Waves on $\mathbb{R}^{2k}$ – the Method of Descent

Michael Taylor

To solve the wave equation

$$\partial_t^2 w - \Delta w = 0, \quad w(0) = f, \quad \partial_t w(0) = g \tag{1}$$

for $w = w(t,x),\ t \in \mathbb{R},\ x \in \mathbb{R}^n,\ n = 2k$, we can use the following device, known as the method of descent. Set

$$F(x, x_{n+1}) = f(x), \quad G(x, x_{n+1}) = g(x), \tag{2}$$

and solve for $W = W(t, x, x_{n+1})$ the wave equation

$$\partial_t^2 W - \Delta_{n+1} W = 0, \quad W(0) = F, \quad \partial_t W(0) = G. \tag{3}$$

Then W is independent of x_{n+1} and

$$w(t,x) = W(t,x,0). \tag{4}$$

In particular,

$$u(t,x) = \cos t\sqrt{-\Delta}\, f(x) = \cos t\sqrt{-\Delta_{n+1}} F(x,0), \tag{5}$$

and

$$v(t,x) = \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} g(x) = \frac{\sin t\sqrt{-\Delta_{n+1}}}{\sqrt{-\Delta_{n+1}}} G(x,0). \tag{6}$$

Using what we know about waves on $\mathbb{R}^{n+1} = \mathbb{R}^{2k+1}$, we have

$$u(t,x) = C_{n+1} t \Bigl(\frac{1}{2t}\frac{d}{dt}\Bigr)^k \bigl[t^{2k-1} f_x^\#(t)\bigr], \tag{7}$$

where

$$\begin{aligned} f_x^\#(r) &= \overline{F}_{(x,0)}(r) \\ &= \frac{1}{A_n} \int\limits_{S^n} F((x,0) + r\omega)\, dS(\omega) \\ &= \frac{2}{A_n} \int\limits_{S^n_+} f(x + r\omega^b)\, dS(\omega), \end{aligned} \tag{8}$$

with

$$S^n_+ = \{\omega = (\omega^b, \omega_{n+1}) \in S^n : \omega_{n+1} \geq 0\}. \tag{9}$$

Here A_n is the n-dimensional area of S^n and, we recall,

$$C_{n+1} = \frac{1}{2}\pi^{-n/2} A_n. \tag{10}$$

Note that $f^\#_x(r) = f^\#_x(-r)$. To proceed, map $B = \{y \in \mathbb{R}^n : |y| \leq 1\}$ to S^n_+ by

$$y \mapsto (y, \psi(y)), \quad \psi(y) = \sqrt{1 - |y|^2}. \tag{11}$$

Then

$$dS(\omega) = \sqrt{1 + |\nabla\psi(y)|^2}\, dy = \frac{dy}{\sqrt{1 - |y|^2}}, \tag{12}$$

and we get

$$f^\#_x(r) = \frac{2}{A_n} \int\limits_{|y|\leq 1} f(x + ry)\, \frac{dy}{\sqrt{1 - |y|^2}}. \tag{13}$$

An alternative formula is

$$f^\#_x(r) = \frac{2}{A_n} \frac{1}{r^{n-1}} \int\limits_{|y|\leq r} \frac{f(x - y)}{\sqrt{r^2 - |y|^2}}\, dy, \tag{14}$$

for $r > 0$. Plugging this in (7), we get, for a function f on $\mathbb{R}^n = \mathbb{R}^{2k}$, $t > 0$,

$$\cos t\sqrt{-\Delta} f(x) = 2\frac{C_{n+1}}{A_n}\, t\Big(\frac{1}{2t}\frac{d}{dt}\Big)^k \int\limits_{|y|\leq t} \frac{f(x - y)}{\sqrt{t^2 - |y|^2}}\, dy. \tag{15}$$

Note that

$$2\frac{C_{n+1}}{A_n} = \pi^{-n/2}. \tag{16}$$

Similarly, we have

$$\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} g(x) = \frac{C_{n+1}}{2}\Big(\frac{1}{2t}\frac{d}{dt}\Big)^{k-1} \big[t^{2k-1} g^\#_x(t)\big], \tag{17}$$

where $g_x^{\#}(|t|)$ is as in (8), with G in place of F. Consequently, for $t > 0$,

$$\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}\, g(x) = \frac{C_{n+1}}{A_n}\Big(\frac{1}{2t}\frac{d}{dt}\Big)^{k-1} \int\limits_{|y|\le t} \frac{f(x-y)}{\sqrt{t^2-|y|^2}}\, dy. \tag{18}$$

and $(C_{n+1}/A_n) = 1/2\pi^{n/2}$.

If we specialize to $n = 2$ ($k = 1$), we get

$$\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}\, g(x) = \frac{1}{2\pi} \int\limits_{|y|\le t} \frac{g(x-y)}{\sqrt{t^2-|y|^2}}\, dy. \tag{19}$$

Note that the formulas (15), (18), and (19) exhibit finite propagation speed: the left sides depend on $f(y)$ and $g(y)$ only for

$$y \in B_{|t|}(x). \tag{20}$$

On the other hand, we do not have the strong Huyghens principle, exhibited for the solution to the wave equation on $\mathbb{R} \times \mathbb{R}^n$ for n odd (and ≥ 3).