4

Sobolev Spaces

Introduction

In this chapter we develop the elements of the theory of Sobolev spaces, a tool that, together with methods of functional analysis, provides for numerous successful attacks on the questions of existence and smoothness of solutions to many of the basic partial differential equations. For a positive integer k, the Sobolev space $H^k(\mathbb{R}^n)$ is the space of functions in $L^2(\mathbb{R}^n)$ such that, for $|\alpha| \leq k$, $D^\alpha u$, regarded a priori as a distribution, belongs to $L^2(\mathbb{R}^n)$. This space can be characterized in terms of the Fourier transform, and such a characterization leads to a notion of $H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$. For $s < 0$, $H^s(\mathbb{R}^n)$ is a space of distributions. There is an invariance under coordinate transformations, permitting an invariant notion of $H^s(M)$ whenever M is a compact manifold. We also define and study $H^s(\Omega)$ when Ω is a compact manifold with boundary.

The tools from Sobolev space theory discussed in this chapter are of great use in the study of linear PDE; this will be illustrated in the following chapter. Chapter 13 will develop further results in Sobolev space theory, which will be seen to be of use in the study of nonlinear PDE.

1. Sobolev spaces on $\mathbb{R}^n$

When $k \geq 0$ is an integer, the Sobolev space $H^k(\mathbb{R}^n)$ is defined as follows:

$$H^k(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : D^\alpha u \in L^2(\mathbb{R}^n) \text{ for } |\alpha| \leq k\}, \tag{1.1}$$

where $D^\alpha u$ is interpreted a priori as a tempered distribution. Results from Chapter 3 on Fourier analysis show that, for such k, if $u \in L^2(\mathbb{R}^n)$, then

$$u \in H^k(\mathbb{R}^n) \Longleftrightarrow \langle \xi \rangle^k \, \hat{u} \in L^2(\mathbb{R}^n). \tag{1.2}$$

Recall that

$$\langle \xi \rangle = \big(1 + |\xi|^2\big)^{1/2}. \tag{1.3}$$

We can produce a definition of the Sobolev space $H^s(\mathbb{R}^n)$ for general $s \in \mathbb{R}$, parallel to (1.2), namely

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \langle\xi\rangle^s \hat{u} \in L^2(\mathbb{R}^n)\}. \tag{1.4}$$

We can define the operator Λ^s on $\mathcal{S}'(\mathbb{R}^n)$ by

$$\Lambda^s u = \mathcal{F}^{-1}\big(\langle\xi\rangle^s \hat{u}\big). \tag{1.5}$$

Then (1.4) is equivalent to

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \Lambda^s u \in L^2(\mathbb{R}^n)\}, \tag{1.6}$$

or $H^s(\mathbb{R}^n) = \Lambda^{-s} L^2(\mathbb{R}^n)$. Each space $H^s(\mathbb{R}^n)$ is a Hilbert space, with inner product

$$\big(u, v\big)_{H^s(\mathbb{R}^n)} = \big(\Lambda^s u, \Lambda^s v\big)_{L^2(\mathbb{R}^n)}. \tag{1.7}$$

We note that the dual of $H^s(\mathbb{R}^n)$ is $H^{-s}(\mathbb{R}^n)$.

Clearly, we have

$$D_j : H^s(\mathbb{R}^n) \longrightarrow H^{s-1}(\mathbb{R}^n), \tag{1.8}$$

and hence

$$D^\alpha : H^s(\mathbb{R}^n) \longrightarrow H^{s-|\alpha|}(\mathbb{R}^n). \tag{1.9}$$

Furthermore, it is easy to see that, given $u \in H^s(\mathbb{R}^n)$,

$$u \in H^{s+1}(\mathbb{R}^n) \Longleftrightarrow D_j u \in H^s(\mathbb{R}^n), \quad \forall\, j. \tag{1.10}$$

We can relate difference quotients to derivatives of elements of Sobolev spaces. Define τ_y, for $y \in \mathbb{R}^n$, by

$$\tau_y u(x) = u(x+y). \tag{1.11}$$

By duality this extends to $\mathcal{S}'(\mathbb{R}^n)$:

$$\langle \tau_{-y} u, v\rangle = \langle u, \tau_y v\rangle.$$

Note that

$$\tau_y v = \mathcal{F}^{-1}\big(e^{iy\cdot\xi}\ \hat{v}\big), \tag{1.12}$$

so it is clear that $\tau_y : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ is norm-preserving for each $s \in \mathbb{R}$, $y \in \mathbb{R}^n$. Also, for each $u \in H^s(\mathbb{R}^n)$, $\tau_y u$ is a continuous fuction of y with values in $H^s(\mathbb{R}^n)$. The following result is of frequent use, as we will see in the next chapter.

Proposition 1.1. *Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{R}^n$; let $u \in H^s(\mathbb{R}^n)$. Then*

$$\sigma^{-1}(\tau_{\sigma e_j} u - u) \text{ is bounded in } H^s(\mathbb{R}^n),$$

for $\sigma \in (0, 1]$, if and only if $D_j u \in H^s(\mathbb{R}^n)$.

Proof. We have $\sigma^{-1}(\tau_{\sigma e_j}u - u) \to iD_j u$ in $H^{s-1}(\mathbb{R}^n)$ as $\sigma \to 0$ if $u \in H^s(\mathbb{R}^n)$. The hypothesis of boundedness implies that there is a sequence $\sigma_\nu \to 0$ such that $\sigma_\nu^{-1}(\tau_{\sigma_\nu e_j}u - u)$ converges weakly to an element of $H^s(\mathbb{R}^n)$; call it w. Since the natural inclusion $H^s(\mathbb{R}^n) \hookrightarrow H^{s-1}(\mathbb{R}^n)$ is easily seen to be continuous, it follows that $w = iD_j u$. Since $w \in H^s(\mathbb{R}^n)$, this gives the desired conclusion.

Corollary 1.2. *Given $u \in H^s(\mathbb{R}^n)$, then u belongs to $H^{s+1}(\mathbb{R}^n)$ if and only if $\tau_y u$ is a Lipschitz-continuous function of y with values in $H^s(\mathbb{R}^n)$.*

Proof. This follows easily, given the observation (1.10).

We now show that elements of $H^s(\mathbb{R}^n)$ are smooth in the classical sense for sufficiently large positive s. This is a Sobolev imbedding theorem.

Proposition 1.3. *If $s > n/2$, then each $u \in H^s(\mathbb{R}^n)$ is bounded and continuous.*

Proof. By the Fourier inversion formula, it suffices to prove that $\hat{u}(\xi)$ belongs to $L^1(\mathbb{R}^n)$. Indeed, using Cauchy's inequality, we get

$$\int |\hat{u}(\xi)|\, d\xi \le \Big(\int |\hat{u}(\xi)|^2 \langle\xi\rangle^{2s}\, d\xi\Big)^{1/2} \cdot \Big(\int \langle\xi\rangle^{-2s}\, d\xi\Big)^{1/2}. \tag{1.13}$$

Since the last integral on the right is finite precisely for $s > n/2$, this completes the proof.

Corollary 1.4. *If $s > n/2 + k$, then $H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$.*

If $s = n/2 + \alpha$, $0 < \alpha < 1$, we can establish Hölder continuity. For $\alpha \in (0,1)$, we say

$$u \in C^\alpha(\mathbb{R}^n) \Longleftrightarrow u \ \text{ bounded and } \ |u(x+y) - u(x)| \le C|y|^\alpha. \tag{1.14}$$

An alternative notation is $\mathrm{Lip}^\alpha(\mathbb{R}^n)$; then the definition above is effective for $\alpha \in (0,1]$.

Proposition 1.5. *If $s = n/2 + \alpha$, $0 < \alpha < 1$, then $H^s(\mathbb{R}^n) \subset C^\alpha(\mathbb{R}^n)$.*

Proof. For $u \in H^s(\mathbb{R}^n)$, use the Fourier inversion formula to write

$$\begin{aligned} |u(x+y) - u(x)| &= (2\pi)^{-n/2}\Big|\int \hat{u}(\xi) e^{ix\cdot\xi}(e^{iy\cdot\xi} - 1)\, d\xi\Big| \\ &\le C\Big(\int |\hat{u}(\xi)|^2 \langle\xi\rangle^{n+2\alpha}\, d\xi\Big)^{1/2} \cdot \Big(\int \big|e^{iy\cdot\xi} - 1\big|^2 \langle\xi\rangle^{-n-2\alpha}\, d\xi\Big)^{1/2}. \end{aligned} \tag{1.15}$$

Now, if $|y| \le 1/2$, write

$$\begin{aligned} &\int |e^{iy\cdot\xi} - 1|^2 \langle\xi\rangle^{-n-2\alpha}\, d\xi \\ &\qquad \le C \int\limits_{|\xi|\le \frac{1}{|y|}} |y|^2|\xi|^2\langle\xi\rangle^{-n-2\alpha}\, d\xi + 4 \int\limits_{|\xi|\ge\frac{1}{|y|}} \langle\xi\rangle^{-n-2\alpha}\, d\xi. \end{aligned} \tag{1.16}$$

If we use polar coordinates, the right side is readily dominated by

$$C|y|^2 + C|y|^2 \frac{|y|^{2\alpha-2} - 1}{2\alpha - 2} + C|y|^{2\alpha}, \tag{1.17}$$

provided $0 < \alpha < 1$. This implies that, for $|y| \le 1/2$,

$$|u(x+y) - u(x)| \le C_\alpha |y|^\alpha, \tag{1.18}$$

given $u \in H^s(\mathbb{R}^n)$, $s = n/2 + \alpha$, and the proof is complete.

We remark that if one took $\alpha = 1$, the middle term in (1.17) would be modified to $C|y|^2 \log(1/|y|)$, so when $u \in H^{n/2+1}(\mathbb{R}^n)$, one gets the estimate

$$|u(x+y) - u(x)| \le C|y| \Bigl(\log \frac{1}{|y|}\Bigr)^{1/2}.$$

Elements of $H^{n/2+1}(\mathbb{R}^n)$ need not be Lipschitz, and elements of $H^{n/2}(\mathbb{R}^n)$ need not be bounded.

We indicate an example of the last phenomenon. Let us define u by

$$\hat{u}(\xi) = \frac{\langle\xi\rangle^{-n}}{1 + \log\langle\xi\rangle}. \tag{1.19}$$

It is easy to show that $u \in H^{n/2}(\mathbb{R}^n)$. But $\hat{u} \notin L^1(\mathbb{R}^n)$. Now one can show that if $\hat{u} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is positive and belongs to $\mathcal{S}'(\mathbb{R}^n)$, but does not belong to $L^1(\mathbb{R}^n)$, then $u \notin L^\infty(\mathbb{R}^n)$; and this is what happens in the case of (1.19). For more on this, see Exercises 2 and 3 below.

A result dual to Proposition 1.3 is

$$\delta \in H^{-n/2-\varepsilon}(\mathbb{R}^n), \quad \text{for all } \varepsilon > 0, \tag{1.20}$$

which follows directly from the definition (1.4) together with the fact that $\mathcal{F}\delta = (2\pi)^{-n/2}$, by the same sort of estimate on $\int \langle\xi\rangle^{-2s} d\xi$ used to prove Proposition 1.3. Consequently,

$$D^\alpha \delta \in H^{-n/2-|\alpha|-\varepsilon}(\mathbb{R}^n), \quad \text{for all } \varepsilon > 0. \tag{1.21}$$

Next we consider the trace map τ, defined initially from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^{n-1})$ by $\tau u = f$, where $f(x') = u(0, x')$ if $x = (x_1, \dots, x_n), x' = (x_2, \dots, x_n)$.

Proposition 1.6. *The map τ extends uniquely to a continuous linear map*

$$\tau : H^s(\mathbb{R}^n) \longrightarrow H^{s-1/2}(\mathbb{R}^{n-1}), \quad \text{for } s > \frac{1}{2}. \tag{1.22}$$

Proof. If $f = \tau u$, we have

$$\hat{f}(\xi') = \frac{1}{\sqrt{2\pi}} \int \hat{u}(\xi)\, d\xi_1, \tag{1.23}$$

as a consequence of the identity $\int g(x_1) e^{-ix_1\xi_1}\, dx_1\, d\xi_1 = 2\pi g(0)$. Thus

$$|\hat{f}(\xi')|^2 \le \frac{1}{2\pi} \Bigl(\int |\hat{u}(\xi)|^2 \langle \xi \rangle^{2s} d\xi_1 \Bigr) \cdot \Bigl(\int \langle \xi \rangle^{-2s} d\xi_1 \Bigr),$$

where the last integral is finite if $s > 1/2$. In such a case, we have

$$\begin{aligned} \int \langle \xi \rangle^{-2s}\, d\xi_1 &= \int \bigl(1 + |\xi'|^2 + \xi_1^2\bigr)^{-s}\, d\xi_1 \\ &= C\bigl(1 + |\xi'|^2\bigr)^{-s+1/2} = C\langle \xi' \rangle^{-2(s-1/2)}. \end{aligned} \tag{1.24}$$

Thus

$$\langle \xi' \rangle^{2(s-1/2)} |\hat{f}(\xi')|^2 \le C \int |\hat{u}(\xi)|^2 \langle \xi \rangle^{2s}\, d\xi_1, \tag{1.25}$$

and integrating with respect to ξ' gives

$$\|f\|^2_{H^{s-1/2}(\mathbb{R}^{n-1})} \le C\|u\|^2_{H^s(\mathbb{R}^n)}. \tag{1.26}$$

Proposition 1.6 has a converse:

Proposition 1.7. *The map (1.22) is surjective, for each $s > 1/2$.*

Proof. If $g \in H^{s-1/2}(\mathbb{R}^{n-1})$, we can let

$$\hat{u}(\xi) = \hat{g}(\xi')\, \frac{\langle \xi' \rangle^{2(s-1/2)}}{\langle \xi \rangle^{2s}}. \tag{1.27}$$

It is easy to verify that this defines an element $u \in H^s(\mathbb{R}^n)$ and $u(0, x') = cg(x')$ for a nonzero constant c, using (1.24) and (1.23); this provides the proof.

In the next section we will develop a tool that establishes the continuity of a number of natural transformations on $H^s(\mathbb{R}^n)$, as an automatic consequence of the (often more easily checked) continuity for integer s. This will be useful for the study of Sobolev spaces on compact manifolds, in §§3 and 4.

Exercises

1. Show that $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ for each s.
2. Assume $v \in \mathcal{S}'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $v(\xi) \ge 0$. Show that if $\hat{v} \in L^\infty(\mathbb{R}^n)$, then $v \in L^1(\mathbb{R}^n)$ and

$$(2\pi)^{n/2} \|\hat{v}\|_{L^\infty} = \|v\|_{L^1}.$$

(*Hint*: Consider $v_k(\xi) = \chi(\xi/k)v(\xi)$, with $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi(0) = 1$.)

3. Verify that (1.19) defines $u \in H^{n/2}(\mathbb{R}^n)$, $u \notin L^\infty(\mathbb{R}^n)$.
4. Show that the pairing

$$\langle u, v\rangle = \int \hat{u}(\xi)\tilde{v}(\xi)\, d\xi = \int \hat{u}(\xi)\langle\xi\rangle^s\tilde{v}(\xi)\langle\xi\rangle^{-s}\, d\xi$$

gives an isomorphism of $H^{-s}(\mathbb{R}^n)$ and the space $H^s(\mathbb{R}^n)'$, dual to $H^s(\mathbb{R}^n)$.

5. Show that the trace map (1.22) satisfies the estimate

$$\|\tau u\|^2_{L^2(\mathbb{R}^{n-1})} \le C\|u\|_{L^2} \cdot \|\nabla u\|_{L^2},$$

given $u \in H^1(\mathbb{R}^n)$, where on the right L^2 means $L^2(\mathbb{R}^n)$.

6. Show that $H^k(\mathbb{R}^n)$ is an algebra for $k > n/2$, that is,

$$u, v \in H^k(\mathbb{R}^n) \Longrightarrow uv \in H^k(\mathbb{R}^n).$$

Reconsider this problem after doing Exercise 5 in §2.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be C^∞, and assume $f(0) = 0$. Show that $u \mapsto f(u)$ defines a continuous map $F : H^k(\mathbb{R}^n) \to H^k(\mathbb{R}^n)$, for $k > n/2$. Show that F is a C^1-map, with $DF(u)v = f'(u)v$. Show that F is a C^∞-map.
8. Show that a continuous map $F : H^{k+m}(\mathbb{R}^n) \to H^k(\mathbb{R}^n)$ is defined by $F(u) = f(D^m u)$, where $D^m u = \{D^\alpha u : |\alpha| \le m\}$, assuming f is smooth in its arguments, $f = 0$ at $u = 0$, and $k > n/2$. Show that F is C^1, and compute $DF(u)$. Show F is a C^∞-map from $H^{k+m}(\mathbb{R}^n)$ to $H^k(\mathbb{R}^n)$.
9. Suppose $P(D)$ is an elliptic differential operator of order m, as in Chapter 3. If $\sigma < s + m$, show that

$$u \in H^\sigma(\mathbb{R}^n), P(D)u = f \in H^s(\mathbb{R}^n) \Longrightarrow u \in H^{s+m}(\mathbb{R}^n).$$

(*Hint*: Estimate $\langle\xi\rangle^{s+m}\hat{u}$ in terms of $\langle\xi\rangle^\sigma\hat{u}$ and $\langle\xi\rangle^s P(\xi)\hat{u}$.)

10. Given $0 < s < 1$ and $u \in L^2(\mathbb{R}^n)$, show that

$$u \in H^s(\mathbb{R}^n) \Longleftrightarrow \int_0^\infty t^{-(2s+1)}\|\tau_{te_j}u - u\|^2_{L^2}\, dt < \infty, \quad 1 \le j \le n, \tag{1.28}$$

where τ_y is as in (1.12).

(*Hint*: Show that the right side of (1.28) is equal to

$$\int\limits_{\mathbb{R}^n} \psi_s(\xi_j)|\hat{u}(\xi)|^2\, d\xi, \tag{1.29}$$

where, for $0 < s < 1$,

$$\psi_s(\xi_j) = 2\int_0^\infty t^{-(2s+1)}\Big(1 - \cos\ t\xi_j\Big)\, dt = C_s|\xi_j|^{2s}.) \tag{1.30}$$

11. The fact that $u \in H^s(\mathbb{R}^n)$ implies that $\sigma^{-1}(\tau_{\sigma e_j}u - u) \to iD_j u$ in $H^{s-1}(\mathbb{R}^n)$ was used in the proof of Proposition 1.1. Give a detailed proof of this. Use it to provide details for a proof of Corollary 1.4.
12. Establish the following, as another approach to justifying Corollary 1.4.

Lemma. *If $u \in C(\mathbb{R}^n)$ and $D_j u \in C(\mathbb{R}^n)$ for each j ($D_j u$ regarded a priori as a distribution), then $u \in C^1(\mathbb{R}^n)$.*

(*Hint*: Consider $\varphi_\varepsilon * u$ for $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\int \varphi\, dx = 1$, and let $\varepsilon \to 0$.)

2. The complex interpolation method

It is easy to see from the product rule that if M_φ is defined by

$$M_\varphi u = \varphi(x)u(x), \tag{2.1}$$

then, for any integer $k \geq 0$,

$$M_\varphi : H^k(\mathbb{R}^n) \longrightarrow H^k(\mathbb{R}^n), \tag{2.2}$$

provided φ is C^∞ and

$$D^\alpha \varphi \in L^\infty(\mathbb{R}^n), \quad \text{for all } \alpha. \tag{2.3}$$

By duality, (2.2) also holds for negative integers. We claim it holds when k is replaced by any real s, but it is not so simple to deduce this directly from the definition (1.4) of $H^s(\mathbb{R}^n)$. Similarly, suppose

$$\chi : \mathbb{R}^n \longrightarrow \mathbb{R}^n \tag{2.4}$$

is a diffeomorphism, which is linear outside some compact set, and define χ^* on functions by

$$\chi^*\, u(x) = u(\chi(x)). \tag{2.5}$$

The chain rule easily gives

$$\chi^* : H^k(\mathbb{R}^n) \longrightarrow H^k(\mathbb{R}^n), \tag{2.6}$$

for any integer $k \geq 0$. Since the adjoint of χ^* is ψ^* composed with the operation of multiplication by $|\det D\psi(x)|$, where $\psi = \chi^{-1}$, we see that (2.6) also holds for negative integers k. Again, it is not so straightforward to deduce (2.6) when k is replaced by any real number s. A convenient tool for proving appropriate generalizations of (2.2) and (2.6) is provided by the complex interpolation method, introduced by A. P. Calderon, which we now discuss.

Let E and F be Banach spaces. We suppose that F is included in E, and the inclusion $F \hookrightarrow E$ is continuous. If Ω is the vertical strip in the complex plane,

$$\Omega = \{z \in \mathbb{C} : 0 < \text{Re } z < 1\}, \tag{2.7}$$

we define
(2.8)

$$\mathcal{H}_{E,F}(\Omega) = \{u(z) \text{ bounded and continuous on } \overline{\Omega} \text{ with values in } E; \\ \text{holomorphic on } \Omega : \|u(1+iy)\|_F \text{ is bounded, for } y \in \mathbb{R}\}.$$

We define the *interpolation* spaces $[E,F]_\theta$ by

$$[E,F]_\theta = \{u(\theta) : u \in \mathcal{H}_{E,F}(\Omega)\}, \quad \theta \in [0,1]. \tag{2.9}$$

We give $[E,F]_\theta$ the Banach space topology, making it isomorphic to the quotient

$$\mathcal{H}_{E,F}(\Omega)/\{u : u(\theta) = 0\}. \tag{2.10}$$

We will also use the convention

$$[F,E]_\theta = [E,F]_{1-\theta}. \tag{2.11}$$

The following result is of basic importance.

Proposition 2.1. *Let E,F be as above; suppose $\widetilde{E},\widetilde{F}$ are Banach spaces with $\widetilde{F}$ continuously injected in $\widetilde{E}$. Suppose $T : E \to \widetilde{E}$ is a continuous linear map, and suppose $T : F \to \widetilde{F}$. Then, for all $\theta \in [0,1]$,*

$$T : [E,F]_\theta \to [\widetilde{E},\widetilde{F}]_\theta. \tag{2.12}$$

Proof. Given $v \in [E,F]_\theta$, let $u \in \mathcal{H}_{E,F}(\Omega)$, $u(\theta) = v$. It follows that $Tu(z) \in \mathcal{H}_{\widetilde{E},\widetilde{F}}(\Omega)$, so $Tv = Tu(\theta) \in [\widetilde{E},\widetilde{F}]_\theta$, as asserted.

We next identify $[H,\mathcal{D}(A)]_\theta$ when H is a Hilbert space and $\mathcal{D}(A)$ is the domain of a positive, self-adjoint operator on H. By the spectral theorem, this means the following. There is a unitary map $U : H \to L^2(X,\mu)$ such that $B = UAU^{-1}$ is a multiplication operator on $L^2(X,\mu)$:

$$Bu(x) = M_b u(x) = b(x)u(x). \tag{2.13}$$

Then $\mathcal{D}(A) = U^{-1}\mathcal{D}(B)$, where

$$\mathcal{D}(B) = \{u \in L^2(X,\mu) : bu \in L^2(X,\mu)\}.$$

We will assume $b(x) \geq 1$, though perhaps b is unbounded. (Of course, if b is bounded, then $\mathcal{D}(B) = L^2(X,\mu)$ and $\mathcal{D}(A) = H$.) This is equivalent to assuming $(Au,u) \geq \|u\|^2$. In such a case, we define A^θ to be $U^{-1}B^\theta U$, where $B^\theta u(x) = b(x)^\theta u(x)$, if $\theta \geq 0$, and $\mathcal{D}(A^\theta) = U^{-1}\mathcal{D}(B^\theta)$, where $\mathcal{D}(B^\theta) = \{u \in L^2(X,\mu) : b^\theta u \in L^2(X,\mu)\}$. We will give a proof of the spectral theorem in Chapter 8. In this chapter we will apply this notion only to operators A for which such a representation is explicitly implemented by a Fourier transform. Our characterization of interpolation spaces $[H,\mathcal{D}(A)]_\theta$ is given as follows.

Proposition 2.2. *For $\theta \in [0,1]$,*

$$[H,\mathcal{D}(A)]_\theta = \mathcal{D}(A^\theta). \tag{2.14}$$

Proof. First suppose $v \in \mathcal{D}(A^\theta)$. We want to write $v = u(\theta)$, for some $u \in \mathcal{H}_{H,\mathcal{D}(A)}(\Omega)$. Let

$$u(z) = A^{-z+\theta} v.$$

Then $u(\theta) = v$, u is bounded with values in H, and furthermore $u(1+iy) = A^{-1}A^{-iy}(A^\theta v)$ is bounded in $\mathcal{D}(A)$.

Conversely, suppose $u(z) \in \mathcal{H}_{H,\mathcal{D}(A)}(\Omega)$. We need to prove that $u(\theta) \in \mathcal{D}(A^\theta)$. Let $\varepsilon > 0$, and note that, by the maximum principle,

$$\begin{aligned} &\|A^z(I+i\varepsilon A)^{-1}u(z)\|_H \\ &\le \sup_{y\in\mathbb{R}} \max\big\{\|(I+i\varepsilon A)^{-1}A^{iy}u(iy)\|_H, \\ &\qquad\qquad \|A^{1+iy}(I+i\varepsilon A)^{-1}u(1+iy)\|_H\big\} \le C, \end{aligned} \tag{2.15}$$

with C independent of ε. This implies $u(\theta) \in \mathcal{D}(A^\theta)$, as desired.

Now the definition of the Sobolev spaces $H^s(\mathbb{R}^n)$ given in §1 makes it clear that, for $s \ge 0$, $H^s(\mathbb{R}^n) = \mathcal{D}(\Lambda^s)$, where Λ^s is the self-adjoint operator on $L^2(\mathbb{R}^n)$ defined by

$$\Lambda^s = \mathcal{F}\, M_{\langle\xi\rangle^s}\, \mathcal{F}^{-1}, \tag{2.16}$$

where $\mathcal{F}$ is the Fourier transform. Thus it follows that, for $k \ge 0$,

$$[L^2(\mathbb{R}^n), H^k(\mathbb{R}^n)]_\theta = H^{k\theta}(\mathbb{R}^n), \quad \theta \in [0,1]. \tag{2.17}$$

In fact, the same sort of reasoning applies more generally. For any $\sigma, s \in \mathbb{R}$,

$$[H^\sigma(\mathbb{R}^n), H^s(\mathbb{R}^n)]_\theta = H^{\theta s+(1-\theta)\sigma}(\mathbb{R}^n), \quad \theta \in [0,1]. \tag{2.18}$$

Consequently Proposition 2.1 is applicable to (2.4) and (2.6), to give

$$M_\varphi : H^s(\mathbb{R}^n) \longrightarrow H^s(\mathbb{R}^n) \tag{2.19}$$

and

$$\chi^* : H^s(\mathbb{R}^n) \longrightarrow H^s(\mathbb{R}^n), \tag{2.20}$$

for all $s \in \mathbb{R}$.

It is often convenient to have a definition of $[E,F]_\theta$ when neither Banach space E nor F is contained in the other. Suppose they are both continuously injected into a locally convex topological vector space V. Then $G = \{e+f : e \in E, f \in F\}$ has a natural structure of a Banach space, with norm

$$\|a\|_G = \inf\{\|e\|_E + \|f\|_F : a = e+f \text{ in } V,\ e \in E, f \in F\}.$$

In fact, G is naturally isomorphic to the quotient $(E \oplus F)/L$ of the Banach space $E \oplus F$, with the product norm, by the closed linear subspace $L =$

$\{(e,-e) : e \in E \cap F \subset V\}$. Generalizing (2.8), we set

$$\begin{aligned}\mathcal{H}_{E,F}(\Omega) = \{&u(z) \text{ bounded and continuous in } \overline{\Omega} \text{ with values in } G;\ \text{holo-}\\ &\text{morphic in } \Omega : \|u(iy)\|_E \text{ and } \|u(1+iy)\|_F \text{ bounded, } y \in \mathbb{R}\},\end{aligned} \tag{2.21}$$

where Ω is the vertical strip (2.7). Then we define the interpolation space $[E,F]_\theta$ by (2.9), as before. In this context, the identity (2.11) is a (simple) proposition rather than a definition.

Typical cases where it is of interest to apply such a construction include $E = L^{p_1}(X,\mu)$, $F = L^{p_2}(X,\mu)$. If (X,μ) is a measure space that is neither finite nor atomic (e.g., $\mathbb{R}^n$ with Lebesgue measure), typically neither of these L^p-spaces is contained in the other. We have the following useful result.

Proposition 2.3. *Take $\theta \in (0,1)$, $p_1 \in [1,\infty)$, $p_2 \geq 1$. Assume either $\mu(x) < \infty$ or $p_2 < \infty$. Then*

$$[L^{p_1}(X,\mu), L^{p_2}(X,\mu)]_\theta = L^q(X,\mu), \tag{2.22}$$

where p_1, p_2, and q are related by

$$\frac{1}{q} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}. \tag{2.23}$$

Proof. Given $f \in L^q$, one can take $c = (q-p_1)/p_1\theta = (p_2-q)/p_2(1-\theta)$ and define

$$u(z) = |f(x)|^{c(\theta-z)}\ f(x), \tag{2.24}$$

by convention zero when $f(x) = 0$. Then u belongs to $\mathcal{H}_{L^{p_1},L^{p_2}}$, which gives $L^q \subset [L^{p_1}, L^{p_2}]_\theta$.

Conversely, suppose that one is given $f \in [L^{p_1}, L^{p_2}]_\theta$; say $f = u(\theta)$ with $u \in \mathcal{H}_{L^{p_1},L^{p_2}}(\Omega)$. For $g \in L^{q'}$, you can define $v(z) = |g(x)|^{b(\theta-z)}\ g(x)$ with $b = (q'-p_1')/p_1'\theta = (p_2'-q')/p_2'(1-\theta)$, chosen so that $v \in \mathcal{H}_{L^{p_1 \prime},L^{p_2 \prime}}(\Omega)$. Then the Hadamard three-lines lemma, applied to $\langle u(z), v(z)\rangle$, implies

$$|\langle f,g\rangle| \leq \Bigl(\sup_{y\in\mathbb{R}} \bigl|\langle u(iy), v(iy)\rangle\bigr|\Bigr)^{1-\theta} \Bigl(\sup_{r\in\mathbb{R}} \bigl|\langle u(1+iy), v(1+iy)\rangle\bigr|\Bigr)^{\theta}, \tag{2.25}$$

for each simple function g. This implies

$$\begin{aligned}\Bigl|\int_X f(x)g(x)\ d\mu(x)\Bigr| &\leq C\Bigl\||g|^{b\theta+1}\Bigr\|_{L^{p_1'}}^{1-\theta} \cdot \Bigl\||g|^{b(\theta-1)+1}\Bigr\|_{L^{p_2'}}^{\theta}\\ &= C\ \|g\|_{L^{q'}},\end{aligned} \tag{2.26}$$

the last identity holding by (2.23) and the identities $b\theta + 1 = q'/p_1'$ and $b(\theta-1)+1 = q'/p_2'$. This implies $f \in L^q$.

If $\mu(X) = \infty$ and $p_2 = \infty$, then (2.24) need not yield an element of $\mathcal{H}_{L^{p_1},L^{p_2}}$, but the argument involving (2.25)–(2.26) still works, to give

$$[L^{p_1}(X,\mu), L^\infty(X,\mu)]_\theta \subset L^q(X,\mu), \quad q = \frac{p_1}{1-\theta}.$$

We record a couple of consequences of Proposition 2.3 and the remark following it, together with Proposition 2.1. Recall that the Fourier transform has the following mapping properties:

$$\mathcal{F} : L^1(\mathbb{R}^n) \longrightarrow L^\infty(\mathbb{R}^n); \quad \mathcal{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n).$$

Thus interpolation yields

$$\mathcal{F} : L^p(\mathbb{R}^n) \longrightarrow L^{p'}(\mathbb{R}^n), \quad \text{for } p \in [1,2], \tag{2.27}$$

where p' is defined by $1/p + 1/p' = 1$. Also, for the convolution product $f * g$, we clearly have

$$L^p * L^1 \subset L^p; \quad L^p * L^{p'} \subset L^\infty.$$

Fixing $f \in L^p$ and interpolating between L^1 and $L^{p'}$ give

$$L^p * L^q \subset L^r, \quad \text{for } q \in [1,p'], \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1. \tag{2.28}$$

We return to Hilbert spaces, and an interpolation result that is more general than Proposition 2.2, in that it involves $\mathcal{D}(A)$ for not necessarily self-adjoint A.

Proposition 2.4. *Let P^t be a uniformly bounded, strongly continuous semigroup on a Hilbert space H_0, whose generator A has domain $\mathcal{D}(A) = H_1$. Let $f \in H_0$, $0 < \theta < 1$. Then the following are equivalent:*

$$f \in [H_0, H_1]_\theta; \tag{2.29}$$

for some u,

$$f = u(0), \quad t^{1/2-\theta} u \in L^2(\mathbb{R}^+, H_1), \quad t^{1/2-\theta} \frac{du}{dt} \in L^2(\mathbb{R}^+, H_0); \tag{2.30}$$

$$\int_0^\infty t^{-(2\theta+1)} \|P^t f - f\|_{H_0}^2 \, dt < \infty. \tag{2.31}$$

Proof. First suppose (2.30) holds; then $u'(t) - Au(t) = g(t)$ satisfies $t^{1/2-\theta} g \in L^2(\mathbb{R}^+, H_0)$. Now, $u(t) = P^t f + \int_0^t P^{t-s} g(s)\, ds$, by Duhamel's principle, so

$$P^t f - f = \bigl(u(t) - f\bigr) - \int_0^t P^{t-s} g(s)\, ds, \tag{2.32}$$

and hence

$$\|t^{-1}(P^t f - f)\|_{H_0} \le \frac{1}{t} \int_0^t \|u'(s)\|_{H_0}\, ds + \frac{C}{t} \int_0^t \|g(s)\|_{H_0}\, ds. \tag{2.33}$$

This implies (2.31), via the elementary inequality (see Exercise 4 below)

$$
\begin{aligned}
&\|\Phi h\|_{L^2(\mathbb{R}^+,t^\beta dt)} \le K\|h\|_{L^2(\mathbb{R}^+,t^\beta dt)}, \quad \beta < 1, \\
&\Phi h(t) = \frac{1}{t}\int_0^t h(s)\, ds,
\end{aligned}
\tag{2.34}
$$

where we set $\beta = 1 - 2\theta$ and take $h(t) = \|u'(t)\|_{H_0}$ or $h(t) = \|g(t)\|_{H_0}$.

Next we show that (2.31) $\Rightarrow$ (2.30). If f satisfies (2.31), set

$$
u(t) = \frac{\varphi(t)}{t}\int_0^t P^s f\, ds,
\tag{2.35}
$$

where $\varphi \in C_0^\infty(\mathbb{R})$ and $\varphi(0) = 1$. Then $u(0) = f$. We need to show that

$$
t^{1/2-\theta} Au \in L^2(\mathbb{R}^+, H_0) \ \text{ and } \ t^{1/2-\theta}u' \in L^2(\mathbb{R}^+, H_0).
\tag{2.36}
$$

Now, $t^{1/2-\theta}Au = \varphi(t)t^{-1/2-\theta}(P^t f - f)$, so the first part of (2.36) follows directly from (2.31). The second part of (2.36) will be proved once we show that $t^{1/2-\theta}v' \in L^2(\mathbb{R}^+, H_0)$, where

$$
v(t) = \frac{1}{t}\int_0^t P^s f\, ds.
\tag{2.37}
$$

Now

$$
v'(t) = \frac{1}{t}\big(P^t f - f\big) - \frac{1}{t^2}\int_0^t \big(P^s f - f\big)\, ds,
\tag{2.38}
$$

and since the first term on the right has been controlled, it suffices to show that

$$
w(t) = t^{1/2-\theta-2}\int_0^t (P^s f - f)\, ds \in L^2(\mathbb{R}^+, H_0).
\tag{2.39}
$$

Indeed, since $s \le t$ in the integrand,

$$
\begin{aligned}
&\|w(t)\|_{H_0} \le \frac{t^{1/2-\theta}}{t}\int_0^t h(s)\, ds, \\
&h(t) = t^{-1}\|P^t f - f\|_{H_0} \in L^2(\mathbb{R}^+, t^{1-2\theta}dt),
\end{aligned}
\tag{2.40}
$$

so (2.39) follows from (2.34).

We now tackle the equivalence (2.29) $\Leftrightarrow$ (2.31). Since we have (2.30) $\Leftrightarrow$ (2.31) and (2.30) is independent of the choice of P^t, it suffices to show that (2.29) $\Leftrightarrow$ (2.31) for a *single* choice of P^t such that $\mathcal{D}(A) = H_1$. Now, we can pick a positive self-adjoint operator B such that $\mathcal{D}(B) = H_1$ (see Exercise 2 below), and take $A = iB$, so $P^t = e^{itB}$ is a unitary group. In such a case, the spectral decomposition yields the identity

$$
\|B^\theta f\|_{H_0}^2 = C_\theta \int_0^\infty t^{-(2\theta+1)}\|e^{itB}f - f\|_{H_0}^2\, dt;
\tag{2.41}
$$

compare (1.28)–(1.30); and the proof is easily completed.

Exercises

1. Show that the class of interpolation spaces $[E, F]_\theta$ defined in (2.9) and (2.15) is unchanged if one replaces various norm bounds $\|u(x+iy)\|$ by bounds on $e^{-K|y|}\|u(x+iy)\|$.

In Exercises 2 and 3, let $H_0 = E$ and $H_1 = F$ be two *Hilbert spaces* satisfying the hypotheses of Proposition 2.1. Assume H_1 is dense in H_0.

2. Show that there is a positive self adjoint operator A on H_0 such that $\mathcal{D}(A) = H_1$. (*Hint*: Use the Friedrichs method.)
3. Let $H_\theta = [H_0, H_1]_\theta$, $0 < \theta < 1$. Show that if $0 \le r < s \le 1$, then

$$[H_r, H_s]_\theta = H_{(1-\theta)r+\theta s}, \quad 0 < \theta < 1.$$

Relate this to (2.18).

4. Prove the estimate (2.34). (*Hint*: Make the change of variable $e^{(\beta-1)\tau/2}h(e^\tau) = \widetilde{h}(\tau)$, and convert Φ into a convolution operator on $L^2(\mathbb{R})$.)
5. Show that, for $0 \le s < n/2$,

$$H^s(\mathbb{R}^n) \subset L^p(\mathbb{R}^n), \quad \forall\, p \in \Big[2, \frac{2n}{n-2s}\Big). \tag{2.42}$$

(*Hint*: Use interpolation.)
Use (2.42) to estimate $(D^\alpha u)(D^\beta v)$, given $u, v \in H^k(\mathbb{R}^n)$, $k > n/2$, $|\alpha| + |\beta| \le k$. Sharper and more general results will be obtained in Chapter 13.

3. Sobolev spaces on compact manifolds

Let M be a compact manifold. If $u \in \mathcal{D}'(M)$, we say $u \in H^s(M)$ provided that, on any coordinate patch $U \subset M$, any $\psi \in C_0^\infty(U)$, the element $\psi u \in \mathcal{E}'(U)$ belongs to $H^s(U)$, if U is identified with its image in $\mathbb{R}^n$. By the invariance under coordinate changes derived in §2, it suffices to work with any single coordinate cover of M. If $s = k$, a nonnegative integer, then $H^k(M)$ is equal to the set of $u \in L^2(M)$ such that, for any ℓ smooth vector fields $X_1, \dots, X_\ell$ on M, $\ell \le k$, $X_1 \cdots X_\ell u \in L^2(M)$. Parallel to (2.17), we have the following result.

Proposition 3.1. *For $k \ge 0$ an integer, $\theta \in [0, 1]$,*

$$[L^2(M), H^k(M)]_\theta = H^{k\theta}(M). \tag{3.1}$$

More generally, for any $\sigma, s \in \mathbb{R}$,

$$[H^\sigma(M), H^s(M)]_\theta = H^{\theta s+(1-\theta)\sigma}(M). \tag{3.2}$$

Proof. These results follow directly from (2.17) and (2.18), with the aid of a partition of unity on M subordinate to a coordinate cover. We leave the details as an exercise.

Similarly, the duality of $H^s(\mathbb{R}^n)$ and $H^{-s}(\mathbb{R}^n)$ can easily be used to establish:

Proposition 3.2. *If M is a compact Riemannian manifold, $s \in \mathbb{R}$, there is a natural isomorphism*

$$H^s(M)^* \approx H^{-s}(M). \tag{3.3}$$

Furthermore, Propositions 1.3–1.5 easily yield:

Proposition 3.3. *If M is a smooth compact manifold of dimension n, and $u \in H^s(M)$, then*

$$u \in C(M) \text{ provided } s > \frac{n}{2}. \tag{3.4}$$

$$u \in C^k(M) \text{ provided } s > \frac{n}{2} + k, \tag{3.5}$$

$$u \in C^\alpha(M) \text{ provided } s = \frac{n}{2} + \alpha,\ \alpha \in (0,1). \tag{3.6}$$

In the case $M = \mathbb{T}^n$, the torus, we know from results on Fourier series given in Chapter 3 that, for $k \geq 0$ an integer,

$$u \in H^k(\mathbb{T}^n) \Longleftrightarrow \sum_{m \in \mathbb{Z}^n} |\hat{u}(m)|^2 \langle m \rangle^{2k} < \infty. \tag{3.7}$$

By duality, this also holds for k a negative integer. Now interpolation, via Proposition 2.2, implies that, for any $s \in \mathbb{R}$,

$$u \in H^s(\mathbb{T}^n) \Longleftrightarrow \sum_{m \in \mathbb{Z}^n} |\hat{u}(m)|^2 \langle m \rangle^{2s} < \infty. \tag{3.8}$$

Alternatively, if we define Λ^s on $\mathcal{D}'(\mathbb{T}^n)$ by

$$\Lambda^s\ u = \sum_{m \in \mathbb{Z}^n} \langle m \rangle^s\ \hat{u}(m)\ e^{im\cdot\theta}, \tag{3.9}$$

then, for $s \in \mathbb{R}$,

$$H^s(\mathbb{T}^n) = \Lambda^{-s} L^2(\mathbb{T}^n). \tag{3.10}$$

Thus, for any $s, \sigma \in \mathbb{R}$,

$$\Lambda^s : H^\sigma(\mathbb{T}^n) \longrightarrow H^{\sigma-s}(\mathbb{T}^n) \tag{3.11}$$

is an isomorphism.

It is clear from (3.9) that, for any $\sigma > 0$,

$$\Lambda^{-\sigma} : H^s(\mathbb{T}^n) \longrightarrow H^s(\mathbb{T}^n)$$

is a norm limit of finite rank operators, hence compact. Consequently, if j denotes the natural injection, we have, for any $s \in \mathbb{R}$,

$$j : H^{s+\sigma}(\mathbb{T}^n) \longrightarrow H^s(\mathbb{T}^n) \text{ compact}, \quad \forall\, \sigma > 0. \tag{3.12}$$

This is a special case of the following result.

Proposition 3.4. *For any compact M, $s \in \mathbb{R}$,*

$$j : H^{s+\sigma}(M) \longrightarrow H^s(M) \text{ is compact}, \quad \forall\, \sigma > 0. \tag{3.13}$$

Proof. This follows easily from (3.12), by using a partition of unity to break up an element of $H^{s+\sigma}(M)$ and transfer it to a finite set of elements of $H^{s+\sigma}(\mathbb{T}^n)$, if $n = \dim M$.

This result is a special case of a theorem of Rellich, which also deals with manifolds with boundary, and will be treated in the next section. Rellich's theorem will play a fundamental role in Chapter 5.

We next mention the following observation, an immediate consequence of (3.8) and Cauchy's inequality, which provides a refinement of Proposition 1.3 of Chapter 3.

Proposition 3.5. *If $u \in H^s(\mathbb{T}^n)$, then the Fourier series of u is absolutely convergent, provided $s > n/2$.*

Exercises

1. Fill in the details in the proofs of Propositions 3.1–3.4.
2. Show that $C^\infty(M)$ is dense in each $H^s(M)$, when M is a compact manifold.
3. Consider the projection P defined by
$$Pf(\theta) = \sum_{n=0}^{\infty} \hat{f}(n) e^{in\theta}.$$
Show that $P : H^s(S^1) \to H^s(S^1)$, for all $s \in \mathbb{R}$.
4. Let $a \in C^\infty(S^1)$, and define M_a by $M_a f(\theta) = a(\theta) f(\theta)$. Thus $M_a : H^s(S^1) \to H^s(S^1)$. Consider the commutator $[P, M_a] = PM_a - M_a P$. Show that
$$[P, M_a]f = \sum_{k \geq 0, m > 0} \hat{a}(k+m)\hat{f}(-m)e^{ik\theta} - \sum_{k > 0, m \geq 0} \hat{a}(-k-m)\hat{f}(m)e^{-ik\theta},$$
and deduce that, for all $s \in \mathbb{R}$,
$$[P, M_a] : H^s(S^1) \longrightarrow C^\infty(S^1).$$
(*Hint*: The Fourier coefficients $\big(\hat{a}(n)\big)$ form a rapidly decreasing sequence.)
5. Let $a_j, b_j \in C^\infty(S^1)$, and consider $T_j = M_{a_j} P + M_{b_j}(I - P)$. Show that
$$T_1 T_2 = M_{a_1 a_2} P + M_{b_1 b_2}(I - P) + R,$$
where, for each $s \in \mathbb{R}$, $R : H^s(S^1) \to C^\infty(S^1)$.
6. Suppose $a, b \in C^\infty(S^1)$ are both nowhere vanishing. Let
$$T = M_a P + M_b (I - P), \quad S = M_{a^{-1}} P + M_{b^{-1}}(I - P).$$

Show that $ST = I + R_1$ and $TS = I + R_2$, where $R_j : H^s(S^1) \to C^\infty(S^1)$, for all $s \in \mathbb{R}$. Deduce that, for each $s \in \mathbb{R}$,

$$T : H^s(S^1) \longrightarrow H^s(S^1) \text{ is Fredholm.}$$

Remark: The theory of Fredholm operators is discussed in §7 of Appendix A, Functional Analysis.

7. Let $e_j(\theta) = e^{ij\theta}$. Describe explicitly the kernel and range of

$$T_{jk} = M_{e_j} P + M_{e_k}(I - P).$$

Hence compute the *index* of T_{jk}. Using this, if a and b are nowhere-vanishing, complex-valued smooth functions on S^1, compute the index of $T_a = M_a P + M_b(I - P)$, in terms of the winding numbers of a and b. (*Hint*: If a and b are homotopic to e_j and e_k, respectively, as maps from S^1 to $\mathbb{C} \setminus 0$, then T and T_{jk} have the same index.)

4. Sobolev spaces on bounded domains

Let $\overline{\Omega}$ be a smooth, compact manifold with boundary $\partial\Omega$ and interior Ω. Our goal is to describe Sobolev spaces $H^s(\Omega)$. In preparation for this, we will consider Sobolev spaces $H^s(\mathbb{R}^n_+)$, where $\mathbb{R}^n_+$ is the half-space

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 > 0\},$$

with closure $\overline{\mathbb{R}^n_+}$. For $k \geq 0$ an integer, we want

$$H^k(\mathbb{R}^n_+) = \{u \in L^2(\mathbb{R}^n_+) : D^\alpha u \in L^2(\mathbb{R}^n_+) \text{ for } |\alpha| \leq k\}. \tag{4.1}$$

Here, $D^\alpha u$ is regarded a priori as a distribution on the interior $\mathbb{R}^n_+$. The space $H^k(\mathbb{R}^n)$ defined above has a natural Hilbert space structure. It is not hard to show that the space $\mathcal{S}(\overline{\mathbb{R}^n_+})$ of restrictions to $\mathbb{R}^n_+$ of elements of $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^k(\mathbb{R}^n_+)$, from the fact that, if $\tau_s u(x) = u(x_1 + s, x_2, \dots, x_n)$, then $\tau_s u \to u$ in $H^k(\mathbb{R}^n_+)$ as $s \searrow 0$, if $u \in H^k(\mathbb{R}^n_+)$. Now, we claim that each $u \in H^k(\mathbb{R}^n_+)$ is the restriction to $\mathbb{R}^n_+$ of an element of $H^k(\mathbb{R}^n)$. To see this, fix an integer N, and let

$$\begin{aligned} Eu(x) &= u(x), && \text{for } x_1 \geq 0, \\ &\sum_{j=1}^N a_j u(-jx_1, x'), && \text{for } x_1 < 0, \end{aligned} \tag{4.2}$$

defined a priori for $u \in \mathcal{S}(\overline{\mathbb{R}^n_+})$. We have the following.

Lemma 4.1. *One can pick $\{a_1, \dots, a_N\}$ such that the map E has a unique continuous extension to*

$$E : H^k(\mathbb{R}^n_+) \longrightarrow H^k(\mathbb{R}^n), \quad \textit{for } k \leq N - 1. \tag{4.3}$$

Proof. Given $u \in \mathcal{S}(\mathbb{R}^n)$, we get an H^k-estimate on Eu provided all the derivatives of Eu of order $\leq N-1$ match up at $x_1 = 0$, that is, provided

$$\sum_{j=1}^{N} (-j)^\ell a_j = 1, \quad \text{for } \ell = 0, 1, \dots, N-1. \tag{4.4}$$

The system (4.4) is a linear system of N equations for the N quantities a_j; its determinant is a Vandermonde determinant that is seen to be nonzero, so appropriate a_j can be found.

Corollary 4.2. *The restriction map*

$$\rho : H^k(\mathbb{R}^n) \longrightarrow H^k(\mathbb{R}^n_+) \tag{4.5}$$

is surjective.

Indeed, this follows from

$$\rho E = I \quad \text{on } H^k(\mathbb{R}^n_+). \tag{4.6}$$

Suppose $s \geq 0$. We can define $H^s(\mathbb{R}^n_+)$ by interpolation:

$$H^s(\mathbb{R}^n_+) = [L^2(\mathbb{R}^n_+), H^k(\mathbb{R}^n_+)]_\theta, \quad k \geq s, s = \theta k. \tag{4.7}$$

We can show that (4.7) is independent of the choice of an integer $k \geq s$. Indeed, interpolation from (4.3) gives

$$E : H^s(\mathbb{R}^n_+) \longrightarrow H^s(\mathbb{R}^n); \tag{4.8}$$

interpolation of (4.5) gives

$$\rho : H^s(\mathbb{R}^n) \longrightarrow H^s(\mathbb{R}^n_+); \tag{4.9}$$

and we have

$$\rho E = I \quad \text{on } H^s(\mathbb{R}^n_+). \tag{4.10}$$

This gives

$$H^s(\mathbb{R}^n_+) \approx H^s(\mathbb{R}^n)/\{u \in H^s(\mathbb{R}^n) : u\big|_{\mathbb{R}^n_+} = 0\}, \tag{4.11}$$

for $s \geq 0$, a characterization that is manifestly independent of the choice of $k \geq s$ in (4.7).

Now let $\overline{\Omega}$ be a smooth, compact manifold with smooth boundary. We can suppose that $\overline{\Omega}$ is imbedded as a submanifold of a compact (boundaryless) manifold M of the same dimension. If $\overline{\Omega} \subset \mathbb{R}^n$, $n = \dim \Omega$, you can arrange this by putting $\overline{\Omega}$ in a large box and identifying opposite sides to get $\overline{\Omega} \subset \mathbb{T}^n$. In the general case, one can construct the "double" of $\overline{\Omega}$, as follows. Using a vector field X on $\partial\Omega$ that points into Ω at each point, that is, X is nowhere vanishing on $\partial\Omega$ and in fact nowhere tangent to $\partial\Omega$, we can extend X to a vector field on a neighborhood of $\partial\Omega$ in $\overline{\Omega}$, and using its integral curves construct a neighborhood of $\partial\Omega$ in $\overline{\Omega}$ diffeomorphic to

$[0,1) \times \partial\Omega$, a so-called "collar neighborhood" of $\partial\Omega$. Using this, one can glue together two copies of $\overline{\Omega}$ along $\partial\Omega$ in such a fashion as to produce a smooth, compact M as desired.

If $k \geq 0$ is an integer, we define $H^k(\Omega)$ to consist of all $u \in L^2(\Omega)$ such that $Pu \in L^2(\Omega)$ for all differential operators P of order $\leq k$ with coefficients in $C^\infty(\overline{\Omega})$. We use Ω to denote $\overline{\Omega}\backslash\partial\Omega$. Similar to the case of $\mathbb{R}^n_+$, one shows that $C^\infty(\overline{\Omega})$ is dense in $H^k(\Omega)$. By covering a neighborhood of $\partial\Omega \subset M$ with coordinate patches and locally using the extension operator E from above, we get, for each finite N, an extension operator

$$E : H^k(\Omega) \longrightarrow H^k(M), \quad 0 \leq k \leq N-1. \tag{4.12}$$

If, for real $s \geq 0$, we define $H^s(\Omega)$ by

$$H^s(\Omega) = [L^2(\Omega), H^k(\Omega)]_\theta, \quad k \geq s,\ s = \theta k, \tag{4.13}$$

we see that

$$E : H^s(\Omega) \longrightarrow H^s(M), \tag{4.14}$$

so the restriction $\rho : H^s(M) \to H^s(\Omega)$ is onto, and

$$H^s(\Omega) \approx H^s(M)/\{u \in H^s(M) : u\big|_\Omega = 0\}, \tag{4.15}$$

which shows that (4.13) is independent of the choice of $k \geq s$.

The characterization (4.15) can be used to define $H^s(\Omega)$ when s is a negative real number. In that case, one wants to show that the space $H^s(\Omega)$ so defined is independent of the inclusion $\Omega \subset M$. We will take care of this point in the next section.

The existence of the extension map (4.14) allows us to draw the following immediate consequence from Proposition 3.3.

Proposition 4.3. *If* $\dim \Omega = n$ *and* $u \in H^s(\Omega)$, *then*

$$\begin{aligned} &u \in C(\overline{\Omega}) \text{ provided } s > \frac{n}{2}; \\ &u \in C^k(\overline{\Omega}) \text{ provided } s > \frac{n}{2} + k; \\ &u \in C^\alpha(\overline{\Omega}) \text{ provided } s = \frac{n}{2} + \alpha,\ \alpha \in (0,1). \end{aligned}$$

We now extend Proposition 3.4, obtaining the full version of Rellich's theorem.

Proposition 4.4. *For any* $s \geq 0, \sigma > 0$, *the natural inclusion*

$$j : H^{s+\sigma}(\Omega) \longrightarrow H^s(\Omega) \text{ is compact.} \tag{4.16}$$

Proof. Using E and ρ, we can factor the map (4.16) through the map (3.9):

$$\begin{array}{ccc} H^{s+\sigma}(\Omega) & \xrightarrow{j} & H^s(\Omega) \\ {\scriptstyle E}\downarrow & & \uparrow{\scriptstyle \rho} \\ H^{s+\sigma}(M) & \xrightarrow{j} & H^s(M) \end{array}$$

which immediately gives (4.16) as a consequence of Proposition 3.4.

The boundary $\partial\Omega$ of Ω is a smooth, compact manifold, on which Sobolev spaces have been defined. By using local coordinate systems flattening out $\partial\Omega$, together with the extension map (4.14) and the trace theorem, Proposition 1.6, we have the following result on the trace map:

$$\tau u = u\big|_{\partial\Omega}. \tag{4.17}$$

Proposition 4.5. *For $s > 1/2$, τ extends uniquely to a continuous map*

$$\tau : H^s(\Omega) \longrightarrow H^{s-1/2}(\partial\Omega). \tag{4.18}$$

We close this section with a consideration of mapping properties on Sobolev spaces of the Poisson integral considered in §2 of Chapter 3:

$$\mathrm{PI} : C(S^1) \longrightarrow C(\overline{\mathcal{D}}), \tag{4.19}$$

where

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}, \tag{4.20}$$

given explicitly by

$$\mathrm{PI}\ f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k + \sum_{k=1}^{\infty} \hat{f}(-k) \overline{z}^k, \tag{4.21}$$

as in (2.4) of Chapter 3, and satisfying the property that

$$u = \mathrm{PI}\ f \Longrightarrow \Delta u = 0 \text{ in } \mathcal{D} \quad \text{and} \quad u\big|_{S^1} = f. \tag{4.22}$$

The following result can be compared with Proposition 2.2 in Chapter 3.

Proposition 4.6. *The Poisson integral gives a continuous map*

$$PI : H^s(S^1) \longrightarrow H^{s+1/2}(\mathcal{D}), \quad \text{for } s \ge -\frac{1}{2}. \tag{4.23}$$

Proof. It suffices to prove this for $s = k - 1/2$, $k = 0, 1, 2, \dots$; this result for general $s \ge -1/2$ will then follow by interpolation. Recall that to say

$f \in H^{k-1/2}(S^1)$ means

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \langle k \rangle^{2k-1} < \infty. \tag{4.24}$$

Now the functions $\{r^{|n|}e^{in\theta} : n \in \mathbb{Z}\}$ are mutually orthogonal in $L^2(\mathcal{D})$, and

$$\iint_{\mathcal{D}} \left| r^{|n|} e^{in\theta} \right|^2 \, dx \, dy = 2\pi \int_0^1 r^{2|n|} r \, dr = \frac{\pi}{|n|+1}. \tag{4.25}$$

In particular, $f \in H^{-1/2}(S^1)$ implies

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \langle n \rangle^{-1} < \infty,$$

which implies PI $f \in L^2(\mathcal{D})$, by (4.25).

Next, if $f \in H^{k-1/2}(S^1)$, then $(\partial/\partial\theta)^\nu f \in H^{-1/2}(S^1)$, for $0 \le \nu \le k$, so $(\partial/\partial\theta)^\nu \mathrm{PI}\, f = \mathrm{PI}(\partial/\partial\theta)^\nu f \in L^2(\mathcal{D})$. We need to show that

$$\Bigl(r\frac{\partial}{\partial r}\Bigr)^\mu \Bigl(\frac{\partial}{\partial \theta}\Bigr)^\nu \mathrm{PI}\, f \in L^2(\mathcal{D}),$$

for $0 \le \mu + \nu \le k$. Indeed, set

$$Nf = \sum_{n=-\infty}^{\infty} |n| \hat{f}(n) e^{in\theta}. \tag{4.26}$$

It follows from Plancherel's theorem that $(\partial/\partial\theta)^\nu N^\mu f \in H^{-1/2}(S^1)$, for $0 \le \mu+\nu \le k$, if $f \in H^{k-1/2}(S^1)$, while, as in (2.18) of Chapter 2, we have

$$\Bigl(r\frac{\partial}{\partial r}\Bigr)^\mu \Bigl(\frac{\partial}{\partial \theta}\Bigr)^\nu \mathrm{PI}\, f = \mathrm{PI}\, \Bigl(\frac{\partial}{\partial \theta}\Bigr)^\nu N^\mu\, f, \tag{4.27}$$

which hence belongs to $L^2(\mathcal{D})$. Since PI f is smooth in a neighborhood of the origin $r = 0$, this finishes the proof.

The Poisson integral taking functions on the sphere S^{n-1} to harmonic functions on the ball in $\mathbb{R}^n$, and more generally the map taking functions on the boundary of $\partial\Omega$ of a compact Riemannian manifold $\overline{\Omega}$ (with boundary), to harmonic functions on Ω, will be studied in Chapter 5.

Exercises

1. Let $\mathcal{D}$ be the unit disk in $\mathbb{R}^2$, with boundary $\partial\mathcal{D} = S^1$. Consider the solution to the Neumann problem

$$\Delta u = 0 \;\text{ on } \mathcal{D}, \quad \frac{\partial u}{\partial r} = g \;\text{ on } S^1, \tag{4.28}$$

studied in Chapter 3, §2, Exercises 1–4. Show that, for $s \geq 1/2$,

$$g \in H^s(S^1) \Longrightarrow u \in H^{s+3/2}(\mathcal{D}). \tag{4.29}$$

(*Hint*: Write $u = \mathrm{PI}\, f$, with $Nf = g$, where N is given by (4.26).)

2. Let $\overline{\Omega}$ be a smooth, compact manifold with boundary. Show that the following versions of the divergence theorem and Green's formula hold:

$$\int_{\Omega} \Big[(\operatorname{div} X)uv + (Xu)v + u(Xv)\Big]\, dV = \int_{\partial\Omega} \langle X, \nu\rangle uv\, dS, \tag{4.30}$$

when, among X, u, and v, one is smooth and two belong to $H^1(\Omega)$. Also show that

$$-(u, \Delta v)_{L^2(\Omega)} = (du, dv)_{L^2(\Omega)} - \int_{\partial\Omega} u\, \frac{\partial \overline{v}}{\partial \nu}\, dS, \tag{4.31}$$

for $u \in H^1(\Omega)$, $v \in H^2(\Omega)$. (*Hint*: Approximate.)

3. Show that if $u \in H^2(\Omega)$ satisfies $\Delta u = 0$ on Ω and $\partial u/\partial\nu = 0$ on $\partial\Omega$, then u must be constant, if Ω is connected. (*Hint*: Use (4.31) with $v = u$.)

Exercises 4–9 deal with the "oblique derivative problem" for the Laplace operator on the disk $\mathcal{D} \subset \mathbb{R}^2$. The oblique derivative problem on higher-dimensional regions is discussed in exercises in §12 of Chapter 5.

4. Consider the oblique derivative problem

$$\Delta u = 0 \text{ on } \mathcal{D}, \quad a\frac{\partial u}{\partial r} + b\frac{\partial u}{\partial \theta} + cu = g \text{ on } S^1, \tag{4.32}$$

where $a, b, c \in C^\infty(S^1)$ are given. If $u = \mathrm{PI}\, f$, show that u is a solution if and only if $Qf = g$, where

$$Q = M_a N + M_b \frac{\partial}{\partial\theta} + M_c : H^{s+1}(S^1) \longrightarrow H^s(S^1). \tag{4.33}$$

5. Recall $\Lambda : H^{s+1}(S^1) \to H^s(S^1)$, defined by

$$\Lambda f(\theta) = \sum \langle k\rangle \hat{f}(k) e^{ik\theta}, \tag{4.34}$$

as in (3.9). Show that Λ is an isomorphism and that

$$\Lambda - N : H^s(S^1) \longrightarrow H^s(S^1). \tag{4.35}$$

6. With Q as in (4.33), show that $Q = T\Lambda$ with

$$T = M_{a+ib}P + M_{a-ib}(I - P) + R : H^s(S^1) \longrightarrow H^s(S^1), \tag{4.36}$$

where

$$R : H^s(S^1) \longrightarrow H^{s+1}(S^1).$$

Here P is as in Exercise 3 of §3. (*Hint*: Note that $\partial/\partial\theta = iPN - i(I-P)N$.)

7. Deduce that the operator Q in (4.33) is Fredholm provided $a + ib$ and $a - ib$ are nowhere vanishing on S^1. In particular, if a and b are real-valued, Q is Fredholm provided a and b have no common zeros on S^1. (*Hint*: Recall Exercises 4–6 of §3.)

8. Let $\mathcal{H} = \{u \in C^2(\mathcal{D}) : \Delta u = 0 \text{ in } \mathcal{D}\}$. Take $s > 0$. Using the commutative diagram

$$\begin{array}{ccc} H^{s+1}(S^1) & \xrightarrow{PI} & H^{s+\frac{3}{2}}(\mathcal{D}) \cap \mathcal{H} \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle B} \\ H^s(S^1) & \xrightarrow{I} & H^s(S^1) \end{array} \tag{4.37}$$

where Q is as in (4.33) and

$$Bu = a\frac{\partial u}{\partial r} + b\frac{\partial u}{\partial \theta} + cu\Big|_{S^1}, \tag{4.38}$$

deduce that B is Fredholm provided $a, b \in C^\infty(S^1)$ are real-valued and have no common zeros on S^1. In such a case, compute the index of B. (*Hint*: Recall Exercise 7 from §3. Also note that the two horizontal arrows in (4.37) are isomorphisms.)

9. Let B be as above; assume $a, b, c \in C^\infty(S^1)$ are all real-valued. Also assume that a is nowhere vanishing on S^1. If $c/a \geq 0$ on S^1, show that Ker B consists at most of constant functions. (*Hint*: See Zaremba's principle, in §2 of Chapter 5.)
   If, in addition, c is not identically zero, show that Ker $B = 0$. Using Exercise 8, show that B has index zero in this case. Draw conclusions about the solvability of the oblique derivative problem (4.32).

10. Prove that $C^\infty(\overline{\Omega})$ is dense in $H^s(\Omega)$ for all $s \geq 0$.
    (*Hint*: With E as in (4.14), approximate Eu by elements of $C^\infty(M)$.)

11. Consider the Vandermonde determinant

$$\Delta_{n+1}(x_0, \dots, x_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_0^n & x_1^n & \cdots & x_n^n \end{vmatrix}.$$

   Show that $\Delta_{n+1}(x_0, \dots, x_{n-1}, t)$ is a polynomial of degree n in t, with roots $x_0, \dots, x_{n-1}$, hence equal to $K(t - x_0) \cdots (t - x_{n-1})$; the coefficient K of t^n is equal to $\Delta_n(x_0, \dots, x_{n-1})$. Deduce by induction that

$$\Delta_{n+1}(x_0, \dots, x_n) = \prod_{0 \leq j < k \leq n} (x_k - x_j).$$

12. Given $0 < s < 1$ and $f \in L^2(\mathbb{R}^+)$, show that

$$f \in H^s(\mathbb{R}^+) \Longleftrightarrow \int_0^\infty t^{-(2s+1)} \|\tau_t f - f\|^2_{L^2(\mathbb{R}^+)}\, dt < \infty, \tag{4.39}$$

   where $\tau_t f(x) = f(x+t)$. (*Hint*: Use Proposition 2.4, with $P^t f(x) = f(x+t)$, whose infinitesimal generator is d/dx, with domain $H^1(\mathbb{R}^+)$. Note that "$\Rightarrow$" also follows from (4.14) plus (1.28).)
   More generally, given $0 < s < 1$ and $f \in L^2(\mathbb{R}^n_+)$, show that

$$f \in H^s(\mathbb{R}^n_+) \Longleftrightarrow \int_0^\infty t^{-(2s+1)} \|\tau_{te_j} f - f\|^2_{L^2(\mathbb{R}^n_+)}\, dt < \infty, \quad 1 \leq j \leq n, \tag{4.40}$$

   where τ_y is as in (1.12).

5. The Sobolev spaces $H_0^s(\Omega)$

Let $\overline{\Omega}$ be a smooth, compact manifold with boundary; we denote the interior by Ω, as before. As before, we can suppose $\overline{\Omega}$ is contained in a compact, smooth manifold M, with $\partial\Omega$ a smooth hypersurface. For $s \geq 0$, we define $H_0^s(\Omega)$ to consist of the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$. For $s = k$ a nonnegative integer, it is not hard to show that

$$H_0^k(\Omega) = \{u \in H^k(M) : \operatorname{supp} u \subset \overline{\Omega}\}. \tag{5.1}$$

This is because a norm giving the topology of $H^k(\Omega)$ can be taken to be the square root of

$$\sum_{j=1}^{K} \|P_j u\|^2_{L^2(\Omega)}, \tag{5.2}$$

for a certain finite number of differential operators P_j of order $\leq k$, which implies that the closure of $C_0^\infty(\Omega)$ in $H^k(\Omega)$ can be identified with its closure in $H^k(M)$. Since the topology of $H^s(M)$ for $s \notin \mathbb{Z}^+$ is not defined in such a localizable fashion, such an argument does not work for general real s. For a general closed set B in M, set

$$H_B^s(M) = \{u \in H^s(M) : \operatorname{supp} u \subset B\}. \tag{5.3}$$

It has been proved in [Fu] that, for $s \geq 0$,

$$H_0^s(\Omega) \approx H_{\overline{\Omega}}^s(M) \ \text{ if } \ s + \frac{1}{2} \notin \mathbb{Z}. \tag{5.4}$$

See the exercises below for some related results.

Recall our characterization of the space $H^s(\Omega)$ given in (4.15), which we rewrite as

$$H^s(\Omega) \approx H^s(M)/H_K^s(\Omega), \quad K = \overline{M \setminus \Omega}. \tag{5.5}$$

This characterization makes sense for any $s \in \mathbb{R}$, not just for $s \geq 0$, and we use it as a definition of $H^s(\Omega)$ for $s < 0$. For $k \in \mathbb{Z}^+$, we can redefine $H^{-k}(\Omega)$ in a fashion intrinsic to $\overline{\Omega}$, making use of the following functional analytic argument.

In general, if E is a Banach space, with dual E^*, and F a closed linear subspace of E, we have a natural isomorphism of dual spaces:

$$F^* \approx E^*/F^\perp, \tag{5.6}$$

where

$$F^\perp = \{u \in E^* : \langle v, u\rangle = 0 \text{ for all } v \in F\}. \tag{5.7}$$

If $E = H^k(M)$, we take $F = H_0^k(\Omega)$, which, as discussed above, we can regard as the closure of $C_0^\infty(\Omega)$ in $H^k(M) = E$. Then it is clear that $F^\perp = H_K^{-k}(M)$, with $K = \overline{M \setminus \Omega}$, so we have proved:

Proposition 5.1. *For Ω open in M with smooth boundary, $k \geq 0$ an integer, we have a natural isomorphism*

$$H_0^k(\Omega)^* \approx H^{-k}(\Omega). \tag{5.8}$$

Let P be a differential operator of order $2k$, with smooth coefficients on $\overline{\Omega}$. Suppose

$$P = \sum_{j=1}^{L} A_j B_j, \tag{5.9}$$

where A_j and B_j are differential operators of order k, with coefficients smooth on $\overline{\Omega}$. Then we have a well-defined continuous linear map

$$P : H_0^k(\Omega) \longrightarrow H^{-k}(\Omega), \tag{5.10}$$

and, if A_j^t denotes the formal adjoint of A_j on $\overline{\Omega}$, endowed with a smooth Riemannian metric, then, for $u, v \in H_0^k(\Omega)$, we have

$$\langle u, Pv \rangle = \sum_{j=1}^{L} (A_j^t u, B_j v)_{L^2(\Omega)}, \tag{5.11}$$

the dual pairing on the left side being that of (5.8). In fact, the formula (5.5) gives

$$P : H^s(\Omega) \longrightarrow H^{s-2k}(\Omega) \tag{5.12}$$

for all real s, and in particular

$$P : H^k(\Omega) \longrightarrow H^{-k}(\Omega), \tag{5.13}$$

and the identity (5.11) holds for $v \in H^k(\Omega)$, provided $u \in H_0^k(\Omega)$. In Chapter 5 we will study in detail properties of the map (5.10) when P is the Laplace operator (so $k = 1$).

The following is an elementary but useful result.

Proposition 5.2. *Suppose $\overline{\Omega}$ is a smooth, connected, compact manifold with boundary, endowed with a Riemannian metric. Suppose $\partial\Omega \neq \emptyset$. Then there exists a constant $C = C(\Omega) < \infty$ such that*

$$\|u\|_{L^2(\Omega)}^2 \leq C \|du\|_{L^2(\Omega)}^2, \quad \text{for } u \in H_0^1(\Omega). \tag{5.14}$$

It suffices to establish (5.14) for $u \in C^\infty(\Omega)$. Given $u\big|_{\partial\Omega} = 0$, one can write

$$u(x) = -\int_{\gamma(x)} du, \tag{5.15}$$

for any $x \in \Omega$, where $\gamma(x)$ is some path from x to $\partial\Omega$. Upon making a reasonable choice of $\gamma(x)$, obtaining (5.14) is an exercise, which we leave to the reader. (See Exercises 4–5 below.)

Finding a sharp value of C such that (5.14) holds is a challenging problem, for which a number of interesting results have been obtained. As will follow from results in Chapter 5, this is equivalent to the problem of estimating the smallest eigenvalue of $-\Delta$ on Ω, with Dirichlet boundary conditions.

Below, there is a sequence of exercises, one of whose implications is that

$$[L^2(\Omega), H_0^1(\Omega)]_s = H_0^s(\Omega) = H^s(\Omega), \quad 0 < s < \frac{1}{2}. \tag{5.16}$$

Here we will establish a result that is useful for the proof.

Proposition 5.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded region with smooth boundary. If $0 \le s < 1/2$, and $Tu = \chi_{\overline{\Omega}} u$, then*

$$T : H^s(\mathbb{R}^n) \longrightarrow H^s(\mathbb{R}^n). \tag{5.17}$$

Proof. It is easy to reduce this to the case $\Omega = \mathbb{R}^n_+$, and then to the case $n = 1$, which we will treat here. Also, the case $s = 0$ is trivial, so we take $0 < s < 1/2$. By (1.28), it suffices to estimate

$$\int_0^\infty t^{-(2s+1)} \|\tau_t \widetilde{u} - \widetilde{u}\|^2_{L^2(\mathbb{R})} \, dt, \tag{5.18}$$

where $\widetilde{u}(x) = Tu(x)$, so, for $t > 0$,

$$\begin{aligned} \tau_t \widetilde{u}(x) - \widetilde{u}(x) = u(t+x) - u(x), &\quad x > 0 \\ u(t+x), &\quad -t < x < 0 \\ 0, &\quad x < -t \end{aligned} \tag{5.19}$$

Hence (5.18) is

$$\le \int_0^\infty t^{-(2s+1)} \|\tau_t u - u\|^2_{L^2(\mathbb{R})} \, dt + \int_0^\infty t^{-(2s+1)} \int_{-t}^0 |u(t+x)|^2 \, dx \, dt. \tag{5.20}$$

The first term in (5.20) is finite for $u \in H^s(\mathbb{R})$, $0 < s < 1$, by (1.28). The last term in (5.20) is equal to

$$\begin{aligned} \int_0^\infty \int_0^t t^{-(2s+1)} |u(t-x)|^2 \, dx \, dt &= \int_0^\infty \int_0^t t^{-(2s+1)} |u(x)|^2 \, dx \, dt \\ &= C_s \int_0^\infty |x|^{-2s} |u(x)|^2 \, dx. \end{aligned} \tag{5.21}$$

The next lemma implies that this is finite for $u \in H^s(\mathbb{R})$, $0 < s < 1/2$.

Lemma 5.4. *If* $0 < s < 1/2$, *then*

$$u \in H^s(\mathbb{R}^n) \Longrightarrow |x_1|^{-s} u \in L^2(\mathbb{R}^n). \tag{5.22}$$

Proof. The general case is easily deduced from the case $n = 1$, which we establish here. Also, it suffices to show that, for $0 < s < 1/2$,

$$u \in H^s(\mathbb{R}) \Longrightarrow x^{-s}\widetilde{u} \in L^2(\mathbb{R}^+), \tag{5.23}$$

where $\widetilde{u} = u\big|_{\mathbb{R}^+}$. Now, for $x > 0$, $u \in C_0^\infty(\mathbb{R})$, set

$$v(x) = \frac{1}{x}\int_0^x \big[u(x) - u(y)\big]\, dy, \quad w(x) = \int_x^\infty \frac{v(y)}{y}\, dy. \tag{5.24}$$

We claim that

$$u(x) = v(x) - w(x), \quad x > 0. \tag{5.25}$$

In fact, if $u \in C_0^\infty(\mathbb{R})$, then $v(x) \to 0$ and $w(x) \to 0$ as $x \to +\infty$, and one verifies easily that $u'(x) = v'(x) - w'(x)$. Thus it suffices to show that, for $0 < s < 1/2$,

$$\|x^{-s}v\|_{L^2(\mathbb{R}^+)} \le C\|u\|_{H^s(\mathbb{R})}, \quad \|x^{-s}w\|_{L^2(\mathbb{R}^+)} \le C\|u\|_{H^s(\mathbb{R})}, \tag{5.26}$$

for $u \in C_0^\infty(\mathbb{R})$.

To verify the first estimate in (5.26), we will use the simple fact that $|v(x)|^2 \le (1/x)\int_0^x |u(x) - u(y)|^2\, dy$. Hence

$$\begin{aligned} \int_0^\infty x^{-s}|v(x)|^2\, dx &\le \int_0^\infty \int_0^x x^{-(2s+1)}\big|u(x) - u(y)\big|^2\, dy\, dx \\ &= \int_0^\infty \int_0^\infty (y+t)^{-(2s+1)}\big|u(y+t) - u(y)\big|^2\, dt\, dy \\ &\le \int_0^\infty y^{-(2s+1)}\|\tau_t u - u\|^2_{L^2(\mathbb{R}^+)}\, dy. \end{aligned} \tag{5.27}$$

Since the $L^2(\mathbb{R}^+)$-norm is less than the $L^2(\mathbb{R})$-norm, it follows from (1.28) that the last integral in (5.27) is dominated by $C\|u\|^2_{H^s(\mathbb{R})}$, for $0 < s < 1$.

Thus, to prove the rest of (5.26), it suffices to show that

$$\|x^{-s}w\|_{L^2(\mathbb{R}^+)} \le C\|x^{-s}v\|_{L^2(\mathbb{R}^+)}, \quad 0 < s < \frac{1}{2}, \tag{5.28}$$

or equivalently, that $\|w\|_{L^2(\mathbb{R}^+, x^{-2s}dx)} \le C\|v\|_{L^2(\mathbb{R}^+, x^{-2s}dx)}$. In turn, this follows from the estimate (2.34), with $\beta = 2s$, since we have $w = \Phi^* v$, where Φ^* acting on $L^2(\mathbb{R}^+, x^{-\beta}dx)$ is the adjoint of Φ in (2.34). This completes the proof of the lemma, hence of Proposition 5.3.

Corollary 5.5. *If* $Sv(x) = v(x)$ *for* $x \in \Omega$, *and* $Sv(x) = 0$ *for* $x \in \mathbb{R}^n \setminus \Omega$, *then*

$$S : H^s(\Omega) \longrightarrow H^s(\mathbb{R}^n), \quad 0 \le s < \frac{1}{2}. \tag{5.29}$$

Proof. Apply Proposition 5.3 to $u = Ev$, where $E : H^s(\Omega) \to H^s(\Omega)$ is any extension operator that works for $0 \le s \le 1$.

Exercises

1. Give the a detailed proof of (5.1).
2. With $\tau u = u\big|_{\partial\Omega}$, as in (4.17), prove that

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : \tau u = 0\}. \tag{5.30}$$

   (*Hint*: Given $u \in H^1(\Omega)$ and $\tau u = 0$, define $\tilde{u} = u(x)$ for $x \in \Omega$, $\tilde{u}(x) = 0$ for $x \in M \setminus \Omega$. Use (4.30) to show that $\tilde{u} \in H^1(M)$.)
3. Let $u \in H^k(\Omega)$. Prove that $u \in H_0^k(\Omega)$ if and only if $\tau(Pu) = 0$ for all differential operators P (with smooth coefficients) of order $\le k - 1$ on M.
4. Give a detailed proof of Proposition 5.2 along the lines suggested, involving (5.15).
5. Give an alternative proof of Proposition 5.2, making use of the compactness of the inclusion $H^1(\Omega) \hookrightarrow L^2(\Omega)$. (*Hint*: If (5.14) is false, take $u_j \in H_0^1(\Omega)$ such that $\|du_j\|_{L^2} \to 0$, $\|u_j\|_{L^2} = 1$. The compactness yields a subsequence $u_j \to v$ in $H^1(\Omega)$. Hence $\|v\|_{L^2} = 1$ while $\|dv\|_{L^2} = 0$.)
6. Suppose $\Omega \subset \mathbb{R}^n$ lies between two parallel hyperplanes, $x_1 = A$ and $x_1 = B$. Show that the estimate (5.14) holds with $C = (B - A)^2/\pi^2$.
   Reconsider this problem after reading §1 of Chapter 5.
7. Show that $C^\infty(\overline{\Omega})$ is dense in $H^{-s}(\Omega)$, for $s \ge 0$. Compare Exercise 10 of §4.
8. Give a detailed proof that (5.11) is true for $u \in H_0^k(\Omega)$, $v \in H^k(\Omega)$.
   (*Hint*: Approximate u by $u_j \in C_0^\infty(\Omega)$ and v by $v_j \in C^\infty(\overline{\Omega})$.)
9. Show that if P^t is the formal adjoint of P, then $\langle u, Pv\rangle = \langle P^t u, v\rangle$ for $u, v \in H_0^k(\Omega)$.

In the following problems, let Ω be an open subset of a compact manifold M, with smooth boundary $\partial\Omega$ and closure $\overline{\Omega}$. Let $\mathcal{O} = M \setminus \overline{\Omega}$.

10. Define $Z : L^2(\Omega) \to L^2(M)$ by $Zu(x) = u(x)$ for $x \in \Omega$, 0 for $x \in \mathcal{O}$. Show that

$$Z : H_0^k(\Omega) \longrightarrow H_{\overline{\Omega}}^k(M), \quad k = 0, 1, 2, \dots \tag{5.31}$$

   and that Z is an isomorphism in these cases. Deduce that

$$Z : [L^2(\Omega), H_0^k(\Omega)]_\theta \longrightarrow H_{\overline{\Omega}}^{k\theta}(M), \quad 0 < \theta < 1,\ k \in \mathbb{Z}^+. \tag{5.32}$$

11. For fixed but large N, let $E : H^s(\mathcal{O}) \to H^s(M)$ be an extension operator, similar to (4.14), for $0 \le s \le N$. Define $Tu = u - ERu$, where $Ru = u\big|_{\mathcal{O}}$. Show that

$$T : H^s(M) \longrightarrow H_{\overline{\Omega}}^s(M), \quad 0 \le s \le N. \tag{5.33}$$

   Note that $Tu = u$ for $u \in H_{\overline{\Omega}}^s(M)$.

12. Set $T^b u = Tu\big|_{\Omega}$, so $T^b : H^k(M) \to H^k_0(\Omega)$, for $0 \le k \le N$, and hence

$$T^b : H^{k\theta}(M) \longrightarrow [L^2(\Omega), H^k_0(\Omega)]_\theta.$$

Show that

$$T^b jZ = id. \ \text{ on } [L^2(\Omega), H^k_0(\Omega)]_\theta,$$

where $j : H^s_{\overline{\Omega}}(M) \hookrightarrow H^s(M)$ is the natural inclusion. Deduce that (5.32) is an isomorphism. Conclude that

$$[L^2(\Omega), H^k_0(\Omega)]_\theta \approx [H^0_{\overline{\Omega}}(M), H^k_{\overline{\Omega}}(M)]_\theta = H^{k\theta}_{\overline{\Omega}}(M), \quad 0 < \theta < 1. \tag{5.34}$$

13. Show that $H^s_{\overline{\Omega}}(M)$ is equal to the closure of $C_0^\infty(\Omega)$ in $H^s(M)$. (This can fail when $\partial\Omega$ is not smooth.) Conclude that there is a natural injective map

$$\kappa : H^s_{\overline{\Omega}}(M) \longrightarrow H^s_0(\Omega), \quad s \ge 0.$$

(*Hint*: Recall that $H^s_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega) \approx H^s(M)/H^s_{\overline{\mathcal{O}}}(M)$.)

14. If Z is defined as in Exercise 10, use Corollary 5.5 to show that

$$Z : H^s_0(\Omega) \longrightarrow H^s(M), \quad 0 \le s < \frac{1}{2}. \tag{5.35}$$

15. If $v \in C^\infty(\overline{\Omega})$, and $w = v$ on Ω, 0 on $\mathcal{O}$, show that $w \in H^s(M)$, for all $s \in [0, 1/2)$. If $v = 1$, show that $w \notin H^{1/2}(M)$.
16. Show that

$$H^s_0(\Omega) = H^s(\Omega), \ \text{ for } 0 \le s \le \frac{1}{2}. \tag{5.36}$$

(*Hint*: To show that $C_0^\infty(\Omega)$ is dense in $H^s(\Omega)$, show that $\{u \in C^\infty(M) : u = 0 \text{ near } \partial\Omega\}$ is dense in $H^s(M)$, for $0 \le s \le 1/2$.)

17. Using the results of Exercises 10–16, show that, for $k \in \mathbb{Z}^+$,

$$[L^2(\Omega), H^k_0(\Omega)]_\theta = H^s_0(\Omega) = H^s(\Omega) \ \text{ if } \ s = k\theta \in [0, \tfrac{1}{2}). \tag{5.37}$$

See [LM], pp. 60–62, for a demonstration that, for $s > 0$,

$$Z : H^s_0(\Omega) \longrightarrow H^s(M) \Longleftrightarrow s - \frac{1}{2} \notin \mathbb{Z},$$

which, by Exercise 12, implies (5.4) and also, for $k \in \mathbb{Z}^+$,

$$[L^2(\Omega), H^k_0(\Omega)]_\theta = H^s_0(\Omega) \ \text{ if } \ s = k\theta \notin \mathbb{Z} + \frac{1}{2}.$$

18. If F is a closed subspace of a Banach space, there is a natural isomorphism $(E/F)^* \approx F^\perp = \{\omega \in E^* : \langle f, \omega \rangle = 0, \forall\, f \in F\}$. Use this to show that

$$H^s(\Omega)^* \approx H^{-s}_{\overline{\Omega}}(M). \tag{5.38}$$

19. Applying (5.6) with $E = H^k(\Omega)$, $F = H^k_0(\Omega)$, in conjunction with (5.8) and (5.38), show that for $k \in \mathbb{N}$,

$$H^{-k}(\Omega) \approx H^{-k}_{\overline{\Omega}}(M)/H^{-k}_{\partial\Omega}(M). \tag{5.39}$$

6. The Schwartz kernel theorem

Let M and N be compact manifolds. Suppose

$$T : C^\infty(M) \longrightarrow \mathcal{D}'(N) \tag{6.1}$$

is a linear map that is continuous. We give $C^\infty(M)$ its usual Fréchet space topology and $\mathcal{D}'(N)$ its weak* topology. Consequently, we have a bilinear map

$$B : C^\infty(M) \times C^\infty(N) \longrightarrow \mathbb{C}, \tag{6.2}$$

separately continuous in each factor, given by

$$B(u, v) = \langle v, Tu \rangle, \quad u \in C^\infty(M),\ v \in C^\infty(N). \tag{6.3}$$

For such u, v, define

$$u \otimes v \in C^\infty(M \times N) \tag{6.4}$$

by

$$(u \otimes v)(x, y) = u(x)v(y), \quad x \in M,\ y \in N. \tag{6.5}$$

We aim to prove the following result, known as the Schwartz kernel theorem.

Theorem 6.1. *Given B as in (6.2), there exists a distribution*

$$\kappa \in \mathcal{D}'(M \times N) \tag{6.6}$$

such that

$$B(u, v) = \langle u \otimes v, \kappa \rangle, \tag{6.7}$$

for all $u \in C^\infty(M), v \in C^\infty(N)$.

We note that the right side of (6.7) defines a bilinear map (6.2) that is continuous in each factor, so Theorem 6.1 establishes an isomorphism between $\mathcal{D}'(M \times N)$ and the space of maps of the form (6.2), or equivalently the space of continuous linear maps (6.1).

The first step in the proof is to elevate the hypothesis of separate continuity to an apparently stronger condition. Generally speaking, let E and F be Fréchet spaces, and let

$$\beta : E \times F \longrightarrow \mathbb{C} \tag{6.8}$$

be a separately continuous bilinear map. Suppose the topology of E is defined by seminorms $p_1 \leq p_2 \leq p_3 \leq \cdots$ and that of F by seminorms $q_1 \leq q_2 \leq q_3 \leq \cdots$. We have the following result.

Proposition 6.2. *If β in (6.8) is separately continuous, then there exist seminorms p_K and q_L and a constant C' such that*

$$|\beta(u, v)| \leq C' p_K(u) q_L(v), \quad u \in E,\ v \in F. \tag{6.9}$$

Proof. This will follow from the Baire category theorem, in analogy with the proof of the uniform boundedness theorem. Let $S_{C,j} \subset E$ consist of $u \in E$ such that

$$|\beta(u,v)| \le C q_j(v), \quad \text{for all } v \in F. \tag{6.10}$$

The hypothesis that β is continuous in v for each u implies

$$\bigcup_{C,j} S_{C,j} = E. \tag{6.11}$$

The hypothesis that β is continuous in u implies that each $S_{C,j}$ is closed. The Baire category theorem implies that some $S_{C,L}$ has nonempty interior. Hence $S_{1/2,L} = (2C)^{-1} S_{C,L}$ has nonempty interior. Since $S_{c,L} = -S_{c,L}$ and $S_{1/2,L} + S_{1/2,L} = S_{1,L}$, it follows that $S_{1,L}$ is a neighborhood of 0 in E. Picking K so large that, for some C_1, the set of $u \in E$ with $p_K(u) \le C_1$ is contained in this neighborhood, we have (6.9) with $C' = C/C_1$. This proves the proposition.

Returning to the bilinear map B of (6.2), we use Sobolev norms to define the topology of $C^\infty(M)$ and of $C^\infty(N)$:

$$p_j(u) = \|u\|_{H^j(M)}, \quad q_j(v) = \|v\|_{H^j(N)}. \tag{6.12}$$

In the case of $M = \mathbb{T}^m$, we can take

$$p_j(u) = \Big(\sum_{|\alpha| \le j} \|D^\alpha u\|^2_{L^2(\mathbb{T}^m)} \Big)^{1/2}, \tag{6.13}$$

and similarly for $p_j(v)$ if $N = \mathbb{T}^n$. Proposition 6.2 implies that there are C, K, L such that

$$|B(u,v)| \le C \|u\|_{H^K(M)} \|v\|_{H^L(N)}. \tag{6.14}$$

Recalling that the dual of $H^L(N)$ is $H^{-L}(N)$, we have the following result.

Proposition 6.3. *Let B be as in Theorem 6.1. Then for some K, L, there is a continuous linear map*

$$T : H^K(M) \longrightarrow H^{-L}(N) \tag{6.15}$$

such that

$$B(u,v) = \langle v, Tu \rangle, \quad \text{for } u \in C^\infty(M),\ v \in C^\infty(N). \tag{6.16}$$

Thus, if a continuous linear map of the form (6.1) is given, it has a continuous linear extension of the form (6.15).

In the next few steps of the proof of Theorem 6.1, it will be convenient to work with the case $M = \mathbb{T}^m$, $N = \mathbb{T}^n$. Once Theorem 6.1 is established in this case, it can readily be extended to the general case.

Recall from (3.7) the isomorphisms

$$\Lambda^s : H^\sigma(\mathbb{T}^m) \longrightarrow H^{\sigma-s}(\mathbb{T}^m), \tag{6.17}$$

for all real s, σ, where $\Lambda^2 = I - \Delta$. It follows from (6.15) that

$$T_{jk} = (I-\Delta)^{-j} T (I-\Delta)^{-k} : L^2(\mathbb{T}^m) \longrightarrow H^s(\mathbb{T}^n) \tag{6.18}$$

as long as $k \geq K/2$ and $j \geq L/2 + s$. Note that

$$T = (I-\Delta)^j T_{jk} (I-\Delta)^k. \tag{6.19}$$

The next step in our analysis will exploit the fact that if j is picked sufficiently large in (6.18), then T_{jk} is a *Hilbert-Schmidt* operator from $L^2(\mathbb{T}^m)$ to $L^2(\mathbb{T}^n)$.

We recall here the notion of a Hilbert-Schmidt operator, which is discussed in detail in §6 of Appendix A. Let H_1 and H_2 be two separable infinite dimensional Hilbert spaces, with orthonormal bases $\{u_j\}$ and $\{v_j\}$, respectively. Then $A : H_1 \to H_2$ is Hilbert-Schmidt if and only if

$$\sum_j \|Au_j\|^2 = \sum_{j,k} |a_{jk}|^2 < \infty, \tag{6.20}$$

where $a_{jk} = (Au_j, v_k)$. The quantity on the left is denoted $\|A\|_{HS}^2$. It is not hard to show that this property is independent of choices of orthonormal bases. Also, if there are bounded operators $V_1 : X_1 \to H_1$ and $V_2 : H_2 \to X_2$ between Hilbert spaces, we have

$$\|V_2 A V_1\|_{HS} \leq \|V_2\| \cdot \|A\|_{HS} \cdot \|V_1\|, \tag{6.21}$$

where of course $\|V_j\|$ are operator norms. If V_j are both unitary, there is identity in (6.21). For short, we call a Hilbert-Schmidt operator an "HS operator."

From the definition, and using the exponential functions for Fourier series as an orthonormal basis, it easily follows that

$$\Lambda^{-s} \text{ is HS on } L^2(\mathbb{T}^n) \Longleftrightarrow s > \frac{n}{2}. \tag{6.22}$$

Consequently, we can say of the operator T_{jk} given by (6.18) that

$$T_{jk} : L^2(\mathbb{T}^m) \longrightarrow L^2(\mathbb{T}^n) \text{ is HS if } 2k \geq K \text{ and } 2j > L + n. \tag{6.23}$$

Our next tool, which we call the Hilbert-Schmidt kernel theorem, is proved in §6 of Appendix A.

Theorem 6.4. *Given a Hilbert-Schmidt operator*

$$T_1 : L^2(X_1, \mu_1) \longrightarrow L^2(X_2, \mu_2),$$

there exists $K \in L^2(X_1 \times X_2, \mu_1 \times \mu_2)$ *such that*

$$(T_1 u, v)_{L^2} = \iint K(x_1, x_2) u(x_1) \overline{v(x_2)}\, d\mu_1(x_1)\, d\mu_2(x_2). \tag{6.24}$$

To proceed with the proof of the Schwartz kernel theorem, we can now establish the following.

Proposition 6.5. *The conclusion of Theorem 6.1 holds when $M = \mathbb{T}^m$ and $N = \mathbb{T}^n$.*

Proof. By Theorem 6.4, there exists $K \in L^2(\mathbb{T}^m \times \mathbb{T}^n)$ such that

$$\langle v, T_{jk}u\rangle = \iint K(x,y)u(x)v(y)\ dx\ dy, \tag{6.25}$$

for $u \in C^\infty(\mathbb{T}^m), v \in C^\infty(\mathbb{T}^n)$, provided T_{jk}, given by (6.18), satisfies (6.23). In view of (6.19), this implies

$$\begin{aligned} \langle v, Tu\rangle &= \langle (I-\Delta)^j v, T_{jk}(I-\Delta)^k u\rangle \\ &= \iint K(x,y)\ (I-\Delta_y)^j v(y)\ (I-\Delta_x)^k u(x)\ dx\ dy, \end{aligned} \tag{6.26}$$

so (6.7) holds with

$$\kappa = (I-\Delta_x)^k (I-\Delta_y)^j K(x,y) \in \mathcal{D}'(\mathbb{T}^m \times \mathbb{T}^n). \tag{6.27}$$

Now Theorem 6.1 for general compact M and N can be proved by writing

$$B(u,v) = \sum_{j,k} B(\varphi_j u, \psi_k v), \tag{6.28}$$

for partitions of unity $\{\varphi_j\}$, $\{\psi_k\}$ subordinate to coordinate covers of M and N, and transferring the problem to the case of tori.

Exercises

1. Extend Theorem 6.1 to treat the case of

$$B : C_0^\infty(M) \times C_0^\infty(N) \longrightarrow \mathbb{C},$$

   when M and N are smooth, paracompact manifolds. State carefully an appropriate continuity hypothesis on B.
2. What is the Schwartz kernel of the identity map $I : C^\infty(\mathbb{T}^n) \to C^\infty(\mathbb{T}^n)$?

7. Sobolev spaces on rough domains

With $\overline{\Omega} \subset M$ as in §§4–5, suppose $\mathcal{O} \subset \Omega$ is an open subset, perhaps with quite rough boundary. As in our definitions of $H^k(\Omega)$ and $H_0^k(\Omega)$, we set, for $k \in \mathbb{Z}^+$,

$$H^k(\mathcal{O}) = \{u \in L^2(\mathcal{O}) : Pu \in L^2(\mathcal{O}),\ \forall\, P \in \mathrm{Diff}^k(M)\}, \tag{7.1}$$

where $\mathrm{Diff}^k(M)$ denotes the set of all differential operators of order $\le k$, with C^∞ coefficients, on M. Then we set

$$H_0^k(\mathcal{O}) = \text{ closure of } C_0^\infty(\mathcal{O}) \text{ in } H^k(\mathcal{O}). \tag{7.2}$$

There exist operators $P_{k1}, \dots, P_{kN} \in \mathrm{Diff}^k(M)$ spanning $\mathrm{Diff}^k(M)$ over $C^\infty(M)$, $N = N(k)$, and we can take

$$\|u\|^2_{H^k(\mathcal{O})} = \sum_{j=1}^{n} \|P_{kj} u\|^2_{L^2(\mathcal{O})}. \tag{7.3}$$

It readily folllows that

$$H_0^k(\mathcal{O}) = \text{ closure of } C_0^\infty(\mathcal{O}) \text{ in } H^k(M), \tag{7.4}$$

with $u \in H_0^k(\mathcal{O})$ extended by 0 off $\mathcal{O}$. We have

$$H_0^k(\mathcal{O}) \subset H^k_{\overline{\mathcal{O}}}(M), \tag{7.5}$$

where

$$H^k_{\overline{\mathcal{O}}}(M) = \{u \in H^k(M) : \operatorname{supp} u \subset \overline{\mathcal{O}}\}. \tag{7.6}$$

Unlike in (5.1), the reverse inclusion can fail for rough $\partial\mathcal{O}$. Here is a condition favorable for such a reverse inclusion.

Proposition 7.1. *If at each point $\partial\mathcal{O}$ is locally the graph of a continuous function, then*

$$H_0^k(\mathcal{O}) = H^k_{\overline{\mathcal{O}}}(M). \tag{7.7}$$

In such a case, given $u \in H^k_{\overline{\mathcal{O}}}(M)$, one can use a partition of unity, slight shifts, and mollifiers to realize u as a limit in $H^k(M)$ of functions in $C_0^\infty(\mathcal{O})$.

A simple example of a domain $\mathcal{O}$ for which (7.7) fails, for all $k \ge 1$, is the slit disk:

$$\mathcal{O} = \{x \in \mathbb{R}^2 : |x| < 1\} \setminus \{(x_1, 0) : 0 \le x_1 < 1\}. \tag{7.8}$$

Another easy consequence of (7.4), plus Proposition 4.4, is that for $k \ge 1$, the natural injection

$$H_0^k(\mathcal{O}) \hookrightarrow L^2(\mathcal{O}) \text{ is compact.} \tag{7.9}$$

Also, the extension of $u \in H_0^k(\mathcal{O})$ by zero off $\mathcal{O}$ gives

$$H_0^k(\mathcal{O}) \hookrightarrow H_0^k(\Omega), \quad \text{closed subspace.} \tag{7.10}$$

Specializing this to $k = 1$ and recalling Proposition 5.2, we have

$$\|u\|^2_{L^2(\mathcal{O})} \le \widetilde{C} \|du\|^2_{L^2(\mathcal{O})}, \quad \forall\, u \in H_0^1(\mathcal{O}), \tag{7.11}$$

with $\widetilde{C} \le C$, where C is as in (5.14).

Recall the restriction map $\rho : H^k(M) \to H^k(\Omega)$, considered in §4. Similarly we have $\rho : H^k(M) \to H^k(\mathcal{O})$, but for rough $\partial\mathcal{O}$ this map might not be onto. There might not be an extension operator $E : H^k(\mathcal{O}) \to H^k(M)$, as in (4.12). Here is one favorable case for the existence of an extension operator.

Proposition 7.2. *If at each point $\partial\mathcal{O}$ is locally the graph of a Lipschitz function, then there exists*

$$E : H^k(\mathcal{O}) \longrightarrow H^k(M), \quad \text{for } \ k = 0, 1, \quad \rho E = I \ \text{ on } \ H^k(\mathcal{O}). \tag{7.12}$$

In such a case, given $u \in H^k(\mathcal{O})$, one can use a partition of unity to reduce the construction to extending u supported on a small neighborhood in $\overline{\mathcal{O}}$ of a point $p_0 \in \partial\mathcal{O}$ and use a bi-Lipschitz map to flatten out $\partial\mathcal{O}$ on this support. Such bi-Lipschitz maps preserve H^k for $k = 0$ and 1, and we can appeal to Lemma 4.1.

If (7.12) holds, then, as in Proposition 4.4, we have

$$H^1(\mathcal{O}) \hookrightarrow L^2(\mathcal{O}) \tag{7.13}$$

compact. However, for rough $\partial\mathcal{O}$, compactness in (7.13) can fail. A simple example of such failure is given by

$$\mathcal{O} = \bigcup_{k=1}^{\infty} \mathcal{O}_k, \quad \mathcal{O}_k = \{x \in \mathbb{R}^2 : |x - (2^{-k}, 0)| < 8^{-k}\}. \tag{7.14}$$

When (7.12) holds, results on

$$H^s(\mathcal{O}) = [L^2(\mathcal{O}), H^1(\mathcal{O})]_s, \quad 0 < s < 1, \tag{7.15}$$

parallel to those presented in §5, hold, as the reader is invited to verify.

Exercises

1. The example (7.8), for which (7.7) fails, is not equal to the interior of its closure. Construct $\mathcal{O} \subset \mathbb{R}^n$, equal to the interior of its closure, for which (7.7) fails.
2. The example (7.14), for which (7.13) is not compact, has infinitely many connected components. Construct a connected, open, bounded $\mathcal{O} \subset \mathbb{R}^n$, such that (7.13) is not compact.

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