Variations on Complex Interpolation

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1. Introduction

Let X and Y be Banach spaces, assumed to be linear subspaces of a Hausdorff locally convex space V (with continuous inclusions). We say (X, Y, V) is a compatible triple. For $\theta \in (0, 1)$, the classical complex interpolation space $[X, Y]_\theta$ is defined as follows. First, $Z = X + Y$ gets a natural norm; for $v \in X + Y$,

$$\|v\|_Z = \inf \{\|v_1\|_X + \|v_2\|_Y : v = v_1 + v_2,\ v_1 \in X,\ v_2 \in Y\}. \tag{1.1}$$

One has $X + Y \approx X \oplus Y/L$, where $L = \{(v, -v) : x \in X \cap Y\}$ is a closed linear subspace, so $X+Y$ is a Banach space. Let $\Omega = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, with closure $\overline{\Omega}$. Define $\mathcal{H}_\Omega(X, Y)$ to be the space of functions $f : \overline{\Omega} \to Z = X + Y$, continuous on $\overline{\Omega}$, holomorphic on Ω (with values in $X + Y$), satisfying $f : \{\operatorname{Im} z = 0\} \to X$ continuous, $f : \{\operatorname{Im} z = 1\} \to Y$ continuous, and

$$\|u(z)\|_Z \le C, \quad \|u(iy)\|_X \le C, \quad \|u(1 + iy)\|_Y \le C, \tag{1.2}$$

for some $C < \infty$, independent of $z \in \overline{\Omega}$ and $y \in \mathbb{R}$. Then, for $\theta \in (0, 1)$,

$$[X, Y]_\theta = \{u(\theta) : u \in \mathcal{H}_\Omega(X, Y)\}. \tag{1.3}$$

One has

$$[X, Y]_\theta \approx \mathcal{H}_\Omega(X, Y)/\{u \in \mathcal{H}_\Omega(X, Y) : u(\theta) = 0\}, \tag{1.4}$$

giving $[X, Y]_\theta$ the sructure of a Banach space. Here

$$\|u\|_{\mathcal{H}_\Omega(X,Y)} = \sup_{z\in\overline{\Omega}} \|u(z)\|_Z + \sup_y \|u(iy)\|_X + \sup_y \|u(1 + iy)\|_Y. \tag{1.5}$$

If I is an interval in $\mathbb{R}$, one says a family of Banach spaces $X_s,\ s \in I$ (subspaces of V) forms a complex interpolation scale provided that for $s, t \in I,\ \theta \in (0, 1)$,

$$[X_s, X_t]_\theta = X_{(1-\theta)s+\theta t}. \tag{1.6}$$

Some very useful tools for analysis arise from the fact that many natural classes of function spaces (and spaces of distributions) form complex interpolation scales. We mention [Cal], [LM], and [BL], where many more references can be found. Examples include L^p-Sobolev spaces $X_s = H^{s,p}(M),\ s \in \mathbb{R}$, where $p \in (1, \infty)$ and

M is a compact Riemannian manifold, as well as many families of Besov spaces, Triebel-Lizorkin spaces, etc.

On the other hand, there are some important families of function spaces that barely fail to form complex interpolation scales as defined above. One class of examples, mentioned in [Tri], is the family of Besov spaces

$$X_s = B^s_{p,\infty}(\mathbb{R}^n), \quad s \in \mathbb{R}, \tag{1.7}$$

given $p \in (1,\infty)$. Here $[X_s, X_t]_\theta$ is not X_τ, with $\tau = (1-\theta)s + \theta t$, but rather the closure of $\mathcal{S}(\mathbb{R}^n)$ in X_τ. Such a phenomenon arises frequently for families X_s of function spaces on $\mathbb{R}^n$ for which $\mathcal{S}(\mathbb{R}^n)$ is not dense in X_s. One sees similar examples for families of function spaces X_s on a compact manifold M for which $C^\infty(M)$ is not dense in X_s.

There are variants of $[X, Y]_\theta$, introduced in [CT] and discussed in [Tri], for which such spaces as (1.7) do form interpolation scales. The specific constructions given there are somewhat special to the context of function spaces on Euclidean spaces, and it seems useful to extend the scope of such generalized complex interpolation scales. There are two natural motivations for doing this. One is to treat function (and distribution) spaces on manifolds other than $\mathbb{R}^n$. The other is that one of the major points of interpolation theory is that if a linear operator is continuous on some spaces X_{s_j}, then it is also continuous on the spaces X_s in between. To produce more results of this flavor, it is desirable to have additional flexibility in producing variants of the complex interpolation scale $[X, Y]_\theta$.

The purpose of this paper is to explore further variants of $[X, Y]_\theta$. In §2 we define $[X, Y]_{\theta;V}$, where (X, Y, V) is a compatible triple as defined above, and we derive some elementary properties, particularly regarding interpolations of continuous linear operators, mapping $X \to X, Y \to Y$, and $V \to V$. In addition, we define a family $[X, Y]^b_\theta$ when $Y \subset X$ is a continuously nested pair of Banach spaces, and record the analogous interpolation mapping property.

We then discuss several families of examples. In §3 we discuss L^p-Sobolev spaces $X_s = H^{s,p}(\mathbb{R}^n)$, with $p \in (1,\infty)$, where these various interpolation spaces coincide. In §4 we discuss $X_s = B^s_{\infty,\infty}(M) = C^s_*(M)$, known as Zygmund spaces, and in §5 we discuss Hardy and bmo-Sobolev spaces. In these cases, spaces of the form $[X, Y]_{\theta;V}$ and $[X, Y]^b_\theta$ yield bigger (and more useful) interpolation spaces than $[X, Y]_\theta$.

2. The spaces $[X, Y]_{\theta;V}$ and $[X, Y]^b_\theta$

Let (X, Y, V) be a compatible triple, as defined in §1, and let $\Omega = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$. We define $\mathcal{H}_\Omega(X, Y, V)$ to be the space of functions $u : \overline{\Omega} \to X + Y = Z$ such that

$$u : \Omega \longrightarrow Z \ \text{ is holomorphic,} \tag{2.1}$$

$$\|u(z)\|_Z \le C, \quad \|u(iy)\|_X \le C, \quad \|u(1+iy)\|_Y \le C, \tag{2.2}$$

and

$$u : \overline{\Omega} \longrightarrow V \text{ is continuous.} \tag{2.3}$$

For such u, we again use the norm (1.5). Note that the only difference with $\mathcal{H}_\Omega(X,Y)$ is that we are relaxing the continuity hypothesis for u on $\overline{\Omega}$. $\mathcal{H}_\Omega(X,Y,V)$ is also a Banach space, and we have a natural isometric inclusion

$$\mathcal{H}_\Omega(X,Y) \hookrightarrow \mathcal{H}_\Omega(X,Y,V). \tag{2.4}$$

Now for $\theta \in (0,1)$ we set

$$[X,Y]_{\theta;V} = \{u(\theta) : u \in \mathcal{H}_\Omega(X,Y,V)\}. \tag{2.5}$$

Again this space gets a Banach space structure, via

$$[X,Y]_{\theta;V} \approx \mathcal{H}_\Omega(X,Y,V)/\{u \in \mathcal{H}_\Omega(X,Y,V) : u(\theta) = 0\}, \tag{2.6}$$

and there is a natural continuous injection

$$[X,Y]_\theta \hookrightarrow [X,Y]_{\theta;V}. \tag{2.7}$$

Sometimes this is an isomorphism. In fact, sometimes $[X,Y]_\theta = [X,Y]_{\theta;V}$ for practically all reasonable choices of V. In §3 we verify this for $X = L^p(\mathbb{R}^n)$, $Y = H^{s,p}(\mathbb{R}^n)$, the L^p-Sobolev space, with $p \in (1,\infty), s \in (0,\infty)$. On the other hand, we devote §§4–5 to some cases where equality in (2.7) does not hold, and where $[X,Y]_{\theta;V}$ is of greater interest than $[X,Y]_\theta$.

We next define $[X,Y]^b_\theta$. In this case we assume X and Y are Banach spaces and $Y \subset X$ (continuously). We take Ω as above, and set $\widetilde{\Omega} = \{z \in \mathbb{C} : 0 < \operatorname{Re} z \le 1\}$, i.e., we throw in the right boundary but not the left boundary. We then define $\mathcal{H}^b_\Omega(X,Y)$ to be the space of functions $u : \widetilde{\Omega} \to X$ such that

$$\begin{gathered} u : \Omega \longrightarrow X \text{ is holomorphic,} \\ \|u(z)\|_X \le C, \quad \|u(1+iy)\|_Y \le C, \\ u : \widetilde{\Omega} \longrightarrow X \text{ is continuous.} \end{gathered} \tag{2.8}$$

Note that the essential difference between $\mathcal{H}_\Omega(X,Y)$, discussed in §1, and the space we have just introduced is that we have completely dropped any continuity requirement at $\{\operatorname{Re} z = 0\}$. We also do not require continuity from $\{\operatorname{Re} z = 1\}$ to Y. The space $\mathcal{H}^b_\Omega(X,Y)$ is a Banach space, with norm

$$\|u\|_{\mathcal{H}^b_\Omega(X,Y)} = \sup_{z\in\widetilde{\Omega}} \|u(z)\|_X + \sup_y \|u(1+iy)\|_Y.$$

Now, for $\theta \in (0,1)$, we set

$$[X,Y]^b_\theta = \{u(\theta) : u \in \mathcal{H}^b_\Omega(X,Y)\}, \tag{2.9}$$

with the same sort of Banach space structure as arose in (1.4) and (2.6). We have continuous injections

$$[X,Y]_\theta \hookrightarrow [X,Y]_{\theta;X} \hookrightarrow [X,Y]^b_\theta. \tag{2.10}$$

To conclude this section, we extend the standard result on operator interpolation from the setting of $[X,Y]_\theta$ to that of $[X,Y]_{\theta;V}$ and $[X,Y]^b_\theta$.

Proposition 2.1. *Let* (X_j, Y_j, V_j) *be compatible triples,* $j = 1, 2$. *Assume that* $T : V_1 \to V_2$ *is continuous and that*

$$T : X_1 \longrightarrow X_2, \quad T : Y_1 \longrightarrow Y_2, \tag{2.11}$$

continuously. (Continuity is automatic, by the closed graph theorem.) Then, for each $\theta \in (0, 1)$,

$$T : [X_1, Y_1]_{\theta;V_1} \longrightarrow [X_2, Y_2]_{\theta;V_2}. \tag{2.12}$$

Furthermore, if $Y_j \subset X_j$ *(continuously) and* T *is a continuous linear map satisfying (2.11), then for each* $\theta \in (0, 1)$,

$$T : [X_1, Y_1]^b_\theta \longrightarrow [X_2, Y_2]^b_\theta. \tag{2.13}$$

Proof. Given $f \in [X_1, Y_2]_{\theta;V}$, pick $u \in \mathcal{H}_\Omega(X_1, Y_1, V_1)$ such that $f = u(\theta)$. Then we have

$$\mathcal{T} : \mathcal{H}_\Omega(X_1, Y_1, V_1) \to \mathcal{H}_\Omega(X_2, Y_2, V_2), \quad (\mathcal{T}u)(z) = Tu(z), \tag{2.14}$$

and hence

$$Tf = (\mathcal{T}u)(\theta) \in [X_2, Y_2]_{\theta;V_2}. \tag{2.15}$$

This proves (2.12). The proof of (2.13) is similar.

Remark. In case $V = X + Y$, with the weak topology, $[X, Y]_{\theta;V}$ is what is denoted $(X, Y)^w_\theta$ in [JJ], and called the weak complex interpolation space. It is stated in [JJ] that $[X, Y]_\theta = (X, Y)^w_\theta$ whenever X is separable, and also that there is equality whenever $Y \subset X$. In light of this, we will see in §§4–5 examples where $(X, Y)^w_\theta \neq [X, Y]^b_\theta$.

3. L^p-Sobolev spaces

Fix $p \in (1, \infty)$ and $s \in (0, \infty)$. For present purposes, we characterize the L^p-Sobolev spaces $H^{s,p}(\mathbb{R}^n)$ as

$$H^{s,p}(\mathbb{R}^n) = \Lambda^{-s} L^p(\mathbb{R}^n), \quad \Lambda = (I - \Delta)^{1/2} \tag{3.1}$$

where Δ is the Laplace operator. In this section we show that when $\theta \in (0, 1)$,

$$[L^p(\mathbb{R}^n), H^{s,p}(\mathbb{R}^n)]_{\theta;V} = H^{\theta s,p}(\mathbb{R}^n), \tag{3.2}$$

for a broad range of spaces V, including

$$L^p(\mathbb{R}^n), \quad \mathcal{S}'(\mathbb{R}^n), \quad \mathcal{D}'(\mathbb{R}^n), \tag{3.3}$$

with either the strong or weak* topology. We will also show that

$$[L^p(\mathbb{R}^n), H^{s,p}(\mathbb{R}^n)]^b_\theta = H^{\theta s,p}(\mathbb{R}^n). \tag{3.4}$$

The proofs of these results are variants of the classical result

$$[L^p(\mathbb{R}^n), H^{s,p}(\mathbb{R}^n)]_\theta = H^{\theta s,p}(\mathbb{R}^n), \tag{3.5}$$

Our goal in this section is largely to provide a perspective for the following sections.

First we show the left side of (3.2) is contained in the right side. As a preliminary, we note via Alaoglu's theorem that if $(f_\alpha : \alpha \in \mathcal{A})$ is any bounded family in $L^p(\mathbb{R}^n)$ indexed by a directed set $\mathcal{A}$ and $f_\alpha \to f$ in any of the spaces mentioned above, then $f_\alpha \to f$ weak* in L^p, so each space $\mathcal{H}_\Omega(L^p, H^{s,p}, V)$ is equal to (or, if $V = L^p$ with the strong topology, contained in) $\mathcal{H}_\Omega(L^p, H^{s,p}, V_0)$, where $V_0 = L^p(\mathbb{R}^n)$ with the weak* topology. So it suffices to consider $f \in [L^p, H^{s,p}]_{\theta,V_0}$ and show that $f \in H^{\theta s,p}(\mathbb{R}^n)$. We have $f = u(\theta)$ for some $u \in \mathcal{H}_\Omega(L^p, H^{s,p}, V_0)$. For $\varepsilon > 0$, look at

$$v_\varepsilon(z) = e^{z^2} e^{-\varepsilon\Lambda} \Lambda^{sz} u(z). \tag{3.6}$$

Bounds of the type (2.2) yield

$$\|v_\varepsilon(iy)\|_{L^p}, \ \|v_\varepsilon(1 + iy)\|_{L^p} \le C, \tag{3.7}$$

with C independent of y and of ε. Then, for each $g \in L^{p'}(\mathbb{R}^n)$, the function $\langle v_\varepsilon(z), g\rangle$ is continuous on $\overline{\Omega}$, holomorphic on Ω, and vanishes at infinity, so the maximum principle gives

$$|\langle v_\varepsilon(\theta), g\rangle| \le C\|g\|_{L^{p'}}, \tag{3.8}$$

with C independent of ε, hence

$$\|e^{-\varepsilon\Lambda} \Lambda^{s\theta} f\|_{L^p} \le C, \tag{3.9}$$

independent of $\varepsilon > 0$. Taking $\varepsilon \searrow 0$ yields $\Lambda^{s\theta} f \in L^p(\mathbb{R}^n)$, hence $f \in H^{s,p}(\mathbb{R}^n)$.

Second we show that the right side of (3.2) is contained in the left side. It sufices to treat $V = L^p(\mathbb{R}^n)$, with the strong topology. This is close to the standard argument proving (3.5), and we recall this argument. If $f \in H^{\theta s,p}(\mathbb{R}^n)$, take

$$u(z) = e^{z^2} \Lambda^{(\theta-z)s} f. \tag{3.10}$$

Then $u(\theta) = e^{\theta^2} f$, and it remains to verify that

$$u \in \mathcal{H}_\Omega(L^p, H^{s,p}, L^p). \tag{3.11}$$

Note that

$$u(x+iy) = e^{(x+iy)^2} \Lambda^{-xs} \Lambda^{-iys} (\Lambda^{\theta s} f). \tag{3.12}$$

This yields the bounds of the form (2.2). The continuity (2.3) in this setting is a consequence of the fact that

$$f \in L^p(\mathbb{R}^n) \Longrightarrow \lim_{x \searrow 0, y \to 0} \Lambda^{-(x+iy)} f = f, \quad \text{in} \quad L^p\text{-norm}. \tag{3.13}$$

This in turn is a consequence of the fact that $\{\Lambda^{-(x+iy)} : 0 < x \le 1, -1 \le y \le 1\}$ has uniformly bounded L^p-operator norm, that (as is easily verified)

$$f \in \mathcal{S}(\mathbb{R}^n) \Longrightarrow \lim_{x \searrow 0, y \to 0} \Lambda^{-(x+iy)} f = f, \quad \text{in} \quad L^p\text{-norm}, \tag{3.14}$$

and that

$$\mathcal{S}(\mathbb{R}^n) \ \text{ is dense in } \ L^p(\mathbb{R}^n). \tag{3.15}$$

In fact, we have all the ingredients to conclude that $u \in \mathcal{H}_\Omega(L^p, H^{s,p})$, so (3.4) is simultaneously proven.

We turn to the proof of (3.4). Since we have (2.10) and (3.5), it remains only to consider $u \in \mathcal{H}^b_\Omega(L^p, H^{s,p})$ and show that $u(\theta) \in H^{\theta s,p}(\mathbb{R}^n)$. To this end, consider v_ε, defined by (3.6). We have $v_\varepsilon : \widetilde{\Omega} \to L^p(\mathbb{R}^n)$, continuous, bounded (with a bound that might depend on ε) for each $\varepsilon \in (0,1]$, and

$$\|v_\varepsilon(1+iy)\|_{L^p} \le C, \tag{3.16}$$

with C independent of $y \in \mathbb{R}$ and of $\varepsilon \in (0,1]$. Also

$$\|v_\varepsilon(\varepsilon+iy)\|_{L^p} \le e^{\varepsilon^2 - y^2} \|e^{-\varepsilon\Lambda} \Lambda^{\varepsilon s} \Lambda^{isy} u(\varepsilon+iy)\|_{L^p}. \tag{3.17}$$

By hypothesis, $\|u(\varepsilon+iy)\|_{L^p} \le C$, independent of y and ε. The L^p-operator norm of Λ^{isy} has an exponential bound in y, which is beaten out by the factor e^{-y^2}. Furthermore,

$$e^{-\varepsilon\Lambda} \Lambda^{\varepsilon s} = e^{-\varepsilon(\Lambda - s\log\Lambda)}, \tag{3.18}$$

and (with $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$),

$$\left\{ e^{-\varepsilon(\langle\xi\rangle - s\log\langle\xi\rangle)} : \varepsilon \in (0,1] \right\} \quad \text{is bounded in} \quad S^0_{1,0}, \tag{3.19}$$

so the family of operators in (3.18) has L^p-opertor norm bounded independent of $\varepsilon \in (0,1]$. Thus

$$\|v_\varepsilon(\varepsilon + iy)\|_{L^p} \le C, \tag{3.20}$$

independent of $y \in \mathbb{R}$ and of $\varepsilon \in (0,1]$. Thus, given $\theta \in (0,1)$, we can apply the maximum principle to obtain

$$\{e^{-\varepsilon\Lambda}\Lambda^{s\theta}u(\theta) : 0 < \varepsilon < \theta\} \quad \text{bounded in} \quad L^p(\mathbb{R}^n). \tag{3.21}$$

This in turn implies $u(\theta) \in H^{\theta s,p}(\mathbb{R}^n)$, and we have (3.4).

For spaces considered in the next two sections, the first part of the argument above goes through, but the analogue of (3.13) fails, basically because the analogue of (3.15) fails. For this reason, $[X,Y]_{\theta;V}$, for various spaces V with a weaker topology than $X+Y$, and also $[X,Y]^b_\theta$, are of special interest.

4. Zygmund spaces

We begin by working on the n-torus $\mathbb{T}^n$. We will define the Zygmund space $C^r_*(\mathbb{T}^n)$, for $r \in \mathbb{R}$, as follows. Take $\varphi \in C^\infty_0(\mathbb{R}^n)$, radial, satisfying $\varphi(\xi) = 1$ for $|\xi| \le 1$. Set $\varphi_k(\xi) = \varphi(2^{-k}\xi)$. Then set $\psi_0 = \varphi$, $\psi_k = \varphi_k - \varphi_{k-1}$ for $k \in \mathbb{N}$, so $\{\psi_k : k \ge 0\}$ is a Littlewood-Paley partition of unity. We define $C^r_*(\mathbb{T}^n)$ to consist of $f \in \mathcal{D}'(\mathbb{T}^n)$ such that

$$\|f\|_{C^r_*} = \sup_{k\ge 0} 2^{kr}\|\psi_k(D)f\|_{L^\infty} < \infty. \tag{4.1}$$

With $\Lambda = (I-\Delta)^{1/2}$ and $s,t \in \mathbb{R}$, we have

$$\Lambda^{s+it} : C^r_*(\mathbb{T}^n) \longrightarrow C^{r-s}_*(\mathbb{T}^n). \tag{4.2}$$

It is classical that

$$r \in \mathbb{R}^+ \setminus \mathbb{Z}^+ \Longrightarrow C^r_*(\mathbb{T}^n) = C^r(\mathbb{T}^n), \tag{4.3}$$

where, if $r = k + \alpha$ with $k \in \mathbb{Z}^+$ and $0 < \alpha < 1$, $C^r(\mathbb{T}^n)$ consists of functions whose derivatives of order $\le k$ are Hölder continuous of exponent α.

We aim to show that if $r < s < t$ and $0 < \theta < 1$, then

$$[C^s_*(\mathbb{T}^n), C^t_*(\mathbb{T}^n)]_{\theta;C^r_*(\mathbb{T}^n)} = C^{(1-\theta)s+\theta t}_*(\mathbb{T}^n), \tag{4.4}$$

and

$$[C^s_*(\mathbb{T}^n), C^t_*(\mathbb{T}^n)]^b_\theta = C^{(1-\theta)s+\theta t}_*(\mathbb{T}^n). \tag{4.5}$$

First, suppose $f \in [C^s_*, C^t_*]_{\theta;C^r_*}$, so $f = u(\theta)$ for some $u \in \mathcal{H}_\Omega(C^s_*, C^t_*, C^r_*)$. Then consider

$$v(z) = e^{z^2} \Lambda^{(t-s)z} \Lambda^s u(z). \tag{4.6}$$

Bounds of the type (2.2) yield

$$\|v(iy)\|_{C^0_*}, \ \|v(1+iy)\|_{C^0_*} \le C,$$

with C independent of $y \in \mathbb{R}$. In other words,

$$\|\psi_k(D)v(z)\|_{L^\infty} \le C, \quad \operatorname{Re} z = 0, 1, \tag{4.7}$$

with C independent of Im z and k. Also, for each $k \in \mathbb{Z}^+$, $\psi_k(D)v : \overline{\Omega} \to L^\infty(\mathbb{T}^n)$ continuously, so the maximum principle implies

$$\|\psi_k(D)\Lambda^{(t-s)\theta}\Lambda^s f\|_{L^\infty} \le C, \tag{4.8}$$

independent of $k \in \mathbb{Z}^+$. This gives $\Lambda^{(1-\theta)s+\theta t} f \in C^0_*$, hence $f \in C^{(1-\theta)s+\theta t}_*(\mathbb{T}^n)$.

Second, suppose $f \in C^{(1-\theta)s+\theta t}_*(\mathbb{T}^n)$. Set

$$u(z) = e^{z^2} \Lambda^{(\theta - z)(t-s)} f. \tag{4.9}$$

Then $u(\theta) = e^{\theta^2} f$. We claim that

$$u \in \mathcal{H}_\Omega(C^s_*, C^t_*, C^r_*), \tag{4.10}$$

as long as $r < s < t$. Once we establish this, we will have the reverse containment in (4.4). Bounds of the form

$$\|u(z)\|_{C^s_*} \le C, \quad \|u(1+iy)\|_{C^t_*} \le C \tag{4.11}$$

are straightforward, and are more than adequate versions of (2.2). It remains to establish that

$$u : \overline{\Omega} \longrightarrow C^r_*(\mathbb{T}^n), \ \text{ continuously.} \tag{4.12}$$

Indeed, we know $u : \overline{\Omega} \to C^s_*(\mathbb{T}^n)$ is bounded. It is readily verified that

$$u : \overline{\Omega} \longrightarrow \mathcal{D}'(\mathbb{T}^n), \ \text{ continuously,} \tag{4.13}$$

and that

$$r < s \Longrightarrow C^s_*(\mathbb{T}^n) \hookrightarrow C^r_*(\mathbb{T}^n) \ \text{ is compact.} \tag{4.14}$$

The result (4.12) follows from these observations. Thus the proof of (4.4) is complete.

We turn to the proof of (4.5). If $u \in \mathcal{H}^b_\Omega(C^s_*, C^t_*)$, form $v(z)$ as in (4.6), and for $\varepsilon \in (0,1]$ set

$$v_\varepsilon(z) = e^{-\varepsilon\Lambda} v(z), \quad v_\varepsilon : \widetilde{\Omega} \to C^0_*(\mathbb{T}^n) \text{ bounded and continuous} \tag{4.15}$$

(with bound that might depend on ε). We have

$$\psi_k(D) v_\varepsilon(\varepsilon + iy) = e^{(\varepsilon+iy)^2} \psi_k(D) e^{-\varepsilon\Lambda} \Lambda^{(t-s)\varepsilon} \Lambda^{i(t-s)y} \Lambda^s u(z). \tag{4.16}$$

Now $\{\Lambda^s u(z) : z \in \widetilde{\Omega}\}$ is bounded in $C^0_*(\mathbb{T}^n)$, and the operator norm of $\Lambda^{i(t-s)y}$ on $C^0_*(\mathbb{T}^n)$ is exponentially bounded in $|y|$. As in (3.19), we have

$$\{e^{-\varepsilon\Lambda} \Lambda^{\varepsilon(t-s)} : 0 < \varepsilon \le 1\} \text{ bounded in } \mathrm{OPS}^0_{1,0}(\mathbb{T}^n), \tag{4.17}$$

hence bounded in operator norm on $C^0_*(\mathbb{T}^n)$. We deduce that

$$\|\psi_k(D) v_\varepsilon(\varepsilon + iy)\|_{L^\infty} \le C, \tag{4.18}$$

independent of $y \in \mathbb{R}$ and $\varepsilon \in (0,1]$. The hypothesis on u also implies

$$\|\psi_k(D) v_\varepsilon(1 + iy)\|_{L^\infty} \le C, \tag{4.19}$$

independent of $y \in \mathbb{R}$ and $\varepsilon \in (0,1]$. Now the maximum principle applies. Given $\theta \in (0,1)$,

$$\|\psi_k(D) e^{-\varepsilon\Lambda} v(\theta)\|_{L^\infty} \le C, \tag{4.20}$$

independent of ε. Taking $\varepsilon \searrow 0$ yields $v(\theta) \in C^0_*(\mathbb{T}^n)$, hence $u(\varepsilon) \in C^{(1-\theta)s+\theta t}_*(\mathbb{T}^n)$.

This proves one inclusion in (4.5). The proof of the reverse inclusion is similar to that for (4.4). Given $f \in C^{(1-\theta)s+\theta t}_*(\mathbb{T}^n)$, take $u(z)$ as in (4.9). The claim is that $u \in \mathcal{H}^b_\Omega(C^s_*, C^t_*)$. We already have (4.11), and the only thing that remains is to check that

$$u : \widetilde{\Omega} \longrightarrow C^s_*(\mathbb{T}^n) \text{ continuously,} \tag{4.21}$$

and this is straightforward. (What fails is continuity of $u : \overline{\Omega} \to C^s_*(\mathbb{T}^n)$ at the left boundary of $\overline{\Omega}$.)

If $\mathrm{OPS}^m_{1,0}(\mathbb{T}^n)$ denotes the class of pseudodifferential operators on $\mathbb{T}^n$ with symbols in $S^m_{1,0}$, then for all $s, m \in \mathbb{R}$,

$$P \in \mathrm{OPS}^m_{1,0}(\mathbb{T}^n) \Longrightarrow P : C^s_*(\mathbb{T}^n) \to C^{s-m}_*(\mathbb{T}^n). \tag{4.22}$$

Cf. [T2], Chapter 13, Proposition 8.6. Using coordinate invariance of $OPS^m_{1,0}$ and of $C^r(\mathbb{T}^n)$ for $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, we deduce invariance of $C^s_*(\mathbb{T}^n)$ under diffeomorphisms, for all $s \in \mathbb{R}$.

If M is a smooth compact manifold (without boundary) we have a natural definition of Zygmund spaces $C^r_*(M)$ for $s \in \mathbb{R}$, using a partition of unity and local coordinate charts (identifying patches on M with patches on $\mathbb{T}^n$), making use of the material given above. We have

$$r \in \mathbb{R}^+ \setminus \mathbb{Z}^+ \Longrightarrow C^r_*(M) = C^r(M), \tag{4.23}$$

and, for all $s, m \in \mathbb{R}$,

$$P \in \mathrm{OPS}^m_{1,0}(M) \Longrightarrow P : C^s_*(M) \to C^{s-m}_*(M). \tag{4.24}$$

In particular, if M is given a smooth Riemannian metric and Laplace-Beltrami operator Δ, then, with $\Lambda = (I - \Delta)^{1/2}$, for each $r, s \in \mathbb{R}$,

$$C^r_*(M) = \Lambda^{s-r} C^s_*(M). \tag{4.25}$$

Furthermore, a standard argument using partitions of unity, local coordinate charts, and (4.4) shows that, if $r < s < t$ and $0 < \theta < 1$, then

$$[C^s_*(M), C^t_*(M)]_{\theta; C^r_*(M)} = C^{(1-\theta)s+\theta t}_*(M). \tag{4.26}$$

Also

$$[C^s_*(M), C^t_*(M)]^b_\theta = C^{(1-\theta)s+\theta t}_*(M). \tag{4.27}$$

Now let $\overline{\Omega}$ be a smooth compact manifold with boundary. Using the material above, we can develop a theory of Zygmund spaces $C^r_*(\overline{\Omega})$. The material below is a corrected version of the presentation in (8.37)–(8.42) of [T2], Chapter 13. As before, if $r = k + \alpha \in \mathbb{R}^+ \setminus \mathbb{Z}^+,\ k \in \mathbb{Z}^+,\ 0 < \alpha < 1$, let $C^r(\overline{\Omega})$ denote the space of functions whose derivatives of order $\le k$ are Hölder continuous of exponent α. Given $r \in (0, \infty)$, the idea is to define $C^r_*(\overline{\Omega})$ by interpolation:

$$C^r_*(\overline{\Omega}) = [C^{s_1}(\overline{\Omega}), C^{s_2}(\overline{\Omega})]_{\theta; C^{s_0}(\overline{\Omega})}, \tag{4.28}$$

where $0 < s_0 < s_1 < r < s_2,\ 0 < \theta < 1,\ r = (1-\theta)s_1 + \theta s_2$ (and $s_j \notin \mathbb{Z}$). It is necessary to show that this is independent of choices of such s_j. To see this, we make use of a classical construction, giving, for each $N \in \mathbb{Z}^+$, an extension operator

$$E : C^s(\overline{\Omega}) \longrightarrow C^s(M), \quad s \in (0, N) \setminus \mathbb{Z}, \tag{4.29}$$

providing a right inverse for the surjective restriction operator

$$\rho : C^s(M) \longrightarrow C^s(\overline{\Omega}). \tag{4.30}$$

Here M is a smooth compact manifold without boundary, containing $\overline{\Omega}$, i.e., the double of $\overline{\Omega}$. We have from (4.23)–(4.26) that

$$C^r_*(M) = [C^{s_1}(M), C^{s_2}(M)]_{\theta; C^{s_0}_*(M)}, \tag{4.31}$$

where r, s_j and θ are as in (4.28). Applying Proposition 2.1, we have

$$E : C^r_*(\overline{\Omega}) \to C^r_*(M), \quad \rho : C^r_*(M) \to C^r_*(\overline{\Omega}), \tag{4.32}$$

if $C^r_*(\overline{\Omega})$ is given by (4.28). One also has $\rho E = I$ on $C^r_*(\overline{\Omega})$. Hence

$$C^r_*(\overline{\Omega}) \approx C^r_*(M)/\{u \in C^r_*(M) : u|_\Omega = 0\}. \tag{4.33}$$

This characterization is clearly independent of the choices made in (4.28). Note also that the right side of (4.33) is meaningful even for $r \le 0$. One can also check that setting

$$C^r_*(\overline{\Omega}) = [C^{s_1}(\overline{\Omega}), C^{s_2}(\overline{\Omega})]^b_\theta \tag{4.34}$$

also leads to (4.33), and hence agrees with (4.28).

As we have seen, $C^r_*(M) = C^r(M)$ for $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, so we deduce that

$$C^r_*(\overline{\Omega}) = C^r(\overline{\Omega}), \quad \text{for} \;\; r \in \mathbb{R}^+ \setminus \mathbb{Z}^+. \tag{4.35}$$

Using the spaces $C^r_*(\overline{\Omega})$, we can fill in the gaps (at $r \in \mathbb{Z}^+$) in various Schauder-type regularity results. The following is an illustrative example. Let $\overline{\Omega}$ have a smooth Riemannian metric and Laplace-Beltrami operator Δ. Then the Dirichlet problem

$$\Delta u = 0, \quad u\big|_{\partial\Omega} = f, \tag{4.36}$$

has a unique solution,

$$u = \mathrm{PI}\, f, \quad \mathrm{PI} : C(\partial\Omega) \to C(\overline{\Omega}). \tag{4.37}$$

The classical Schauder regularity results give

$$\mathrm{PI} : C^s(\partial\Omega) \longrightarrow C^s(\overline{\Omega}), \quad s \in \mathbb{R}^+ \setminus \mathbb{Z}^+. \tag{4.38}$$

The extension is:

Proposition 4.1. *Given $r > 0$,*

$$\text{PI} : C^r_*(\partial\Omega) \longrightarrow C^r_*(\overline{\Omega}). \tag{4.39}$$

Proof. Pick $s_j \in \mathbb{R}^+ \setminus \mathbb{Z}^+$, $0 < s_0 < s_1 < r < s_2$, and $\theta \in (0,1)$ such that $r = (1-\theta)s_1 + \theta s_2$. By (4.26) we have

$$C^r_*(\partial\Omega) = [C^{s_1}(\partial\Omega), C^{s_2}(\partial\Omega)]_{\theta;C^{s_0}(\partial\Omega)},$$

and we also have (4.28). Hence (4.39) follows from (4.38) (with $s = s_j$) and Proposition 2.1. Instead of using (4.26) and (4.28), we could use (4.27) and (4.34).

Remark. One can also use real interpolation to develop material on the spaces $C^r_*(\overline{\Omega})$. Cf. [Tri], §4.3.4. However, it was our goal here to show that a version of complex interpolation does the job.

5. Hardy and bmo-Sobolev spaces

Again we start with functions on the n-dimensional torus $\mathbb{T}^n$. One way to define the Hardy space $\mathfrak{h}^1(\mathbb{T}^n)$ is via a Littlewood-Paley characterization:

$$\|f\|_{\mathfrak{h}^1} = \Big\| \Big\{ \sum_{k\ge 0} |\psi_k(D) f|^2 \Big\}^{1/2} \Big\|_{L^1}, \tag{5.1}$$

where $\psi_k(D)$ are as in (4.1). There is an equivalent characterization in terms of an atomic decomposition, which provides a convenient path to show that $\mathfrak{h}^1(\mathbb{T}^n)$ is invariant under $C^{1+\delta}$ diffeomorphisms of $\mathbb{T}^n$, given $\delta > 0$. It is well known that

$$P \in \text{OPS}^0_{1,0}(\mathbb{T}^n) \Longrightarrow P : \mathfrak{h}^1(\mathbb{T}^n) \to \mathfrak{h}^1(\mathbb{T}^n). \tag{5.2}$$

Furthermore, $\mathfrak{h}^1(\mathbb{T}^n)$ is a module over the algebra of Hölder continuous functions on $\mathbb{T}^n$, under pointwise multiplication. Another characterization of $\mathfrak{h}^1(\mathbb{T}^n)$ (cf. [G]) is that

$$\|f\|_{\mathfrak{h}^1} \approx \|f\|_{L^1} + \sum_{k=1}^n \|R_k f\|_{L^1}, \tag{5.3}$$

where $R_k = r_k(D)$, $r_k(\xi) = \xi_k |\xi|^{-1}$ on $\mathbb{Z}^n \setminus 0$. Using (5.3) one sees that if $\varphi \in C^\infty_0(\mathbb{R}^n)$, $\varphi(\xi) = 1$ for $|\xi| \le 1$, then

$$f \in \mathfrak{h}^1(\mathbb{T}^n) \Longrightarrow \lim_{\varepsilon\to 0} \|f - \varphi(\varepsilon D) f\|_{\mathfrak{h}^1} = 0, \tag{5.4}$$

hence

$$C^\infty(\mathbb{T}^n) \ \text{ is dense in } \ \mathfrak{h}^1(\mathbb{T}^n). \tag{5.5}$$

The space $\mathrm{bmo}(\mathbb{T}^n)$ consists of functions $f \in L^1(\mathbb{T}^n)$ such that

$$\|f\|_* = \sup_R \frac{1}{m(R)} \int |f - f_R|\, dx < \infty, \tag{5.6}$$

where R ranges over the set of n-cubes in $\mathbb{T}^n$, $m(R)$ is the volume of R, and $f_R = m(R)^{-1} \int f\, dx$. One sets $\|f\|_{\mathrm{bmo}} = \|f\|_{L^1} + \|f\|_*$. A central result from [FS] (with complements in [D]) is that

$$\mathfrak{h}^1(\mathbb{T}^n)^* = \mathrm{bmo}(\mathbb{T}^n). \tag{5.7}$$

It follows that

$$P \in \mathrm{OPS}^0_{1,0}(\mathbb{T}^n) \Longrightarrow P : \mathrm{bmo}(\mathbb{T}^n) \to \mathrm{bmo}(\mathbb{T}^n), \tag{5.8}$$

and that $\mathrm{bmo}(\mathbb{T}^n)$ is invariant under $C^{1+\delta}$ diffeomorphisms of $\mathbb{T}^n$. The space $C^\infty(\mathbb{T}^n)$ is not dense in $\mathrm{bmo}(\mathbb{T}^n)$; rather the space

$$\mathrm{vmo}(\mathbb{T}^n) = \ \text{closure of } \ C^\infty(\mathbb{T}^n) \ \text{ in } \ \mathrm{bmo}(\mathbb{T}^n) \tag{5.9}$$

is of interest. An important counterpart to (5.7) is that

$$\mathrm{vmo}(\mathbb{T}^n)^* = \mathfrak{h}^1(\mathbb{T}^n). \tag{5.10}$$

We define Hardy and bmo-Sobolev spaces as follows. As before, set $\Lambda = (I - \Delta)^{1/2}$, where Δ is the Laplace operator on $\mathbb{T}^n$. Then for $s \in \mathbb{R}$ we set

$$\begin{aligned} \mathfrak{h}^{s,1}(\mathbb{T}^n) &= \Lambda^{-s}\mathfrak{h}^1(\mathbb{T}^n), \\ \mathrm{bmo}^s(\mathbb{T}^n) &= \Lambda^{-s}\mathrm{bmo}(\mathbb{T}^n). \end{aligned} \tag{5.11}$$

In light of (5.9), it is also useful to consider

$$\mathrm{vmo}^s(\mathbb{T}^n) = \Lambda^{-s}\mathrm{vmo}(\mathbb{T}^n). \tag{5.12}$$

We note that $s < t \Rightarrow \mathrm{bmo}^t(\mathbb{T}^n) \subset \mathrm{vmo}^s(\mathbb{T}^n)$.

Classical complex interpolation applies smoothly to the spaces $\mathfrak{h}^{s,1}(\mathbb{T}^n)$, yielding

$$[\mathfrak{h}^{s,1}(\mathbb{T}^n), \mathfrak{h}^{t,1}(\mathbb{T}^n)]_\theta = \mathfrak{h}^{(1-\theta)s+\theta t,1}(\mathbb{T}^n), \tag{5.13}$$

for $\theta \in (0,1)$, $s, t \in \mathbb{R}$. The proof works just as for L^p-Sobolev spaces in §3, with (5.5) taking the place of (3.15). For bmo-Sobolev spaces, the situation is different. The next two propositions provide a correction to Proposition 3.1(d) of [Str].

Proposition 5.1. *Given $s < t$ and $0 < \theta < 1$,*

$$[\mathrm{bmo}^s(\mathbb{T}^n), \mathrm{bmo}^t(\mathbb{T}^n)]_\theta = \mathrm{vmo}^{(1-\theta)s+\theta t}(\mathbb{T}^n). \tag{5.14}$$

Proof. First we show that the right side of (5.14) is contained in the left side. For starters, take $f \in \mathrm{bmo}^{(1-\theta)s+\theta t}(\mathbb{T}^n)$. (We will specialize shortly.) Then, as in (4.9), consider

$$u(z) = e^{z^2} \Lambda^{(\theta-z)(t-s)} f. \tag{5.15}$$

Then $u(\theta) = e^{\theta^2} f$. We check whether $u \in \mathcal{H}_\Omega(\mathrm{bmo}^s, \mathrm{bmo}^t)$. Bounds of the form

$$\|u(z)\|_{\mathrm{bmo}^s} \le C, \quad \|u(1+iy)\|_{\mathrm{bmo}^t} \le C \tag{5.16}$$

are straightforward. It remains to show that

$$u : \overline{\Omega} \longrightarrow \mathrm{bmo}^s(\mathbb{T}^n), \quad \text{continuously}, \tag{5.17}$$

and also $u : \{\mathrm{Re}\, z = 1\} \to \mathrm{bmo}^t(\mathbb{T}^n)$, continuously. This comes down to showing that

$$\lim_{x \searrow 0} \Lambda^{-x} g = g \quad \text{and} \quad \lim_{y \to 0} \Lambda^{iy} g = g, \quad \text{in bmo-norm}, \tag{5.18}$$

given $g = \Lambda^{(1-\theta)s+\theta t} f$. Our current hypothesis on f implies $g \in \mathrm{bmo}(\mathbb{T}^n)$. However, (5.18) does not hold for every $g \in \mathrm{bmo}$. For example, for each $x > 0$, $\Lambda^{-x} g \in \mathrm{vmo}$. We do have (5.18) whenever $g \in \mathrm{vmo}$. This gives one containment in (5.14).

Next we establish the reverse inclusion in (5.14). Suppose $f = u(\theta)$ for some $u \in \mathcal{H}_\Omega(\mathrm{bmo}^s, \mathrm{bmo}^t)$, and consider

$$v(z) = e^{z^2} \Lambda^{(t-s)z} \Lambda^s u(z). \tag{5.19}$$

Bounds of the form (2.2) yield

$$\|v(iy)\|_{\mathrm{bmo}}, \ \|v(1+iy)\|_{\mathrm{bmo}} \le C, \tag{5.20}$$

with C independent of $y \in \mathbb{R}$. The maximum principle then gives $\|e^{-\varepsilon\Lambda} v(z)\|_{\mathrm{bmo}} \le C$, for each $\varepsilon > 0$, with C independent of $z \in \overline{\Omega}$, and of ε. Taking $\varepsilon \searrow 0$ then gives $v(z) \in \mathrm{bmo}$ for each $z \in \overline{\Omega}$ and

$$\|v(z)\|_{\mathrm{bmo}} \le C. \tag{5.21}$$

This implies $u(\theta) = e^{-\theta^2} \Lambda^{(s-t)\theta} \Lambda^{-s} v(\theta) \in \mathrm{bmo}^{(1-\theta)s+\theta t}(\mathbb{T}^n)$. It remains to show that actually $v(\theta) \in \mathrm{vmo}(\mathbb{T}^n)$.

Indeed, since $u \in \mathcal{H}_\Omega(\mathrm{bmo}^s, \mathrm{bmo}^t)$ and $s < t$, we have $u(x+iy) \to u(iy)$ in bmo^s-norm as $x \searrow 0$, and by the results established above, $u(x+iy) \in \mathrm{bmo}^{(1-x)s+xt}(\mathbb{T}^n)$. This implies $u(iy) \in \mathrm{vmo}^s(\mathbb{T}^n)$, for each $y \in \mathbb{R}$. In other words, actually

$$u : \overline{\Omega} \longrightarrow \mathrm{vmo}^s(\mathbb{T}^n), \quad \text{continuously,} \tag{5.22}$$

as well as $u : \{\mathrm{Re}\, z = 1\} \to \mathrm{bmo}^t(\mathbb{T}^n)$, boundedly and continuously. Thus we have

$$\|(I - e^{-\varepsilon\Lambda})v(z)\|_{\mathrm{bmo}} \le C, \quad \forall\, z \in \overline{\Omega},\ \varepsilon \in (0,1], \tag{5.23}$$

and

$$\|(I - e^{-\varepsilon\Lambda})v(iy)\|_{\mathrm{bmo}} \le \delta(\varepsilon) \to 0 \ \text{ as } \ \varepsilon \to 0. \tag{5.24}$$

Then Hadamard's three lines lemma applies, to give

$$\|(I - e^{-\varepsilon\Lambda})v(\theta)\|_{\mathrm{bmo}} \le C^\theta \delta(\varepsilon)^{1-\theta} \to 0, \quad \text{as } \ \varepsilon \to 0. \tag{5.25}$$

This implies that $v(\theta) \in \mathrm{vmo}(\mathbb{T}^n)$, hence $u(\theta) \in \mathrm{vmo}^{(1-\theta)s+\theta t}(\mathbb{T}^n)$. The proof of Proposition 5.1 is complete.

The following result complements Proposition 5.1.

Proposition 5.2. *Given $s < t$ and $0 < \theta < 1$,*

$$[\mathrm{bmo}^s(\mathbb{T}^n), \mathrm{bmo}^t(\mathbb{T}^n)]_{\theta;V} = \mathrm{bmo}^{(1-\theta)s+\theta t}(\mathbb{T}^n), \tag{5.26}$$

when V is any one of the following spaces:

$$\begin{aligned} &\mathrm{bmo}^r(\mathbb{T}^n), && r < s, \\ &H^{r,p}(\mathbb{T}^n), && p < \infty,\ r \le s, \\ &\mathrm{bmo}^s(\mathbb{T}^n), && \textit{with the weak}^* \textit{ topology.} \end{aligned} \tag{5.27}$$

Furthermore,

$$[\mathrm{bmo}^s(\mathbb{T}^n), \mathrm{bmo}^t(\mathbb{T}^n)]^b_\theta = \mathrm{bmo}^{(1-\theta)s+\theta t}(\mathbb{T}^n). \tag{5.28}$$

Proof. The argument is parallel to the proof of Proposition 5.1. To show the right side of (5.26) is contained in the left side, take $f \in \mathrm{bmo}^{(1-\theta)s+\theta t}(\mathbb{T}^n)$ and form $u(z)$ as in (5.15). The estimates (5.16) arise as before, and it remains to show that

$$u : \overline{\Omega} \longrightarrow V, \quad \text{continuously,} \tag{5.29}$$

for each space V listed in (5.27). For $V = \mathrm{bmo}^r,\ r < s$, (5.29) holds whenever $f \in \mathrm{vmo}^{(1-\theta)r+\theta t}(\mathbb{T}^n) \supset \mathrm{bmo}^{(1-\theta)s+\theta t}(\mathbb{T}^n)$. For $V = H^{r,p},\ p \in (1,\infty),\ r \le s$, (5.29) holds whenever $f \in H^{(1-\theta)s+\theta t,p}(\mathbb{T}^n) \supset \mathrm{bmo}^{(1-\theta)s+\theta t}(\mathbb{T}^n)$. For $V = \mathrm{bmo}^s(\mathbb{T}^n)$ with the weak* topology, we are claiming that

$$g \in \mathfrak{h}^{-s,1}(\mathbb{T}^n),\ z_k \to z \in \overline{\Omega} \Longrightarrow \langle u(z_k), g\rangle \to \langle u(z), g\rangle. \tag{5.30}$$

We have $\{u(z_k)\}$ bounded in $\mathrm{bmo}^s(\mathbb{T}^n)$, and by the observations just made $u(z_k) \to u(z)$ in $H^{s,p}(\mathbb{T}^n)$, provided $p < \infty$. Hence (5.30) certainly holds for each $g \in C^\infty(\mathbb{T}^n)$. Since $C^\infty(\mathbb{T}^n)$ is dense in $\mathfrak{h}^{-s,1}(\mathbb{T}^n)$, by (5.5), we have (5.30).

To establish the reverse inclusion in (5.26), we suppose $f = u(\theta)$ for some $u \in \mathcal{H}_\Omega(\mathrm{bmo}^s, \mathrm{bmo}^t, V)$, with V as in (5.27). It suffices to consider $V = H^{r,p}(\mathbb{T}^n)$. Again we form $v(z)$ as in (5.19), and we obtain the estimates (5.20)–(5.21), and hence the desired result that $f \in \mathrm{bmo}^{(1-\theta)s+\theta t}(\mathbb{T}^n)$.

We turn to the proof of (5.28). First, if $f \in \mathrm{bmo}^{(1-\theta)s+\theta t}(\mathbb{T}^n)$, we again form $u(z)$ as in (5.15), and this time it remains to show that

$$u : \widetilde{\Omega} \longrightarrow \mathrm{bmo}^s(\mathbb{T}^n), \quad \text{continuously}, \tag{5.31}$$

which is straightforward (as in (4.21), what fails is the continuity of $u : \overline{\Omega} \to \mathrm{bmo}^s(\mathbb{T}^n)$ at the left boundary of $\overline{\Omega}$). To establish the reverse inclusion in (5.28), we argue as in (3.16) and (4.15). Namely, given $u \in \mathcal{H}^b_\Omega(\mathrm{bmo}^s, \mathrm{bmo}^t)$, we set

$$v_\varepsilon(z) = e^{-\varepsilon\Lambda} v(z), \tag{5.32}$$

with $v(z)$ given by (5.19). Thus for each $\varepsilon \in (0,1]$, $v_\varepsilon : \widetilde{\Omega} \to \mathrm{bmo}(\mathbb{T}^n)$ is continuous and bounded (with a bound that might depend on ε), we have

$$\|v_\varepsilon(1+iy)\|_{\mathrm{bmo}} \le C, \tag{5.33}$$

with C independent of $y \in \mathbb{R}$ and $\varepsilon \in (0,1]$. By the bound (4.17), $\{e^{-\varepsilon\Lambda}\Lambda^{(t-s)\varepsilon} : 0 < \varepsilon \le 1\}$ has uniformly bounded bmo-operator norm. Furthermore $\{\Lambda^{i(t-s)y} : y \in \mathbb{R}\}$ has bmo-operator norm bounded by $Ae^{B|y|}$. Since

$$v_\varepsilon(\varepsilon+iy) = e^{(\varepsilon+iy)^2} e^{-\varepsilon\Lambda}\Lambda^{(t-s)\varepsilon}\Lambda^{i(t-s)y}\Lambda^s u(z), \tag{5.34}$$

we have

$$\|v_\varepsilon(\varepsilon+iy)\|_{\mathrm{bmo}} \le C, \tag{5.35}$$

independent of $y \in \mathbb{R}$ and $\varepsilon \in (0,1]$. Hence the maximum principle gives for each $\theta \in (0,1)$,

$$\{e^{-\varepsilon\Lambda}\Lambda^{(t-s)\theta}\Lambda^s u(\theta) : 0 < \varepsilon < \theta\} \ \text{ bounded in } \ \mathrm{bmo}(\mathbb{T}^n), \tag{5.36}$$

and taking $\varepsilon \searrow 0$ then gives $\Lambda^{(t-s)\theta}\Lambda^s u(\theta) \in \mathrm{bmo}(\mathbb{T}^n)$, hence $u(\theta) \in \mathrm{bmo}^{(1-\theta)s+\theta t}(\mathbb{T}^n)$. This finishes the proof of (5.28).

If M is a compact smooth manifold, one can extend the results given above to $\mathrm{bmo}^s(M)$, etc., via arguments using partitions of unity and local coordinate charts. This works essentially as sketched in [Str], so we will not dwell on it here.

The next proposition illustrates the results established in this section. We give two proofs, the first using Proposition 5.2 and the second using Proposition 5.1. This result in turn is of use in work on commutator estimates in [T4].

Proposition 5.3. *Assume $s < t$ and $p \in (1, \infty)$. Suppose we have a continuous map*

$$T : \mathrm{bmo}^s(\mathbb{T}^n) \to H^{s,p}(\mathbb{T}^n), \quad T : \mathrm{bmo}^t(\mathbb{T}^n) \to H^{t,p}(\mathbb{T}^n). \tag{5.37}$$

Then, if $s < r < t$,

$$T : \mathrm{bmo}^r(\mathbb{T}^n) \to H^{r,p}(\mathbb{T}^n). \tag{5.38}$$

Proof. Take $\theta \in (0, 1)$ such that $r = (1 - \theta)s + \theta t$. Then (5.28) implies

$$[\mathrm{bmo}^s(\mathbb{T}^n), \mathrm{bmo}^t(\mathbb{T}^n)]^b_\theta = \mathrm{bmo}^r(\mathbb{T}^n). \tag{5.39}$$

We also have

$$[H^{s,p}(\mathbb{T}^n), H^{t,p}(\mathbb{T}^n)]^b_\theta = H^{r,p}(\mathbb{T}^n), \tag{5.40}$$

via the same sort of argument used for (3.4). Now the result (5.38) follows from Proposition 2.1.

For a second proof, we use the interpolation functor $[X, Y]_\theta$ and Proposition 5.1 to deduce from (5.37) that

$$T : \mathrm{vmo}^r(\mathbb{T}^n) \longrightarrow H^{r,p}(\mathbb{T}^n). \tag{5.41}$$

This result is weaker than (5.38), but we can recover the full strength of (5.38) as follows. Given $f \in \mathrm{bmo}^r(\mathbb{T}^n)$, we have

$$\begin{aligned} &\{e^{-\varepsilon\Lambda} f : 0 < \varepsilon \le 1\} \ \text{ bounded in } \ \mathrm{vmo}^r(\mathbb{T}^n), \\ &\lim_{\varepsilon \to 0} e^{-\varepsilon\Lambda} f = f \ \text{ in } \ \mathrm{bmo}^s\text{-norm.} \end{aligned} \tag{5.42}$$

Hence, given (5.37) plus (5.41), we have

$$\begin{aligned} &\{Te^{-\varepsilon\Lambda} f : 0 < \varepsilon \le 1\} \ \text{ bounded in } \ H^{r,p}(\mathbb{T}^n), \\ &\lim_{\varepsilon \to 0} Te^{-\varepsilon\Lambda} f = Tf \ \text{ in } \ H^{s,p}\text{-norm,} \end{aligned} \tag{5.43}$$

which then gives $Tf \in H^{r,p}(\mathbb{T}^n)$.

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