1(\mathbb{R}^n). \tag{8.48}$$

An argument parallel to the proof of Proposition 8.11 gives estimates

$$\|u\|_{C^k(M)} \le C\varepsilon^\delta\|u\|_{H^s(M)} + C\Big(\log\frac{1}{\varepsilon}\Big)\|u\|_{C^k_*(M)}, \tag{8.49}$$

given $k \in \mathbb{Z}^+,\ s > n/2 + k + \delta$, and consequently

$$\|u\|_{C^k(M)} \le C\|u\|_{C^k_*(M)}\Big[1 + \log\Big(\frac{\|u\|_{H^s}}{\|u\|_{C^k_*}}\Big)\Big] \tag{8.50}$$

when M is a compact manifold without boundary.

We can also establish such an estimate for the $C^k(\overline{\Omega})$-norm when $\overline{\Omega}$ is a compact manifold with boundary. If $\overline{\Omega} \subset M$ as above, this follows easily from (8.50), via:

Lemma 8.12. *For any $r \in (0, N)$,*

$$\|u\|_{C^r_*(\overline{\Omega})} \approx \|Eu\|_{C^r_*(M)}. \tag{8.51}$$

Proof. If $Eu_j \to v$ in $C^r_*(M)$, then $\rho Eu_j \to \rho v$ in $C^r_*(\overline{\Omega})$, that is, $u_j \to \rho v$ in $C^r_*(\overline{\Omega})$, since $\rho Eu_j = u_j$. Thus $v = E\rho v$, in this case. This proves the lemma, which is also equivalent to the statement that E in (8.40) has closed range.

We also have such a result for Sobolev spaces:

$$\|u\|_{H^{r,p}(\Omega)} \approx \|Eu\|_{H^{r,p}(M)}, \quad 1 < p < \infty. \tag{8.52}$$

Thus (8.50) yields

$$\|u\|_{C^k(\overline{\Omega})} \le C\|u\|_{C^k_*(\overline{\Omega})}\Big[1 + \log\Big(\frac{\|u\|_{H^s(\Omega)}}{\|u\|_{C^k_*(\overline{\Omega})}}\Big)\Big], \tag{8.53}$$

provided $s > n/2 + k$.

Exercises

1. Extend the estimates of Theorem 8.9 and Proposition 8.10 to solutions of

$$Pu = f \text{ on } \Omega, \quad B_j u = g_j \text{ on } \partial\Omega. \tag{8.54}$$

   Show that, for $r \in (\mu, \infty)$, $\mu = \max(m_j)$,

$$f \in C_*^{r-m}(\overline{\Omega}),\ g_j \in C_*^{r-m_j}(\partial\Omega) \Longrightarrow u \in C_*^r(\overline{\Omega}). \tag{8.55}$$

   Note that we allow $r - m < 0$, in which case $C_*^{r-m}(\overline{\Omega})$ is defined by the right side of (8.41) (with r replaced by $r - m$).

2. Establish the following result, similar to (8.44):

$$\|u\|_{L^\infty} \le C\varepsilon^\delta \|u\|_{H^{s,p}} + C\Bigl(\log\ \frac{1}{\varepsilon}\Bigr)^{1-1/q} \|u\|_{H^{n/q,q}}, \tag{8.56}$$

   where $s > n/p + \delta$, $q \in [2, \infty)$, and a similar estimate for $q \in (1, 2]$, using $\bigl(\log\ 1/\varepsilon\bigr)^{1/q}$. (See [BrG] and [BrW].)

3. From (8.15) it follows that $H^{1,p}(\mathbb{R}^n) \subset C^r(\mathbb{R}^n)$ if $p > n$, $r = 1 - n/p$. Demonstrate the following more precise result:

$$|u(x) - u(y)| \le C|x - y|^{1-n/p}\ \|\nabla u\|_{L^p(B_{xy})}, \quad p > n, \tag{8.57}$$

   where $B_{xy} = B_{|x-y|}(x) \cap B_{|x-y|}(y)$.
   (*Hint*: Apply scaling to (2.16) to obtain

$$|v(re_1) - v(0)| \le Cr^{p-n} \int\limits_{B_r(0)} |\nabla v(x)|^p\ dx.$$

   To pass from $B_{|x-y|}(x)$ to B_{xy} in (8.57), note what the support of φ is in Exercise 5 of §2.) There is a stronger estimate, known as *Morrey's inequality*. See Chapter 14 for more on this.

9. Pseudodifferential operators with nonregular symbols

We establish here some results on Hölder and Sobolev space continuity for pseudodifferential operators $p(x, D)$ with symbols $p(x, \xi)$ which are somewhat more ill behaved than those for which we had L^2-Sobolev estimates in Chapter 7 or L^p-Sobolev estimates and Hölder estimates in §§5 and 8 of this chapter. These results will be very useful in the analysis of nonlinear, elliptic PDE in Chapter 14 and will also be used in Chapters 15 and 16.

Let $r \in (0, \infty)$. We say $p(x, \xi) \in C_*^r S_{1,\delta}^m(\mathbb{R}^n)$ provided

$$|D_\xi^\alpha p(x, \xi)| \le C_\alpha \langle \xi \rangle^{m-|\alpha|} \tag{9.1}$$

and

$$\|D_\xi^\alpha p(\cdot, \xi)\|_{C_*^r(\mathbb{R}^n)} \le C_\alpha \langle \xi \rangle^{m-|\alpha|+\delta r}. \tag{9.2}$$

Here $\delta \in [0,1]$. The following rather strong result is due to G. Bourdaud [Bou], following work of E. Stein [S2].

Theorem 9.1. *If $r > 0$ and $p \in (1,\infty)$, then, for $p(x,\xi) \in C^r_* S^m_{1,1}$,*

$$p(x,D) : H^{s+m,p} \longrightarrow H^{s,p}, \tag{9.3}$$

provided $0 < s < r$. Furthermore, under these hypotheses,

$$p(x,D) : C^{s+m}_* \longrightarrow C^s_*. \tag{9.4}$$

Before giving the proof of this result, we record some implications. Note that any $p(x,\xi) \in S^m_{1,1}$ satisfies the hypotheses for all $r > 0$. Since operators in $OPS^m_{1,\delta}$ possess good multiplicative properties for $\delta \in [0,1)$, we have the following:

Corollary 9.2. *If $p(x,\xi) \in S^m_{1,\delta},\ 0 \le \delta < 1$, we have the mapping properties (9.3) and (9.4) for all $s \in \mathbb{R}$.*

It is known that elements of $OPS^0_{1,1}$ need not be bounded on L^p, even for $p = 2$, but by duality and interpolation we have the following:

Corollary 9.3. *If $p(x,D)$ and $p(x,D)^*$ belong to $OPS^m_{1,1}$, then (9.3) holds for all $s \in \mathbb{R}$.*

We prepare to prove Theorem 9.1. It suffices to treat the case $m = 0$. Following [Bou] and also [Ma2], we make use of the following results from Littlewood-Paley theory. These results follow from (6.23) and (6.25), respectively.

Lemma 9.4. *Let $f_k \in \mathcal{S}'(\mathbb{R}^n)$ be such that, for some $A > 0$,*

$$\text{supp}\ \hat{f}_k \subset \{\xi : A \cdot 2^{k-1} \le |\xi| \le A \cdot 2^{k+1}\}, \quad k \ge 1. \tag{9.5}$$

Say $\hat{f}_0$ has compact support. Then, for $p \in (1,\infty),\ s \in \mathbb{R}$, we have

$$\Big\|\sum_{k=0}^{\infty} f_k\Big\|_{H^{s,p}} \le C\Big\|\Big\{\sum_{k=0}^{\infty} 4^{ks}|f_k|^2\Big\}^{1/2}\Big\|_{L^p}. \tag{9.6}$$

If $f_k = \varphi_k(D)f$ with φ_k supported in the shell defined in (9.5) and bounded in $S^0_{1,0}$, then the converse of the estimate (9.6) also holds.

Lemma 9.5. *Let $f_k \in \mathcal{S}'(\mathbb{R}^n)$ be such that*

$$\text{supp}\ \hat{f}_k \subset \{\xi : |\xi| \le A \cdot 2^{k+1}\}, \quad k \ge 0. \tag{9.7}$$

Then, for $p \in (1,\infty),\ s > 0$, we have

$$\Big\|\sum_{k=0}^{\infty} f_k\Big\|_{H^{s,p}} \le C\Big\|\Big\{\sum_{k=0}^{\infty} 4^{ks}|f_k|^2\Big\}^{1/2}\Big\|_{L^p}. \tag{9.8}$$

The next ingredient is a symbol decomposition. We begin with the Littlewood-Paley partition of unity (5.37),

$$1 = \sum \varphi_j(\xi)^2 = \sum \psi_j(\xi), \tag{9.9}$$

and with

$$p(x,\xi) = \sum_{j=0}^{\infty} p(x,\xi)\psi_j(\xi) = \sum_{j=0}^{\infty} p_j(x,\xi). \tag{9.10}$$

Now, let us take a basis of $L^2\big(|\xi_j| < \pi\big)$ of the form

$$e_\alpha(\xi) = e^{i\alpha\cdot\xi},$$

and write (for $j \ge 1$)

$$p_j(x,\xi) = \sum_\alpha p_{j\alpha}(x) e_\alpha(2^{-j}\xi)\psi_j^\#(\xi), \tag{9.11}$$

where $\psi_1^\#(\xi)$ has support on $1/2 < |\xi| < 2$ and is 1 on supp ψ_1, $\psi_j^\#(\xi) = \psi_1^\#(2^{-j+1}\xi)$, with an analogous decomposition for $p_0(\xi)$. Inserting these decompositions into (9.10) and summing over j, we obtain $p(x,\xi)$ as a sum of a rapidly decreasing sequence of elementary symbols.

By definition, an elementary symbol in $C_*^r S_{1,\delta}^0$ is of the form

$$q(x,\xi) = \sum_{k=0}^{\infty} Q_k(x)\varphi_k(\xi), \tag{9.12}$$

where φ_k is supported on $\langle\xi\rangle \sim 2^k$ and bounded in S_1^0—in fact, $\varphi_k(\xi) = \varphi_1(2^{-k+1}\xi)$, for $k \ge 2$—and $Q_k(x)$ satisfies

$$|Q_k(x)| \le C, \ \ \|Q_k\|_{C_*^r} \le C \cdot 2^{kr\delta}. \tag{9.13}$$

For the purpose of proving Theorem 9.1, we take $\delta = 1$. It suffices to estimate the $H^{r,p}$-operator norm of $q(x,D)$ when $q(x,\xi)$ is such an elementary symbol.

Set $Q_{kj}(x) = \psi_j(D)Q_k(x)$, with $\{\psi_j\}$ the partition of unity described in (9.9). Set

$$\begin{aligned} q(x,\xi) &= \sum_k \Big\{\sum_{j=0}^{k-4} Q_{kj}(x) + \sum_{j=k-3}^{k+3} Q_{kj}(x) + \sum_{j=k+4}^{\infty} Q_{kj}(x)\Big\}\varphi_k(\xi) \\ &= q_1(x,\xi) + q_2(x,\xi) + q_3(x,\xi). \end{aligned} \tag{9.14}$$

We will perform separate estimates of these three pieces. Set $f_k = \varphi_k(D)f$.

First we estimate $q_1(x,D)f$. By Lemma 9.4, since $\langle\xi\rangle \sim 2^j$ on the spectrum of Q_{kj},

$$
\begin{aligned}
\|q_1(x,D)f\|_{H^{s,p}} &\le C\Big\|\Big\{\sum_{k=4}^{\infty} 4^{ks}\Big|\sum_{j=0}^{k-4} Q_{kj}f_k\Big|^2\Big\}^{1/2}\Big\|_{L^p} \\
&\le C\Big\|\Big\{\sum_{k=4}^{\infty} 4^{ks}\|Q_k\|_{L^\infty}^2\,|f_k|^2\Big\}^{1/2}\Big\|_{L^p} \qquad (9.15)\\
&\le C\|f\|_{H^{s,p}},
\end{aligned}
$$

for all $s \in \mathbb{R}$.

To estimate $q_2(x,D)f$, note that $\|Q_{kj}\|_{L^\infty} \le C\cdot 2^{-jr+kr}$. Then Lemma 9.5 implies

$$
\|q_2(x,D)f\|_{H^{s,p}} \le C\big\|\big\{\sum_{k=0}^{\infty} 4^{ks}|f_k|^2\big\}^{1/2}\big\|_{L^p} \le C\|f\|_{H^{s,p}}, \tag{9.16}
$$

for $s > 0$.

To estimate $q_3(x,D)f$, we apply Lemma 9.4 to $h_j = \sum_{k=0}^{j-4} Q_{kj}f_k$, to obtain

$$
\begin{aligned}
\|q_3(x,D)f\|_{H^{s,p}} &\le C\Big\|\Big\{\sum_{j=4}^{\infty} 4^{js}\Big|\sum_{k=0}^{j-4} Q_{kj}f_k\Big|^2\Big\}^{1/2}\Big\|_{L^p} \qquad (9.17)\\
&\le C\Big\|\Big\{\sum_{j=4}^{\infty} 4^{j(s-r)}\Big(\sum_{k=0}^{j-4} 2^{kr}|f_k|\Big)^2\Big\}^{1/2}\Big\|_{L^p}.
\end{aligned}
$$

Now, if we set $g_j = \sum_{k=0}^{j-4} 2^{(k-j)r}|f_k|$ and then set $G_j = 2^{js}g_j$ and $F_j = 2^{js}|f_j|$, we see that

$$
G_j = \sum_{k=0}^{j-4} 2^{(k-j)(r-s)}F_k.
$$

As long as $r > s$, Young's inequality (see Exercise 1 at the end of this section) yields $\|(G_j)\|_{\ell^2} \le C\|(F_j)\|_{\ell^2}$, so the last line in (9.17) is bounded by

$$
C\big\|\big\{\sum_{j=0}^{\infty} 4^{js}|f_j|^2\big\}^{1/2}\big\|_{L^p} \le C\|f\|_{H^{s,p}}.
$$

This proves (9.3).

The proof of (9.4) is similar. We replace (9.6) by

$$
\|f\|_{C^r_*} \sim \sup_{k\ge 0}\ 2^{kr}\|\psi_k(D)f\|_{L^\infty}, \quad r > 0. \tag{9.18}
$$

We also need an analogue of Lemma 9.5:

Lemma 9.6. *If $f_k \in \mathcal{S}'(\mathbb{R}^n)$ and supp $\hat{f}_k \subset \{\xi : |\xi| \le A \cdot 2^{k+1}\}$, then, for $r > 0$,*

$$\Big\|\sum_{k=0}^{\infty} f_k\Big\|_{C_*^r} \le C \sup_{k\ge 0}\ 2^{kr}\|f_k\|_{L^\infty}. \tag{9.19}$$

Proof. For some finite N, we have $\psi_j(D)\sum_{k\ge 0} f_k = \psi_j(D)\sum_{k\ge j-N} f_k$. Suppose $\sup_k\ 2^{kr}\|f_k\|_{L^\infty} = S$. Then

$$\Big\|\psi_j(D)\sum_{k\ge 0} f_k\Big\|_{L^\infty} \le CS \sum_{k\ge j-N} 2^{-kr} \le C' S 2^{-jr}.$$

This proves (9.19).

Now, to prove (9.4), as before it suffices to consider elementary symbols, of the form (9.12)–(9.13), and we use again the decomposition $q(x,\xi) = q_1 + q_2 + q_3$ of (9.14). Thus it remains to obtain analogues of the estimates (9.15)–(9.17).

Parallel to (9.15), using the fact that $\sum_{j=0}^{k-4} Q_{kj}(x) f_k$ has spectrum in the shell $\langle\xi\rangle \sim 2^k$, and $\|Q_k\|_{L^\infty} \le C$, we obtain

$$\begin{aligned}\|q_1(x,D)f\|_{C_*^s} &\le C \sup_{k\ge 0}\ 2^{ks}\Big\|\sum_{j=0}^{k-4} Q_{kj} f_k\Big\|_{L^\infty}\\ &\le C \sup_{k\ge 0}\ 2^{ks}\|f_k\|_{L^\infty}\\ &\le C\|f\|_{C_*^s},\end{aligned} \tag{9.20}$$

for all $s \in \mathbb{R}$. Parallel to (9.16), using $\|Q_{kj}\|_{L^\infty} \le C \cdot 2^{-jr+kr}$ and Lemma 9.6, we have

$$\begin{aligned}\|q_2(x,D)f\|_{C_*^s} &\le \Big\|\sum_{k=0}^{\infty} g_k\Big\|_{C_*^s}\\ &\le C \sup_{k\ge 0}\ 2^{ks}\|g_k\|_{L^\infty}\\ &\le C \sup_{k\ge 0}\ 2^{ks}\|f_k\|_{L^\infty} \le C\|f\|_{C_*^s},\end{aligned} \tag{9.21}$$

for all $s > 0$, where the sum of seven terms

$$g_k = \sum_{j=k-3}^{k+3} Q_{kj}(x) f_k$$

has spectrum contained in $|\xi| \le C \cdot 2^k$, and $\|g_k\|_{L^\infty} \le C\|f_k\|_{L^\infty}$.

Finally, parallel to (9.17), since $\sum_{k=0}^{j-4} Q_{kj} f_k$ has spectrum in the shell $\langle\xi\rangle \sim 2^j$, we have

$$\begin{aligned} \|q_3(x,D)f\|_{C^s_*} &\le C \sup_{j\ge 0}\ 2^{js} \Big\| \sum_{k=0}^{j-4} Q_{kj} f_k \Big\|_{L^\infty} \\ &\le C \sup_{j\ge 0}\ 2^{j(s-r)} \sum_{k=0}^{j-4} 2^{kr} \|f_k\|_{L^\infty}. \end{aligned} \tag{9.22}$$

If we bound this last sum by

$$\Big[\sum_{k=0}^{j-4} 2^{k(r-s)} \Big] \sup_k\ 2^{ks} \|f_k\|_{L^\infty}, \tag{9.23}$$

then

$$\|q_3(x,D)f\|_{C^s_*} \le C \Big[\sup_{j\ge 0}\ 2^{j(s-r)} \sum_{k=0}^{j-4} 2^{k(r-s)} \Big] \|f\|_{C^s_*}, \tag{9.24}$$

and the factor in brackets is finite as long as $s < r$. The proof of Theorem 9.1 is complete.

Things barely blow up in (9.24) when $s = r$. We will establish the following result here. A sharper result (for $p(x,\xi) \in C^r_* S^m_{1,\delta}$ with $\delta < 1$) is given in (9.43).

Proposition 9.7. *If* $p(x,\xi) \in C^r_* S^m_{1,1}$*, then*

$$p(x,D) : C^{m+r+\varepsilon}_* \longrightarrow C^r_*, \quad \text{for all } \varepsilon > 0. \tag{9.25}$$

Proof. It suffices to treat the case $m = 0$. We follow the proof of (9.4). The estimates (9.20) and (9.21) continue to work; (9.22) yields

$$\begin{aligned} \|q_3(x,D)f\|_{C^r_*} &\le C \sup_{j\ge 0} \sum_{k=0}^{j-4} 2^{kr} \|f_k\|_{L^\infty} \\ &= C \sum_{k=0}^{\infty} 2^{kr} \|f_k\|_{L^\infty} \\ &\le C \sum_{k=0}^{\infty} 2^{kr} \cdot 2^{-kr-k\varepsilon} \|f\|_{C^{r+\varepsilon}_*}, \end{aligned} \tag{9.26}$$

which proves (9.25).

The way symbols in $C^r_* S^m_{1,\delta}$ most frequently arise is the following. One has in hand a symbol $p(x,\xi) \in C^r_* S^m_{1,0}$, such as the symbol of a *differential* operator, with Hölder-continuous coefficients. One is then motivated to decompose $p(x,\xi)$ as a sum

$$p(x,\xi) = p^{\#}(x,\xi) + p^b(x,\xi), \tag{9.27}$$

where $p^{\#}(x,\xi) \in S^m_{1,\delta}$, for some $\delta \in (0,1)$, and there is a good operator calculus for $p^{\#}(x,D)$, while $p^b(x,\xi) \in C^r_* S^{\mu}_{1,\delta}$ (for some $\mu < m$) is treated as a remainder term, to be estimated. We will refer to this construction as *symbol smoothing*.

The symbol decomposition (9.27) is constructed as follows. Use the partition of unity $\psi_j(\xi)$ of (9.9). Given $p(x,\xi) \in C^r_* S^m_{1,0}$, choose $\delta \in (0,1]$ and set

$$p^{\#}(x,\xi) = \sum_{j=0}^{\infty} J_{\varepsilon_j} p(x,\xi)\ \psi_j(\xi), \tag{9.28}$$

where J_ε is a smoothing operator on functions of x, namely

$$J_\varepsilon f(x) = \phi(\varepsilon D) f(x), \tag{9.29}$$

with $\phi \in C_0^\infty(\mathbb{R}^n),\ \phi(\xi) = 1$ for $|\xi| \le 1$ (e.g., $\phi = \psi_0$), and we take

$$\varepsilon_j = 2^{-j\delta}. \tag{9.30}$$

We then define $p^b(x,\xi)$ to be $p(x,\xi) - p^{\#}(x,\xi)$, yielding (9.27).

To analyze these terms, we use the following simple result.

Lemma 9.8. *For* $\varepsilon \in (0,1]$,

$$\|D_x^\beta J_\varepsilon f\|_{C^s_*} \le C_\beta\ \varepsilon^{-|\beta|} \|f\|_{C^s_*} \tag{9.31}$$

and

$$\|f - J_\varepsilon f\|_{C^{s-t}_*} \le C\varepsilon^t \|f\|_{C^s_*}, \quad \text{for } t \ge 0. \tag{9.32}$$

Furthermore, if $s > 0$,

$$\|f - J_\varepsilon f\|_{L^\infty} \le C_s \varepsilon^s \|f\|_{C^s_*}. \tag{9.33}$$

Proof. The estimate (9.31) follows from the fact that, for each $\beta \ge 0$,

$$\varepsilon^{|\beta|} D_x^\beta \phi(\varepsilon D) \text{ is bounded in } OPS^0_{1,0},$$

and the estimate (9.32) follows from the fact that, with $\Lambda = (1-\Delta)^{1/2}$,

$$\Lambda^t : C^s_* \longrightarrow C^{s-t}_* \text{ isomorphically,}$$

plus the fact that

$$\varepsilon^{-t}\Lambda^{-t}(1-\phi(\varepsilon D)) \text{ is bounded in } OPS^0_{1,0},$$

for $0 < \varepsilon \le 1$. As for (9.33), if $\varepsilon \sim 2^{-j}$, we have

$$\|\big(1-\phi(\varepsilon D)\big)f\|_{L^\infty} \le \sum_{\ell \ge j} \|\psi_\ell(D) f\|_{L^\infty} \le C \sum_{\ell \ge j} 2^{-\ell s} \|f\|_{C^s_*},$$

and since $\sum_{\ell \ge j} 2^{-\ell s} \le C_s 2^{-js}$ for $s > 0$, (9.33) follows.

Using this, we easily derive the following conclusion:

Proposition 9.9. *If $p(x,\xi) \in C^r_* S^m_{1,0}$, then, in the decomposition (9.27),*

$$p^\#(x,\xi) \in S^m_{1,\delta} \tag{9.34}$$

and

$$p^b(x,\xi) \in C^r_* S^{m-r\delta}_{1,\delta}. \tag{9.35}$$

Proof. The estimate (9.31) yields

$$\|D^\beta_x D^\alpha_\xi p^\#(\cdot,\xi)\|_{C^r_*} \le C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|+\delta|\beta|}, \tag{9.36}$$

which implies (9.34).

That $p^b(x,\xi)$ satisfies an estimate of the form (9.2), with m replaced by $m - r\delta$, follows from (9.32), with $t = 0$. That it satisfies (9.1), with m replaced by $m - r\delta$, is a consequence of the estimate (9.33).

It will also be useful to smooth out a symbol $p(x,\xi) \in C^r_* S^m_{1,\delta}$, for $\delta \in (0,1)$. Pick $\gamma \in (\delta, 1)$, and apply (9.28), with $\varepsilon_j = 2^{-j(\gamma-\delta)}$, obtaining $p^\#(x,\xi)$ and hence a decomposition of the form (9.27). In this case, we obtain

$$p(x,\xi) \in C^r_* S^m_{1,\delta} \Longrightarrow p^\#(x,\xi) \in S^m_{1,\gamma}, \quad p^b(x,\xi) \in C^r_* S^{m-(\gamma-\delta)r}_{1,\gamma}. \tag{9.37}$$

We use the symbol decomposition (9.27) to establish the following variant of Theorem 9.1, which will be most useful in Chapter 14.

Proposition 9.10. *If $\delta \in [0,1)$ and $p(x,\xi) \in C^r_* S^m_{1,\delta}$, then*

$$\begin{aligned} p(x,D) &: H^{s+m,p} \longrightarrow H^{s,p}, \\ p(x,D) &: C^{s+m}_* \longrightarrow C^s_*, \end{aligned} \tag{9.38}$$

provided $p \in (1,\infty)$ and

$$-(1-\delta)r < s < r. \tag{9.39}$$

Proof. The result follows directly from Theorem 9.1 if $0 < s < r$, so it remains to consider $s \in \big(-(1-\delta)r, 0\big]$. Use the decomposition (9.27), $p = p^\# + p^b$, with (9.37) holding. Thus $p^\#(x,D)$ has the mapping property (9.38) for all $s \in \mathbb{R}$. Applying Theorem 9.1 to $p^b(x,D)$ yields mapping properties such as

$$p^b(x,D) : H^{\sigma+m-(\gamma-\delta)r,p} \longrightarrow H^{\sigma,p}, \quad \sigma > 0,$$

or, setting $s = \sigma - (\gamma - \delta)r$,

$$p^b(x, D) : H^{s+m,p} \longrightarrow H^{s+(\gamma-\delta)r,p} \subset H^{s,p}, \quad s > -(\gamma - \delta)r,$$

and similar results on C_*^{s+m}. Then letting $\gamma \nearrow 1$ completes the proof of (9.38).

Recall that, for $r \in (0, \infty)$, we have defined $p(x, \xi)$ to belong to the space $C_*^r S_{1,\delta}^m(\mathbb{R}^n)$ provided the estimates (9.1) and (9.2) hold. If $r \in [0, \infty)$, we will say that $p(x, \xi) \in C^r S_{1,\delta}^m(\mathbb{R}^n)$ provided that (9.1)–(9.2) hold and, additionally,

$$\|D_\xi^\alpha p(\cdot, \xi)\|_{C^j(\mathbb{R}^n)} \le C_\alpha \langle\xi\rangle^{m-|\alpha|+j\delta}, \quad 0 \le j \le r, \quad j \in \mathbb{Z}. \tag{9.40}$$

In particular, we make a semantic distinction between $C_*^r S_{1,\delta}^m$ and $C^r S_{1,\delta}^m$ even when $r \notin \mathbb{Z}^+$, in which cases C_*^r and C^r coincide. The differences between the two symbol classes are minor, especially when $r \notin \mathbb{Z}$, but natural examples of symbols often do have this additional property, and we sometimes use the symbol classes just defined to record this fact.

Exercises

1. Young's inequality implies

$$\|f * g\|_{\ell^q} \le \|f\|_{\ell^1} \|g\|_{\ell^q},$$

where $f = (f_j)$, $g = (g_j)$, and $(f * g)_j = \sum_k f_{j-k} g_k$. Show how this applies (with $q = 2$) to the estimate of (9.17).

2. Supplement Lemma 9.8 with the estimates

$$\|D_x^\beta J_\varepsilon f\|_{L^\infty} \le \begin{cases} C\|f\|_{C^s}, & |\beta| \le s, \\ C\varepsilon^{-(|\beta|-s)} \|f\|_{C_*^s}, & |\beta| > s, \end{cases} \tag{9.41}$$

given $s > 0$.

3. Show that if $p(x, \xi) \in C_*^r S_{1,0}^m$ has the decomposition (9.27), then

$$D_x^\beta p^\#(x, \xi) \in \begin{cases} S_{1,\delta}^m, & \text{for } |\beta| < r, \\ S_{1,\delta}^{m+\delta(|\beta|-r)}, & \text{for } |\beta| > r. \end{cases} \tag{9.42}$$

4. Strengthen part of Proposition 9.10 to obtain, for $\delta \in [0, 1)$, $r > 0$,

$$p(x, \xi) \in C_*^r S_{1,\delta}^m \Longrightarrow p(x, D) : C_*^{s+m} \longrightarrow C_*^s, \text{ for } -(1-\delta)r < s \le r. \tag{9.43}$$

(*Hint*: Apply Proposition 9.7 to $p^b(x, D)$, arising in (9.37).)

5. Given $s \in \mathbb{R}$, $1 \le p, q \le \infty$, we say $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the *Triebel space* $F_{p,q}^s(\mathbb{R}^n)$ provided

$$\|f\|_{F_{p,q}^s} = \Big\| \{2^{js} \psi_j(D) f\} \Big\|_{L^p(\mathbb{R}^n, \ell^q)} < \infty, \tag{9.44}$$

where $\{\psi_j\}$ is the partition of unity (9.9). Note that $F_{p,2}^s = H^{s,p}$ if $1 < p < \infty$, by Lemma 9.4. Also, we say that $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the *Besov space*

$B^s_{p,q}(\mathbb{R}^n)$ provided

$$\|f\|_{B^s_{p,q}} = \Big\| \{2^{js}\psi_j(D)f\} \Big\|_{\ell^q(L^p(\mathbb{R}^n))} < \infty. \tag{9.45}$$

Note that $B^s_{\infty,\infty} = C^s_*$. Also, $B^s_{2,2} = H^s$, since $\ell^2(L^2(\mathbb{R}^n)) = L^2(\mathbb{R}^n, \ell^2)$. Extend Theorem 9.1 to results of the form

$$p(x,D) : F^{s+m}_{p,q} \to F^s_{p,q}, \quad p(x,D) : B^{s+m}_{p,q} \to B^s_{p,q}.$$

(See [Ma1].)

6. We define the symbol class $C^r_* S^m_{cl}$ to consist of $p(x,\xi) \in C^r_* S^m_{1,0}$ such that

$$p(x,\xi) \sim \sum_{j\ge 0} p_j(x,\xi) \tag{9.46}$$

where $p_j(x,\xi) \in C^r_* S^m_{1,0}$ is homogeneous of degree $m-j$ in ξ, for $|\xi| \ge 1$, and (9.46) means that the difference between the left side and the sum over $0 \le j < N$ belongs to $C^r_* S^{m-N}_{1,0}$. If $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, we also denote the symbol class by $C^r S^m_{cl}$. Show that estimates of the form (9.3) and (9.4) have simpler proofs in this case, derived from expansions of the form

$$p_j(x,\xi) = \sum_\nu p_{j\nu}(x)|\xi|^{m-j}\omega_\nu\Big(|\xi|^{-1}\xi\Big), \tag{9.47}$$

for $|\xi| \ge 1$, where $\{\omega_\nu\}$ is an orthonormal basis of $L^2(S^{n-1})$ consisting of eigenfunctions of the Laplace operator.

10. Paradifferential operators

Here we develop the paradifferential operator calculus, introduced by J.-M. Bony in [Bon]. We begin with Y. Meyer's ingenious formula for $F(u)$ as $M(x,D)u + R$ where F is smooth in its argument(s), u belongs to a Hölder or Sobolev space, $M(x,D)$ is a pseudodifferential operator of type $(1,1)$, and R is smooth. From there, one applies symbol smoothing to $M(x,\xi)$ and makes use of results established in §9.

Following [Mey], we discuss the connection between $F(u)$, for smooth nonlinear F, and the action on u of certain pseudodifferential operators of type $(1,1)$. Let $\psi_j(\xi) = \varphi_j(\xi)^2$ be the Littlewood-Paley partition of unity (5.37), and set $\Psi_k(\xi) = \sum_{j\le k} \psi_j(\xi)$. Given u (e.g., in $C^r(\mathbb{R}^n)$), set

$$u_k = \Psi_k(D)u, \tag{10.1}$$

and write

$$F(u) = F(u_0) + [F(u_1) - F(u_0)] + \cdots + [F(u_{k+1}) - F(u_k)] + \cdots. \tag{10.2}$$

Then write

$$\begin{aligned} F(u_{k+1}) - F(u_k) &= F\big(u_k + \psi_{k+1}(D)u\big) - F(u_k) \\ &= m_k(x)\psi_{k+1}(D)u, \end{aligned} \tag{10.3}$$

where

$$m_k(x) = \int_0^1 F'\big(\Psi_k(D)u + t\psi_{k+1}(D)u\big)\,dt. \tag{10.4}$$

Consequently, we have

$$\begin{aligned} F(u) &= F(u_0) + \sum_{k=0}^{\infty} m_k(x)\psi_{k+1}(D)u \\ &= M(x,D)u + F(u_0), \end{aligned} \tag{10.5}$$

where

$$M(x,\xi) = \sum_{k=0}^{\infty} m_k(x)\psi_{k+1}(\xi) = M_F(u;x,\xi). \tag{10.6}$$

We claim

$$M(x,\xi) \in S^0_{1,1}, \tag{10.7}$$

provided u is continuous. To estimate $M(x,\xi)$, note first that by (10.4)

$$\|m_k\|_{L^\infty} \le \sup|F'(\lambda)|. \tag{10.8}$$

To estimate higher derivatives, we use the elementary estimate

$$\|D^\ell g(h)\|_{L^\infty} \le C \sum_{\ell_1+\cdots+\ell_\nu \le \ell} \|g'\|_{C^{\nu-1}} \|D^{\ell_1}h\|_{L^\infty} \cdots \|D^{\ell_\nu}h\|_{L^\infty} \tag{10.9}$$

to obtain

$$\|D_x^\ell m_k\|_{L^\infty} \le C_\ell \|F''\|_{C^{\ell-1}} \langle \|u\|_{L^\infty}\rangle^{\ell-1} \cdot 2^{k\ell}, \tag{10.10}$$

granted the following estimates, which hold for all $u \in L^\infty$:

$$\|\Psi_k(D)u + t\psi_{k+1}(D)u\|_{L^\infty} \le C\|u\|_{L^\infty} \tag{10.11}$$

and

$$\|D^\ell[\Psi_k(D)u + t\psi_{k+1}(D)u]\|_{L^\infty} \le C_\ell 2^{k\ell}\|u\|_{L^\infty} \tag{10.12}$$

for $t \in [0,1]$. Consequently, (10.6) yields

$$|D_\xi^\alpha M(x,\xi)| \le C_\alpha \sup_\lambda |F'(\lambda)| \langle\xi\rangle^{-|\alpha|} \tag{10.13}$$

and, for $|\beta| \ge 1$,

$$|D_x^\beta D_\xi^\alpha M(x,\xi)| \le C_{\alpha\beta} \|F''\|_{C^{|\beta|-1}} \langle \|u\|_{L^\infty}\rangle^{|\beta|-1} \langle\xi\rangle^{|\beta|-|\alpha|}. \tag{10.14}$$

We give a formal statement of the result just established.

Proposition 10.1. *If F is C^∞ and $u \in C^r$ with $r \ge 0$, then*

$$F(u) = M_F(u;x,D)u + R(u), \tag{10.15}$$

where

$$R(u) = F(\psi_0(D)u) \in C^\infty$$

and

$$M_F(u;x,\xi) = M(x,\xi) \in S^0_{1,1}. \tag{10.16}$$

Following [Bon] and [Mey], we call $M_F(u;x,D)$ a *paradifferential operator.*

Applying Theorem 9.1, we have

$$\|M(x,D)f\|_{H^{s,p}} \le K\|f\|_{H^{s,p}}, \tag{10.17}$$

for $p \in (1,\infty)$, $s > 0$, with

$$K = K_N(F,u) = C\|F'\|_{C^N}[1 + \|u\|^N_{L^\infty}], \tag{10.18}$$

provided $0 < s < N$, and similarly

$$\|M(x,D)f\|_{C^s_*} \le K\|f\|_{C^s_*}. \tag{10.19}$$

Using $f = u$, we have the following important Moser-type estimates, extending Proposition 3.9:

Proposition 10.2. *If F is smooth with $\|F'\|_{C^N(\mathbb{R})} < \infty$, and $0 < s < N$, then*

$$\|F(u)\|_{H^{s,p}} \le K_N(F,u)\|u\|_{H^{s,p}} + \|R(u)\|_{H^{s,p}} \tag{10.20}$$

and

$$\|F(u)\|_{C^s_*} \le K_N(F,u)\|u\|_{C^s_*} + \|R(u)\|_{C^s_*}, \tag{10.21}$$

given $1 < p < \infty$, with $K_N(F,u)$ as in (10.18).

This expression for $K_N(F,u)$ involves the L^∞-norm of u, and one can use $\|F'\|_{C^N(I)}$, where I contains the range of u. Note that if $F(u) = u^2$, then $F'(u) = 2u$, and higher powers of $\|u\|_{L^\infty}$ do not arise; hence we obtain the estimate

$$\|u^2\|_{H^{s,p}} \le C_s\|u\|_{L^\infty} \cdot \|u\|_{H^{s,p}}, \quad s > 0, \tag{10.22}$$

and a similar estimate on $\|u^2\|_{C^s_*}$.

It will be useful to have further estimates on the symbol $M(x,\xi) = M_F(u;x,\xi)$ when $u \in C^r$ with $r > 0$. The estimate (10.12) extends to

$$\left\|D^\ell\left[\Psi_k(D)f + t\psi_{k+1}(D)f\right]\right\|_{L^\infty} \le \begin{cases} C_\ell\|f\|_{C^r}, & \ell \le r, \\ C_\ell 2^{k(\ell-r)}\|f\|_{C^r}, & \ell > r, \end{cases} \tag{10.23}$$

so we have, when $u \in C^r$,

$$|D_x^\beta D_\xi^\alpha M(x,\xi)| \le \begin{cases} K_{\alpha\beta}\langle\xi\rangle^{-|\alpha|}, & |\beta| \le r, \\ K_{\alpha\beta}\langle\xi\rangle^{-|\alpha|+|\beta|-r}, & |\beta| > r, \end{cases} \tag{10.24}$$

with

$$K_{\alpha\beta} = K_{\alpha\beta}(F,u) = C_{\alpha\beta}\|F'\|_{C^{|\beta|}}[1 + \|u\|_{C^r}^{|\beta|}]. \tag{10.25}$$

Also, since $\Psi_k(D) + t\psi_{k+1}(D)$ is uniformly bounded on C^r, for $t \in [0,1]$ and $k \ge 0$, we have

$$\|D_\xi^\alpha M(\cdot,\xi)\|_{C^r} \le K_{\alpha r}\langle\xi\rangle^{-|\alpha|}, \tag{10.26}$$

where $K_{\alpha r}$ is as in (10.25), with $|\beta| = [r]+1$. This last estimate shows that

$$u \in C^r \Longrightarrow M_F(u;x,\xi) \in C^r S^0_{1,0}. \tag{10.27}$$

This is useful additional information; for example, (10.17) and (10.19) hold for $s > -r$, and of course we can apply the symbol smoothing of §9.

It will be useful to have terminology expressing the structure of the symbols we produce. Given $r \ge 0$, we say

$$\begin{aligned} p(x,\xi) \in \mathcal{A}^r S^m_{1,\delta} \Longleftrightarrow \|D_\xi^\alpha p(\cdot,\xi)\|_{C^r} &\le C_\alpha\langle\xi\rangle^{m-|\alpha|} \quad \text{and} \\ |D_x^\beta D_\xi^\alpha p(x,\xi)| &\le C_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|+\delta(|\beta|-r)}, \quad |\beta| > r. \end{aligned} \tag{10.28}$$

Thus (10.24)–(10.26) yield

$$M(x,\xi) \in \mathcal{A}^r S^0_{1,1} \tag{10.29}$$

for the $M(x,\xi)$ of Proposition 10.1. If $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, the class $\mathcal{A}^r S^m_{1,1}$ coincides with the symbol class denoted by $\mathcal{A}^m_r$ by Meyer [Mey]. Clearly, $\mathcal{A}^0 S^m_{1,\delta} = S^m_{1,\delta}$, and

$$\mathcal{A}^r S^m_{1,\delta} \subset C^r S^m_{1,0} \cap S^m_{1,\delta}.$$

Also, from the definition we see that

$$p(x,\xi) \in \mathcal{A}^r S^m_{1,\delta} \Longrightarrow D_x^\beta p(x,\xi) \in \begin{cases} S^m_{1,\delta}, & \text{for } |\beta| \le r, \\ S^{m+\delta(|\beta|-r)}_{1,\delta}, & \text{for } |\beta| \ge r. \end{cases} \tag{10.30}$$

It is also natural to consider a slightly smaller symbol class:

$$p(x,\xi) \in \mathcal{A}^r_0 S^m_{1,\delta} \Longleftrightarrow \|D_\xi^\alpha p(\cdot,\xi)\|_{C^{r+s}} \le C_{\alpha s}\langle\xi\rangle^{m-|\alpha|+\delta s}, \quad s \ge 0. \tag{10.31}$$

Considering the cases $s = 0$ and $s = |\beta| - r$, we see that

$$\mathcal{A}^r_0 S^m_{1,\delta} \subset \mathcal{A}^r S^m_{1,\delta}.$$

We also say

$$p(x,\xi) \in {}^r S^m_{1,\delta} \Longleftrightarrow \text{ the right side of (10.30) holds,} \tag{10.32}$$

so

$$\mathcal{A}^r S^m_{1,\delta} \subset {}^r S^m_{1,\delta}.$$

The following result refines (10.29).

Proposition 10.3. *For the symbol $M(x,\xi) = M_F(u;x,\xi)$ of Proposition 10.1, we have*

$$M(x,\xi) \in \mathcal{A}^r_0 S^0_{1,1}, \tag{10.33}$$

provided $u \in C^r,\ r \geq 0$.

Proof. For this, we need

$$\|m_k\|_{C^{r+s}} \leq C \cdot 2^{ks}. \tag{10.34}$$

Now, extending (10.9), we have

$$\|g(h)\|_{C^{r+s}} \leq C\|g\|_{C^N}[1 + \|h\|^N_{L^\infty}](\|h\|_{C^{r+s}} + 1), \tag{10.35}$$

with $N = [r+s]+1$, as a consequence of (10.21) when $r+s$ is not an integer, and by (10.9) when it is. This gives, via (10.4),

$$\|m_k\|_{C^{r+s}} \leq C(\|u\|_{L^\infty}) \sup_{t\in I} \|(\Psi_k + t\psi_{k+1})u\|_{C^{r+s}}, \tag{10.36}$$

where $I = [0,1]$. However,

$$\|(\Psi_k + t\psi_{k+1})u\|_{C^{r+s}} \leq C \cdot 2^{ks}\|u\|_{C^r}. \tag{10.37}$$

For $r+s \in \mathbb{Z}^+$, this follows from (9.41); for $r+s \notin \mathbb{Z}^+$, it follows as in the proof of Lemma 9.8, since

$$2^{-ks}\Lambda^s(\Psi_k + t\psi_{k+1}) \text{ is bounded in } OPS^0_{1,0}. \tag{10.38}$$

This establishes (10.34), and hence (10.33) is proved.

Returning to symbol smoothing, if we use the method of §9 to write

$$M(x,\xi) = M^\#(x,\xi) + M^b(x,\xi), \tag{10.39}$$

then (10.27) implies

$$M^\#(x,\xi) \in S^m_{1,\delta}, \quad M^b(x,\xi) \in C^r S^{m-r\delta}_{1,\delta}. \tag{10.40}$$

We now refine these results; for $M^\#$ we have a general result:

Proposition 10.4. *For the symbol decomposition defined by the formulas (9.27)–(9.30),*

$$p(x,\xi) \in C^r S^m_{1,0} \Longrightarrow p^\#(x,\xi) \in \mathcal{A}^r_0 S^m_{1,\delta}. \tag{10.41}$$

Proof. This is a simple modification of (9.42) which essentially says that $p^{\#}(x,\xi) \in \mathcal{A}^r S^m_{1,\delta}$; we simply supplement (9.41) with

$$\|J_\varepsilon f\|_{C_*^{r+s}} \le C\, \varepsilon^{-s} \|f\|_{C_*^r}, \;\; s \ge 0, \tag{10.42}$$

which is basically the same as (10.37).

To treat $M^b(x,\xi)$, we have, for $\delta \le \gamma$,

$$p(x,\xi) \in \mathcal{A}_0^r S^m_{1,\gamma} \Longrightarrow p^b(x,\xi) \in C^r S^{m-\delta r}_{1,\delta} \cap \mathcal{A}_0^r S^m_{1,\gamma} \subset S^{m-\delta r}_{1,\gamma}, \tag{10.43}$$

where containment in $C^r S^{m-\delta r}_{1,\delta}$ follows from (9.35). To see the last inclusion, note that for $p^b(x,\xi)$ to belong to the intersection above implies

$$\begin{aligned} \|D_\xi^\alpha p^b(\cdot,\xi)\|_{C^s} \le C\langle\xi\rangle^{m-|\alpha|-\delta r+\delta s}&, \quad \text{for } 0 \le s \le r, \\ C\langle\xi\rangle^{m-|\alpha|+(s-r)\gamma}&, \quad \text{for } s \ge r. \end{aligned} \tag{10.44}$$

In particular, these estimates imply $p^b(x,\xi) \in S^{m-r\delta}_{1,\gamma}$. This proves the following:

Proposition 10.5. *For the symbol $M(x,\xi) = M_F(u;x,\xi)$ with decomposition* (10.39),

$$u \in C^r \Longrightarrow M^b(x,\xi) \in S^{-r\delta}_{1,1}. \tag{10.45}$$

Results discussed above extend easily to the case of a function F of several variables, say $u = (u_1, \dots, u_L)$. Directly extending (10.2)–(10.6), we have

$$F(u) = \sum_{j=1}^{L} M_j(x,D)u_j + F(\Psi_0(D)u), \tag{10.46}$$

with

$$M_j(x,\xi) = \sum_k m_k^j(x)\psi_{k+1}(\xi), \tag{10.47}$$

where

$$m_k^j(x) = \int_0^1 (\partial_j F)\big(\Psi_k(D)u + t\psi_{k+1}(D)u\big)\, dt. \tag{10.48}$$

Clearly, the results established above apply to the $M_j(x,\xi)$ here; for example,

$$u \in C^r \Longrightarrow M_j(x,\xi) \in \mathcal{A}_0^r S^m_{1,1}. \tag{10.49}$$

In the particular case $F(u,v) = uv$, we obtain

$$uv = A(u;x,D)v + A(v;x,D)u + \Psi_0(D)u \cdot \Psi_0(D)v, \tag{10.50}$$

where

$$A(u;x,\xi)=\sum_{k=1}^{\infty}\Big[\Psi_k(D)u+\frac{1}{2}\psi_{k+1}(D)u\Big]\psi_{k+1}(\xi). \tag{10.51}$$

Since this symbol belongs to $S^0_{1,1}$ for $u\in L^\infty$, we obtain the following extension of (10.22), which generalizes the Moser estimate (3.21):

Corollary 10.6. *For $s>0,\ 1<p<\infty$, we have*

$$\|uv\|_{H^{s,p}}\le C\big[\|u\|_{L^\infty}\|v\|_{H^{s,p}}+\|u\|_{H^{s,p}}\|v\|_{L^\infty}\big]. \tag{10.52}$$

We now analyze a nonlinear differential operator in terms of a paradifferential operator. If F is smooth in its arguments, in analogy with (10.46)–(10.48) we have

$$F(x,D^m u)=\sum_{|\alpha|\le m}M_\alpha(x,D)D^\alpha u+F(x,D^m\Psi_0(D)u), \tag{10.53}$$

where $F(x,D^m\Psi_0(D)u)\in C^\infty$ and

$$M_\alpha(x,\xi)=\sum_k m_k^\alpha(x)\psi_{k+1}(\xi), \tag{10.54}$$

with

$$m_k^\alpha(x)=\int_0^1(\partial F/\partial\zeta_\alpha)\big(\Psi_k(D)D^m u+t\psi_{k+1}(D)D^m u\big)\,dt. \tag{10.55}$$

As in Propositions 10.1 and 10.3, we have, for $r\ge 0$,

$$u\in C^{m+r}\Longrightarrow M_\alpha(x,\xi)\in\mathcal{A}_0^r S^0_{1,1}\subset S^0_{1,1}\cap C^r S^0_{1,0}. \tag{10.56}$$

In other words, if we set

$$M(u;x,D)=\sum_{|\alpha|\le m}M_\alpha(x,D)D^\alpha, \tag{10.57}$$

we obtain

Proposition 10.7. *If $u\in C^{m+r},\ r\ge 0$, then*

$$F(x,D^m u)=M(u;x,D)u+R, \tag{10.58}$$

with $R\in C^\infty$ and

$$M(u;x,\xi)\in\mathcal{A}_0^r S^m_{1,1}\subset S^m_{1,1}\cap C^r S^m_{1,0}. \tag{10.59}$$

As in Propositions 10.4 and 10.5, in this case symbol smoothing yields

$$M(u;x,\xi)=M^\#(x,\xi)+M^b(x,\xi), \tag{10.60}$$

with

$$M^{\#}(x,\xi) \in \mathcal{A}_0^r S^m_{1,\delta}, \quad M^b(x,\xi) \in S^{m-r\delta}_{1,1}. \tag{10.61}$$

A specific choice for symbol smoothing which leads to paradifferential operators of [Bon] and [Mey] is the following operation on $M(x,\xi)$:

$$M^{\#}(x,\xi) = \sum_k \Psi_{k-5} M(x,\xi)\ \psi_k(\xi), \tag{10.62}$$

where, as in (9.28), Ψ_{k-5} acts on $M(x,\xi)$ as a function of x. We use $\Psi_{k-5} = \Psi_{k-5}(D)$, with $\Psi_\ell(\xi) = \sum_{j\le \ell} \psi_j(\xi)$. We have

$$M(x,\xi) \in L^\infty S^m_{1,0} \Longrightarrow M^{\#}(x,\xi) \in \mathcal{B}_\rho S^m_{1,1}, \tag{10.63}$$

with $\rho = 1/16$, where we define $\mathcal{B}_\rho S^m_{1,1}$ for $\rho < 1$ to be

$$\mathcal{B}_\rho S^m_{1,1} = \{b(x,\xi) \in S^m_{1,1} : \hat{b}(\eta,\xi) \ \text{ supported in } |\eta| \le \rho|\xi|\}, \tag{10.64}$$

and where $\hat{b}(\eta,\xi) = \int b(x,\xi) e^{-i\eta\cdot x}\, dx$. Set $\mathcal{B}S^m_{1,1} = \cup_{\rho<1} \mathcal{B}_\rho S^m_{1,1}$.

Most of the applications of the material of this section made in the following chapters of this book will involve symbol smoothing, (10.60)–(10.61), with $\delta < 1$. However, we will establish some basic results on operator calculus for symbols of the form (10.64).

We will analyze products $a(x,D)b(x,D) = p(x,D)$ when we are given $a(x,\xi) \in S^\mu_{1,1}(\mathbb{R}^n)$ and $b(x,\xi) \in \mathcal{B}S^m_{1,1}(\mathbb{R}^n)$. We are particularly interested in estimating the remainder $r_\nu(x,\xi)$, arising in

$$a(x,D)b(x,D) = p_\nu(x,D) + r_\nu(x,D), \tag{10.65}$$

where

$$p_\nu(x,\xi) = \sum_{|\alpha|\le\nu} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha a(x,\xi)\cdot \partial_x^\alpha b(x,\xi). \tag{10.66}$$

Proposition 10.8 below is a variant of results of [Bon] and [Mey], established in [AT].

To begin the analysis, we have the formula

$$r_\nu(x,\xi) = \frac{1}{(2\pi)^n} \int \Big[a(x,\xi+\eta) - \sum_{|\alpha|\le\nu} \frac{\eta^\alpha}{\alpha!}\, \partial_\xi^\alpha a(x,\xi)\Big] e^{ix\cdot\eta} \hat{b}(\eta,\xi)\, d\eta. \tag{10.67}$$

Write

$$r_\nu(x,\xi) = \sum_{j\ge 0} r_{\nu j}(x,\xi), \tag{10.68}$$

with

$$\begin{aligned} r_{\nu j}(x,\xi) &= \int \widehat{A}_{\nu j}(x,\xi,\eta) \widehat{B}_j(x,\xi,\eta)\ d\eta \\ &= \int A_{\nu j}(x,\xi,y) B_j(x,\xi,-y)\ dy, \end{aligned} \tag{10.69}$$

where the terms in these integrands are defined as follows. Pick $\vartheta > 1$, and take a Littlewood-Paley partition of unity $\{\varphi_j^2 : j \geq 0\}$, such that $\varphi_0(\eta)$ is supported in $|\eta| \leq 1$, while for $j \geq 1$, $\varphi_j(\eta)$ is supported in $\vartheta^{j-1} \leq |\eta| \leq \vartheta^{j+1}$. Then we set

$$\begin{aligned} \widehat{A}_{\nu j}(x,\xi,\eta) &= \frac{1}{(2\pi)^n}\Big[a(x,\xi+\eta) - \sum_{|\alpha|\leq\nu} \frac{\eta^\alpha}{\alpha!}\partial_\xi^\alpha a(x,\xi)\Big]\varphi_j(\eta), \\ \widehat{B}_j(x,\xi,\eta) &= \hat{b}(\eta,\xi)\varphi_j(\eta)e^{ix\cdot\eta}. \end{aligned} \tag{10.70}$$

Note that

$$B_j(x,\xi,y) = \varphi_j(D_y)b(x+y,\xi). \tag{10.71}$$

Thus

$$\|B_j(x,\xi,\cdot)\|_{L^\infty} \leq C\vartheta^{-rj}\|b(\cdot,\xi)\|_{C_*^r}. \tag{10.72}$$

Also,

$$\operatorname{supp}\hat{b}(\eta,\xi) \subset \{|\eta| < \rho|\xi|\} \Longrightarrow B_j(x,\xi,y) = 0, \quad \text{for } \vartheta^{j-1} \geq \rho|\xi|. \tag{10.73}$$

We next estimate the L^1-norm of $A_{\nu j}(x,\xi,\cdot)$. Now, by a standard proof of Sobolev's imbedding theorem, given $K > n/2$, we have

$$\|A_{\nu j}(x,\xi,\cdot)\|_{L^1} \leq C\|\Gamma_j \widehat{A}_{\nu j}(x,\xi,\cdot)\|_{H^K}, \tag{10.74}$$

where $\Gamma_j f(\eta) = f(\vartheta^j\eta)$, so $\Gamma_j\widehat{A}_{\nu j}$ is supported in $|\eta| \leq \vartheta$. Let us use the integral formula for the remainder term in the power-series expansion to write

$$\begin{aligned} &\widehat{A}_{\nu j}(x,\xi,\vartheta^j\eta) = \\ &\frac{\varphi_j(\vartheta^j\eta)}{(2\pi)^n}\sum_{|\alpha|=\nu+1}\frac{\nu+1}{\alpha!}\Big(\int_0^1 (1-s)^{\nu+1}\partial_\xi^\alpha a(x,\xi+s\vartheta^j\eta)\,ds\Big)\vartheta^{j|\alpha|}\eta^\alpha. \end{aligned} \tag{10.75}$$

Since $|\eta| \leq \vartheta$ on the support of $\Gamma_j\widehat{A}_{\nu j}$, if also $\vartheta^{j-1} < \rho|\xi|$, then $|\vartheta^j\eta| < \rho\vartheta^2|\xi|$. Now, given $\rho \in (0,1)$, choose $\vartheta > 1$ such that $\rho\vartheta^3 < 1$. This implies $\langle\xi\rangle \sim \langle\xi + s\vartheta^j\eta\rangle$, for all $s \in [0,1]$. We deduce that the hypothesis

$$|\partial_\xi^\alpha a(x,\xi)| \leq C_\alpha\langle\xi\rangle^{\mu_2-|\alpha|}, \quad \text{for } |\alpha| \geq \nu+1, \tag{10.76}$$

implies

$$\|A_{\nu j}(x,\xi,\cdot)\|_{L^1} \leq C_\nu\vartheta^{j(\nu+1)}\langle\xi\rangle^{\mu_2-\nu-1}, \quad \text{for } \vartheta^{j-1} < \rho|\xi|. \tag{10.77}$$

Now, when (10.72) and (10.77) hold, we have

$$|r_{\nu j}(x,\xi)| \leq C_\nu\vartheta^{j(\nu+1-r)}\langle\xi\rangle^{\mu_2-\nu-1}\|b(\cdot,\xi)\|_{C_*^r}, \tag{10.78}$$

and if (10.73) also applies, we have

$$|r_\nu(x,\xi)| \leq C_\nu\langle\xi\rangle^{\mu_2-r}\|b(\cdot,\xi)\|_{C_*^r} \quad \text{if } \nu+1 > r, \tag{10.79}$$

since

$$\sum_{\vartheta^{j-1}<\rho|\xi|} \vartheta^{j(\nu+1-r)} \le C|\xi|^{\nu+1-r}$$

in such a case.

To estimate derivatives of $r_\nu(x,\xi)$, we can write

$$D_x^\beta D_\xi^\gamma r_{\nu j}(x,\xi) =$$

(10.80)

$$\sum_{\beta_1+\beta_2=\beta} \sum_{\gamma_1+\gamma_2=\gamma} \binom{\beta}{\beta_1}\binom{\gamma}{\gamma_1} \int D_x^{\beta_1} D_\xi^{\gamma_1} A_{\nu j}(x,\xi,y) \cdot D_x^{\beta_2} D_\xi^{\gamma_2} B_j(x,\xi,-y)\, dy.$$

Now $D_x^{\beta_1} D_\xi^{\gamma_1} A_{\nu j}(x,\xi,y)$ is produced just like $A_{\nu j}(x,\xi,y)$, with the symbol $a(x,\xi)$ replaced by $D_x^{\beta_1} D_\xi^{\gamma_1} a(x,\xi)$, and $D_x^{\beta_2} D_\xi^{\gamma_2} B_j(x,\xi,-y)$ is produced just like $B_j(x,\xi,-y)$, with $b(x,\xi)$ replaced by $D_x^{\beta_2} D_\xi^{\gamma_2} b(x,\xi)$. Thus, if we strengthen the hypothesis (10.76) to

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le C_{\alpha\beta} \langle\xi\rangle^{\mu_2-|\alpha|+|\beta|}, \quad \text{for} \ \ |\alpha| \ge \nu+1, \tag{10.81}$$

we have

$$\|D_x^{\beta_1} D_\xi^{\gamma_1} A_{\nu j}(x,\xi,\cdot)\|_{L^1} \le C_\nu \vartheta^{j(\nu+1)} \langle\xi\rangle^{\mu_2-|\gamma_1|+|\beta_1|-\nu-1}, \tag{10.82}$$

for $\vartheta^{j-1} < \rho|\xi|$. Furthermore, extending (10.72), we have

$$\|D_x^{\beta_2} D_\xi^{\gamma_2} B_j(x,\xi,\cdot)\|_{L^\infty} \le C\vartheta^{(|\beta_2|-r)j} \|D_\xi^{\gamma_2} b(\cdot,\xi)\|_{C_*^r}. \tag{10.83}$$

Now

$$\sum_{\vartheta^{j-1}<\rho|\xi|} \vartheta^{j(\nu+1+|\beta_2|-r)} \le C|\xi|^{\nu+1+|\beta_2|-r} \tag{10.84}$$

if $\nu+1>r$, so as long as (10.73) applies, (10.82) and (10.83) yield

$$|D_x^\beta D_\xi^\gamma r_\nu(x,\xi)| \le C \sum_{\gamma_1+\gamma_2=\gamma} \langle\xi\rangle^{\mu_2+|\beta|-|\gamma_1|-r} \|D_\xi^{\gamma_2} b(\cdot,\xi)\|_{C_*^r} \tag{10.85}$$

if $\nu+1>r$. These estimates lead to the following result:

Proposition 10.8. *Assume*

$$a(x,\xi) \in S^\mu_{1,1}, \quad b(x,\xi) \in \mathcal{B}S^m_{1,1}. \tag{10.86}$$

Then

$$a(x,D)b(x,D) = p(x,D) \in OPS^{\mu+m}_{1,1}. \tag{10.87}$$

Assume furthermore that

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \le C_{\alpha\beta} \langle\xi\rangle^{\mu_2-|\alpha|+|\beta|}, \quad \textit{for} \ \ |\alpha| \ge \nu+1, \tag{10.88}$$

with $\mu_2 \le \mu$, *and that*

$$\|D_\xi^\alpha b(\cdot,\xi)\|_{C_*^r} \le C_\alpha \langle \xi \rangle^{m_2-|\alpha|}. \tag{10.89}$$

Then, if $\nu + 1 > r$, *we have (10.65)–(10.66), with*

$$r_\nu(x,D) \in OPS_{1,1}^{\mu_2+m_2-r}. \tag{10.90}$$

The following is a commonly encountered special case of Proposition 10.8.

Corollary 10.9. *In Proposition 10.8, replace the hypothesis (10.89) by*

$$D_x^\beta b(x,\xi) \in S_{1,1}^{m_2}, \quad \text{for} \ \ |\beta| = K, \tag{10.91}$$

where $K \in \{1,2,3,\dots\}$ *is given. Then we have (10.65)–(10.66), with*

$$r_\nu(x,\xi) \in OPS_{1,1}^{\mu_2+m_2-K} \quad \text{ if } \ \nu \ge K. \tag{10.92}$$

Proof. The hypothesis (10.91) implies (10.89), with $r = K$.

We can also deduce from Proposition 10.8 that $a(x,D)b(x,D)$ has a complete asymptotic expansion if $b(x,\xi)$ is a symbol of type $(1,\delta)$ with $\delta < 1$.

Corollary 10.10. *If* $0 \le \delta < 1$ *and*

$$a(x,\xi) \in S_{1,1}^\mu, \quad b(x,\xi) \in S_{1,\delta}^m, \tag{10.93}$$

then $a(x,D)b(x,D) \in OPS_{1,1}^{\mu+m}$, *and we have (10.65)–(10.66), with*

$$r_\nu(x,D) \in OPS_{1,1}^{\mu+m-\nu(1-\delta)}. \tag{10.94}$$

Proof. Altering $b(x,\xi)$ by an element of $S_{1,0}^{-\infty}$, one can arrange that the condition (10.73) on supp $\hat{b}(\eta,\xi)$ hold. Then, apply Corollary 10.9, with $m_2 = m + K\delta$, so $m_2 - K = m - K(1-\delta)$, and take $K = \nu$.

Note that, under the hypotheses of Corollary 10.10,

$$\sum_{|\alpha|=\nu} \frac{1}{\alpha!}\, \partial_\xi^\alpha a(x,\xi) \cdot \partial_x^\alpha b(x,\xi) \in S_{1,1}^{\mu+m-\nu(1-\delta)}, \tag{10.95}$$

so we actually have

$$r_{\nu-1}(x,D) \in OPS_{1,1}^{\mu+m-\nu(1-\delta)}. \tag{10.96}$$

The family $\cup_m OP\mathcal{B}S_{1,1}^m$ does not form an algebra, but the following result is a useful substitute:

Proposition 10.11. *If* $p_j(x,\xi) \in \mathcal{B}_{\rho_j} S^{m_j}_{1,1}$ *and* $\rho = \rho_1 + \rho_2 + \rho_1\rho_2 < 1$, *then*

$$\begin{gathered} p_1(x,\xi)p_2(x,\xi) \in \mathcal{B}_\rho S^{m_1+m_2}_{1,1}, \\ p_1(x,D)p_2(x,D) \in OP\mathcal{B}_\rho S^{m_1+m_2}_{1,1}. \end{gathered} \tag{10.97}$$

Proof. The result for the symbol product is obvious; in fact, one can replace ρ by $\rho_1 + \rho_2$. As for $A(x,D) = p_1(x,D)p_2(x,D)$, we already have from Proposition 10.8 that $A(x,\xi) \in S^{m_1+m_2}_{1,1}$; we merely need to check the support of $\widehat{A}(\eta,\xi)$. We can do this using the formula

$$\widehat{A}(\eta,\xi) = \int \hat{p}_1(\eta-\zeta,\xi+\zeta)\hat{p}_2(\zeta,\xi)\, d\zeta. \tag{10.98}$$

Note that given (η,ξ), if there exists $\zeta \in \mathbb{R}^n$ such that $\hat{p}_1(\eta-\zeta,\xi+\zeta) \neq 0$ and $\hat{p}_2(\zeta,\xi) \neq 0$, then

$$|\eta-\zeta| \le \rho_1|\xi+\zeta|, \quad |\zeta| \le \rho_2|\xi|,$$

so

$$|\eta| \le \rho_1|\xi+\zeta| + |\zeta| \le \rho_1|\xi| + \rho_1|\zeta| + \rho_2|\xi| \le (\rho_1+\rho_2+\rho_1\rho_2)|\xi|.$$

This completes the proof.

Exercises

1. Prove the commutator property:

$$[OPS^\mu, OP\mathcal{A}^r_0 S^m_{1,\delta}] \subset OPS^{m+\mu-r}_{1,\delta}, \quad 0 \le r < 1,\ 0 \le \delta < 1. \tag{10.99}$$

2. Prove that, for $0 \le \delta < 1$,

$$P \in OP\mathcal{A}^r_0 S^m_{1,\delta} \Longrightarrow P^* \in OP\mathcal{A}^r_0 S^m_{1,\delta}. \tag{10.100}$$

(*Hint*: Use $P(x,D)^* = P^*(x,D)$, with $P^*(x,\xi) \sim \sum D^\alpha_x D^\alpha_\xi \overline{p(x,\xi)}$. Show that

$$p(x,\xi) \in \mathcal{A}^r_0 S^m_{1,\delta} \Longrightarrow D^\alpha_x D^\alpha_\xi p(x,\xi) \in \mathcal{A}^r_0 S^{m-(1-\delta)|\alpha|}_{1,\delta}.)$$

3. Show that

$$\sum_{|\alpha|\le m} a_\alpha(x, D^{m-1}u) D^\alpha u = M(u;x,D)u + R, \tag{10.101}$$

where $R \in C^\infty$ and, for $0 < r < 1$,

$$u \in C^{m-1+r} \Longrightarrow M(u;x,\xi) \in \mathcal{A}^r_0 S^m_{1,1} + S^{m-r}_{1,1}. \tag{10.102}$$

Deduce that you can write

$$M(u;x,\xi) = M^\#(x,\xi) + M^b(x,\xi), \tag{10.103}$$

with

$$M^\#(x,\xi) \in \mathcal{A}^r_0 S^m_{1,\delta}, \quad M^b(x,\xi) \in S^{m-r\delta}_{1,1}. \tag{10.104}$$

Note that the hypothesis on u is weaker than in Proposition 10.7.

4. The estimate (10.9) follows from the formula

$$D^\alpha g(h) = \sum_{\alpha_1+\cdots+\alpha_\nu=\alpha} C(\alpha_1,\dots,\alpha_\nu) h^{(\alpha_1)}\cdots h^{(\alpha_\nu)} g^{(\nu)}(h),$$

which is a consequence of the chain rule. Show that the following Moser-type estimate holds:

$$\|D^\ell g(h)\|_{L^\infty} \le C \sum_{1\le\nu\le\ell} \|g'\|_{C^{\nu-1}} \|h\|_{L^\infty}^{\nu-1} \|D^\ell h\|_{L^\infty}. \tag{10.105}$$

5. The paraproduct of J.-M. Bony [Bon] is defined by applying symbol smoothing to the multiplication operator, $M_f u = fu$. One takes

$$T_f u = \sum_k \Psi_{k-5}(D) f \cdot \psi_k(D) u, \tag{10.106}$$

where, as in (10.62), $\Psi_\ell(\xi) = \sum_{j\le\ell} \psi_j(\xi)$. Show that, with $T_f = F(x,D)$,

$$f \in L^\infty(\mathbb{R}^n) \Longrightarrow F(x,\xi) \in S^0_{1,1}(\mathbb{R}^n). \tag{10.107}$$

Show that, for any $r \in \mathbb{R}$,

$$f \in C^r_*(\mathbb{R}^n) \Longrightarrow |D_x^\beta D_\xi^\alpha F(x,\xi)| \le C_{\alpha\beta} \|f\|_{C^r_*} \langle\xi\rangle^{-r-|\alpha|+|\beta|}, \quad \text{for } |\alpha| \ge 1. \tag{10.108}$$

6. Using Propositions 10.8–10.11, show that if $p(x,\xi) \in \mathcal{B}_{1/2} S^m_{1,1}$, then

$$f \in C^0_* \Longrightarrow [T_f, p(x,D)] \in OP\mathcal{B}S^m_{1,1}. \tag{10.109}$$

Applications of this are given in [AT].

7. Show that $p(x,\xi) \in \mathcal{B}S^m_{1,1}$ implies $p(x,D)^* \in OPS^m_{1,1}$, and, if ρ is sufficiently small,

$$p(x,\xi) \in \mathcal{B}_\rho S^m_{1,1} \Longrightarrow p(x,D)^* \in OP\mathcal{B}S^m_{1,1}. \tag{10.110}$$

8. Investigate properties of operators with symbols in

$$\mathcal{B}^r S^m_{1,1} = \mathcal{B}S^m_{1,1} \cap \mathcal{A}^r_0 S^m_{1,1}. \tag{10.111}$$

11. Young measures and fuzzy functions

Limits in the weak* topology of sequences $f_j \in L^p(\Omega)$ are often not well behaved under the pointwise application of nonlinear functions. For example,

$$\sin\, nx \to 0 \quad \text{weak}^* \text{ in } L^\infty\big([0,\pi]\big), \tag{11.1}$$

while

$$\sin^2\, nx \to \frac{1}{2} \quad \text{weak}^* \text{ in } L^\infty\big([0,\pi]\big) \tag{11.2}$$

(see Fig. 11.1). A *fuzzy function* is endowed with an extra piece of structure, allowing for convergence under nonlinear mappings.

FIGURE 11.1

Assume Ω is an open set in $\mathbb{R}^n$. Given $1 \le p \le \infty$, we define an element of $Y^p(\Omega)$ to be a pair (f, λ), where $f \in L^p(\Omega)$ and λ is a positive Borel measure on $\Omega \times \overline{\mathbb{R}}$ ($\overline{\mathbb{R}} = [-\infty, \infty]$), having the properties

$$y \in L^p\big(\Omega \times \overline{\mathbb{R}}, d\lambda(x,y)\big), \tag{11.3}$$

(so, in particular, $\Omega \times \{\pm\infty\}$ has measure zero),

$$\lambda(E \times \mathbb{R}) = \mathcal{L}^n(E), \tag{11.4}$$

for Borel sets $E \subset \Omega$, where $\mathcal{L}^n$ is Lebesgue measure on Ω, and

$$\iint\limits_{E \times \mathbb{R}} y \; d\lambda(x,y) = \int\limits_E f(x) \; dx, \tag{11.5}$$

for each Borel set $E \subset \Omega$. We can equivalently state (11.4) and (11.5) as

$$\iint \varphi(x) \; d\lambda(x,y) = \int \varphi(x) \; dx \tag{11.6}$$

and

$$\iint \varphi(x) y \; d\lambda(x,y) = \int \varphi(x) f(x) \; dx, \tag{11.7}$$

for $\varphi \in C_0(\Omega)$, that is, for continuous and compactly supported φ.

Note that (11.5) implies

$$\int\limits_E |f(x)| \; dx \le \iint\limits_{E \times \mathbb{R}} |y| \; d\lambda(x,y), \tag{11.8}$$

since we can write $E = E_1 \cup E_2$ with $f \ge 0$ on E_1 and $f < 0$ on E_2. If we partition E into tiny sets, on each of which f is nearly constant, we obtain

$$\int\limits_E |f(x)|^p \; dx \le \iint\limits_{E \times \mathbb{R}} |y|^p \; d\lambda(x,y). \tag{11.9}$$

We say that (f, λ) is a fuzzy function, and λ is a *Young measure*, representing f.

A special case of such λ is γ_f, defined by

$$\iint \psi(x, y)\ d\gamma_f(x, y) = \int \psi\big(x, f(x)\big)\ dx, \tag{11.10}$$

for $\psi \in C_0(\Omega \times \mathbb{R})$. We say (f, γ_f) is sharply defined.

Fuzzy functions arise as limits of sharply defined functions in the following sense. Suppose $f_j \in L^p(\Omega)$, $1 < p \leq \infty$, and $(f, \lambda) \in Y^p(\Omega)$. We say

$$f_j \to (f, \lambda) \quad \text{in } \ Y^p(\Omega), \tag{11.11}$$

provided

$$f_j \to f \quad \text{weak}^* \text{ in } \ L^p(\Omega) \tag{11.12}$$

and

$$\gamma_{f_j} \to \lambda \quad \text{weak}^* \text{ in } \ \mathcal{M}(\Omega \times \overline{\mathbb{R}}), \tag{11.13}$$

and furthermore,

$$\|y\|_{L^p(\Omega \times \overline{\mathbb{R}}, d\gamma_{f_j})} \leq C < \infty. \tag{11.14}$$

Actually, (11.12) is a consequence of (11.13) and (11.14), thanks to (11.9).

To take an example, if $\Omega = (0, \pi)$ and $f_n(x) = \sin\ nx$, as in (11.1), it is easily seen that

$$f_n \to (0, \lambda_0) \quad \text{in } \ Y^\infty(\Omega), \tag{11.15}$$

where

$$d\lambda_0(x, y) = \chi_{[-1,1]}(y)\ \frac{2\ dx\ dy}{\sqrt{1 - y^2}}. \tag{11.16}$$

Also,

$$f_n^2 \to \Big(\frac{1}{2}, \lambda_1\Big) \quad \text{in } \ Y^\infty(\Omega), \tag{11.17}$$

where

$$d\lambda_1(x, y) = \chi_{[0,1]}(y)\ \frac{2\ dx\ dy}{\sqrt{y(y - 1)}}. \tag{11.18}$$

The following result illustrates the use of $Y^p(\Omega)$ in controlling the behavior of nonlinear maps. We make rather restrictive hypotheses for this first result, to keep the argument short and reveal its basic simplicity.

Proposition 11.1. *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be continuous. If $f_j \to (f, \lambda)$ in $Y^\infty(\Omega)$, then*

$$\Phi(f_j) \to g \quad \textit{weak}^* \textit{ in } \ L^\infty(\Omega), \tag{11.19}$$

where $g \in L^\infty(\Omega)$ is specified by

$$\int g(x)\varphi(x)\,dx = \iint \Phi(y)\varphi(x)\,d\lambda(x,y), \quad \varphi \in C_0(\Omega). \tag{11.20}$$

Proof. We need to check the behavior of $\int \Phi(f_j)\varphi\,dx$. Since $\Phi(f_j)$ is bounded in $L^\infty(\Omega)$, it suffices to take φ in $C_0(\Omega)$, which is dense in $L^1(\Omega)$. Let I be a compact interval in $(-\infty,\infty)$, containing the range of each function $\Phi(f_j)$. Now, for any $\varphi \in C_0(\Omega)$,

$$\begin{aligned} \int_\Omega \Phi(f_j)\varphi\,dx &= \iint_{\Omega\times I} \varphi(x)\Phi(y)\,d\gamma_{f_j}(x,y) \\ &\to \iint_{\Omega\times I} \varphi(x)\Phi(y)\,d\lambda(x,y), \end{aligned} \tag{11.21}$$

since $\gamma_{f_j} \to \lambda$ weak* in $\mathcal{M}(\Omega\times I)$. This proves the proposition.

Under the hypotheses of Proposition 11.1, we see that, more precisely than (11.19),

$$\Phi(f_j) \to (g,\nu) \quad \text{in} \;\; Y^\infty(\Omega), \tag{11.22}$$

where g is given by (11.20) and ν is specified by

$$\iint \psi(x,y)\,d\nu(x,y) = \iint \psi\big(x,\Phi(y)\big)\,d\lambda(x,y), \quad \psi \in C_0(\Omega\times\overline{\mathbb{R}}). \tag{11.23}$$

Thus ν is the natural image of λ under the map $\widetilde{\Phi}(x,y) = \big(x,\Phi(y)\big)$ of $\Omega\times I \to \Omega\times\overline{\mathbb{R}}$. One often writes $\nu = \widetilde{\Phi}_*\lambda$. The extra information carried by (11.22) is that $\gamma_{\Phi(f_j)} \to \nu$, weak* in $\mathcal{M}(\Omega\times\overline{\mathbb{R}})$, which follows from

$$\begin{aligned} \iint \psi(x,y)\,d\gamma_{\Phi(f_j)}(x,y) &= \iint \psi\big(x,\Phi(y)\big)\,d\gamma_{f_j}(x,y) \\ &\to \iint \psi\big(x,\Phi(y)\big)\,d\lambda(x,y). \end{aligned} \tag{11.24}$$

We can extend Proposition 11.1 and its refinement (11.22) to

$$f_j \to (f,\lambda) \text{ in } Y^p(\Omega) \Longrightarrow \Phi(f_j) \to (g,\nu) \text{ in } Y^q(\Omega), \tag{11.25}$$

with $1 < p,q < \infty$, where g and ν are given by the same formulas as above, provided that $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|\Phi(y)| \le C|y|^{p/q}. \tag{11.26}$$

We need this only for large $|y|$ if Ω has finite measure.

This result suggests defining the action of Φ on a fuzzy function (f,λ) by

$$\Phi(f,\lambda) = (g,\nu), \tag{11.27}$$

where g and ν are given by the formulas (11.20) and (11.23). Thus (11.22) can be restated as

$$f_j \to (f,\lambda) \text{ in } Y^\infty(\Omega) \Longrightarrow \Phi(f_j) \to \Phi(f,\lambda) \text{ in } Y^\infty(\Omega). \tag{11.28}$$

It is now natural to extend the notion of convergence $f_j \to (f,\lambda)$ in $Y^p(\Omega)$ to $(f_j,\lambda_j) \to (f,\lambda)$ in $Y^p(\Omega)$, provided all these objects belong to $Y^p(\Omega)$ and we have, parallel to (11.12)–(11.14),

$$f_j \to f \quad \text{weak}^* \text{ in } L^p(\Omega), \tag{11.29}$$

$$\lambda_j \to \lambda \quad \text{weak}^* \text{ in } \mathcal{M}(\Omega \times \overline{\mathbb{R}}), \tag{11.30}$$

and

$$\|y\|_{L^p(\Omega\times\overline{\mathbb{R}},d\lambda_j)} \le C < \infty. \tag{11.31}$$

As before, (11.29) is actually a consequence of (11.30) and (11.31). Now (11.28) is easily extended to

$$(f_j,\lambda_j) \to (f,\lambda) \text{ in } Y^\infty(\Omega) \Longrightarrow \Phi(f_j,\lambda_j) \to \Phi(f,\lambda) \text{ in } Y^\infty(\Omega), \tag{11.32}$$

for continuous $\Phi : \mathbb{R} \to \mathbb{R}$. There is a similar extension of (11.25), granted the bound (11.26) on $\Phi(y)$.

We say that f_j (or more generally (f_j,λ_j)) converges *sharply* in $Y^p(\Omega)$, if it converges, in the sense defined above, to (f,λ) with $\lambda = \gamma_f$. It is of interest to specify conditions under which we can guarantee sharp convergence. We will establish some results in that direction a bit later.

When one has a fuzzy function (f,λ), it can be conceptually useful to pass from the measure λ on $\Omega \times \overline{\mathbb{R}}$ to a family of probability measures λ_x on $\overline{\mathbb{R}}$, defined for a.e. $x \in \Omega$. We discuss how this can be done. From (11.4) we have

$$\Bigl| \iint\limits_{E\times\overline{\mathbb{R}}} \psi(y)\, d\lambda(x,y) \Bigr| \le \sup\, |\psi|\, \mathcal{L}^n(E), \tag{11.33}$$

and hence

$$\Bigl| \iint\limits_{\Omega\times\overline{\mathbb{R}}} \varphi(x)\psi(y)\, d\lambda(x,y) \Bigr| \le \sup\, |\psi| \cdot \|\varphi\|_{L^1(\Omega)}. \tag{11.34}$$

It follows that there is a linear transformation

$$T : C(\overline{\mathbb{R}}) \longrightarrow L^\infty(\Omega), \quad \|T\psi\|_{L^\infty(\Omega)} \le \sup\, |\psi|, \tag{11.35}$$

such that

$$\iint\limits_{\Omega\times\overline{\mathbb{R}}} \varphi(x)\psi(y)\, d\lambda(x,y) = \int\limits_\Omega \varphi(x) T\psi(x)\, dx. \tag{11.36}$$

Using the separability of $C(\overline{\mathbb{R}})$, we can deduce that there is a set $S \subset \Omega$, of Lebesgue measure zero, such that, for *all* $\psi \in C(\overline{\mathbb{R}})$, $T\psi(x)$ is defined *pointwise*, for $x \in \Omega \setminus S$. Note that T is positivity preserving and $T(1) = 1$. Thus for each $x \in \Omega \setminus S$, there is a probability measure λ_x on $\overline{\mathbb{R}}$ such that

$$T\psi(x) = \int_{\overline{\mathbb{R}}} \psi(y)\ d\lambda_x(y). \tag{11.37}$$

Hence

$$\iint_{\Omega\times\overline{\mathbb{R}}} \varphi(x)\psi(y)\ d\lambda(x,y) = \int_{\Omega} \left(\int_{\overline{\mathbb{R}}} \varphi(x)\psi(y)\ d\lambda_x(y) \right) dx. \tag{11.38}$$

From this it follows that

$$\iint_{\Omega\times\overline{\mathbb{R}}} \psi(x,y)\ d\lambda(x,y) = \int_{\Omega} \left(\int_{\overline{\mathbb{R}}} \psi(x,y)\ d\lambda_x(y) \right) dx, \tag{11.39}$$

for any Borel-measurable function ψ that is either positive or integrable with respect to $d\lambda$. Thus we can reformulate Proposition 11.1:

Corollary 11.2. *If $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous and $f_j \to (f, \lambda)$ in $Y^\infty(\Omega)$, then*

$$\Phi(f_j) \to g \quad \textit{weak}^* \textit{ in } L^\infty(\Omega), \tag{11.40}$$

where

$$g(x) = \int_{\mathbb{R}} \Phi(y)\ d\lambda_x(y), \quad \textit{a.e.} \quad x \in \Omega. \tag{11.41}$$

One key feature of the notion of convergence of a sequence of fuzzy functions is that, while it is preserved under nonlinear maps, we also retain the sort of compactness property that weak* convergence has.

Proposition 11.3. *Let $(f_j, \lambda_j) \in Y^\infty(\Omega)$, and assume $\|f_j\|_{L^\infty(\Omega)} \le M$. Then there exist $(f, \lambda) \in Y^\infty(\Omega)$ and a subsequence $(f_{j_\nu}, \lambda_{j_\nu})$ such that*

$$(f_{j_\nu}, \lambda_{j_\nu}) \longrightarrow (f, \lambda). \tag{11.42}$$

Proof. The well-known weak* compactness (and metrizability) of $\{g \in L^\infty(\Omega) : \|g\|_{L^\infty} \le M\}$ implies that one can pass to a subsequence (which we continue to denote by (f_j, λ_j)) such that $f_j \to f$ weak* in $L^\infty(\Omega)$.

Each measure λ_j is supported on $\Omega \times I,\ I = [-M, M]$. Now we exploit the weak* compactness and metrizability of $\{\mu \in \mathcal{M}(K \times I) : \|\mu\| \le \mathcal{L}^n(K)\}$, for each compact $K \subset \Omega$, together with a standard diagonal argument, to obtain a further subsequence such that $\lambda_{j_\nu} \to \lambda$ weak* in $\mathcal{M}(\Omega \times I)$. The identities (11.6) and (11.7) are preserved under passage to such a limit, so the proposition is proved.

So far we have dealt with real-valued fuzzy functions, but we can as easily consider fuzzy functions with values in a finite-dimensional, normed vector space V. We define $Y^p(\Omega, V)$ to consist of pairs (f, λ), where $f \in L^p(\Omega, V)$ is a V-valued L^p function and λ is a positive Borel measure on $\Omega \times \overline{V}$ ($\overline{V} = V$ plus the sphere S_∞ at infinity), having the properties

$$|y| \in L^p\big(\Omega \times \overline{V}, d\lambda(x, y)\big), \tag{11.43}$$

so in particular $\Omega \times S_\infty$ has measure zero,

$$\lambda(E \times V) = \mathcal{L}^n(E), \tag{11.44}$$

for Borel sets $E \subset \Omega$, and

$$\iint\limits_{E \times V} y \; d\lambda(x, y) = \int\limits_E f(x) \; dx \in V, \tag{11.45}$$

for each Borel set $E \subset \Omega$.

All of the preceding results of this section extend painlessly to this case. Instead of considering $\Phi : \mathbb{R} \to \mathbb{R}$, we take $\Phi : V_1 \to V_2$, where V_j are two normed finite-dimensional vector spaces. This time, a Young measure λ "disintegrates" into a family λ_x of probability measures on $\overline{V}$.

There is a natural map

$$\& : Y^\infty(\Omega, V_1) \times Y^\infty(\Omega, V_2) \longrightarrow Y^\infty(\Omega, V_1 \oplus V_2) \tag{11.46}$$

defined by

$$(f_1, \lambda_1) \& (f_2, \lambda_2) = (f_1 \oplus f_2, \nu), \tag{11.47}$$

where, for a.e. $x \in \Omega$, Borel $F_j \subset V_j$,

$$\nu_x(F_1 \times F_2) = \lambda_{1x}(F_1) \lambda_{2x}(F_2). \tag{11.48}$$

Using this, we can define an "addition" on elements of $Y^\infty(\Omega, V)$:

$$(f_1, \lambda_1) + (f_2, \lambda_2) = S\big((f_1, \lambda_1) \& (f_2, \lambda_2)\big), \tag{11.49}$$

where $S : V \oplus V \to V$ is given by $S(v, w) = v + w$, and we extend S to a map $S : Y^\infty(\Omega, V \oplus V) \to Y^\infty(\Omega, V)$ by the same process as used in (11.27).

Of course, multiplication by a scalar $a \in \mathbb{R}$, $M_a : V \to V$, induces a map M_a on $Y^\infty(\Omega, V)$, so we have what one might call a "fuzzy linear structure" on $Y^\infty(\Omega, V)$. It is not truly a linear structure since certain basic requirements on vector space operations do not hold here. For example (in the case $V = \mathbb{R}$), $(f, \lambda) \in Y^\infty(\Omega)$ has a natural "negative," namely $(-f, \check{\lambda})$, where $\check{\lambda}(E) = \lambda(-E)$. However, $(f, \lambda) + (-f, \check{\lambda}) \neq (0, \gamma_0)$ unless (f, λ) is sharply defined. Similarly, $(f, \lambda) + (f, \lambda) \neq 2(f, \lambda)$ unless (f, λ) is sharply defined, so the distributive law fails.

We now derive some conditions under which, for a given sequence $u_j \to (u, \lambda)$ in $Y^\infty(\Omega)$ and a given nonlinear function F, we also have $F(u_j) \to$

$F(u)$ weak* in $L^\infty(\Omega)$, which is the same here as $F(u) = \overline{F}$. The following result is of the nature that weak* convergence of the dot product of the $\mathbb{R}^2$-valued functions $\big(u_j, F(u_j)\big)$ with a certain family of $\mathbb{R}^2$-valued functions $V(u_j)$ to $(u, \overline{F}) \cdot \overline{V}$ will imply $\overline{F} = F(u)$. The specific choice of $V(u_j)$ will perhaps look curious; we will explain below how this choice arises.

Proposition 11.4. *Suppose* $u_j \to (u, \lambda)$ *in* $Y^\infty(\Omega)$, *and let* $F : \mathbb{R} \to \mathbb{R}$ *be* C^1. *Suppose you know that*

$$u_j q(u_j) - F(u_j)\eta(u_j) \longrightarrow u\overline{q} - \overline{F}\overline{\eta} \quad \text{weak}^* \text{ in } L^\infty(\Omega), \tag{11.50}$$

for every convex function $\eta : \mathbb{R} \to \mathbb{R}$, *with* q *given by*

$$q(y) = \int_c^y \eta'(s) F'(s)\, ds, \tag{11.51}$$

and where

$$q(u, \lambda) = (\overline{q}, \nu_1), \quad F(u, \lambda) = (\overline{F}, \nu_2), \quad \eta(u, \lambda) = (\overline{\eta}, \nu_3). \tag{11.52}$$

Then

$$F(u_j) \to F(u) \quad \text{weak}^* \text{ in } L^\infty(\Omega). \tag{11.53}$$

Proof. It suffices to prove that $\overline{F} = F(u)$ a.e. on Ω. Now, applying Corollary 11.2 to $\Phi(y) = yq(y) - F(y)\eta(y)$, we have the left side of (11.50) converging weak* in $L^\infty(\Omega)$ to

$$v(x) = \int \big[yq(y) - F(y)\eta(y) \big]\, d\lambda_x(y),$$

so the hypothesis (11.50) implies

$$v = u\overline{q} - \overline{F}\overline{\eta}, \quad \text{a.e. on } \Omega.$$

Rewrite this as

$$\int \Big\{ \big(F(y) - \overline{F}(x)\big)\eta(y) - \big(u(x) - y\big)q(y) \Big\}\, d\lambda_x(y) = 0, \quad \text{a.e. } x \in \Omega. \tag{11.54}$$

Now we make the following special choices of functions η and q:

$$\eta_a(y) = |y - a|, \quad q_a(y) = \operatorname{sgn}(y - a)\, \big(F(y) - F(a)\big). \tag{11.55}$$

We use these in (11.54), with $a = u(x)$, obtaining, after some cancellation,

$$\big(F\big(u(x)\big) - \overline{F}(x)\big) \int |y - u(x)|\, d\lambda_x(y) = 0, \quad \text{a.e. } x \in \Omega. \tag{11.56}$$

Thus, for a.e. $x \in \Omega$, either $\overline{F}(x) = F\big(u(x)\big)$ or $\lambda_x = \delta_{u(x)}$, which also implies $\overline{F}(x) = F\big(u(x)\big)$. The proof is complete.

Why is one motivated to work with such functions $\eta(u)$ and $q(u)$? They arise in the study of solutions to some nonlinear PDE on $\Omega \subset \mathbb{R}^2$. Let us use coordinates (t, x) on Ω. As long as u is a Lipschitz-continuous, real-valued function on Ω, it follows from the chain rule that

$$u_t + F(u)_x = 0 \Longrightarrow \eta(u)_t + q(u)_x = 0, \tag{11.57}$$

provided $q'(y) = \eta'(y)F'(y)$, that is, q is given by (11.52). (For general $u \in L^\infty(\Omega)$, the implication (11.57) does not hold.) Our next goal is to establish the following:

Proposition 11.5. *Assume $u_j \in L^\infty(\Omega)$, of norm $\le M < \infty$. Assume also that*

$$\partial_t u_j + \partial_x F(u_j) \to 0 \quad \text{in } H^{-1}_{\text{loc}}(\Omega) \tag{11.58}$$

and

$$\partial_t \eta(u_j) + \partial_x q(u_j) \quad \text{precompact in } H^{-1}_{\text{loc}}(\Omega), \tag{11.59}$$

for each convex function $\eta : \mathbb{R} \to \mathbb{R}$, with q given by (11.51). If $u_j \to u$ weak in $L^\infty(\Omega)$, then*

$$\partial_t u + \partial_x F(u) = 0. \tag{11.60}$$

Proof. By Proposition 11.3, passing to a subsequence, we have $u_j \to (u, \lambda)$ in $Y^\infty(\Omega)$. Then, by Proposition 11.1, $F(u_j) \to \overline{F}$, $q(u_j) \to \overline{q}$, and $\eta(u_j) \to \overline{\eta}$ weak* in $L^\infty(\Omega)$. Consider the vector-valued functions

$$v_j = \big(u_j, F(u_j)\big), \quad w_j = \big(q(u_j), -\eta(u_j)\big). \tag{11.61}$$

Thus $v_j \to (u, \overline{F})$, $w_j \to (\overline{q}, -\overline{\eta})$ weak* in $L^\infty(\Omega)$. The hypotheses (11.58)–(11.59) are equivalent to

$$\text{div } v_j, \quad \text{rot } w_j \quad \text{precompact in } H^{-1}_{\text{loc}}(\Omega). \tag{11.62}$$

Also, the hypothesis on $\|u_j\|_{L^\infty}$ implies that v_j and w_j are bounded in $L^\infty(\Omega)$, and a fortiori in $L^2_{\text{loc}}(\Omega)$. The div-curl lemma hence implies that

$$v_j \cdot w_j \to v \cdot w \quad \text{in } \mathcal{D}'(\Omega), \quad v = (u, \overline{F}), \quad w = (\overline{q}, -\overline{\eta}). \tag{11.63}$$

In view of the L^∞-bounds, we hence have

$$u_j q(u_j) - F(u_j)\eta(u_j) \longrightarrow u\overline{q} - \overline{F}\overline{\eta} \quad \text{weak}^* \text{ in } L^\infty(\Omega). \tag{11.64}$$

Since this is the hypothesis (11.50) of Proposition 11.4, we deduce that

$$F(u_j) \longrightarrow F(u) \quad \text{weak}^* \text{ in } L^\infty(\Omega). \tag{11.65}$$

Hence $\partial_t u_j + \partial_x F(u_j) \to \partial_t u + \partial_x F(u)$ in $\mathcal{D}'(\Omega)$, so we have (11.60).

One of the most important cases leading to the situation dealt with in Proposition 11.5 is the following; for $\varepsilon \in (0, 1]$, consider the PDE

$$\partial_t u_\varepsilon + \partial_x F(u_\varepsilon) = \varepsilon \partial_x^2 u_\varepsilon \quad \text{on } \Omega = (0, \infty) \times \mathbb{R}, \quad u_\varepsilon(0) = f. \tag{11.66}$$

Say $f \in L^\infty(\mathbb{R})$. The unique solvability of (11.66), for $t \in [0,\infty)$, for each $\varepsilon > 0$, will be established in Chapter 15, and results there imply

$$u_\varepsilon \in C^\infty(\Omega), \tag{11.67}$$

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \le \|f\|_{L^\infty}, \tag{11.68}$$

and

$$\varepsilon \int_0^\infty \int_{-\infty}^\infty (\partial_x u_\varepsilon)^2 \, dx \, dt \le \frac{1}{2}\|f\|_{L^2}^2. \tag{11.69}$$

The last result implies that $\sqrt{\varepsilon}\partial_x u_\varepsilon$ is bounded in $L^2(\Omega)$. Hence $\varepsilon\partial_x^2 u_\varepsilon \to 0$ in $H^{-1}(\Omega)$, as $\varepsilon \to 0$. Thus, if u_{ε_j} is relabeled u_j, with $\varepsilon_j \to 0$, we have hypothesis (11.58) of Proposition 11.5. We next check hypothesis (11.59).

Using the chain rule and (11.66), we have

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \varepsilon\partial_x^2 \eta(u_\varepsilon) - \varepsilon\eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2, \tag{11.70}$$

at least when η is C^2 and q satisfies (11.52). Parallel to (11.69), we have

$$\varepsilon \int_0^T \int \eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2 \, dx \, dt = \int \eta\big(f(x)\big) \, dx - \int \eta\big(u_\varepsilon(T,x)\big) \, dx. \tag{11.71}$$

A simple approximation argument, taking smooth $\eta_\delta \to \eta$, shows that whenever η is nonnegative and convex, C^2 or not,

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \varepsilon\partial_x^2 \eta(u_\varepsilon) - R_\varepsilon, \tag{11.72}$$

with

$$R_\varepsilon \ \text{ bounded in } \ \mathcal{M}(\Omega). \tag{11.73}$$

Since $\partial_x \eta(u_\varepsilon) = \eta'(u_\varepsilon)\partial_x u_\varepsilon$, and any convex η is locally Lipschitz, we deduce from (11.68) and (11.69) that $\sqrt{\varepsilon}\partial_x \eta(u_\varepsilon)$ is bounded in $L^2(\Omega)$. Hence

$$\varepsilon\partial_x^2 \eta(u_\varepsilon) \to 0 \ \text{ in } \ H^{-1}(\Omega), \qquad \text{as } \ \varepsilon \to 0. \tag{11.74}$$

We thus have certain bounds on the right side of (11.72), by (11.73) and (11.74). Meanwhile, the left side of (11.72) is certainly bounded in $H^{-1,p}_{\text{loc}}(\Omega)$, $\forall \ p < \infty$. This situation is treated by the following lemma of F. Murat.

Lemma 11.6. *Suppose $\mathcal{F}$ is bounded in $H^{-1,p}_{\text{loc}}(\Omega)$, for some $p > 2$, and $\mathcal{F} \subset \mathcal{G} + \mathcal{H}$, where $\mathcal{G}$ is precompact in $H^{-1}_{\text{loc}}(\Omega)$ and $\mathcal{H}$ is bounded in $\mathcal{M}_{\text{loc}}(\Omega)$. Then $\mathcal{F}$ is precompact in $H^{-1}_{\text{loc}}(\Omega)$.*

Proof. Multiplying by a cut-off $\chi \in C_0^\infty(\Omega)$, we reduce to the case where all $f \in \mathcal{F}$ are supported in some compact K, and the decomposition $f = g + h$, $g \in \mathcal{G}$, $h \in \mathcal{H}$ also has g, h supported in K. Putting K in a box and identifying opposite sides, we are reduced to establishing an analogue of the lemma when Ω is replaced by $\mathbb{T}^n$.

Now Sobolev imbedding theorems imply

$$\mathcal{M}(\mathbb{T}^n) \subset H^{-s,q}(\mathbb{T}^n), \quad s \in (0,n), \quad q \in \Big(1, \frac{n}{n-s}\Big).$$

Via Rellich's compactness result (6.9), it follows that

$$\iota : \mathcal{M}(\mathbb{T}^n) \hookrightarrow H^{-1,q}(\mathbb{T}^n), \quad \text{compact} \ \ \forall\, q \in \Big(1, \frac{n}{n-1}\Big). \tag{11.75}$$

Hence $\mathcal{H}$ is precompact in $H^{-1,q}(\mathbb{T}^n)$, for any $q < n/(n-1)$, so we have

$$\mathcal{F} \ \text{ precompact in } \ H^{-1,q}(\mathbb{T}^n), \quad \text{bounded in } \ H^{-1,p}(\mathbb{T}^n), \quad p > 2. \tag{11.76}$$

By a simple interpolation argument, (11.76) implies that $\mathcal{F}$ is precompact in $H^{-1}(\mathbb{T}^n)$, so the lemma is proved.

We deduce that if the family $\{u_\varepsilon : 0 < \varepsilon \le 1\}$ of solutions to (11.66) satisfies (11.67)–(11.69), then

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) \quad \text{precompact in } \ H^{-1}_{\text{loc}}(\Omega), \tag{11.77}$$

which is hypothesis (11.59) of Proposition 11.5. Therefore, we have the following:

Proposition 11.7. *Given solutions u_ε, $0 < \varepsilon \le 1$ to (11.66), satisfying (11.67)–(11.69), a weak* limit u in $L^\infty(\Omega)$, as $\varepsilon = \varepsilon_j \to 0$, satisfies*

$$\partial_t u + \partial_x F(u) = 0. \tag{11.78}$$

The approach to the solvability of (11.78) used above is given in [Tar]. In Chapter 16, §6, we will obtain global existence results containing that of Proposition 11.7, using different methods, involving uniform estimates of $\|\partial_x u_\varepsilon(t)\|_{L^1(\mathbb{R})}$. On the other hand, in §9 of Chapter 16 we will make use of techniques involving fuzzy functions and the div-curl lemma to establish some global solvability results for certain 2×2 hyperbolic systems of conservation laws, following work of R. DiPerna [DiP].

The notion of fuzzy function suggests the following notion of a "fuzzy solution" to a PDE, of the form

$$\sum_j \frac{\partial}{\partial x_j}\, F_j(u) = 0. \tag{11.79}$$

Namely, $(u, \lambda) \in Y^\infty(\Omega)$ is a fuzzy solution to (11.79) if

$$\sum_j \frac{\partial}{\partial x_j} \overline{F}_j = 0 \ \text{ in } \ \mathcal{D}'(\Omega), \quad \overline{F}_j(x) = \int F_j(y)\, d\lambda_x(y). \tag{11.80}$$

This notion was introduced in [DiP], where (u, λ) is called a "measure-valued solution" to (11.79). Given $|F_j(y)| \le C\langle y\rangle^p$, we can also consider the concept of a fuzzy solution $(u, \lambda) \in Y^p(\Omega)$. Contrast the following simple result with Proposition 11.5:

Proposition 11.8. *Assume* $(u_j, \lambda_j) \in Y^\infty(\Omega)$, $\|u_j\|_{L^\infty} \le M$, *and* $(u_j, \lambda_j) \to (u, \lambda)$ *in* $Y^\infty(\Omega)$. *If*

$$\sum_k \partial_k F_k(u_j) \to 0 \quad \text{in } \mathcal{D}'(\Omega), \tag{11.81}$$

as $j \to \infty$, *then* u *is a fuzzy solution to (11.79).*

Proof. By Proposition 11.1, $F_k(u_j) \to \overline{F}_k$ weak* in $L^\infty(\Omega)$. The result follows immediately from this.

In [DiP] there are some results on when one can say that, when $(u, \lambda) \in Y^\infty(\Omega)$ is a fuzzy solution to (11.79), then $u \in L^\infty(\Omega)$ is a weak solution to (11.79), results that in particular lead to another proof of Proposition 11.7.

Exercises

1. If $f_j \to (f, \lambda)$ in $Y^\infty(\Omega)$, we say the convergence is *sharp* provided $\lambda = \gamma_f$. Show that sharp convergence implies
$$f_j \to f \quad \text{in} \quad L^2(\Omega_0),$$
for any $\Omega_0 \subset\subset \Omega$.
(*Hint*: Sharp convergence implies $|f_j|^2 \to |f|^2$ weak* in $L^\infty(\Omega)$. Thus $f_j \to f$ weakly in L^2 and also $\|f_j\|_{L^2(\Omega_0)} \to \|f\|_{L^2(\Omega_0)}$.)
2. Deduce that, given $f_j \to (f, \lambda)$ in $Y^\infty(\Omega)$, the convergence is sharp if and only if, for some subsequence, $f_{j_\nu} \to f$ a.e. on Ω.
3. Given $(f, \lambda) \in Y^\infty(\Omega)$ and the associated family of probability measures λ_x, $x \in \Omega$, as in (11.37)–(11.39), show that $\lambda = \gamma_f$ if and only if, for a.e. $x \in \Omega$, λ_x is a point mass.
4. Complete the interpolation argument cited in the proof of Lemma 11.6. Show that (with $X = \Lambda^{-1}(\mathcal{F})$) if $q < 2 < p$,
$$X \text{ precompact in } L^q(\mathbb{T}^n), \text{ bounded in } L^p(\mathbb{T}^n) \Longrightarrow X \text{ precompact in } L^2(\mathbb{T}^n).$$
(*Hint*: If $f_n \in X$, $f_n \to f$ in $L^q(\mathbb{T}^n)$, use
$$\|f_n - f\|_{L^2} \le \|f_n - f\|_{L^q}^{\alpha} \|f_n - f\|_{L^p}^{1-\alpha}.)$$
5. Extend various propositions of this section from $Y^\infty(\Omega)$ to $Y^p(\Omega)$, $1 < p \le \infty$.

12. Hardy spaces

The Hardy space $\mathfrak{H}^1(\mathbb{R}^n)$ is a subspace of $L^1(\mathbb{R}^n)$ defined as follows. Set

$$(\mathcal{G}f)(x) = \sup\{|\varphi_t * f(x)| : \varphi \in \mathcal{F},\ t > 0\}, \tag{12.1}$$

where $\varphi_t(x) = t^{-n}\varphi(x/t)$ and

$$\mathcal{F} = \big\{\varphi \in C_0^\infty(\mathbb{R}^n) : \varphi(x) = 0 \text{ for } |x| \geq 1,\ \|\nabla\varphi\|_{L^\infty} \leq 1\big\}. \tag{12.2}$$

This is called the *grand maximal function* of f. Then we define

$$\mathfrak{H}^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : \mathcal{G}f \in L^1(\mathbb{R}^n)\}. \tag{12.3}$$

A related (but slightly larger) space is $\mathfrak{h}^1(\mathbb{R}^n)$, defined as follows. Set

$$(\mathcal{G}^b f)(x) = \sup\big\{|\varphi_t * f(x)| : \varphi \in \mathcal{F},\ 0 < t \leq 1\big\}, \tag{12.4}$$

and define

$$\mathfrak{h}^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : \mathcal{G}^b f \in L^1(\mathbb{R}^n)\}. \tag{12.5}$$

An important tool in the study of Hardy spaces is another maximal function, the *Hardy-Littlewood maximal function*, defined by

$$\mathcal{M}(f)(x) = \sup_{r>0} \frac{1}{\text{vol}(B_r)} \int\limits_{B_r(x)} |f(y)|\, dy. \tag{12.6}$$

The basic estimate on this maximal function is the following weak type-(1,1) estimate:

Proposition 12.1. *There is a constant $C = C(n)$ such that, for any $\lambda > 0,\ f \in L^1(\mathbb{R}^n)$, we have the estimate*

$$\textit{meas}\big(\{x \in \mathbb{R}^n : \mathcal{M}(f)(x) > \lambda\}\big) \leq \frac{C}{\lambda}\|f\|_{L^1}. \tag{12.7}$$

Note that the estimate

$$\text{meas}\big(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}\big) \leq \frac{1}{\lambda}\|f\|_{L^1}$$

follows by integrating the inequality $|f| \geq \lambda\chi_{S_\lambda}$, where $S_\lambda = \{|f| > \lambda\}$.

To begin the proof of Proposition 12.1, let

$$F_\lambda = \{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\}. \tag{12.8}$$

We remark that, for any $f \in L^1(\mathbb{R}^n)$ and any $\lambda > 0,\ F_\lambda$ is open. Given $x \in F_\lambda$, pick $r = r_x$ such that $A_r|f|(x) > \lambda$, and let $B_x = B_{r_x}(x)$. Thus $\{B_x : x \in F_\lambda\}$ is a covering of F_λ by balls. We will be able to obtain the estimate (12.7) from the following "covering lemma," due to N. Wiener.

Lemma 12.2. *If $\mathcal{C} = \{B_\alpha : \alpha \in \mathfrak{A}\}$ is a collection of open balls in $\mathbb{R}^n$, with union U, and if $m_0 <$ meas(U), then there is a finite collection of disjoint balls $B_j \in \mathcal{C},\ 1 \leq j \leq K$, such that*

$$\sum \textit{meas}(B_j) > 3^{-n} m_0. \tag{12.9}$$

We show how the lemma allows us to prove (12.7). In this case, let $\mathcal{C} = \{\overset{\circ}{B}_x : x \in F_\lambda\}$. Thus, if $m_0 <$ meas(F_λ), there exist disjoint balls $B_j = \overset{\circ}{B}_{r_j}(x_j)$ such that meas$(\cup B_j) > 3^{-n} m_0$. This implies

$$m_0 < 3^n \sum \text{meas}(B_j) \leq \frac{3^n}{\lambda} \sum \int_{B_j} |f(x)|\, dx \leq \frac{3^n}{\lambda} \int |f(x)|\, dx, \tag{12.10}$$

for all $m_0 <$ meas(F_λ), which yields (12.7), with $C = 3^n$.

We now turn to the proof of Lemma 12.2. We can pick a compact $K \subset U$ such that $m(K) > m_0$. Then the covering $\mathcal{C}$ yields a finite covering of K, say $A_1, \dots, A_N$. Let B_1 be the ball A_j of the largest radius. Throw out all A_ℓ that meet B_1, and let B_2 be the remaining ball of largest radius. Continue until $\{A_1, \dots, A_N\}$ is exhausted. One gets disjoint balls $B_1, \dots, B_K$ in $\mathcal{C}$. Now each A_j meets some B_ℓ, having the property that the radius of B_ℓ is $\geq$ the radius of A_j. Thus, if $\widehat{B}_j$ is the ball concentric with B_j, with 3 times the radius, we have

$$\bigcup_{j=1}^{K} \widehat{B}_j \supset \bigcup_{\ell=1}^{N} A_\ell \supset K.$$

This yields (12.9).

Note that clearly

$$f \in L^\infty(\mathbb{R}^n) \Longrightarrow \|\mathcal{M}(f)\|_{L^\infty} \leq \|f\|_{L^\infty}. \tag{12.11}$$

Now the method of proof of the Marcinkiewicz interpolation theorem, Proposition 5.4, yields the following.

Corollary 12.3. *If $1 < p < \infty$, then*

$$\|\mathcal{M}(f)\|_{L^p} \leq C_p \|f\|_{L^p}. \tag{12.12}$$

Our first result on Hardy spaces is the following, relating $\mathfrak{h}^1(\mathbb{R}^n)$ to the smaller space $\mathfrak{H}^1(\mathbb{R}^n)$.

Proposition 12.4. *If $u \in \mathfrak{h}^1(\mathbb{R}^n)$ has compact support and $\int u(x)\, dx = 0$, then $u \in \mathfrak{H}^1(\mathbb{R}^n)$.*

Proof. It suffices to show that

$$v(x) = \sup\{|\varphi_t * u(x)| : \varphi \in \mathcal{F}, t \geq 1\} \tag{12.13}$$

belongs to $L^1(\mathbb{R}^n)$. Clearly, v is bounded. Also, if supp $u \subset \{|x| \leq R\}$, then we can write $u = \sum \partial_j u_j,\ u_j \in L^1(B_R)$. Then

(12.14)

$$\varphi_t * u(x) = \sum_j t^{-1} \psi_{jt} * u_j(x), \quad \psi_{jt}(x) = t^{-n} \psi_j(t^{-1}x),\ \psi_j(x) = \partial_j \varphi(x).$$

If $|x| = R + 1 + \rho$, then $\psi_{jt} * u_j(x) = 0$ for $t < \rho$, so

$$v(x) \le C\rho^{-1} \sum_j \mathcal{M}(u_j)(x). \tag{12.15}$$

The weak (1,1) bound (12.7) on $\mathcal{M}$ now readily yields an L^1-bound on $v(x)$.

One advantage of $\mathfrak{h}^1(\mathbb{R}^n)$ is its localizability. We have the following useful result:

Proposition 12.5. *If $r > 0$ and $g \in C^r(\mathbb{R}^n)$ has compact support, then*

$$u \in \mathfrak{h}^1(\mathbb{R}^n) \Longrightarrow gu \in \mathfrak{h}^1(\mathbb{R}^n). \tag{12.16}$$

Proof. If $g \in C^r$ and $0 < r \le 1$, we have, for all $\varphi \in \mathcal{F}$,

$$\bigl|\varphi_t * (gu)(x) - g(x)\varphi_t * u(x)\bigr| \le Ct^{r-n} \int\limits_{B_t(x)} |u(y)|\, dy. \tag{12.17}$$

Hence it suffices to show that

$$v(x) = \sup_{0<t\le 1} \; t^{r-n} \int\limits_{B_t(x)} |u(y)|\, dy \tag{12.18}$$

belongs to $L^1(\mathbb{R}^n)$. Since

$$v(x) \le \int \frac{\chi(x-y)}{|x-y|^{n-r}} |u(y)|\, dy, \tag{12.19}$$

where $\chi(x)$ is the characteristic function of $\{|x| \le 1\}$, this is clear.

Given $\Omega \subset \mathbb{R}^n$ open, $u \in L^1_{\text{loc}}(\Omega)$, we say

$$u \in \mathfrak{H}^1_{\text{loc}}(\Omega) \Longleftrightarrow gu \in \mathfrak{h}^1(\mathbb{R}^n), \quad \forall\, g \in C_0^\infty(\Omega). \tag{12.20}$$

This is equivalent to the statement that, for any compact $K \subset \Omega$, there is a $v \in \mathfrak{H}^1(\mathbb{R}^n)$ such that $u = v$ on a neighborhood of K. To see this, note that if $u \in \mathfrak{H}^1_{\text{loc}}(\Omega)$ and $g \in C_0^\infty(\Omega)$, $g = 1$ on a neighborhood of K, then $gu \in \mathfrak{h}^1(\mathbb{R}^n)$. Now take $v = gu + h$, where $h \in C_0^\infty(\mathbb{R}^n)$ has support disjoint from supp g, and $\int h(x)\, dx = -\int g(x)u(x)\, dx$. By Proposition 12.4, $v \in \mathfrak{H}^1(\mathbb{R}^n)$. The converse is established similarly.

Not every compactly supported element of $L^1(\mathbb{R}^n)$ belongs to $\mathfrak{h}^1(\mathbb{R}^n)$, but we do have the following.

Proposition 12.6. *If $p > 1$ and $u \in L^p(\mathbb{R}^n)$ has compact support, then $u \in \mathfrak{h}^1(\mathbb{R}^n)$.*

Proof. We have

$$(\mathcal{G}^b f)(x) \le (\mathcal{G} f)(x) \le C\mathcal{M}f(x). \tag{12.21}$$

Hence, given $p > 1,\ u \in L^p(\mathbb{R}^n) \Rightarrow \mathcal{G}^b u \in L^p(\mathbb{R}^n)$. Also, $\mathcal{G}^b u$ has support in $|x| \le R + 1$ if supp $u \subset \{|x| \le R\}$, so $\mathcal{G}^b u \in L^1(\mathbb{R}^n)$.

The spaces $\mathfrak{H}^1(\mathbb{R}^n)$ and $\mathfrak{h}^1(\mathbb{R}^n)$ are Banach spaces, with norms

$$\|u\|_{\mathfrak{H}^1} = \|\mathcal{G}u\|_{L^1}, \quad \|u\|_{\mathfrak{h}^1} = \|\mathcal{G}^b u\|_{L^1}. \tag{12.22}$$

It is useful to have the following approximation result.

Proposition 12.7. *Fix $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\int \psi(x)\ dx = 1$. If $u \in \mathfrak{H}^1(\mathbb{R}^n)$, then*

$$\|\psi_\varepsilon * u - u\|_{\mathfrak{H}^1} \longrightarrow 0, \quad \textit{as}\ \ \varepsilon \to 0. \tag{12.23}$$

Proof. One easily verifies from the definition that, for some $C < \infty$, $\mathcal{G}(\psi_\varepsilon * u)(x) \le C\mathcal{G}u(x), \forall\ x, \forall\ \varepsilon \in (0,1]$. Hence, by the dominated convergence theorem, it suffices to show that

$$\mathcal{G}(\psi_\varepsilon * u - u)(x) \longrightarrow 0, \quad \text{a.e. } x, \quad \text{as } \ \varepsilon \to 0; \tag{12.24}$$

that is,

$$\sup_{t>0,\varphi\in\mathcal{F}} \bigl|(\varphi_t * \psi_\varepsilon * u - \varphi_t * u)(x)\bigr| \to 0, \quad \text{a.e. } x, \quad \text{as } \ \varepsilon \to 0.$$

To prove this, it suffices to show that

$$\lim_{\varepsilon,\delta\to 0}\ \sup_{0<t\le\delta}\ \sup_{\varphi\in\mathcal{F}} \bigl|(\varphi_t * \psi_\varepsilon * u - \varphi_t * u)(x)\bigr| = 0, \quad \text{a.e. } x, \tag{12.25}$$

and that, for each $\delta > 0$,

$$\lim_{\varepsilon\to 0}\ \sup_{t\ge\delta}\ \sup_{\varphi\in\mathcal{F}} \bigl|(\varphi_t * \psi_\varepsilon - \varphi_t) * u(x)\bigr| = 0. \tag{12.26}$$

In fact, (12.25) holds whenever x is a *Lebesgue point* for u (see the exercises for more on this), and (12.26) holds for all $x \in \mathbb{R}^n$, since $u \in L^1(\mathbb{R}^n)$ and, for all $\varphi \in \mathcal{F}$, we have $\|\varphi_t * \psi_\varepsilon - \varphi_t\|_{L^\infty} \le C\varepsilon t^{-n-1}$.

Corollary 12.8. *Let $T_y u(x) = u(x + y)$. Then, for $u \in \mathfrak{H}^1(\mathbb{R}^n)$,*

$$\|T_y u - u\|_{\mathfrak{H}^1} \longrightarrow 0, \quad \textit{as}\ \ |y| \to 0. \tag{12.27}$$

Proof. Since $\|T\|_{\mathcal{L}(\mathfrak{H}^1)} = 1$ for all y, it suffices to show that (12.27) holds for u in a dense subspace of $\mathfrak{H}^1(\mathbb{R}^n)$. Thus it suffices to show that, for each $\varepsilon > 0,\ u \in \mathfrak{H}^1(\mathbb{R}^n)$,

$$\lim_{|y|\to 0} \|T_y(\psi_\varepsilon * u) - \psi_\varepsilon * u\|_{\mathfrak{H}^1} = 0. \tag{12.28}$$

But $T_y(\psi_\varepsilon * u) - \psi_\varepsilon * u = (\psi_{\varepsilon y} - \psi_\varepsilon) * u$, where

$$\psi_{\varepsilon y}(x) - \psi_\varepsilon(x) = \varepsilon^{-n}\big[\psi(\varepsilon^{-1}(x+y)) - \psi(\varepsilon^{-1}x)\big]. \tag{12.29}$$

Thus

$$\begin{aligned} \|T_y(\psi_\varepsilon * u) - \psi_\varepsilon * u\|_{\mathfrak{H}^1} &= \sup_{t>0,\varphi\in\mathcal{F}} \|(\psi_{\varepsilon y} - \psi_\varepsilon) * \varphi_t * u\|_{L^1} \\ &\le \|\psi_{\varepsilon y} - \psi_\varepsilon\|_{L^\infty} \|u\|_{\mathfrak{H}^1} \\ &\le C|y|\varepsilon^{-n-1}\|u\|_{\mathfrak{H}^1}, \end{aligned} \tag{12.30}$$

which finishes the proof.

It is clear that we can replace $\mathfrak{H}^1$ by $\mathfrak{h}^1$ in Proposition 12.7 and Corollary 12.8, obtaining, for $u \in \mathfrak{h}^1(\mathbb{R}^n)$,

$$\|\psi_\varepsilon * u - u\|_{\mathfrak{h}^1} \longrightarrow 0, \tag{12.31}$$

as $\varepsilon \to 0$, and

$$\|T_y u - u\|_{\mathfrak{h}^1} \longrightarrow 0, \tag{12.32}$$

as $|y| \to 0$.

We can also approximate by cut-offs:

Proposition 12.9. *Fix $\chi \in C_0^\infty(\mathbb{R}^n)$, so that $\chi(x) = 1$ for $|x| \le 1$, 0 for $|x| \ge 2$, and $0 \le \chi \le 1$. Set $\chi_R(x) = \chi(x/R)$. Then, given $u \in \mathfrak{h}^1(\mathbb{R}^n)$, we have*

$$\lim_{R\to\infty} \|u - \chi_R u\|_{\mathfrak{h}^1} = 0. \tag{12.33}$$

Proof. Clearly, $\mathcal{G}^b(u - \chi_R u)(x) = 0$, for $|x| \le R-1$, so

$$\lim_{R\to\infty} G^b(u - \chi_R u)(x) = 0, \quad \forall\, x \in \mathbb{R}^n.$$

To get (12.33), we would like to appeal to the dominated convergence theorem. In fact, the estimates (12.17)–(12.19) (with $g = 1 - \chi_R$) give

$$\mathcal{G}^b(u - \chi_R u)(x) \le \mathcal{G}^b u(x) + Av(x), \quad \forall\, R \ge 1, \tag{12.34}$$

where $A = \|\nabla\chi\|_{L^\infty}$, and $v(x)$ is given by (12.19), with $r = 1$, so $v \in L^1(\mathbb{R}^n)$. Thus dominated convergence does give

$$\lim_{R\to\infty} \|\mathcal{G}^b(u - \chi_R u)\|_{L^1} = 0, \tag{12.35}$$

and the proof is done.

Together with (12.31), this gives

Corollary 12.10. *The space $C_0^\infty(\mathbb{R}^n)$ is dense in $\mathfrak{h}^1(\mathbb{R}^n)$.*

A slightly more elaborate argument shows that

$$\mathcal{D}_0 = \Big\{ u \in C_0^\infty(\mathbb{R}^n) : \int u(x)\, dx = 0 \Big\} \tag{12.36}$$

is dense in $\mathfrak{H}^1(\mathbb{R}^n)$; see [Sem].

One significant measure of how much smaller $\mathfrak{H}^1(\mathbb{R}^n)$ is than $L^1(\mathbb{R}^n)$ is the following identification of an element of the dual of $\mathfrak{H}^1(\mathbb{R}^n)$ that does not belong to $L^\infty(\mathbb{R}^n)$.

Proposition 12.11. *We have*

$$\Big| \int f(x)\, \log|x|\, dx \Big| \le C \|f\|_{\mathfrak{H}^1}. \tag{12.37}$$

Proof. Let $\lambda(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $\lambda(x) = 1$ for $|x| \le 1$, $\lambda(x) = 0$ for $|x| \ge 2$. Set

$$\ell(x) = -\sum_{j=1}^\infty \lambda(2^j x) + \sum_{j=0}^\infty \big(1 - \lambda(2^{-j} x)\big). \tag{12.38}$$

It is easy to check that

$$\log|x| - (\log 2)\ell(x) \in L^\infty(\mathbb{R}^n). \tag{12.39}$$

Thus it suffices to estimate $\int f(x)\ell(x)\, dx$. We have

$$\Big| \int f(x)\ell(x)\, dx \Big| \le \sum_{j=-\infty}^\infty \Big| \int f(x)\lambda(2^j x)\, dx \Big|. \tag{12.40}$$

We claim that, for each $j \in \mathbb{Z}$,

$$\Big| \int f(x)\lambda(2^j x)\, dx \Big| \le C 2^{-jn} \inf_{B_{2^{-j}}(0)} \mathcal{G}f. \tag{12.41}$$

In fact, given $j \in \mathbb{Z},\ z \in B_{2^{-j}}(0)$, we can write

$$\int f(x) 2^{jn} \lambda(2^j x)\, dx = K \varphi_r * f(z), \tag{12.42}$$

with $r = 2^{2-j},\ K = K(\lambda, n)$, for some $\varphi \in \mathcal{F}$; say $\varphi(x)$ is a multiple of a translate of $\lambda(4x)$. Consequently, with $S_j = B_{2^{-j}}(0)$, we have

$$\Big| \int f(x)\ell(x)\, dx \Big| \le C \sum_{j=-\infty}^\infty \int_{S_j \setminus S_{j+1}} \mathcal{G}f = C \|f\|_{\mathfrak{H}^1}. \tag{12.43}$$

By Corollary 12.8, we have the following:

Corollary 12.12. *Given* $f \in \mathfrak{H}^1(\mathbb{R}^n)$,

$$\log * f \in C(\mathbb{R}^n). \tag{12.44}$$

The result (12.37) is a very special case of the fact that the dual of $\mathfrak{H}^1(\mathbb{R}^n)$ is naturally isomorphic to a space of functions called $\mathrm{BMO}(\mathbb{R}^n)$. This was established in [FS]. The special case given above is the only case we will use in this book. More about this duality and its implications for analysis can be found in the treatise [S3]. Also, [S3] has other important information about Hardy spaces, including a study of singular integral operators on these spaces.

The next result is a variant of the div-curl lemma (discussed in Exercises for §6), due to [CLMS]. It states that a certain function that obviously belongs to $L^1(\mathbb{R}^n)$ actually belongs to $\mathfrak{H}^1(\mathbb{R}^n)$. Together with Corollary 12.12, this produces a useful tool for PDE. An application will be given in §12B of Chapter 14. The proof below follows one of L. Evans and S. Muller, given in [Ev2].

Proposition 12.13. *If $u \in L^2(\mathbb{R}^n, \mathbb{R}^n)$, $v \in H^1(\mathbb{R}^n)$, and $\operatorname{div} u = 0$, then $u \cdot \nabla v \in \mathfrak{H}^1(\mathbb{R}^n)$.*

Proof. Clearly, $u \cdot \nabla v \in L^1(\mathbb{R}^n)$. Now, with $\varphi \in C_0^\infty(\mathbb{R}^n)$, supported in the unit ball, set $\varphi_r(y) = r^{-n}\varphi\big(r^{-1}(x-y)\big)$. We have

$$\int (u \cdot \nabla v)\varphi_r \, dy = -\int\limits_{B_r(x)} (v - v_{x,r}) u \cdot \nabla \varphi_r \, dy, \tag{12.45}$$

since $\operatorname{div} u = 0$. Thus, with $C_0 = \|\nabla \varphi\|_{L^\infty}$,

$$\Big| \int (u \cdot \nabla v)\varphi_r \, dy \Big| \le \frac{C_0}{r} \int\limits_{B_r(x)} |u - v_{x,r}| \cdot |u| \, dy. \tag{12.46}$$

Take

$$p \in \Big(2, \frac{2n}{n-2}\Big), \quad q = \frac{p}{p-1} \in (1,2). \tag{12.47}$$

Then

$$\begin{aligned} \Big| \int (u \cdot \nabla v)\varphi_r \, dy \Big| &\le \frac{C_0}{r} \Big(\int\limits_{B_r(x)} |v - v_{x,r}|^p \, dy \Big)^{1/p} \Big(\int\limits_{B_r(x)} |u|^q \, dy \Big)^{1/q} \\ &\le \frac{C_0}{r^a} \Big(\int\limits_{B_r(x)} |\nabla v|^\rho \, dy \Big)^{1/\rho} \Big(\int\limits_{B_r(x)} |u|^q \, dy \Big)^{1/q}, \end{aligned} \tag{12.48}$$

where $\rho = pn/(p+n) < 2$ and $a = n+1$. Consequently,

$$\begin{aligned} \Big| \int (u \cdot \nabla v)\varphi_r \, dy \Big| &\le C_0 \mathcal{M}\big(|\nabla v|^\rho\big)^{1/\rho} \mathcal{M}\big(|u|^q\big)^{1/q} \\ &\le C_0 \big\{ \mathcal{M}\big(|\nabla v|^\rho\big)^{2/\rho} + \mathcal{M}\big(|u|^q\big)^{2/q} \big\}. \end{aligned} \tag{12.49}$$

By Corollary 12.3, we have $\|\mathcal{M}(|\nabla v|^\rho)\|_{L^{2/\rho}} \le C\||\nabla v|^\rho\|_{L^{2/\rho}}$, and so

$$\int \mathcal{M}(|\nabla v|^\rho)^{2/\rho}\,dx \le C\int |\nabla v|^2\,dx.$$

Similarly,

$$\int \mathcal{M}(|u|^q)^{2/q}\,dx \le C\int |u|^2\,dx.$$

Hence

$$\|u\cdot\nabla v\|_{\mathfrak{H}^1} = \sup_{\varphi\in\mathcal{F},r>0}\Big\|\int (u\cdot\nabla v)\varphi_r\,dy\Big\|_{L^1} \le C\big(\|\nabla v\|^2+\|u\|_{L^2}^2\big). \tag{12.50}$$

We next establish a localized version of Proposition 12.13.

Proposition 12.14. *Let $\Omega\subset\mathbb{R}^n$ be open. If $u\in L^2(\Omega,\mathbb{R}^n)$, div $u=0$, and $v\in H^1(\Omega)$, then $u\cdot\nabla v\in\mathfrak{H}^1_{\rm loc}(\Omega)$.*

Proof. We may as well suppose $n>1$. Take any $\overline{\mathcal{O}}\subset\Omega$, diffeomorphic to a ball. It suffices to show that $u\cdot\nabla v$ is equal on $\overline{\mathcal{O}}$ to an element of $\mathfrak{H}^1(\mathbb{R}^n)$. Say $\overline{\mathcal{O}}\subset\subset\overline{U}\subset\subset\Omega$, with $\overline{U}$ also diffeomorphic to a ball. Pick $\chi\in C_0^\infty(U)$, $\chi=1$ on $\overline{\mathcal{O}}$.

Let $\tilde u\in L^2(\Omega,\Lambda^{n-1})$ correspond to u via the volume element on Ω. Then $d\tilde u=0$. We use the Hodge decomposition of $L^2(U,\Lambda^{n-1})$, with absolute boundary condition:

$$\tilde u = d\delta G^A\tilde u + \delta d G^A\tilde u + P_h^A\tilde u \quad \text{on } u. \tag{12.51}$$

Since $d\tilde u=0$, we have by (9.48) of Chapter 5 that $\delta dG^A\tilde u=0$. Also, given $n>1$, $\mathcal{H}^{n-1}(\overline{U})=0$, so $P_h^A\tilde u=0$, too, and so

$$\tilde u = d\tilde w,\quad \tilde w\in H^1(U,\Lambda^{n-2}). \tag{12.52}$$

Having this, we define a vector field u_0 on $\mathbb{R}^n$ so that $\tilde u_0 = d(\chi\tilde w)$, and we set $v_0=\chi v$. It follows that $u_0,\ v_0$ satisfy the hypotheses of Proposition 12.13, so $u_0\cdot\nabla v_0\in\mathfrak{H}^1(\mathbb{R}^n)$. But $u_0\cdot\nabla v_0 = u\cdot\nabla v$ on $\overline{\mathcal{O}}$, so the proof is done.

Let us finally mention that while we have only briefly alluded to the space BMO, it has also proven to be of central importance, especially since the work of [FS]. More about the role of BMO in paradifferential operator calculus can be found in [T2]. Also, Proposition 12.13 can be deduced from a commutator estimate involving BMO, as explained in [CLMS]; see also [AT].

Exercises

We say $x \in \mathbb{R}^n$ is a Lebesgue point for $f \in L^1(\mathbb{R}^n)$ provided

$$\lim_{r\to 0} \frac{1}{\operatorname{vol}(B_r)} \int_{B_r(x)} |f(y) - f(x)| \, dy = 0.$$

In Exercises 1 and 2, we establish that, given $f \in L^1(\mathbb{R}^n)$, a.e. $x \in \mathbb{R}^n$ is a Lebesgue point of f.

1. Set

$$\widetilde{\mathcal{M}}(f)(x) = \sup_{r>0} \frac{1}{\operatorname{vol}(B_r)} \int_{B_r(x)} |f(y) - f(x)| \, dy.$$

Show that, for all $x \in \mathbb{R}^n$,

$$\widetilde{\mathcal{M}}(f)(x) \le \mathcal{M}(f)(x) + |f(x)|.$$

2. Given $\lambda > 0$, let

$$E_\lambda = \Big\{ x \in \mathbb{R}^n : \limsup_{r\to 0} \frac{1}{\operatorname{vol}(B_r)} \int_{B_r(x)} |f(y) - f(x)| \, dy > \lambda \Big\}.$$

Take $\varepsilon > 0$, and take $g \in C_0^\infty(\mathbb{R}^n)$ so that $\|f - g\|_{L^1} < \varepsilon$. Show that E_λ is unchanged if f is replaced by $f - g$. Deduce that

$$E_\lambda \subset \Big\{ x : \mathcal{M}(f-g)(x) > \frac{1}{2}\lambda \Big\} \cup \Big\{ x : |f(x) - g(x)| > \frac{1}{2}\lambda \Big\},$$

and hence, via Proposition 12.1,

$$\operatorname{meas}(E_\lambda) \le \frac{C}{\lambda} \|f - g\|_{L^1} \le \frac{C\varepsilon}{\lambda}.$$

Deduce that $\operatorname{meas}(E_\lambda) = 0, \forall\, \lambda > 0$, and hence a.e. $x \in \mathbb{R}^n$ is a Lebesgue point for f.

3. Now verify that (12.25) holds whenever x is a Lebesgue point of u.
4. If $u : \mathbb{R}^2 \to \mathbb{R}^2$, show that

$$u \in H^1(\mathbb{R}^2) \Longrightarrow \det Du \in \mathfrak{H}^1(\mathbb{R}^2).$$

(*Hint*: Compute div w, when $w = (\partial_y u_1, -\partial_x u_2)$.)

5. If $u : \mathbb{R}^2 \to \mathbb{R}^3$, show that

$$u \in H^1(\mathbb{R}^2) \Longrightarrow u_x \times u_y \in \mathfrak{H}^1(\mathbb{R}^2).$$

(*Hint*. Show that the first argument of $u_x \times u_y$ is det Dv, where $v = (u_2, u_3)$.)

A. Variations on complex interpolation

Let X and Y be Banach spaces, assumed to be linear subspaces of a Hausdorff locally convex space V (with continuous inclusions). We say (X, Y, V) is a compatible triple. For $\theta \in (0, 1)$, the classical complex interpolation space $[X,Y]_\theta$, introduced in Chapter 4 and much used in this chapter, is defined as follows. First, $Z = X + Y$ gets a natural norm; for $v \in X + Y$,

$$\|v\|_Z = \inf\{\|v_1\|_X + \|v_2\|_Y : v = v_1 + v_2,\ v_1 \in X,\ v_2 \in Y\}. \tag{A.1}$$

One has $X + Y \approx X \oplus Y/L$, where $L = \{(v,-v) : x \in X \cap Y\}$ is a closed linear subspace, so $X + Y$ is a Banach space. Let $\Omega = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, with closure $\overline{\Omega}$. Define $\mathcal{H}_\Omega(X,Y)$ to be the space of functions $f : \overline{\Omega} \to Z = X + Y$, continuous on $\overline{\Omega}$, holomorphic on Ω (with values in $X + Y$), satisfying $f : \{\operatorname{Im} z = 0\} \to X$ continuous, $f : \{\operatorname{Im} z = 1\} \to Y$ continuous, and

$$\|u(z)\|_Z \le C, \quad \|u(iy)\|_X \le C, \quad \|u(1+iy)\|_Y \le C, \tag{A.2}$$

for some $C < \infty$, independent of $z \in \overline{\Omega}$ and $y \in \mathbb{R}$. Then, for $\theta \in (0,1)$,

$$[X,Y]_\theta = \{u(\theta) : u \in \mathcal{H}_\Omega(X,Y)\}. \tag{A.3}$$

One has

$$[X,Y]_\theta \approx \mathcal{H}_\Omega(X,Y)/\{u \in \mathcal{H}_\Omega(X,Y) : u(\theta) = 0\}, \tag{A.4}$$

giving $[X,Y]_\theta$ the sructure of a Banach space. Here

$$\|u\|_{\mathcal{H}_\Omega(X,Y)} = \sup_{z \in \overline{\Omega}} \|u(z)\|_Z + \sup_y \|u(iy)\|_X + \sup_y \|u(1+iy)\|_Y. \tag{A.5}$$

If I is an interval in $\mathbb{R}$, we say a family of Banach spaces $X_s,\ s \in I$ (subspaces of V) forms a complex interpolation scale provided that for $s,t \in I,\ \theta \in (0,1)$,

$$[X_s, X_t]_\theta = X_{(1-\theta)s+\theta t}. \tag{A.6}$$

Examples of such scales include L^p-Sobolev spaces $X_s = H^{s,p}(M),\ s \in \mathbb{R}$, provided $p \in (1,\infty)$, as shown in §6 of this chapter, the case $p = 2$ having been done in Chapter 4. It turns out that (A.6) fails for Zygmund spaces $X_s = C^s_*(M)$, but an analogous identity holds for some closely related interpolation functors, which we proceed to introduce.

If (X,Y,V) is a compatible triple, as defined in above, we define $\mathcal{H}_\Omega(X,Y,V)$ to be the space of functions $u : \overline{\Omega} \to X + Y = Z$ such that

$$u : \Omega \longrightarrow Z \ \text{ is holomorphic,} \tag{A.7}$$

$$\|u(z)\|_Z \le C, \quad \|u(iy)\|_X \le C, \quad \|u(1+iy)\|_Y \le C, \tag{A.8}$$

and

$$u : \overline{\Omega} \longrightarrow V \ \text{ is continuous.} \tag{A.9}$$

For such u, we again use the norm (A.5). Note that the only difference with $\mathcal{H}_\Omega(X,Y)$ is that we are relaxing the continuity hypothesis for u on $\overline{\Omega}$. $\mathcal{H}_\Omega(X,Y,V)$ is also a Banach space, and we have a natural isometric inclusion

$$\mathcal{H}_\Omega(X,Y) \hookrightarrow \mathcal{H}_\Omega(X,Y,V). \tag{A.10}$$

Now for $\theta \in (0,1)$ we set

$$[X,Y]_{\theta;V} = \{u(\theta) : u \in \mathcal{H}_\Omega(X,Y,V)\}. \tag{A.11}$$

Again this space gets a Banach space structure, via

$$[X,Y]_{\theta;V} \approx \mathcal{H}_\Omega(X,Y,V)/\{u \in \mathcal{H}_\Omega(X,Y,V) : u(\theta) = 0\}, \tag{A.12}$$

and there is a natural continuous injection

$$[X,Y]_\theta \hookrightarrow [X,Y]_{\theta;V}. \tag{A.13}$$

Sometimes this is an isomorphism. In fact, sometimes $[X,Y]_\theta = [X,Y]_{\theta;V}$ for practically all reasonable choices of V. For example, one can verify this for $X = L^p(\mathbb{R}^n)$, $Y = H^{s,p}(\mathbb{R}^n)$, the L^p-Sobolev space, with $p \in (1,\infty)$, $s \in (0,\infty)$. On the other hand, there are cases where equality in (A.10) does not hold, and where $[X,Y]_{\theta;V}$ is of greater interest than $[X,Y]_\theta$.

We next define $[X,Y]^b_\theta$. In this case we assume X and Y are Banach spaces and $Y \subset X$ (continuously). We take Ω as above, and set $\widetilde{\Omega} = \{z \in \mathbb{C} : 0 < \operatorname{Re} z \le 1\}$, i.e., we throw in the right boundary but not the left boundary. We then define $\mathcal{H}^b_\Omega(X,Y)$ to be the space of functions $u : \widetilde{\Omega} \to X$ such that

$$\begin{gathered} u : \Omega \longrightarrow X \quad \text{is holomorphic}, \\ \|u(z)\|_X \le C, \quad \|u(1+iy)\|_Y \le C, \\ u : \widetilde{\Omega} \longrightarrow X \quad \text{is continuous}. \end{gathered} \tag{A.14}$$

Note that the essential difference between $\mathcal{H}_\Omega(X,Y)$ and the space we have just introduced is that we have completely dropped any continuity requirement at $\{\operatorname{Re} z = 0\}$. We also do not require continuity from $\{\operatorname{Re} z = 1\}$ to Y. The space $\mathcal{H}^b_\Omega(X,Y)$ is a Banach space, with norm

$$\|u\|_{\mathcal{H}^b_\Omega(X,Y)} = \sup_{z \in \widetilde{\Omega}} \|u(z)\|_X + \sup_y \|u(1+iy)\|_Y. \tag{A.15}$$

Now, for $\theta \in (0,1)$, we set

$$[X,Y]^b_\theta = \{u(\theta) : u \in \mathcal{H}^b_\Omega(X,Y)\}, \tag{A.16}$$

with the same sort of Banach space structure as arose in (A.4) and (A.12). We have continuous injections

$$[X,Y]_\theta \hookrightarrow [X,Y]_{\theta;X} \hookrightarrow [X,Y]^b_\theta. \tag{A.17}$$

Our next task is to extend the standard result on operator interpolation from the setting of $[X,Y]_\theta$ to that of $[X,Y]_{\theta;V}$ and $[X,Y]^b_\theta$.

Proposition A.1. *Let (X_j,Y_j,V_j) be compatible triples, $j = 1,2$. Assume that $T : V_1 \to V_2$ is continuous and that*

$$T : X_1 \longrightarrow X_2, \quad T : Y_1 \longrightarrow Y_2, \tag{A.18}$$

continuously. (Continuity is automatic, by the closed graph theorem.) Then, for each $\theta \in (0,1)$,

$$T : [X_1, Y_1]_{\theta;V_1} \longrightarrow [X_2, Y_2]_{\theta;V_2}. \tag{A.19}$$

Furthermore, if $Y_j \subset X_j$ *(continuously) and* T *is a continuous linear map satisfying (A.18), then for each* $\theta \in (0,1)$,

$$T : [X_1, Y_1]^b_\theta \longrightarrow [X_2, Y_2]^b_\theta. \tag{A.20}$$

Proof. Given $f \in [X_1, Y_2]_{\theta;V}$, pick $u \in \mathcal{H}_\Omega(X_1, Y_1, V_1)$ such that $f = u(\theta)$. Then we have

$$\mathcal{T} : \mathcal{H}_\Omega(X_1, Y_1, V_1) \to \mathcal{H}_\Omega(X_2, Y_2, V_2), \quad (\mathcal{T}u)(z) = Tu(z), \tag{A.21}$$

and hence

$$Tf = (\mathcal{T}u)(\theta) \in [X_2, Y_2]_{\theta;V_2}. \tag{A.22}$$

This proves (A.19). The proof of (A.20) is similar.

Remark. In case $V = X + Y$, with the weak topology, $[X, Y]_{\theta;V}$ is what is denoted $(X, Y)^w_\theta$ in [JJ], and called the weak complex interpolation space.

Alternatives to (A.6) for a family X_s of Banach spaces include

$$[X_s, X_t]_{\theta;V} = X_{(1-\theta)s+\theta t} \tag{A.23}$$

and

$$[X_s, X_t]^b_\theta = X_{(1-\theta)s+\theta t}. \tag{A.24}$$

Here, as before, we take $\theta \in (0,1)$. It is an exercise, using results of §6, to show that both (A.23) and (A.24), as well as (A.6), hold when $X_s = H^{s,p}(M)$, given $p \in (1,\infty)$, where M can be $\mathbb{R}^n$ or a compact Riemannian manifold. We now discuss the situation for Zygmund spaces.

We start with Zygmund spaces on the torus $\mathbb{T}^n$. We recall from §8 that the Zygmund space $C^r_*(\mathbb{T}^n)$ is defined for $r \in \mathbb{R}$, as follows. Take $\varphi \in C^\infty_0(\mathbb{R}^n)$, radial, satisfying $\varphi(\xi) = 1$ for $|\xi| \le 1$. Set $\varphi_k(\xi) = \varphi(2^{-k}\xi)$. Then set $\psi_0 = \varphi$, $\psi_k = \varphi_k - \varphi_{k-1}$ for $k \in \mathbb{N}$, so $\{\psi_k : k \ge 0\}$ is a Littlewood-Paley partition of unity. We define $C^r_*(\mathbb{T}^n)$ to consist of $f \in \mathcal{D}'(\mathbb{T}^n)$ such that

$$\|f\|_{C^r_*} = \sup_{k \ge 0} 2^{kr} \|\psi_k(D)f\|_{L^\infty} < \infty. \tag{A.25}$$

With $\Lambda = (I - \Delta)^{1/2}$ and $s, t \in \mathbb{R}$, we have

$$\Lambda^{s+it} : C^r_*(\mathbb{T}^n) \longrightarrow C^{r-s}_*(\mathbb{T}^n). \tag{A.26}$$

By material developed in §8,

$$r \in \mathbb{R}^+ \setminus \mathbb{Z}^+ \Longrightarrow C_*^r(\mathbb{T}^n) = C^r(\mathbb{T}^n), \tag{A.27}$$

where, if $r = k+\alpha$ with $k \in \mathbb{Z}^+$ and $0 < \alpha < 1$, $C^r(\mathbb{T}^n)$ consists of functions whose derivatives of order $\le k$ are Hölder continuous of exponent α.

We aim to show the following.

Proposition A.2. *If $r < s < t$ and $0 < \theta < 1$, then*

$$[C_*^s(\mathbb{T}^n), C_*^t(\mathbb{T}^n)]_{\theta;C_*^r(\mathbb{T}^n)} = C_*^{(1-\theta)s+\theta t}(\mathbb{T}^n), \tag{A.28}$$

and

$$[C_*^s(\mathbb{T}^n), C_*^t(\mathbb{T}^n)]_\theta^b = C_*^{(1-\theta)s+\theta t}(\mathbb{T}^n). \tag{A.29}$$

Proof. First, suppose $f \in [C_*^s, C_*^t]_{\theta;C_*^r}$, so $f = u(\theta)$ for some $u \in \mathcal{H}_\Omega(C_*^s, C_*^t, C_*^r)$. Then consider

$$v(z) = e^{z^2} \Lambda^{(t-s)z} \Lambda^s u(z). \tag{A.30}$$

Bounds of the type (A.8) on u, together with (8.13) in the torus setting, yield

$$\|v(iy)\|_{C_*^0},\ \|v(1+iy)\|_{C_*^0} \le C, \tag{A.31}$$

with C independent of $y \in \mathbb{R}$. In other words,

$$\|\psi_k(D)v(z)\|_{L^\infty} \le C, \quad \operatorname{Re} z = 0, 1, \tag{A.32}$$

with C independent of Im z and k. Also, for each $k \in \mathbb{Z}^+$, $\psi_k(D)v : \overline{\Omega} \to L^\infty(\mathbb{T}^n)$ continuously, so the maximum principle implies

$$\|\psi_k(D)\Lambda^{(t-s)\theta}\Lambda^s f\|_{L^\infty} \le C, \tag{A.33}$$

independent of $k \in \mathbb{Z}^+$. This gives $\Lambda^{(1-\theta)s+\theta t} f \in C_*^0$, hence $f \in C_*^{(1-\theta)s+\theta t}(\mathbb{T}^n)$.

Second, suppose $f \in C_*^{(1-\theta)s+\theta t}(\mathbb{T}^n)$. Set

$$u(z) = e^{z^2} \Lambda^{(\theta-z)(t-s)} f. \tag{A.34}$$

Then $u(\theta) = e^{\theta^2} f$. We claim that

$$u \in \mathcal{H}_\Omega(C_*^s, C_*^t, C_*^r), \tag{A.35}$$

as long as $r < s < t$. Once we establish this, we will have the reverse containment in (A.28). Bounds of the form

$$\|u(z)\|_{C_*^s} \le C, \quad \|u(1+iy)\|_{C_*^t} \le C \tag{A.36}$$

follow from (8.13), and are more than adequate versions of (A.8). It remains to establish that

$$u : \overline{\Omega} \longrightarrow C_*^r(\mathbb{T}^n), \quad \text{continuously}. \tag{A.37}$$

Indeed, we know $u : \overline{\Omega} \to C^s_*(\mathbb{T}^n)$ is bounded. It is readily verified that

$$u : \overline{\Omega} \longrightarrow \mathcal{D}'(\mathbb{T}^n), \quad \text{continuously,} \tag{A.38}$$

and that

$$r < s \Longrightarrow C^s_*(\mathbb{T}^n) \hookrightarrow C^r_*(\mathbb{T}^n) \ \text{ is compact.} \tag{A.39}$$

The result (A.37) follows from these observations. Thus the proof of (A.28) is complete.

We turn to the proof of (A.29). If $u \in \mathcal{H}^b_\Omega(C^s_*, C^t_*)$, form $v(z)$ as in (A.30), and for $\varepsilon \in (0,1]$ set

$$v_\varepsilon(z) = e^{-\varepsilon\Lambda} v(z), \quad v_\varepsilon : \widetilde{\Omega} \to C^0_*(\mathbb{T}^n) \ \text{ bounded and continuous} \tag{A.40}$$

(with bound that might depend on ε). We have

$$\psi_k(D) v_\varepsilon(\varepsilon + iy) = e^{(\varepsilon+iy)^2} \psi_k(D) e^{-\varepsilon\Lambda} \Lambda^{(t-s)\varepsilon} \Lambda^{i(t-s)y} \Lambda^s u(z). \tag{A.41}$$

Now $\{\Lambda^s u(z) : z \in \widetilde{\Omega}\}$ is bounded in $C^0_*(\mathbb{T}^n)$, and the operator norm of $\Lambda^{i(t-s)y}$ on $C^0_*(\mathbb{T}^n)$ is exponentially bounded in $|y|$. We have

$$\{e^{-\varepsilon\Lambda} \Lambda^{\varepsilon(t-s)} : 0 < \varepsilon \le 1\} \ \text{ bounded in } \ \mathrm{OPS}^0_{1,0}(\mathbb{T}^n), \tag{A.42}$$

hence bounded in operator norm on $C^0_*(\mathbb{T}^n)$. We deduce that

$$\|\psi_k(D) v_\varepsilon(\varepsilon + iy)\|_{L^\infty} \le C, \tag{A.43}$$

independent of $y \in \mathbb{R}$ and $\varepsilon \in (0,1]$. The hypothesis on u also implies

$$\|\psi_k(D) v_\varepsilon(1 + iy)\|_{L^\infty} \le C, \tag{A.44}$$

independent of $y \in \mathbb{R}$ and $\varepsilon \in (0,1]$. Now the maximum principle applies. Given $\theta \in (0,1)$,

$$\|\psi_k(D) e^{-\varepsilon\Lambda} v(\theta)\|_{L^\infty} \le C, \tag{A.45}$$

independent of ε. Taking $\varepsilon \searrow 0$ yields $v(\theta) \in C^0_*(\mathbb{T}^n)$, hence $u(\varepsilon) \in C^{(1-\theta)s+\theta t}_*(\mathbb{T}^n)$.

This proves one inclusion in (A.29). The proof of the reverse inclusion is similar to that for (A.28). Given $f \in C^{(1-\theta)s+\theta t}_*(\mathbb{T}^n)$, take $u(z)$ as in (A.34). The claim is that $u \in \mathcal{H}^b_\Omega(C^s_*, C^t_*)$. We already have (A.36), and the only thing that remains is to check that

$$u : \widetilde{\Omega} \longrightarrow C^s_*(\mathbb{T}^n) \ \text{ continuously,} \tag{A.46}$$

and this is straightforward. (What fails is continuity of $u : \overline{\Omega} \to C^s_*(\mathbb{T}^n)$ at the left boundary of $\overline{\Omega}$.)

Remark. In contrast to (A.28)–(A.29), one has

$$[C^s_*(\mathbb{T}^n), C^t_*(\mathbb{T}^n)]_\theta = \ \text{closure of } C^\infty(\mathbb{T}^n) \text{ in } C^{(1-\theta)s+\theta t}_*(\mathbb{T}^n). \tag{A.47}$$

Related results are given in [Tri].

If $\mathrm{OPS}^m_{1,0}(\mathbb{T}^n)$ denotes the class of pseudodifferential operators on $\mathbb{T}^n$ with symbols in $S^m_{1,0}$, then for all $s, m \in \mathbb{R}$,

$$P \in \mathrm{OPS}^m_{1,0}(\mathbb{T}^n) \Longrightarrow P : C^s_*(\mathbb{T}^n) \to C^{s-m}_*(\mathbb{T}^n). \tag{A.48}$$

Cf. Proposition 8.6. Using coordinate invariance of $OPS^m_{1,0}$ and of $C^r(\mathbb{T}^n)$ for $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, we deduce invariance of $C^s_*(\mathbb{T}^n)$ under diffeomorphisms, for all $s \in \mathbb{R}$.

From here, we can develop the spaces $C^s_*(M)$ on a compact Riemannian manifold M and the spaces $C^s_*(\overline{M})$ on a compact manifold with boundary. These developments are done in §8.

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