Jacobi's Generalization of Cramer's Formula

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If A is an invertible $n \times n$ matrix, Cramer's formula gives A^{-1} in terms of $\det A$ and the $(n-1) \times (n-1)$ minors of A (or better, of A^t). There is a generalization, due to Jacobi, relating the $k \times k$ minors of A^{-1} to the $(n-k) \times (n-k)$ minors of A^t and $\det A$, which we derive here.

We take an invariant point of view. Let V be an n-dimensional vector space, over a field $\mathbb{F}$ (typically $\mathbb{R}$ or $\mathbb{C}$), with dual V'. Let $A : V \to V$ be linear, with transpose $A^t : V' \to V'$. Take $k \in \{1, \dots, n-1\}$. We bring in the isomorphism

$$\kappa : \Lambda^k V \otimes \Lambda^n V' \xrightarrow{\approx} \Lambda^{n-k} V', \tag{1}$$

given by

$$\kappa(v_1 \wedge \cdots \wedge v_k \otimes \alpha)(w_1, \dots, w_{n-k}) = \alpha(w_1, \dots, w_{n-k}, v_1, \dots, v_k), \tag{2}$$

where an element of $\Lambda^k V'$ is viewed as a k-multilinear antisymmetric functional on V.

Remark. A choice of basis of V yields isomorphisms $V \approx V'$ and $\Lambda^n V' \approx \mathbb{F}$, and then κ becomes essentially the Hodge star operator.

We aim to prove the following.

Proposition. *If A is invertible, then*

$$(\det A)\, \Lambda^k A^{-1} \otimes I = \kappa^{-1} \circ \Lambda^{n-k} A^t \circ \kappa, \tag{3}$$

in $\operatorname{End}(\Lambda^k V \otimes \Lambda^n V')$.

Proof. Since

$$\Lambda^n A^t = (\det A) I \quad \text{in} \quad \operatorname{End}(\Lambda^n V'), \tag{4}$$

the desired identity (3) is equivalent to

$$(\Lambda^{n-k} A^t) \circ \kappa = \kappa \circ \big(\Lambda^k A^{-1} \otimes \Lambda^n A^t\big), \tag{5}$$

in $\operatorname{Hom}(\Lambda^k V \otimes \Lambda^n V', \Lambda^{n-k} V')$. Note that $\Lambda^{n-k} A^t \in \operatorname{End}(\Lambda^{n-k} V')$ is defined by

$$(\Lambda^{n-k} A^t)\beta(w_1, \dots, w_{n-k}) = \beta(Aw_1, \dots, Aw_{n-k}). \tag{6}$$

Hence, if we take $v_1 \wedge \cdots \wedge v_k \otimes \alpha \in \Lambda^k V \otimes \Lambda^n V'$, we get

$$\begin{aligned}
(\Lambda^{n-k} A^t)\kappa(v_1 \wedge \cdots \wedge v_k \otimes \alpha)(w_1, \ldots, w_{n-k}) \\
= \kappa(v_1 \wedge \cdots \wedge v_k \otimes \alpha)(Aw_1, \ldots, Aw_{n-k}) \\
= \alpha(Aw_1, \ldots, Aw_{n-k}, v_1, \ldots, v_k).
\end{aligned} \tag{7}$$

On the other hand, since

$$(\Lambda^k A^{-1} \otimes \Lambda^n A^t)(v_1 \wedge \cdots \wedge v_k \otimes \alpha) = (A^{-1}v_1 \wedge \cdots \wedge A^{-1}V_k) \otimes (\Lambda^n A^t \alpha), \tag{8}$$

we have

$$\begin{aligned}
&\kappa \circ (\Lambda^k A^{-1} \otimes \Lambda^n A^t)(v_1 \wedge \cdots \wedge v_k \otimes \alpha)(w_1, \ldots, w_{n-k}) \\
&\quad = \kappa(A^{-1}v_1 \wedge \cdots \wedge A^{-1}v_k \otimes \Lambda^n A^t \alpha)(w_1, \ldots, w_{n-k}) \\
&\quad = (\Lambda^n A^t \alpha)(w_1, \ldots, w_{n-k}, A^{-1}v_1, \ldots, A^{-1}v_k) \\
&\quad = \alpha(Aw_1, \ldots, Aw_{n-k}, v_1, \ldots, v_k),
\end{aligned} \tag{9}$$

which agrees with the right side of (7). This completes the proof.

Remark. For a classical approach (I'm not saying a digestible approach), see [G], pp. 21–22. For a further generalization, see [MB].

References

[G] F. Gantmacher, The Theory of Matrices, Vol. 1, Chelsea, 1959.
[MB] J. Miao and A. Ben-Israel, Minors of the Moore-Penrose inverse, Linear Algebra Appl. 195 (1993), 191–208.