Wave Decay on Manifolds with Bounded Ricci Tensor, and Related Estimates

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Abstract

We investigate energy decay for solutions to the wave equation $\partial_t^2 u + a(x)\partial_t u - \Delta u = 0$, with damping coefficient $a \geq 0$, where Δ is the Laplace-Beltrami operator on a compact Riemannian manifold M. We make a weak regularity hypothesis on the metric tensor of M, though one that guarantees the unique existence of the geodesic flow. We then establish exponential energy decay under the natural hypothesis that all sufficiently long geodesics pass through a region where $a(x) \geq a_0 > 0$, extending the scope of previous work done in the setting of a smooth metric tensor.

1 Introduction

We examine solutions to the wave equation with dissipation

$$Lu = \partial_t^2 u + a(x)\partial_t u - \Delta u = 0, \tag{1.1}$$

on $\mathbb{R} \times M$, where M is a compact, connected Riemannian manifold, with Laplace-Beltrami operator Δ, and $a \in L^\infty(M)$ is ≥ 0. We seek conditions that guarantee exponential decay of the energy

$$E(u(t)) = \frac{1}{2} \int_M \left\{ |\nabla_x u(t,x)|^2 + |u_t(t,x)|^2 \right\} dV(x), \tag{1.2}$$

as $t \nearrow +\infty$, given

$$u(0) = f \in H^1(M), \quad \partial_t u(0) = g \in L^2(M). \tag{1.3}$$

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As is well known, (1.1), with initial data (1.3), has a unique solution

$$u \in C(\mathbb{R}, H^1(M)) \cap C^1(\mathbb{R}, L^2(M)), \tag{1.4}$$

and we have the dissipation identity

$$\begin{aligned} \frac{d}{dt} E(u(t)) &= \operatorname{Re} \int\limits_M \{\nabla_x u \cdot \nabla_x u_t + u_t u_{tt}\} \, dV(x) \\ &= -\int\limits_M a(x) |\partial_t u(t,x)|^2 \, dV(x). \end{aligned} \tag{1.5}$$

One approach to global solvability of (1.1) is to take

$$\Lambda = \sqrt{-\Delta + 1}, \quad \Lambda_0 = -\Delta \Lambda^{-1} = \Lambda - \Lambda^{-1}, \tag{1.6}$$

and set

$$V = \begin{pmatrix} \Lambda u \\ \partial_t u \end{pmatrix}, \tag{1.7}$$

so (1.1) becomes

$$\partial_t V = GV, \tag{1.8}$$

with

$$G = \begin{pmatrix} 0 & \Lambda \\ -\Lambda_0 & -a \end{pmatrix}, \tag{1.9}$$

which is a bounded perturbation of the skew-adjoint operator

$$G_0 = \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix}, \tag{1.10}$$

and hence generates a strongly continuous group of operators e^{tG} on $L^2(M)$, so

$$V(t) = e^{tG} \begin{pmatrix} \Lambda f \\ g \end{pmatrix}. \tag{1.11}$$

It is fairly easy to show that $E(u(t))$ decays exponentially as $t \nearrow +\infty$ for each $(f, g) \in H^1(M) \oplus L^2(M)$, provided $a \in L^\infty(M)$ satisfies $a(x) \geq a_0 > 0$ for all $x \in M$. For this, it suffices to assume the metric tensor of M is continuous. (The proof of such decay is contained in the proof of a more general decay result given in §2.) It is of interest to assume instead that

$$a(x) \geq a_0 > 0, \quad \forall\, x \in U, \tag{1.12}$$

where U is some open subset of M, and see when this condition implies exponential energy decay. This was treated in [21] in the case of smooth coefficients, i.e., when M has a smooth metric tensor and $a \in C^\infty(M)$ is ≥ 0 on M and satisfies (1.12). In such a setting, [21] showed one has exponential energy decay provided the following condition holds:

$$\begin{gathered}\textbf{Control condition: } \text{There exists } T_0 < \infty \text{ such that} \\ \text{each geodesic in } M \text{ of length } T_0 \text{ intersects } U.\end{gathered} \tag{1.13}$$

The necessity of such a condition follows from work of [19]. In rough outline, the argument of [21] goes as follows. First, propagation of singularity results of [13], applied to $\partial_t u$, which also solves (1.1), yield

$$\begin{aligned}&\int_0^{T_0} \int\limits_M |\partial_t u(s,x)|^2 \, dV(x) \, ds \\ &\leq C \int_0^{T_0} \int\limits_U |\partial_t u(s,x)|^2 \, dV(x) \, ds + C\|u_t\|_{H^{-1}([0,T_0]\times M)}.\end{aligned} \tag{1.14}$$

Then an argument incorporating functional analysis and unique continuation allows one to drop the last term on the right side of (1.14), after perhaps replacing T_0 by a larger (finite) quantity T_1 (and perhaps also expanding U slightly). Then, via (1.5), one obtains

$$E(u(T_1)) \leq E(u(0)) - C_1 \|\partial_t u\|^2_{L^2([0,T_1]\times M)}, \tag{1.15}$$

for all solutions to (1.1) and (1.3), with C_1 independent of f and g. From here, a further argument, which we will present in a more general context in §2, allows one to pass to

$$E(u(T_1)) \leq E(u(0)) - C_2 \int_0^{T_1} E(u(s)) \, ds, \tag{1.16}$$

and since $E(u(s)) \searrow$, by (1.5), this implies

$$E(u(T_1)) \leq (1 + C_2 T_1)^{-1} E(u(0)), \tag{1.17}$$

hence

$$E(u(kT_1)) \leq (1 + C_2 T_1)^{-k} E(u(0)), \tag{1.18}$$

yielding exponential energy decay.

In [22], another proof was given of such exponential energy decay, making use of Fourier integral operators. In this proof, the hypothesis $a \geq 0$

was dropped, and (1.13) was modified to a positivity condition on averages of $a(\gamma(t))$ over sufficiently long geodesics. Both papers worked under the hypothesis of smooth coefficients.

Our goal here is to establish exponential energy decay, for $a \geq 0$ satisfying (1.12), assuming (1.13), under weak smoothness hypotheses on the metric tensor on M and on the coefficient $a(x)$. We succeed in this for a geometrically significant class of Riemannian manifolds whose metric tensors are rougher than C^2.

In order to work with (1.13), we will want the geodesic flow on M to be well defined. Now this flow can be regarded as a flow on the cotangent bundle T^*M generated by the Hamiltonian vector field

$$X(x, \xi) = (\nabla_\xi \Phi, -\nabla_x \Phi), \tag{1.19}$$

with $\Phi : T^*M \to \mathbb{R}$ given, in local coordinates, by

$$\Phi(x, \xi) = g^{jk}(x)\xi_j \xi_k. \tag{1.20}$$

Here, (g_{jk}) is the metric tensor on M (giving an inner product on tangent vectors) and (g^{jk}) its inverse (giving an inner product on cotangent vectors). If the metric tensor is C^2 in local coordinates, then X is C^1, so it generates a well defined flow. More generally, if $g_{jk} \in C^{1,1}$, then X is Lipschitz and it generates a flow. Still more generally, if M has a coordinate cover on which (g_{jk}) has gradient with a log-Lipschitz modulus of continuity, i.e.,

$$|\nabla g_{jk}(x) - \nabla g_{jk}(y)| \leq C|x - y| \log \frac{1}{|x - y|}, \tag{1.21}$$

for $|x - y|$ small, then X has a log-Lipschitz modulus of continuity, and, by the classical Osgood theorem, it generates a flow. This latter situation applies to the following important class of Riemannian manifolds:

$$\begin{gathered} \text{If } (M, g) \text{ has a bounded Ricci tensor,} \\ \text{then (1.21) holds in local harmonic coordinates.} \end{gathered} \tag{1.22}$$

Appendix B recalls basic results about the Ricci tensor. One point to emphasize is that (1.22) provides a natural, coordinate-independent condition yielding the property (1.21). The class of Riemannian manifolds with Ricci tensor bounds is of fundamental significance to geometric analysis. A treatment of key results can be found in [6]. We mention [7] for other material connecting Ricci tensor bounds and analysis on Riemannian manifolds.

Our principal goal in this paper is to prove the following.

Theorem 1.1 *Let M be a compact manifold with a metric tensor satisfying* (1.21)*, and assume a satisfies*

$$a \in C^{r-1}(M), \tag{1.23}$$

for some $r > 1$. Also assume $a \geq 0$ on M and that (1.12) *holds. If the control condition* (1.13) *holds, then there is a uniform exponential rate of decay of energy $E(u(t))$, for all solutions to* (1.1) *with initial data as in* (1.3).

Our approach to the proof of Theorem 1.1 will make use of propagation of singularities results of the sort established in [28], Chapter 3, §11. That work dealt with propagation of microlocal regularity for differential and pseudodifferential operators with coefficients whose gradients had a log-Lipschitz modulus of continuity. Here, we tweak those results, to allow for rougher lower order terms.

Other works on wave propagation with rough coefficients include [23], [18], and [17], concentrating on $C^{1,1}$ and C^2 coefficients, and [24], working on a Riemannian manifold with bounded Riemann curvature tensor. The paper [1] studied wave motion on manifolds with bounded Ricci tensor, and nonempty boundary, with applications to some inverse spectral problems. We also mention the recent paper [11], dealing with metric tensors with singularities of a special type (conormal), regular of class $C^{1+\alpha}$, $\alpha \in (0,1)$.

The rest of this paper is organized as follows. In §2 we treat a general setting in which an estimate of the form (1.15) leads to the energy estimate (1.16), hence to decay (1.18). Section 3 discusses the necessary modifications of the propagation of singularity results of [28] needed in the proof of Theorem 1.1. The additional technical problems that need to be addressed arise from the first order term $a\partial_t$, with a coefficient more singular than the metric tensor.

In §4 we show how the results of §3 lead to the estimate (1.15), under the hypotheses of Theorem 1.1, and hence complete the proof of Theorem 1.1.

We have three appendices. Appendix A shows how estimates on $\|u\|_{L^2([0,T]\times M)}$ for a solution u to (1.1) lead automatically to stronger estimates, on a larger time interval, and similarly for $\|u\|_{H^1([0,T]\times M)}$. To be precise, Propositions A.3 and A.4 yield

$$\|u\|_{C(J,L^2(M))} + \|\partial_t u\|_{C(J,H^{-1}(M))} \leq C\|u\|_{L^2(I\times M)}, \tag{1.24}$$

and

$$\|u\|_{C(J,H^1(M))} + \|\partial_t u\|_{C(J,L^2(M))} \le C\|u\|_{H^1(I\times M)}, \tag{1.25}$$

respectively, where $I = [0,T]$ and J is a (possibly larger) bounded interval. These estimates are technically useful for arguments in Sections 2 and 4. Appendix B recalls results on the Ricci tensor, leading to (1.21). Readers interested in wave motion who are not so familiar with Riemannian geometry might find identities recorded there valuable guides to why Ricci tensor bounds have significant consequences for analysis on Riemannian manifolds. Appendix C records some results involving the operation of "symbol smoothing," applied to symbols of pseudodifferential operators with rough coefficients, of use in §3.

We mention some related works, which suggest further problems to tackle. The pioneering work [4] dealt with manifolds with nonempty boundary, in the smooth category. The papers [15] and [14] take this study further. In these papers, energy dissipations include both $a\partial_t$ and dissipation resulting from boundary conditions. Results include both exponential decay, under hypotheses that are variants of (1.13), and weaker decay results, in the absence of such a control condition.

In [8], there is a further study of cases where (1.13) fails. In this setting, geodesics trapped in the region where $a = 0$ satisfy a hyperbolicity condition, forcing generic perturbations into the damping region. Conditions are given yielding decay, with loss of derivatives, and at a subexponential rate, of the form

$$E(u(t)) \le Ce^{-\alpha\sqrt{t}}\big(\|f\|^2_{H^{1+\delta}(M)} + \|g\|^2_{H^\delta(M)}\big), \tag{1.26}$$

for some $\alpha, \delta > 0$. It would be interesting to consider such results for rougher metrics. We note that, if one has a metric tensor rougher than C^2, the issue of what hyperbolicity might mean could become subtle.

It is natural to wonder if the regularity hypothesis (1.23) on $a(x)$ can be weakened further, to include more general elements of $L^\infty(M)$, perhaps with the requirement of acting as a multiplier on $H^s(M)$ for small $|s|$. Regarding bolder looks at rougher coefficients a, we mention an analysis of a singular limit of dissipative terms in [3], for vibrating strings, which one might pursue in higher dimensions.

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2 Preliminary energy decay estimates

We assume u satisfies (1.1), with initial data as in (1.3). We assume $a \in L^\infty(M)$ and $a \geq 0$, and consider the energy $E(u(t))$, given by (1.2). By (1.5), $E(u(t))$ is monotonically decreasing in t. We make the following hypothesis:

$$\text{There exist } T_1 < \infty,\ A_0 > 0 \text{ such that}$$
$$E(u(T_1)) \leq E(u(0)) - A_0 \int_0^{T_1} \int_M |u_t(s,x)|^2 \, dV(x) \, ds, \tag{2.1}$$

for all solutions $u \in C(\mathbb{R}, H^1(M)) \cap C^1(\mathbb{R}, L^2(M))$ to (1.1). Clearly (2.1) holds for all $T_1 > 0$ if $a \geq A_0$ on all of M, by (1.5). As advertised in §1, we will establish (2.1) in other settings, involving (1.12)–(1.13), in subsequent sections. Our goal here is to prove exponential decay of $E(u(t))$ under the hypotheses just stated, including (2.1). We fix such T_1.

One ingredient in our analysis is an extension result, established in Appendix A. More precisely, we make use of the following, which is a special case of Proposition A.4.

Lemma 2.1 *Let $I \subset J \subset \mathbb{R}$ be bounded intervals. Given $u \in H^1(I \times M)$, solving* (1.1)*, u extends as a solution to $\mathbb{R} \times M$, and there exists $C = C(I,J)$ such that*

$$\|u\|_{H^1(J\times M)} \leq C \|u\|_{H^1(I\times M)}. \tag{2.2}$$

To proceed, given u satisfying (1.1) and (1.3), set

$$f(t,x) = u_t(t,x), \tag{2.3}$$

and use (1.1) to write

$$\partial_t^2 u + \Delta u = 2\partial_t f + a(x) f. \tag{2.4}$$

Elliptic regularity gives

$$\begin{aligned}
\int_{T_1/3}^{2T_1/3} \int_M &\left\{ |u_t(s,x)|^2 + |\nabla_x u(s,x)|^2 + |u(s,x)|^2 \right\} dV(x) \, ds \\
&\leq C \int_0^{T_1} \int_M \left\{ |f(s,x)|^2 + |u(s,x)|^2 \right\} dV(x) \, ds \\
&= C \int_0^{T_1} \int_M \left\{ |u_t(s,x)|^2 + |u(s,x)|^2 \right\} dV \, ds.
\end{aligned} \tag{2.5}$$

It follows from Lemma 2.1 that

$$\int_0^{T_1} \int_M \left\{ |u_t(s,x)|^2 + |\nabla_x u(s,x)|^2 + |u(s,x)|^2 \right\} dV\, ds \tag{2.6}$$

is bounded by a constant times the left side of (2.5). This implies

$$\|u\|^2_{H^1([0,T_1]\times M)} \le C\|\partial_t u\|^2_{L^2([0,T_1]\times M)} + C\|u\|^2_{L^2([0,T_1]\times M)}, \tag{2.7}$$

for all $u \in E$, where

$$E = \{u \in H^1([0,T_1]\times M) : Lu = 0\}. \tag{2.8}$$

Hence (2.7) takes the form

$$\|u\|^2_E \le C\|Tu\|^2_V + C\|Ku\|^2_W, \quad \forall\, u \in E, \tag{2.9}$$

with E as in (2.8),

$$V = W = L^2([0,T_1]\times M), \tag{2.10}$$

$$Tu = \partial_t u, \tag{2.11}$$

and $K : E \hookrightarrow L^2([0,T_1]\times M)$ is the inclusion. Thus

$$T : E \longrightarrow V \ \text{ is continuous, and } \ K : E \longrightarrow W \text{ is compact.} \tag{2.12}$$

Given this structure, it is a standard Fredholm-type result (cf. [16], p. 171) that

$$T : E \longrightarrow V \ \text{ has closed range.} \tag{2.13}$$

Also such T has finite dimensional kernel. In the current setting,

$$\operatorname{Ker} T = \{\varphi(x) : \Delta\varphi = 0\} = \mathbb{C}, \tag{2.14}$$

the set of constant functions. From this, we obtain the estimate

$$\begin{aligned} &\int_0^{T_1} \int_M \left\{ |u_t(s,x)|^2 + |\nabla_x u(s,x)|^2 \right\} dV\, ds \\ &\qquad \le C \int_0^{T_1} \int_M |u_t(s,x)|^2\, dV\, ds, \quad \forall\, u \in E. \end{aligned} \tag{2.15}$$

Plugging this into (2.1) yields

$$E(u(T_1)) \le E(u(0)) - A_1 \int_0^{T_1} E(s)\, ds. \tag{2.16}$$

Since $E(u(s)) \searrow$, this gives

$$E(u(T_1)) \le (1 + A_1 T_1)^{-1} E(u(0)), \tag{2.17}$$

whenever $u \in C(\mathbb{R}, H^1(M)) \cap C^1(\mathbb{R}, L^2(M))$ solves (1.1). Iterating gives

$$E(u(kT_1)) \le (1 + A_1 T_1)^{-k} E(u(0)), \tag{2.18}$$

for all such u. This gives the asserted decay result. We record this formally.

Proposition 2.2 *Let M be a compact, connected Riemannian manifold with a continuous metric tensor, and let $a \in L^\infty(M)$ satisfy $a \ge 0$. Assume (2.1) holds for all solutions $u \in C(\mathbb{R}, H^1(M)) \cap C(\mathbb{R}, L^2(M))$ to (1.1). Then there exist $T_1 < \infty$ and $\alpha > 0$ such that, for all such u,*

$$E(u(t)) \le e^{-\alpha t} E(u(0)), \quad \forall\, t \ge T_1. \tag{2.19}$$

We do not assert that $u(t) \to 0$ in $L^2(M)$ as $t \to +\infty$. In fact, $u \equiv 1$ solves (1.1); its energy is $\equiv 0$, but obviously its amplitude does not decay. In light of this observation, the following result is natural.

Proposition 2.3 *In the setting of Proposition 2.2, given $u \in C(\mathbb{R}, H^1(M)) \cap C^1(\mathbb{R}, L^2(M))$ solving (1.1), there exists $\kappa(u) \in \mathbb{C}$ such that*

$$\|u(t) - \kappa(u)\|_{L^2(M)} \le C e^{-\alpha t/2} E(u(0)), \tag{2.20}$$

for $t \ge T_1$.

Proof. We have

$$\|u((k+1)T_1) - u(kT_1)\|_{L^2} \le \int_{kT_1}^{(k+1)T_1} \|\partial_t u(s)\|_{L^2}\, ds, \tag{2.21}$$

hence

$$\begin{aligned} \|u((k+1)T_1) - u(kT_1)\|_{L^2}^2 &\le T_1 \int_{kT_1}^{(k+1)T_1} \|\partial_t u(s)\|_{L^2}^2\, ds \\ &\le T_1 \int_{kT_1}^{(k+1)T_1} E(u(s))\, ds \\ &\le T_1^2 e^{-\alpha k T_1} E(u(0)). \end{aligned} \tag{2.22}$$

Thus $u_k = u(kT_1)$ is a Cauchy sequence in $L^2(M)$ as $k \to +\infty$, so there exists $\kappa(u) \in L^2(M)$ such that (2.20) holds. Since $E(u_k) \to 0$, it follows that (u_k) is bounded in $H^1(M)$, so a subsequence converges weak* in $H^1(M)$ to $\kappa(u)$, which must be constant. □

3 Propagation of singularities results

In Chapter 3, §11 of [28] there are several results on propagation of singularities for differential and pseudodifferential operators with rough coefficients, both in nondivergence form and in divergence form. Here, before discussing a needed extension, we recall Proposition 11.4 of that chapter. It deals with a second order, divergence form operator

$$Pu = \partial_j A^{jk}(x)\partial_k u. \tag{3.1}$$

Note that the wave equation $(\partial_t^2 - \Delta)u = 0$, which is (1.1) without the dissipative term $a(x)\partial_t u$, can be written in local coordinates as

$$\partial_t g^{1/2}\partial_t u - \partial_j(g^{1/2}g^{jk}\partial_k u) = 0, \tag{3.2}$$

the left side having the form (3.1), with $\partial_0 = \partial_t$.

We assume Ω is an open subset of $\mathbb{R}^{n+1}$, e.g., for (3.2), $\Omega = (a,b)\times U$, where (a,b) is a t-interval and U is a coordinate patch on the Riemannian manifold M. As in Proposition 11.4 of [28], we assume A^{jk} are real valued, $A^{jk} = A^{kj}$, and

$$A^{jk} \in C^r(\Omega), \quad r \in (1,2]. \tag{3.3}$$

We assume

$$u \in H^{1+\sigma-r\delta}_{\mathrm{loc}}(\Omega), \tag{3.4}$$

where

$$\delta \in (0,1), \quad \delta r > 1, \tag{3.5}$$

and

$$-(1-\delta)r < \sigma < r. \tag{3.6}$$

We assume $\mathcal{O}$, $\widetilde{\Gamma}$, and Γ are open conic subsets of $T^*\Omega \setminus 0$, and that

$$u \in H^\sigma_{\mathrm{mcl}}(\mathcal{O}), \quad Pu \in H^{\sigma-1}_{\mathrm{mcl}}(\widetilde{\Gamma}), \quad \mathcal{O}, \overline{\Gamma} \subset \widetilde{\Gamma}. \tag{3.7}$$

We characterize the space $H^\sigma_{\mathrm{mcl}}(\mathcal{O})$ as follows.

Definition. A distribution u on Ω belongs to $H^\sigma_{\mathrm{mcl}}(\mathcal{O})$ provided $\varphi(x,D)u \in H^\sigma$ for each $\varphi(x,D) \in OPS^0_{1,0}$ with total symbol supported in $\mathcal{O}$.

Proposition 11.4 of [28] gives conditions under which one can conclude that

$$u \in H^\sigma_{\mathrm{mcl}}(\Gamma), \tag{3.8}$$

which is a result on propagation of microlocal regularity from $\mathcal{O}$, along Γ.

We describe the conditions relating $\mathcal{O}$, $\widetilde{\Gamma}$, and Γ, leading to the propagation result (3.7) $\Rightarrow$ (3.8). Let

$$p_1(x,\xi) = A^{jk}\xi_j\xi_k|\xi|^{-1}. \tag{3.9}$$

We assume we have smooth symbols

$$d(x,\xi) \in S^{\mu}_{\text{cl}}, \quad f(x,\xi) \in S^{0}_{\text{cl}}, \quad g(x,\xi) \in S^{1}_{\text{cl}}, \tag{3.10}$$

all homogeneous in ξ for $|\xi|$ large, with

$$\mu > 0, \quad \mu \le \frac{r-1}{r+1}, \quad \mu \le \frac{1-\delta}{2}. \tag{3.11}$$

We assume

$$\operatorname{supp} d \subset \widetilde{\Gamma}, \quad g(x,\xi) \ge c|\xi| > 0 \ \text{ on } \ \widetilde{\Gamma}, \tag{3.12}$$

and

$$H_{p_1} d \ge 0 \ \text{ on } \ \widetilde{\Gamma} \setminus \mathcal{O}, \quad H_{p_1} d \ge C|\xi|^{\mu} > 0 \text{ on } \ \overline{\Gamma \setminus \mathcal{O}}, \tag{3.13}$$

where $H_{p_1} d = \{p_1, d\}$ is the Poisson bracket. Furthermore, we assume

$$H_{p_1} f \ge 1, \quad H_{p_1} g \le 0 \ \text{ on } \ \widetilde{\Gamma}. \tag{3.14}$$

The content of Proposition 11.4 of [28], Chapter 3, is that, under these hypotheses, one has the propagation result (3.7) $\Rightarrow$ (3.8).

Note that this result works with coefficients somewhat less regular than advertised in (1.21), which here takes the form

$$|\nabla A^{jk}(x) - \nabla A^{jk}(y)| \le C|x-y| \log \frac{1}{|x-y|}, \tag{3.15}$$

for $|x-y|$ small. Where this additional regularity plays a role is in replacing the conditions (3.10)–(3.14) by natural geometrical conditions on $\mathcal{O}$, Γ, and $\widetilde{\Gamma}$, involving the flow generated by H_{p_1}, which is well defined if (3.15) holds.

To be more precise, let $(x_0,\xi_0) \in T^*\Omega \setminus 0$, and assume

$$(x_0,\xi_0) \in \operatorname{Char}(P) = \{(x,\xi) : p_1(x,\xi) = 0\}. \tag{3.16}$$

Assume $\nabla_{x,\xi} p_1(x_0,\xi_0) \neq 0$. Assuming H_{p_1} generates a flow, let γ be its orbit through (x_0,ξ_0), and assume $Pu \in H^{\sigma-1}$, microlocally on some conic neighborhood $\widetilde{\Gamma}$ of γ, and that $u \in H^{\sigma}_{\text{mcl}}(\mathcal{O})$, for some open conic neighborhood $\mathcal{O}$ of (x_0,ξ_0). We want to deduce that $u \in H^{\sigma}$, microlocally on a conic neighborhood of γ. This comes down to the following. Let I be an open interval in γ and assume $u \in H^\sigma$, microlocally on a conic neighborhood of I. Then we want to conclude that $u \in H^\sigma$ microlocally on a conic neighborhood of each endpoint of I, assuming (3.7) holds. As shown in Chapter 3, §11 of [28], one can make constructions of d, f, and g, and deduce this microlocal regularity from the implication (3.7) $\Rightarrow$ (3.8), provided the coefficients A^{jk} satisfy (3.15). The argument proceeds in stages, first treating the case $A^{jk} \in C^2$, then $A^{jk} \in C^{1,1}$, and then A^{jk} satisfying (3.15).

Our goal here is to extend such microlocal propagation results to certain first order perturbations of P. Before getting to this, we find it convenient to recall some techniques that were applied to P alone in [28], involving symbol smoothing. The reader can refer to Appendix C for basic definitions, including the symbol classes ${}^rS^m_{1,\delta}$ and $C^rS^m_{1,\delta}$, used below.

To start, we take

$$\begin{gathered} A_j = A^{jk}\partial_k = A_j^\# + A_j^b, \\ A_j^\# \in OP\,{}^rS^1_{1,\delta}, \quad A_j^b \in OPC^rS^{1-r\delta}_{1,\delta}. \end{gathered} \tag{3.17}$$

Then

$$\begin{aligned} P &= \partial_j A_j^\# + \partial_j A_j^b \\ &= A_j^\# \partial_j + [\partial_j, A_j^\#] + \partial_j A_j^b. \end{aligned} \tag{3.18}$$

We have

$$A_j^\# \partial_j \in OP\,{}^rS^2_{1,\delta}, \quad [\partial_j, A_j^\#] \in OP\,{}^{r-1}S^1_{1,\delta}. \tag{3.19}$$

Note that the total symbol of $A_j^\#\partial_j$ is real, and that of $[\partial_j, A_j^\#]$ is purely imaginary. Also, (3.4)–(3.6) imply $A_j^b u \in H^\sigma_{\text{loc}}$, hence

$$\partial_j A_j^b u \in H^{\sigma-1}_{\text{loc}}, \tag{3.20}$$

so if also (3.7) holds,

$$(A_j^\# \partial_j + [\partial_j, A_j^\#])u \in H^{\sigma-1}_{\text{mcl}}(\widetilde{\Gamma}). \tag{3.21}$$

It is convenient to switch to first order pseudodifferential operators, so, with $(\Lambda u)\hat{}(\xi) = (1+|\xi|^2)^{1/2}\hat{u}(\xi) = \langle\xi\rangle\hat{u}(\xi)$, we set

$$P_1^\# = A_j^\#\partial_j\Lambda^{-1}, \quad P_0^\# = [\partial_j, A_j^\#]\Lambda^{-1}, \quad P^\# = P_1^\# + P_0^\#, \tag{3.22}$$

so

$$P_1^\# \in OP\,{}^rS^1_{1,\delta}, \quad P_0^\# \in OP\,{}^{r-1}S^0_{1,\delta}, \tag{3.23}$$

and the total symbol of $P_1^\#$ is real, while that of $P_0^\#$ is purely imaginary. In fact,

$$\sigma_{P_1^\#}(x,\xi) = A_j^\#(x,\xi)\xi_j\langle\xi\rangle^{-1}, \quad \sigma_{P_0^\#}(x,\xi) = i\partial_{x_j}A_j^\#(x,\xi)\langle\xi\rangle^{-1}. \tag{3.24}$$

Now we set

$$v = \Lambda u, \quad \text{so } \partial_j A_j^\# u = P^\# v, \tag{3.25}$$

and the hypotheses (3.4) and (3.7) on u imply (in light of (3.21))

$$v \in H^{\sigma-r\delta}_{\text{loc}}(\Omega), \quad v \in H^{\sigma-1}_{\text{mcl}}(\mathcal{O}), \quad P^\# v \in H^{\sigma-1}_{\text{mcl}}(\widetilde{\Gamma}), \tag{3.26}$$

and the desired conclusion (3.8) is equivalent to

$$v \in H^{\sigma-1}_{\text{mcl}}(\Gamma). \tag{3.27}$$

We note that, at this point, the restriction on σ given in (3.6), which was used to pass from (3.4) and (3.7) to (3.26), via (3.20)–(3.21), is no longer needed. We can rephrase the hypotheses in (3.26) as

$$v \in \mathcal{D}'(\Omega), \quad v \in H^t_{\text{mcl}}(\mathcal{O}), \quad P^\# v \in H^t_{\text{mcl}}(\widetilde{\Gamma}), \tag{3.28}$$

and the desired conclusion (3.27) is

$$v \in H^t_{\text{mcl}}(\Gamma). \tag{3.29}$$

The content of Proposition 11.1 in [28], Chapter 3, is that (3.28) $\Rightarrow$ (3.29), provided (3.10)–(3.14) hold, with $t \in \mathbb{R}$ arbitrary.

In order to set up the positive commutator argument used to prove Proposition 11.1 of [28], we decompose $P^\#$ into its self-adjoint and skew-adjoint parts:

$$P^\# = A + iB, \quad A = A^*, \ B = B^*. \tag{3.30}$$

Recall the results (3.22)–(3.24) on the symbol of $P^\#$. We have the symbol expansions for the adjoints:

$$\sigma_{(P_j^\#)^*}(x,\xi) \sim \overline{\sigma_{P_j^\#}(x,\xi)} + \sum_{|\beta|\ge 1} \frac{i^{|\beta|}}{\beta!} D_\xi^\beta D_x^\beta \overline{\sigma_{P_j^\#}(x,\xi)}, \tag{3.31}$$

with $j = 1, 0$. We deduce that

$$A \in OP^{\,r}S^1_{1,\delta}, \quad B \in OP^{\,r-1}S^0_{1,\delta}, \tag{3.32}$$

and also, with p_1 as in (3.9),

$$a(x,\xi) - p_1(x,\xi) \in C^r S^0_{1,\delta}. \tag{3.33}$$

The proof of Proposition 11.1 of [28] uses the positive commutator method, adapted from [13]. It starts with the following basic commutator identity:

$$\mathrm{Im}(CP^{\#}v, Cv) = \mathrm{Re}(\{-iC^*[C,A] + C^*BC + C^*[B,C]\}v, v), \tag{3.34}$$

with A and B as in (3.30)–(3.32), and $C \in OPS^{\mu}_{1,0}$ to be described below. Since B has order 0, we obtain from (3.34) the basic commutator inequality,

$$\mathrm{Re}(\{-iC^*[C,A] - MC^*C\}v, v) \le \|CP^{\#}v\|^2 + |(Wv,v)|, \tag{3.35}$$

where

$$M = \|B\| + \frac{1}{4}, \quad W = \mathrm{Re}\ C^*[B,C], \tag{3.36}$$

with $\mathrm{Re}\,T = (T+T^*)/2$. For C, we actually use a family of operators $C_\varepsilon = c_\varepsilon(x,D)$, with

$$c_\varepsilon(x,\xi) = d(x,\xi)e^{\lambda f(x,\xi)}\big(1+\varepsilon^2 g(x,\xi)^2\big)^{-1/2}, \tag{3.37}$$

where $\lambda > 0$ is taken sufficiently large, $\varepsilon \searrow 0$, and $d(x,\xi)$, $f(x,\xi)$, and $g(x,\xi)$ are as in (3.10)–(3.14). For details on how to achieve the implication (3.28) $\Rightarrow$ (3.29), see pp. 205–209 of [28]. We mention that a version of the sharp Gårding inequality (given on p. 208) is involved.

We now tackle first order perturbations of P. We start with

$$Pu + Qu = \partial_j A^{jk}\partial_k u + b^j\partial_j u, \tag{3.38}$$

with $A^{jk} = A^{kj}$ real valued and satisfying (3.3), and b^j real valued, satisfying

$$b^j \in C^r. \tag{3.39}$$

We show how this more general set-up can be given a parallel treatment. We augment the decomposition (3.18) of P by

$$Q = Q^{\#} + Q^b, \quad Q^{\#} \in OP\,{}^rS^1_{1,\delta}, \quad Q^b \in OPC^rS^{1-r\delta}_{1,\delta}. \tag{3.40}$$

Note that the total symbol of $Q^{\#}$ is purely imaginary. Our hypotheses are now

$$u \in H^{1+\sigma-r\delta}_{\mathrm{loc}}(\Omega), \quad u \in H^{\sigma}_{\mathrm{mcl}}(\mathcal{O}), \quad Pu + Qu \in H^{\sigma-1}_{\mathrm{mcl}}(\widetilde{\Gamma}). \tag{3.41}$$

Now, in place of (3.20), we have

$$\partial_j A_j^b u, \quad Q^b u \in H^{\sigma-1}_{\mathrm{loc}}, \tag{3.42}$$

so, in place of (3.21),

$$(A^{\#}\partial_j + [\partial_j, A_j^{\#}] + Q^{\#})u \in H^{\sigma-1}_{\mathrm{mcl}}(\widetilde{\Gamma}). \tag{3.43}$$

Consequently, in place of (3.22), we can set

$$P_1^{\#} = A_j^{\#}\partial_j\Lambda^{-1}, \quad P_0^{\#} = [\partial_j, A_j^{\#}]\Lambda^{-1} + Q^{\#}\Lambda^{-1}, \quad P^{\#} = P_1^{\#} + P_0^{\#}, \tag{3.44}$$

so $P_1^{\#}$ (actually unchanged from (3.22)) and $P_0^{\#}$ again satisfy (3.23), with, respectively, real and purely imaginary total symbols. For $v = \Lambda u$, we again get (3.26), and desire to conclude that (3.27) holds.

We are now in the same setting as discussed above, so the positive commutator argument involving (3.30)–(3.37) applies, with no further change. We state the following resulting propagation result, which will be applied in §4.

Proposition 3.1 *Assume $u \in H^{1+\sigma-r\delta}_{\mathrm{loc}}(\Omega)$ solves*

$$\partial_j A^{jk}\partial_k u + b^j\partial_j u = 0 \tag{3.45}$$

on Ω, where $A^{jk} = A^{kj}$ are real and (3.15) holds, and b^j are real and $b^j \in C^r(\Omega)$, for some $r > 1$. Assume σ and δ satisfy (3.5)–(3.6). Assume $\mathcal{O} \subset T^*\Omega \setminus 0$ is a conic open set, and

$$u \in H^{\sigma}_{\mathrm{mcl}}(\mathcal{O}). \tag{3.46}$$

Take $p_1(x,\xi)$ as in (3.17), and let $(x_0,\xi_0) \in \mathcal{O}$ satisfy $p_1(x_0,\xi_0) = 0$. Let γ be the orbit of the Hamiltonian vector field H_{p_1} through (x_0,ξ_0). Then there is a conic neighborhood Γ of γ such that

$$u \in H^{\sigma}_{\mathrm{mcl}}(\Gamma). \tag{3.47}$$

Proposition 3.1 requires more smoothness on b^j than we want for a result that is applicable to the proof of Theorem 1.1. We would like to replace the hypothesis $b^j \in C^r$ by

$$b^j \in C^{r-1}. \tag{3.48}$$

In such a case, (3.40) is replaced by

$$Q = Q^{\#} + Q^b, \quad Q^{\#} \in OP^{r-1}S^1_{1,\delta}, \tag{3.49}$$

and

$$Q^b \in OPC^{r-1}S^{1-(r-1)\delta}_{1,\delta}. \tag{3.50}$$

Now $P^\# = P_1^\# + P_0^\#$, given by (3.44), continues to satisfy (3.23). (The extra regularity of $Q^\#$ in (3.40) is overkill.) The problem is with $Q^b u$. For $u \in H^{1+\sigma-r\delta}_{\text{loc}}$, $\sigma = 0$, δ close to 1, and $r-1>0$ small, we cannot conclude that $Q^b u$ belongs to $H^{\sigma-1}_{\text{loc}}$, or is even well defined (an issue related to whether $b^j\partial_j u$ is well defined).

To get an improved result, we make use of some special structure for (1.1). In that setting, we have $b^j = 0$ for $j \neq 0$ and $b^0 = g^{1/2}a$, which is independent of $x_0 = t$. Hence we can switch from (3.38) to

$$Pu + Qu = \partial_j A^{jk}\partial_k u + \partial_j(b^j u). \tag{3.51}$$

Now symbol smoothing yields (3.49), with

$$Q^b = \partial_j B^b_j, \quad B^b_j \in OPC^{r-1}S^{-(r-1)\delta}_{1,\delta}. \tag{3.52}$$

To proceed, we strengthen the hypothesis (3.6) to

$$-(1-\delta)(r-1) < \sigma < r-1. \tag{3.53}$$

Consequently

$$\begin{aligned} u \in H^{1+\sigma-r\delta}_{\text{loc}} &\Longrightarrow B^b_j u \in H^{\sigma}_{\text{loc}} \\ &\Longrightarrow Q^b u \in H^{\sigma-1}_{\text{loc}}. \end{aligned} \tag{3.54}$$

We are again in the setting (3.41)–(3.44), with $P^\# = P_1^\# + P_0^\#$ satisfying (3.23). As before, $P_1^\#$ has real total symbol; this time $P_0^\#$ has purely imaginary principal symbol. This still leads to (3.30)–(3.32) and subsequent arguments, yielding the following variant of Proposition 3.1.

Proposition 3.2 *Assume $u \in H^{1+\sigma-r\delta}_{\text{loc}}(\Omega)$ solves*

$$\partial_j A^{jk}\partial_k u + \partial_j(b^j u) = 0 \tag{3.55}$$

on Ω, where $A^{jk} = A^{kj}$ are real and (3.15) holds, and b^j are real and $b^j \in C^{r-1}(\Omega)$, for some $r > 1$. Assume δ satisfies (3.5) and σ satisfies (3.53). Assume $\mathcal{O} \subset T^*\Omega \setminus 0$ is a conic open set and

$$u \in H^{\sigma}_{\text{mcl}}(\mathcal{O}). \tag{3.56}$$

Form γ as in Proposition 3.1. Then there is a conic neighborhood Γ of γ such that

$$u \in H^{\sigma}_{\text{mcl}}(\Gamma). \tag{3.57}$$

In particular, this conclusion holds for

$$\sigma = 0. \tag{3.58}$$

Remark. Microlocal elliptic regularity applies on

$$\mathcal{N} = \{(x, \xi) \in T^*\Omega \setminus 0 : p_1(x, \xi) \neq 0\}. \tag{3.59}$$

Given that (3.45) holds, $A^{jk}, b^j \in C^r,\ r > 1$, or that (3.55) holds, $A^{jk} \in C^r,\ b^j \in C^{r-1}$, we have $P^{\#}\Lambda u \in H^{\sigma-1}_{\text{loc}}$, and hence

$$u \in H^{1+\sigma}_{\text{mcl}}(\mathcal{N}). \tag{3.60}$$

Compare remarks below Proposition 11.4 of [28].

4 Proof of Theorem 1.1

Under the hypotheses of Theorem 1.1, and given $T \geq T_0 + 2$, we can use Proposition 3.2 to deduce that there exists $\sigma_0 > 0$ such that, if $0 < \sigma < \sigma_0$ and $v \in H^{-\sigma}([0, T] \times M)$ solves $Lv = 0$, then, with U as in (1.12)–(1.13),

$$v \in H^{-\sigma}([0, T] \times M) \cap L^2([0, T] \times U) \Longrightarrow v \in L^2([1, 2] \times M), \tag{4.1}$$

with associated estimate

$$\|v\|_{L^2([1,2]\times M)} \leq C\|v\|_{L^2([0,T]\times U)} + C\|v\|_{H^{-\sigma}([0,T]\times M)}. \tag{4.2}$$

We now bring in the extension result of Appendix A, Proposition A.3, which implies that we have an estimate

$$\|v\|_{L^2([0,T]\times M)} \leq C_T\|v\|_{L^2([1,2]\times M)}. \tag{4.3}$$

Hence

$$\|v\|_{L^2([0,T]\times M)} \leq C_T\|v\|_{L^2([0,T]\times U)} + C_T\|v\|_{H^{-\sigma}([0,T]\times M)}. \tag{4.4}$$

We can write this as

$$\|v\|_E \leq C_T\|Rv\|_V + C_T\|Kv\|_W, \quad \forall\, v \in E, \tag{4.5}$$

where

$$\begin{gathered} E = \{v \in L^2([0, T] \times M) : Lv = 0\}, \quad V = L^2([0, T] \times U), \\ W = H^{-\sigma}([0, T] \times M), \end{gathered} \tag{4.6}$$

and

$$R : E \to V \text{ is restriction}, \quad K : E \to W \text{ is inclusion}. \tag{4.7}$$

Since R is continuous and K is compact, standard Fredholm theory (compare (2.9)–(2.13)) implies

$$R : E \longrightarrow V \text{ has closed range, and finite dimensional kernel}. \tag{4.8}$$

Note that

$$\mathcal{K}_T = \operatorname{Ker} R = \{v \in L^2([0,T] \times M) : Lv = 0,\ v = 0 \text{ on } [0,T] \times U\}. \tag{4.9}$$

To proceed, let us set

$$\mathcal{O} = \{x \in M : a(x) > 0\}, \tag{4.10}$$

so, with a_0 as in (1.12),

$$\mathcal{O} = \bigcup_{j \ge 1} U_j, \quad U_j = \{x \in M : a(x) > 2^{-j} a_0\}. \tag{4.11}$$

In fact,

$$U \subset U_1 \subset U_2 \subset \cdots \subset U_j \nearrow \mathcal{O}. \tag{4.12}$$

We can replace U by U_j in (4.4), for the same range of T and constants C_T, obtaining

$$\|v\|_E \le C_T \|Rv\|_{V_j} + C_T \|Kv\|_W, \tag{4.13}$$

with E and W as in (4.6), and

$$V_j = L^2([0,T] \times U_j). \tag{4.14}$$

Parallel to (4.8)–(4.10), we have

$$R : E \longrightarrow V_j \text{ has closed range, and finite dimensional kernel}, \tag{4.15}$$

with

$$\mathcal{K}_{T,j} = \operatorname{Ker} R = \{v \in L^2([0,T] : Lv = 0,\ v = 0 \text{ on } [0,T] \times U_j\}. \tag{4.16}$$

Lemma 4.1 *There exist $T_1 < \infty$ and $\ell \in \mathbb{N}$ such that $\mathcal{K}_{T_1,\ell} = 0$.*

Proof. For each $j \in \mathbb{N}$, the family $\{\mathcal{K}_{T,j} : T \geq T_0 + 2\}$ is a decreasing family of finite dimensional spaces ($T < \widetilde{T} \Rightarrow \mathcal{K}_{\widetilde{T},j} \subset \mathcal{K}_{T,j}$). Hence this family stabilizes. There exist $S_j < \infty$ such that

$$\mathcal{K}_{T,j} = \mathcal{K}_{S_j,j}, \quad \forall\, T \geq S_j, \tag{4.17}$$

in the sense that each $v \in \mathcal{K}_{S_j,j}$, continued as an element of $C(\mathbb{R}, L^2(M))$, via (A.45), restricts to an element of $\mathcal{K}_{T,j}$ on $[0,T] \times M$. Thus we have the natural identification

$$\mathcal{K}_{S_j,j} = \mathcal{L}_j = \{v \in C(\mathbb{R}, L^2(M)) : Lv = 0,\ v = 0 \text{ on } [0,\infty) \times U_j\}, \tag{4.18}$$

and each such space is finite dimensional. In turn, $\{\mathcal{L}_j : j \in \mathbb{N}\}$ is a decreasing family of spaces ($j < k \Rightarrow \mathcal{L}_k \subset \mathcal{L}_j$). Hence this family stabilizes. There exists $\ell \in \mathbb{N}$ such that $\mathcal{L}_j = \mathcal{L}_\ell$ for all $j \geq \ell$. Hence

$$\mathcal{K}_{S_\ell,\ell} = \mathcal{L}_\ell = \{v \in C(\mathbb{R}, L^2(M)) : Lv = 0,\ v = 0 \text{ on } [0,\infty) \times \mathcal{O}\}. \tag{4.19}$$

Set $T_1 = S_\ell$.

To proceed, we have the group of translations $\tau_s v(t) = v(t+s)$ acting on $C(\mathbb{R}, L^2(M))$, and $\tau_s : \mathcal{L}_\ell \to \mathcal{L}_\ell$ for each $s \geq 0$. Since each such operator is injective, it is an isomorphism on $\mathcal{L}_\ell$, and taking $s \nearrow +\infty$, we deduce that

$$\mathcal{K}_{T_1,\ell} = \mathcal{L}_\ell = \{v \in C(\mathbb{R}, L^2(M)) : Lv = 0,\ v = 0 \text{ on } \mathbb{R} \times \mathcal{O}\}. \tag{4.20}$$

We now bring in an argument from [20] and, for $\varepsilon > 0,\ v \in \mathcal{L}_\ell$, set

$$v_\varepsilon(t) = (4\pi\varepsilon)^{-1/2} \int_{-\infty}^{\infty} e^{-(t-s)^2/4\varepsilon} v(s)\, ds, \tag{4.21}$$

which is well defined due to exponential bounds on $\|v(s)\|_{L^2(M)}$ (cf. (A.32)). Clearly $v_\varepsilon \in \mathcal{L}_\ell$ for each $\varepsilon > 0$, and

$$v_\varepsilon \longrightarrow v \ \text{ in } \ C(\mathbb{R}, L^2(M)), \quad \text{as } \ \varepsilon \to 0. \tag{4.22}$$

Now (4.20) is actually an entire holomorphic function of $t \in \mathbb{C}$, with values in $L^2(M)$. We have

$$w_\varepsilon(t) = v_\varepsilon(it) \ \text{ solves } \ (\partial_t^2 + \Delta) w_\varepsilon = 0 \ \text{ on } \ \mathbb{R} \times M, \tag{4.23}$$

since $w_\varepsilon = 0$ on $\mathbb{R} \times \mathcal{O}$, hence $a(x)\partial_t w_\varepsilon = 0$. Unique continuation results of [2] and [10] imply $w_\varepsilon \equiv 0$, hence $v_\varepsilon \equiv 0$, for each $\varepsilon > 0$. Then, by (4.15), $v \equiv 0$ for all $v \in \mathcal{L}_\ell$, proving the lemma. ∎

Remark. An alternative proof of Lemma 4.1, giving a more precise value of T_1, can be given, using the unique continuation result of [25].

We conclude from (4.15)–(4.16) and Lemma 4.1 that

$$R : E \longrightarrow L^2([0,T_1] \times U_\ell) \ \text{ is an isomorphism onto its range,} \tag{4.24}$$

which implies

$$\int_0^{T_1} \int_M |v(s,x)|^2 \, dV \, ds \le C \int_0^{T_1} \int_{U_\ell} |v(s,x)|^2 \, dV \, ds, \tag{4.25}$$

for all $v \in L^2([0,T_1] \times M)$ solving $Lv = 0$.

We now tackle the proof of Theorem 1.1. If $u \in C(\mathbb{R}, H^1(M)) \cap C^1(\mathbb{R}, L^2(M))$ solves (1.1), we take $v = \partial_t u$, which also solves $Lv = 0$, and apply (4.25), to get

$$\int_0^{T_1} \int_M |u_t(s,x)|^2 \, dV \, ds \le C \int_0^{T_1} \int_{U_\ell} |u_t(s,x)|^2 \, dV \, ds. \tag{4.26}$$

By (1.5) and the fact that $a(x) \ge 2^{-\ell} a_0$ on U_ℓ, we have

$$E(u(T_1)) \le E(u(0)) - A_0 \int_0^{T_1} \int_M |u_t(s,x)|^2 \, dV \, ds, \tag{4.27}$$

which is the hypothesis (2.1). The results of §2, collected in Proposition 2.2, finish the proof of Theorem 1.1.

A Extension of solutions in $L^2(I \times M)$

Given a bounded interval $I = [a,b]$ and a compact, n-dimensional Riemannian manifold M, we seek to establish that a solution $u \in L^2(I \times M)$ to

$$Lu = \partial_t^2 u + a(x)\partial_t u - Au = 0, \quad A = \Delta - \lambda \ (\lambda \in \mathbb{R}), \tag{A.1}$$

extends to $\mathbb{R} \times M$ and belongs to $C(\mathbb{R}, L^2(M)) \cap C^1(\mathbb{R}, H^{-1}(M))$. We also want to establish estimates of the form

$$\|u\|_{C(J,L^2(M))} + \|\partial_t u\|_{C(J,H^{-1}(M))} \le C\|u\|_{L^2(I\times M)}, \tag{A.2}$$

for all such solutions to $Lu = 0$, where J is a (possibly larger) bounded interval in $\mathbb{R}$.

Since we want to deal with non-smooth coefficients, it is natural to write $Lu = 0$ in a local coordinate patch $U \subset M$ as

$$\partial_t^2(g^{1/2}u) + \partial_t(ag^{1/2}u) - \partial_j(g^{1/2}g^{jk}\partial_k u) + \lambda g^{1/2}u = 0. \tag{A.3}$$

The first and second terms on the left side of (A.3) are well defined distributions provided $g_{jk}, a \in L^\infty(U)$. The third is well defined provided $g^{1/2}g^{jk}$ is a multiplier in $H^1(U)$, hence on $H^{-1}(U)$, which holds provided

$$g_{jk} \in H^{1,q}(U), \quad \text{for some} \quad q > n, \tag{A.4}$$

a condition that implies Hölder continuity of (g_{jk}).

To proceed, we take $u \in L^2(I \times M)$, solving (A.1), and "smooth it out" as follows. Take $\delta << (b-a)/4$ and $\psi \in C_0(-\delta, \delta)$, smooth on $(-\delta, \delta) \setminus 0$, equal to $-|x|/2$ on $(-\delta/2, \delta/2)$, and let

$$w(t) = \psi * u(t) = \int \psi(s)u(t-s)\, ds, \quad \text{for} \quad t \in J = [a + 2\delta, b - 2\delta]. \tag{A.5}$$

Then $Lw = 0$ on $J \times M$,

$$w, \partial_t w, \partial_t^2 w \in L^2(J \times M), \tag{A.6}$$

with norms bounded by $C\|u\|_{L^2(I\times M)}$, and

$$u = \partial_t^2 w + \varphi * u \quad \text{on} \quad J \times M, \quad \varphi \in C_0^\infty(-\delta, \delta). \tag{A.7}$$

Note that

$$Aw = \partial_t^2 w + a\partial_t w \in L^2(J, L^2(M)), \tag{A.8}$$

so

$$w \in L^2(J, \mathcal{D}(\Delta)). \tag{A.9}$$

Lemma A.1 *After possibly shrinking J,*

$$\partial_t w \in L^2(J, \mathcal{D}((-\Delta)^{1/2})) = L^2(J, H^1(M)), \tag{A.10}$$

with norm bounded by $C\|u\|_{L^2(I\times M)}$.

Proof. Multiplying w by a cutoff $\beta(t)$ and periodizing, we reduce the task of proving (A.10) to that of showing that

$$\begin{aligned} w \in L^2(\mathbb{T}^1, \mathcal{D}(\Delta)),\ \partial_t^2 w \in L^2(\mathbb{T}^1, L^2(M)) \\ \Rightarrow \partial_t w \in L^2(\mathbb{T}^1, \mathcal{D}((-\Delta)^{1/2})). \end{aligned} \tag{A.11}$$

If $\hat{w}(k) = \int_{\mathbb{T}^1} w(t) e^{-ikt}\, dt$, the hypotheses of (A.11) are equivalent to

$$\sum_{k=-\infty}^{\infty} \|\hat{w}(k)\|^2_{\mathcal{D}(\Delta)},\ \sum_{k=-\infty}^{\infty} k^4 \|\hat{w}(k)\|^2_{L^2(M)} \le CB^2, \tag{A.12}$$

where we can take $B = \|u\|_{L^2(I\times M)}$. Now

$$\begin{aligned} k^2 \|\hat{w}(k)\|^2_{\mathcal{D}((-\Delta)^{1/2})} &\le k^2 \|\hat{w}(k)\|_{\mathcal{D}(\Delta)} \|\hat{w}(k)\|_{L^2(M)} \\ &\le \frac{1}{2} \|\hat{w}(k)\|^2_{\mathcal{D}(\Delta)} + \frac{1}{2} k^4 \|\hat{w}(k)\|^2_{L^2(M)}, \end{aligned} \tag{A.13}$$

so (A.12) implies

$$\sum_k k^2 \|\hat{w}(k)\|^2_{\mathcal{D}((-\Delta)^{1/2})} \le CB^2, \tag{A.14}$$

and we have the implication (A.11). ☐

Now consider

$$v = \partial_t w. \tag{A.15}$$

We have

$$v \in L^2(J, H^1(M)), \quad \partial_t v \in L^2(J, L^2(M)), \quad Lv = 0 \ \text{ on } \ J \times M, \tag{A.16}$$

and, with φ as in (A.7),

$$u = \partial_t v + \varphi * u. \tag{A.17}$$

The following is a major step towards the proof of (A.2).

Lemma A.2 *After possibly shrinking J further,*

$$v \in C(J, H^1(M)), \quad \partial_t v \in C(J, L^2(M)), \tag{A.18}$$

with norms bounded by $C\|u\|_{L^2(I\times M)}$.

Proof. Take a non-negative $\xi \in C_0^\infty(-1,1)$ such that $\int \xi(t)\,dt = 1$, set $\xi_\varepsilon(t) = \varepsilon^{-1}\xi(t/\varepsilon)$, and then set

$$v_\varepsilon(t) = \xi_\varepsilon * v(t) = \int \xi_\varepsilon(s)v(t-s)\,ds, \tag{A.19}$$

so, on a slightly shrunken J,

$$v_\varepsilon \in C^\infty(J, H^1(M)), \quad Lv_\varepsilon = 0. \tag{A.20}$$

If we set

$$\Lambda = \sqrt{-\Delta + 1}, \quad \Lambda_1 = -A\Lambda^{-1} = \Lambda + (\lambda - 1)\Lambda^{-1}, \tag{A.21}$$

and then set

$$V_\varepsilon = \begin{pmatrix} \Lambda v_\varepsilon \\ \partial_t v_\varepsilon \end{pmatrix}, \tag{A.22}$$

we have $V_\varepsilon \in C^\infty(J, L^2(M))$, and, as $\varepsilon \to 0$,

$$V_\varepsilon \longrightarrow V = \begin{pmatrix} \Lambda v \\ \partial_t v \end{pmatrix} \text{ in } L^2(J, L^2(M)). \tag{A.23}$$

Furthermore, for $\varepsilon > 0$ sufficiently small, we have

$$V_\varepsilon(t) = U(t - t_0)V_\varepsilon(t_0), \quad t, t_0 \in J, \tag{A.24}$$

where

$$U(s) = e^{sG}, \quad G = \begin{pmatrix} 0 & \Lambda \\ -\Lambda_1 & -a \end{pmatrix}, \tag{A.25}$$

is a strongly continuous group of operators on $L^2(M)$, since, for $a \in L^\infty(M)$, G is a bounded perturbation of the skew-adjoint operator

$$G_0 = \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix}. \tag{A.26}$$

The standard vector-valued version of Hardy-Littlewood theory implies that $V_\varepsilon \to V$, in $L^2(M)$-norm, almost everywhere in $t \in J$.

In more detail, if $V \in L^2(J, L^2(M))$ and $V_\varepsilon = \xi_\varepsilon * V$, then, given $\varepsilon_0 > 0$,

$$\mathcal{M}(V)(t) = \sup_{0<\varepsilon\le\varepsilon_0} \left|\xi_\varepsilon * V(t)\right| \le CM(\|V\|)(t),$$

where M is the Hardy-Littlewood maximal function. The associated classical estimates on $M(\|V\|)$ together with the denseness of $C(J, L^2(M))$ in

$L^2(J, L^2(M))$ yield the pointwise a.e. convergence, just as in the case of scalar functions.

Consequently, there exist $t_0 \in J$ and $V_0 \in L^2(M)$ such that

$$V_\varepsilon(t_0) \longrightarrow V_0 \ \text{ in } \ L^2(M)\text{-norm, as } \ \varepsilon \to 0. \tag{A.27}$$

By (A.23)–(A.24), we have

$$V(t) = U(t - t_0)V_0, \tag{A.28}$$

hence

$$V \in C(J, L^2(M)), \tag{A.29}$$

which implies (A.18). In (A.27), we can pick $t_0 \in J$ such that

$$\|V_0\|^2_{L^2(M)} \le \frac{1}{\ell(J)} \int_J \|V(t)\|^2_{L^2(M)}\, dt, \tag{A.30}$$

so we have the asserted bounds on v and $\partial_t v$ in (A.18). ☐

Note that the formula (A.28) extends V to

$$V \in C(\mathbb{R}, L^2(M)), \tag{A.31}$$

satisfying

$$\|V(t)\|_{L^2} \le e^{K|t-t_0|}\|V_0\|_{L^2}, \tag{A.32}$$

for some $K < \infty$. (We can omit the exponential factor when $t \ge t_0$, provided $\lambda = 1$ and $a \ge 0$.) In turn (A.31) leads to the global extension

$$v \in C(\mathbb{R}, H^1(M)), \quad \partial_t v \in C(\mathbb{R}, L^2(M)), \tag{A.33}$$

satisfying analogous estimates.

The formula (A.17) immediately gives, for $u \in L^2(I \times M)$ satisfying (A.1),

$$u \in C(J, L^2(M)), \tag{A.34}$$

estimable by $C\|u\|_{L^2(I\times M)}$. We also have

$$\partial_t u = \partial_t^2 v + \varphi' * u, \tag{A.35}$$

and, since $Lv = 0$,

$$\partial_t^2 v = Av - a\partial_t v \in C(J, H^{-1}(M)), \tag{A.36}$$

by (A.18), so

$$\partial_t u \in C(J, H^{-1}(M)), \tag{A.37}$$

estimable by $C\|u\|_{L^2(I\times M)}$.

This establishes (A.2), except for having to shrink I slightly. We now continue $u(t)$ to all $t \in \mathbb{R}$, again using (A.17), which writes u as a sum of two terms. For the first term, $\partial_t v$, we already have (A.33), and hence we can expand (A.36) to

$$\partial_t^2 v \in C(\mathbb{R}, H^{-1}(M)). \tag{A.38}$$

Since $u(t) = \partial_t v(t) + \varphi * u(t)$ for $t \in J$, it remains to extend

$$u_2(t) = \varphi * u(t) \tag{A.39}$$

from $t \in J$ to $t \in \mathbb{R}$. This is easily accomplished, since

$$u_2 \in C^\infty(J, L^2(M)), \quad Lu_2 = 0, \tag{A.40}$$

hence

$$Au_2 = \partial_t^2 u_2 + a\partial_t u_2 \in C^\infty(J, L^2(M)), \tag{A.41}$$

so

$$u_2 \in C^\infty(J, \mathcal{D}(\Delta)) \subset C^\infty(J, H^1(M)). \tag{A.42}$$

Thus we can pick $t_0 \in J$ and set

$$V_2(t_0) = \begin{pmatrix} \Lambda u_2(t_0) \\ \partial_t u_2(t_0) \end{pmatrix}, \tag{A.43}$$

and then

$$V_2(t) = e^{(t-t_0)G} V_2(t_0) = \begin{pmatrix} \Lambda u_2(t) \\ \partial_t u_2(t) \end{pmatrix} \tag{A.44}$$

extends u_2 to

$$u_2 \in C(\mathbb{R}, H^1(M)), \quad \partial_t u_2 \in C(\mathbb{R}, L^2(M)), \tag{A.45}$$

so we have extended u from $L^2(I \times M)$ to

$$u \in C(\mathbb{R}, L^2(M)), \quad \partial_t u \in C(\mathbb{R}, H^{-1}(M)), \tag{A.46}$$

as long as g_{jk} satisfies (A.4) and $a \in L^\infty(M)$. We formally state the result just established.

Proposition A.3 *Assume* $u \in L^2(I \times M)$ *solves* (A.1). *Assume that* $a \in L^\infty(M)$ *and that the metric tensor satisfies* (A.4). *Then* u *extends to a solution to* (A.1) *on* $\mathbb{R} \times M$*, satisfying* (A.46). *Furthermore, for each bounded interval* $J \subset \mathbb{R}$*, there is an estimate*

$$\|u\|_{C(J,L^2(M))} + \|\partial_t u\|_{C(J,H^{-1}(M))} \le C(J)\|u\|_{L^2(I\times M)}. \tag{A.47}$$

Remark. The hypothesis $a \in L^\infty(M)$ is weaker than one would require for well posedness of (A.1) given arbitrary

$$u(0) = f \in L^2(M), \quad \partial_t u(0) = g \in H^{-1}(M). \tag{A.48}$$

A sufficient condition for $U(s)$, given by (A.25), to yield a strongly continuous group of operators on $H^{-1}(M)$ would be for multiplication by a to preserve $H^1(M)$, hence $H^{-1}(M)$, which would hold if (parallel to (A.4)) we had $a \in H^{1,q}(M)$ for some $q > n$. It is interesting that, if we have $u \in L^2(I \times M)$, satisfying (A.1), we do not need this extra regularity of a (beyond $a \in L^\infty(M)$) to pass to (A.46).

We complement Proposition A.3 with the following slightly easier result, which is of use in §2.

Proposition A.4 *Take* L *as in the previous proposition. Given* $u \in H^1(I \times M)$*, solving (A.1), it follows that* u *extends to* $\mathbb{R} \times M$*, solving (A.1). Furthermore, given a bounded interval* $J \subset \mathbb{R}$*, there exists* $C = C(J)$ *such that*

$$\|u\|_{C(J,H^1(M))} + \|\partial_t u\|_{C(J,L^2(M))} \le C\|u\|_{H^1(I\times M)}. \tag{A.49}$$

In fact, taking $u_\varepsilon = \xi_\varepsilon * u$ (as in (A.19)) gives $u_\varepsilon \in C^\infty(J, H^1(M))$, with slightly shrunken $J \subset I$, and for $V_\varepsilon = (\Lambda u_\varepsilon, \partial_t u_\varepsilon)$, we have (A.24). This leads to (A.27) for almost every $t_0 \in J$, and hence to (A.28), for $V = (\Lambda u, \partial_t u)$, hence to (A.29)–(A.30), and then to (A.49). Once we have (A.49) for $J \subset I$, we can extend it to larger J, via (A.28).

B Curvature of rough metric tensors

Here we recall some results on the Riemann tensor and Ricci tensor on a Riemannian manifold M with rough metric tensor. Details can be found in [28], Chapter 3, §10, with complements in §2 of [1].

To begin, let us asume that, on a coordinate patch $U \subset M$, we have a metric tensor (g_{jk}) satisfying

$$g_{jk} \in C(U) \cap H^{1,2}(U). \tag{B.1}$$

We claim the Riemann tensor and Ricci tensor are well defined distributions on U. To see this, we start with the formula for the connection 1-form $\Gamma = \sum_j \Gamma_j \, dx_j$, where Γ_j is an $n \times n$ matrix $(\Gamma^a{}_{bj})$, if $n = \dim M$:

$$\Gamma^a{}_{bj} = \frac{1}{2} g^{am} (\partial_j g_{bm} + \partial_b g_{jm} - \partial_m g_{bj}). \tag{B.2}$$

The hypothesis (B.1) implies

$$\Gamma \in L^2(U). \tag{B.3}$$

The Riemann tensor is then given by

$$\mathcal{R} = d\Gamma + \Gamma \wedge \Gamma. \tag{B.4}$$

It is a matrix valued 2-form, with components $R^a{}_{bjk}$. From (B.3), we have

$$R^a{}_{bjk} \in H^{-1,2}(U) + L^1(U). \tag{B.5}$$

Hence the Ricci tensor, with components

$$\mathrm{Ric}_{bk} = R^j{}_{bjk}, \tag{B.6}$$

is well defined:

$$\mathrm{Ric}_{bk} \in H^{-1,2}(U) + L^1(U). \tag{B.7}$$

To proceed, we strengthen the hypothesis (B.1) slightly, to

$$g_{jk} \in C^s(U) \cap H^{1,2}(U), \tag{B.8}$$

with $s > 0$. In such a case, local harmonic coordinates (of class $C^{1+s} \cap H^{2,2}$) exist on U, and in such coordinates (B.8) still holds. In such coordinates, the formula for Ric_{bk} simplifies substantially. One has

$$2 \; \mathrm{Ric}_{\ell m} = -\Delta g_{\ell m} + B_{\ell m}(g, \nabla g), \tag{B.9}$$

where $B_{\ell m}$ is a quadratic form in ∇g with coefficients that are rational functions of g_{jk}, and Δ acts componentwise on $g_{\ell m}$ as

$$\Delta u = g^{-1/2} \partial_j (g^{1/2} g^{jk} u), \quad g = \det(g_{jk}). \tag{B.10}$$

As noted in [12], one can turn (B.9) around, and regard it as a semilinear elliptic PDE for the metric tensor:

$$\Delta g_{\ell m} = B_{\ell m}(g, \nabla g) - 2 \; \mathrm{Ric}_{\ell m} \, . \tag{B.11}$$

These results apply as follows. Suppose that, in original coordinates, one has (B.8) and that the Ricci tensor is bounded. Then, in new harmonic coordinates, the transformation law of tensor fields implies the Ricci tensor is also bounded, so (B.11) holds with $\mathrm{Ric}_{\ell m} \in L^\infty(U)$. Now generally, solutions to a second order semilinear elliptic PDE tend to have two orders of regularity more than the source term. This works for L^p-Sobolev space regularity when $p \in (1, \infty)$, but it can fail when $p = \infty$. As shown in Proposition 10.2 in Chapter 3 of [28], one has the following.

Proposition B.1 *If U has a metric tensor satisfying (B.8), for which the Ricci tensor is bounded, then, in local harmonic coordinates,*

$$\partial^2 g_{jk} \in \mathrm{bmo}\,. \tag{B.12}$$

Here bmo denotes the localized John-Nirenberg space (cf. [28], p. 27). It is classical that (B.12) implies

$$|\nabla g_{jk}(x) - \nabla g_{jk}(y)| \le C|x-y| \, \log \frac{1}{|x-y|}, \tag{B.13}$$

for $|x-y|$ small, which leads us to (1.21).

C Rough symbols and symbol smoothing

We give definitions of various symbol classes and state some mapping properties of associated pseudodifferential operators. Proofs of these results, and references to basic papers, can be found in Chapter 13 of [27], or in Chapters 1–2 of [26].

First, we recall some classes of smooth symbols on Euclidean space $\mathbb{R}^d$. If $p : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is smooth, we say

$$p(x,\xi) \in S^m_{1,\delta} \Leftrightarrow |D^\beta_x D^\alpha_\xi p(x,\xi)| \le C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|+\delta|\beta|}. \tag{C.1}$$

Here, $m \in \mathbb{R}$, $\delta \in [0,1]$, and $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$. We say

$$p(x,\xi) \in S^m_{\mathrm{cl}}(\mathbb{R}^d) \tag{C.2}$$

if $p(x,\xi) \in S^m_{1,0}$ and it has an asymptotic expansion as $|\xi| \to \infty$,

$$p(x,\xi) \sim \sum_{j \ge 0} p_j(x,\xi), \tag{C.3}$$

with each $p_j(x,\xi)$ homogeneous of degree $m-j$, for $|\xi|$ large. The relation "$\sim$" means that $p(x,\xi) - \sum_{j=0}^{N} p_j(x,\xi) \in S_{1,0}^{m-N-1}$. If $p(x,\xi) \in S_{1,\delta}^m$, we set

$$p(x,D)f = (2\pi)^{-d/2} \int p(x,\xi)\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi, \tag{C.4}$$

where $\hat{f}(\xi)$ is the Fourier transform of f. We say

$$p(x,D) \in OPS_{1,\delta}^m. \tag{C.5}$$

A basic mapping property on L^p-Sobolev spaces is that

$$\begin{aligned} &p(x,D) \in OPS_{1,\delta}^m,\ m \in \mathbb{R},\ \delta \in [0,1) \\ &\Longrightarrow p(x,D) : H^{s+m,p}(\mathbb{R}^d) \to H^{s,p}(\mathbb{R}^d), \end{aligned} \tag{C.6}$$

for all $s \in \mathbb{R},\ p \in (1,\infty)$.

The most basic classes of symbols with limited smoothness we deal with in §3 are the classes $C^r S_{1,\delta}^m$, with $r \in (0,\infty),\ m \in \mathbb{R},\ \delta \in [0,1]$, defined as follows.

$$\begin{aligned} p(x,\xi) \in C^r S_{1,\delta}^m \Leftrightarrow\ &|D_\xi^\alpha p(x,\xi)| \le C_\alpha \langle\xi\rangle^{m-|\alpha|}, \\ &\|D_\xi^\alpha p(\cdot,\xi)\|_{C^r} \le C_\alpha \langle\xi\rangle^{m-|\alpha|+r\delta}, \text{ and} \\ &\|D_\xi^\alpha p(\cdot,\xi)\|_{C^j} \le C_\alpha \langle\xi\rangle^{m-|\alpha|+j\delta}, \text{ for } 0 \le j \le r, \end{aligned} \tag{C.7}$$

the last condition in effect provided $r \ge 1$.

In order to deal with the operator $p(x,D)$ associated to such symbols $p(x,\xi)$, it is convenient to split $p(x,\xi)$ into two pieces:

$$p(x,\xi) = p^\#(x,\xi) + p^b(x,\xi), \tag{C.8}$$

where $p^\#(x,\xi)$ is obtained from $p(x,\xi)$ by "symbol smoothing," and $p^b(x,\xi)$ is the remainder. We define $p^\#(x,\xi)$ as follows. Let $\{\psi_k(\xi) : k \ge 0\}$ denote a Littlewood-Paley partition of unity. Set

$$p^\#(x,\xi) = \sum_{k=0}^{\infty} J_{\varepsilon_k} p(x,\xi)\psi_k(\xi), \tag{C.9}$$

where

$$J_\varepsilon f(x) = \psi_0(\varepsilon D)f(x). \tag{C.10}$$

Take $\delta \in (0,1]$, and set

$$\varepsilon_k = 2^{-k\delta}, \quad (\text{or, if } \delta = 1), \varepsilon_k = 2^{-(k-3)}. \tag{C.11}$$

In §3, we take $\delta \in (0,1)$. The first basic symbol smoothing result is

$$p(x,\xi) \in C^r S^m_{1,0} \Rightarrow p^\#(x,\xi) \in S^m_{1,\delta}, \quad p^b(x,\xi) \in C^r S^{m-r\delta}_{1,\delta}. \tag{C.12}$$

A more precise result on $p^\#$ is that

$$p(x,\xi) \in C^r S^m_{1,0} \Longrightarrow p^\#(x,\xi) \in {}^r S^m_{1,\delta}, \tag{C.13}$$

where, given $r > 0$, we say

$$q(x,\xi) \in {}^r S^m_{1,\delta} \Leftrightarrow D^\beta_x q(x,\xi) \in \begin{matrix} S^m_{1,\delta}, & |\beta| \le r, \\ S^{m+\delta(|\beta|-r)}_{1,\delta}, & |\beta| > r. \end{matrix} \tag{C.14}$$

These results on $p^\#(x,\xi)$ and $p^b(x,\xi)$ are crucial to the analysis described in §3. We also use the $p = 2$ case of the following mapping property.

Proposition C.1 *If $\delta \in [0,1)$, $r > 0$, and $p(x,\xi) \in C^r S^m_{1,\delta}$, then*

$$p(x,D) : H^{s+m,p}(\mathbb{R}^d) \longrightarrow H^{s,p}(\mathbb{R}^d), \tag{C.15}$$

provided $1 < p < \infty$, $m \in \mathbb{R}$, *and*

$$-(1-\delta)r < s < r. \tag{C.16}$$

Mapping properties for the case $\delta = 1$ are more subtle, and will not be mentioned here, though they are of fundamental use in problems in PDE, arising, for example, in the pioneering work of [5]. Such results can also be found in [26] and [27].

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