## Waves on $\mathbb{R}^{2 k}$ - the Method of Descent <br> Michael Taylor

To solve the wave equation

$$
\begin{equation*}
\partial_{t}^{2} w-\Delta w=0, \quad w(0)=f, \quad \partial_{t} w(0)=g \tag{1}
\end{equation*}
$$

for $w=w(t, x), t \in \mathbb{R}, x \in \mathbb{R}^{n}, n=2 k$, we can use the following device, known as the method of descent. Set

$$
\begin{equation*}
F\left(x, x_{n+1}\right)=f(x), \quad G\left(x, x_{n+1}\right)=g(x), \tag{2}
\end{equation*}
$$

and solve for $W=W\left(t, x, x_{n+1}\right)$ the wave equation

$$
\begin{equation*}
\partial_{t}^{2} W-\Delta_{n+1} W=0, \quad W(0)=F, \quad \partial_{t} W(0)=G \tag{3}
\end{equation*}
$$

Then $W$ is independent of $x_{n+1}$ and

$$
\begin{equation*}
w(t, x)=W(t, x, 0) \tag{4}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
u(t, x)=\cos t \sqrt{-\Delta} f(x)=\cos t \sqrt{-\Delta_{n+1}} F(x, 0) \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
v(t, x)=\frac{\sin t \sqrt{-\Delta}}{\sqrt{-\Delta}} g(x)=\frac{\sin t \sqrt{-\Delta_{n+1}}}{\sqrt{-\Delta_{n+1}}} G(x, 0) . \tag{6}
\end{equation*}
$$

Using what we know about waves on $\mathbb{R}^{n+1}=\mathbb{R}^{2 k+1}$, we have

$$
\begin{equation*}
u(t, x)=C_{n+1} t\left(\frac{1}{2 t} \frac{d}{d t}\right)^{k}\left[t^{2 k-1} f_{x}^{\#}(t)\right] \tag{7}
\end{equation*}
$$

where

$$
\begin{align*}
f_{x}^{\#}(r) & =\bar{F}_{(x, 0)}(r) \\
& =\frac{1}{A_{n}} \int_{S^{n}} F((x, 0)+r \omega) d S(\omega)  \tag{8}\\
& =\frac{2}{A_{n}} \int_{S_{+}^{n}} f\left(x+r \omega^{b}\right) d S(\omega),
\end{align*}
$$

with

$$
\begin{equation*}
S_{+}^{n}=\left\{\omega=\left(\omega^{b}, \omega_{n+1}\right) \in S^{n}: \omega_{n+1} \geq 0\right\} \tag{9}
\end{equation*}
$$

Here $A_{n}$ is the $n$-dimensional area of $S^{n}$ and, we recall,

$$
\begin{equation*}
C_{n+1}=\frac{1}{2} \pi^{-n / 2} A_{n} . \tag{10}
\end{equation*}
$$

Note that $f_{x}^{\#}(r)=f_{x}^{\#}(-r)$. To proceed, map $B=\left\{y \in \mathbb{R}^{n}:|y| \leq 1\right\}$ to $S_{+}^{n}$ by

$$
\begin{equation*}
y \mapsto(y, \psi(y)), \quad \psi(y)=\sqrt{1-|y|^{2}} . \tag{11}
\end{equation*}
$$

Then

$$
\begin{equation*}
d S(\omega)=\sqrt{1+|\nabla \psi(y)|^{2}} d y=\frac{d y}{\sqrt{1-|y|^{2}}} \tag{12}
\end{equation*}
$$

and we get

$$
\begin{equation*}
f_{x}^{\#}(r)=\frac{2}{A_{n}} \int_{|y| \leq 1} f(x+r y) \frac{d y}{\sqrt{1-|y|^{2}}} . \tag{13}
\end{equation*}
$$

An alternative formula is

$$
\begin{equation*}
f_{x}^{\#}(r)=\frac{2}{A_{n}} \frac{1}{r^{n-1}} \int_{|y| \leq r} \frac{f(x-y)}{\sqrt{r^{2}-|y|^{2}}} d y \tag{14}
\end{equation*}
$$

for $r>0$. Plugging this in (7), we get, for a function $f$ on $\mathbb{R}^{n}=\mathbb{R}^{2 k}, t>0$,

$$
\begin{equation*}
\cos t \sqrt{-\Delta} f(x)=2 \frac{C_{n+1}}{A_{n}} t\left(\frac{1}{2 t} \frac{d}{d t}\right)^{k} \int_{|y| \leq t} \frac{f(x-y)}{\sqrt{t^{2}-|y|^{2}}} d y \tag{15}
\end{equation*}
$$

Note that

$$
\begin{equation*}
2 \frac{C_{n+1}}{A_{n}}=\pi^{-n / 2} . \tag{16}
\end{equation*}
$$

Similarly, we have

$$
\begin{equation*}
\frac{\sin t \sqrt{-\Delta}}{\sqrt{-\Delta}} g(x)=\frac{C_{n+1}}{2}\left(\frac{1}{2 t} \frac{d}{d t}\right)^{k-1}\left[t^{2 k-1} g_{x}^{\#}(t)\right] \tag{17}
\end{equation*}
$$

where $g_{x}^{\#}(|t|)$ is as in (8), with $G$ in place of $F$. Consequently, for $t>0$,

$$
\begin{equation*}
\frac{\sin t \sqrt{-\Delta}}{\sqrt{-\Delta}} g(x)=\frac{C_{n+1}}{A_{n}}\left(\frac{1}{2 t} \frac{d}{d t}\right)^{k-1} \int_{|y| \leq t} \frac{f(x-y)}{\sqrt{t^{2}-|y|^{2}}} d y . \tag{18}
\end{equation*}
$$

and $\left(C_{n+1} / A_{n}\right)=1 / 2 \pi^{n / 2}$.
If we specialize to $n=2(k=1)$, we get

$$
\begin{equation*}
\frac{\sin t \sqrt{-\Delta}}{\sqrt{-\Delta}} g(x)=\frac{1}{2 \pi} \int_{|y| \leq t} \frac{g(x-y)}{\sqrt{t^{2}-|y|^{2}}} d y \tag{19}
\end{equation*}
$$

Note that the formulas (15), (18), and (19) exhibit finite propagation speed: the left sides depend on $f(y)$ and $g(y)$ only for

$$
\begin{equation*}
y \in B_{|t|}(x) . \tag{20}
\end{equation*}
$$

On the other hand, we do not have the strong Huyghens principle, exhibited for the solution to the wave equation on $\mathbb{R} \times \mathbb{R}^{n}$ for $n$ odd (and $\geq 3$ ).

