The Dirichlet Problem on the Hyperbolic Ball

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1. Introduction

Let $B^n \subset \mathbb{R}^n$ be the unit ball, endowed with the hyperbolic metric tensor

$$ds^2 = \frac{4}{(1-|x|^2)^2} \sum dx_j^2. \tag{1.1}$$

This is a complete Riemannian manifold of constant sectional curvature -1. We want to construct the hyperbolic Poisson integral

$$\mathrm{PI}_{\mathcal{H}} : C(S^{n-1}) \longrightarrow C(\overline{B}^n) \cap C^\infty(B^n), \tag{1.2}$$

such that $u = \mathrm{PI}_{\mathcal{H}} f$ solves

$$\Delta_{\mathcal{H}} u = 0 \ \text{ on } \ B^n, \quad u\big|_{S^{n-1}} = f. \tag{1.2A}$$

Here, $\Delta_{\mathcal{H}}$ is the Laplace-Beltrami operator on B^n, with metric tensor (1.1). We will establish further regularity on $u = \mathrm{PI}_{\mathcal{H}} f$ when f has some further smoothness on S^{n-1}, and estimate $du(x)$, in the hyperbolic metric, as $x \to \partial B^n$.

If $n = 2$, then $\Delta_{\mathcal{H}} u = 0$ if and only if $\Delta u = 0$, where $\Delta = \partial_1^2 + \partial_2^2$ is the Euclidean Laplacian. In that case, $\mathrm{PI}_{\mathcal{H}}$ coincides with the Euclidean Poisson integral

$$\mathrm{PI} : C(S^1) \longrightarrow C(\overline{B}^2) \cap C^\infty(B^2). \tag{1.3}$$

As is well known, if $\delta > 0$,

$$\mathrm{PI} : C^{1+\delta}(S^1) \longrightarrow C^1(\overline{B}^2). \tag{1.4}$$

Hence

$$\begin{aligned} f \in C^{1+\delta}(S^1),\ u = \mathrm{PI}_{\mathcal{H}} f &\Longrightarrow |du(x)|_{\mathcal{E}} \le C \\ &\Longrightarrow |du(x)|_{\mathcal{H}} \le C(1-|x|), \end{aligned} \tag{1.5}$$

where $|du(x)|_{\mathcal{E}}$ is the Euclidean norm of $du(x)$ and $|du(x)|_{\mathcal{H}}$ is its norm in the hyperbolic metric (on cotangent vectors). The latter implication holds by (1.1), which implies

$$|du(x)|_{\mathcal{H}} = \frac{1}{2}(1-|x|^2)|du(x)|_{\mathcal{E}}. \tag{1.6}$$

If we let $\rho(x)$ denote the distance from 0 to x in the hyperbolic metric, (1.1) gives

$$\rho(x) = \int_0^{|x|} \frac{2}{1-r^2}\, dr = \log \frac{1+|x|}{1-|x|}, \tag{1.7}$$

so

$$1 - |x| \sim e^{-\rho(x)} \quad \text{as} \quad x \to \partial B^n. \tag{1.8}$$

Hence, by (1.5),

$$f \in C^{1+\delta}(S^1),\ u = \mathrm{PI}_{\mathcal{H}}\, f \Longrightarrow |du(x)|_{\mathcal{H}} \le C e^{-\rho(x)}. \tag{1.9}$$

Among other things, we want to get such an estimate on $d\,\mathrm{PI}_{\mathcal{H}}\, f(x)$ for higher dimensional hyperbolic space.

In §2, we give a geometrical construction of $\mathrm{PI}_{\mathcal{H}}$, prove that, given $f \in C(S^{n-1})$, $u = \mathrm{PI}_{\mathcal{H}}\, f$ so constructed satisfies (1.2A), and estimate the angular derivatives of u when $f \in C^1(S^{n-1})$.

As the reader can see, the key formula (2.3) for $\mathrm{PI}_{\mathcal{H}}\, f$ is very quickly and simply produced. We mention that there is a literature on more sophisticated generalizations. In particular, [Fur] gives a Poisson integral formula for bounded harmonic functions on a general symmetric space G/K of noncompact type, with boundary data on $G/MAN = K/M$. Further work, with an emphasis on pointwise a.e. convergence for $f \in L^p(K/M)$, has been done in a number of papers, including [Kn], [K1], [K2], [HK], and [Hel]. (See also [K3] and [Ter] for expositions.) The basic formulas (2.3) and (3.19) are special cases of results given in those papers. See also [J] and [GJ] for regularity results that have some overlap with regularity results discussed here. Further comments on results on other symmetric spaces are given in Appendix E of this paper.

Studies of the Dirichlet problem at infinity for general complete, simply connected Riemannian manifolds with sectional curvature bounded between $-b^2$ and $-a^2 < 0$ (which holds for rank-one symmetric spaces of noncompact type) appear in [A] and [Sul], and a simplification of the approach of [A] in [AS]. In this setting, [CC] establish the bound

$$|du(x)| \le C_\gamma e^{-\gamma\rho(x)}, \tag{1.9A}$$

for all $\gamma < a$, given boundary data $f \in C^1(S^{n-1})$. In the setting of hyperbolic space, $a = 1$, and the bound on $|du(x)|$ in (1.9A) is slightly weaker than that in (1.9). On the other hand, even for 2D hyperbolic space, (1.9) fails if $C^{1+\delta}$ is replaced by C^1; extra smoothness is required. See Appendix A for more on this.

To proceed with our quest to extend (1.9), it is convenient to move to the hyperbolic upper half plane $\mathbb{R}^n_+$, with metric tensor

$$ds^2 = x_n^{-2} \sum dx_j^2, \tag{1.10}$$

and associated Poisson integral

$$\mathrm{PI}_{\mathfrak{h}} : C_0(\mathbb{R}^{n-1}) \longrightarrow C(\overline{\mathbb{R}}^n_+). \tag{1.15}$$

We do this in §3. We produce an explicit formula for $\mathrm{PI}_{\mathfrak{h}}\, f$, of convolution type. We use this to estimate derivatives of $\mathrm{PI}_{\mathfrak{h}}\, f$ tangent to $\partial\mathbb{R}^n_+$ when $f \in C^1_0(\mathbb{R}^{n-1})$.

In §4, we give a Fourier integral representation of $\mathrm{PI}_{\mathfrak{h}}\, f$, of the form

$$\mathrm{PI}_{\mathfrak{h}}\, f(x + te_n) = (2\pi)^{-(n-1)} \int\limits_{\mathbb{R}^{n-1}} \hat{p}_{\mathfrak{h}}(t\xi)\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi, \tag{1.16}$$

where $\hat{p}_{\mathfrak{h}}(\xi)$ is the Fourier transform of

$$p_{\mathfrak{h}}(y) = C(|y|^2 + 1)^{-(n-1)}. \tag{1.17}$$

We discuss some qualitative results on $\hat{p}_{\mathfrak{h}}(\xi)$ and use them to analyze (1.16), in particular for $f \in \mathcal{S}(\mathbb{R}^{n-1})$. With these results, we prove the following partial extension of (1.9):

$$f \in C^\infty(S^{n-1}),\ u = \mathrm{PI}_{\mathcal{H}}\, f \Longrightarrow |du(x)|_{\mathcal{H}} \le Ce^{-\rho(x)}. \tag{1.18}$$

We also establish the following partial extension of (1.4):

$$\mathrm{PI}_{\mathcal{H}} : C^\infty(S^{n-1}) \longrightarrow C^1(\overline{B}^n), \tag{1.19}$$

for $n \ge 3$, and, for $n \ge 4$,

$$\mathrm{PI}_{\mathcal{H}} : C^\infty(S^{n-1}) \longrightarrow C^2(\overline{B}^n). \tag{1.20}$$

In §5 we use the fact that

$$\hat{p}_{\mathfrak{h}}(\xi) = C|\xi|^{(n-1)/2} K_{(n-1)/2}(|\xi|), \tag{1.21}$$

where $K_\nu(r)$ is the modified Bessel function known as MacDonald's function, to produce finer results. These include the demonstration that if $n \ge 2$ is *even*,

$$\mathrm{PI}_{\mathcal{H}} : C^\infty(S^{n-1}) \longrightarrow C^\infty(\overline{B}^n). \tag{1.22}$$

We show that (1.20) fails for $n = 3$, but just barely; cf. (5.22).

In §6, we take a still closer look at $\mathrm{PI}_{\mathfrak{h}}$, and extend (1.5) from S^1 to S^{n-1}. We go further, showing that

$$\mathrm{PI}_{\mathcal{H}} : B^1_{\infty,1}(S^{n-1}) \longrightarrow C^1(\overline{B}^n). \tag{1.23}$$

In Appendix A, we give further results along the lines of (1.3)–(1.9) for $f \in C^1(S^1)$, and more generally for $f \in C^1_*(S^1)$ (the Zygmund space). We show that in place of (1.9), we have

$$f \in C^1_*(S^1),\ u = \mathrm{PI}_{\mathcal{H}}\, f \Longrightarrow |du(x)|_{\mathcal{H}} \le C(1+\rho(x))e^{-\rho(x)}, \tag{1.24}$$

which is sharper than (1.9A) in this context.

In Appendix B, we further improve the analysis of $\mathrm{PI}_{\mathfrak{h}}$ and $\mathrm{PI}_{\mathcal{H}}$ when n is even, showing they share the same regularity properties as the Euclidean Poisson integral in this case. See Proposition B.1. Appendix C gives further results when n is odd.

In Appendix D, we record an asymptotic analysis of a Fourier transform that completes the proof of Lemma 6.2.

In Appendix E, we discuss work that has been done on other symmetric spaces, especially in rank one.

In Appendix F, we discuss further convergence results, to the effect that, if $P^r_{\mathcal{H}} f(\omega) = \mathrm{PI}_{\mathcal{H}}\, f(r\omega)$, then, for a variety of banach spaces X of functions on S^{n-1} (mainly L^p-Sobolev spaces),

$$f \in X \Longrightarrow P^r_{\mathcal{H}} f \to f \ \text{ in } \ X. \tag{1.25}$$

In Appendix G, we establish some results of Fatou type, namely, with $u_r(\omega) = u(r\omega)$, if $\Delta_{\mathcal{H}} u = 0$ on B^n, and if $\{u_r : 0 \le r < 1\}$ is bounded in X (as above), then there exists $f \in X$ (or perhaps a larger space) such that $u = \mathrm{PI}_{\mathcal{H}}\, f$.

2. Geometric construction of $\mathrm{PI}_{\mathcal{H}}$

If the solution operator in (1.2) exists, rotational symmetry requires

$$\mathrm{PI}_{\mathcal{H}} f(0) = \frac{1}{A_{n-1}} \int\limits_{S^{n-1}} f\, dS, \tag{2.1}$$

where dS is the standard element of surface area of S^{n-1} and A_{n-1} is its total area. Given $x \in B^n$, let

$$\mathcal{C}_x : B^n \longrightarrow B^n, \quad \mathcal{C}_x(0) = x \tag{2.2}$$

be a conformal diffeomorphism taking 0 to x. (Such a map extends to a diffeomorphism of $\overline{B}^n$ onto itself.) The map $\mathcal{C}_x$ is an *isometry* for the hyperbolic metric (1.1), so $v_x = u \circ \mathcal{C}_x$ also solves $\Delta_{\mathcal{H}} v_x = 0$ if $\Delta_{\mathcal{H}} u = 0$. Hence, if $\mathrm{PI}_{\mathcal{H}}$ exists, it must satisfy

$$\mathrm{PI}_{\mathcal{H}} f(x) = v_x(0) = \frac{1}{A_{n-1}} \int\limits_{S^{n-1}} f \circ \mathcal{C}_x\, dS. \tag{2.3}$$

Note that $\mathcal{C}_x$ in (2.2) is well defined only up to a rotation:

$$\mathcal{C}'_x = \mathcal{C}_x \circ R, \quad R \in SO(n), \tag{2.3A}$$

but altering $\mathcal{C}_x$ by such a factor leaves the right side of (2.3) unchanged.

Note that if $x \in B^n$ and

$$x \to \omega \in S^{n-1} \tag{2.4}$$

(convergence in the topology of $\mathbb{R}^n$), then, for an appropriate choice of $\mathcal{C}_x$,

$$\mathcal{C}_x(\sigma) \longrightarrow \omega \ \text{ for } \ \sigma \in S^{n-1} \setminus \{-\omega\}, \ \text{locally uniformly}, \tag{2.5}$$

so, as defined by (2.3), $\mathrm{PI}_{\mathcal{H}} f(x) \to f(\omega)$ if $f \in C(S^{n-1})$, and hence $\mathrm{PI}_{\mathcal{H}} : C(S^{n-1}) \to C(\overline{B}^n)$.

To show that $u = \mathrm{PI}_{\mathcal{H}} f$ satisfies (1.2A), we start with the following.

Lemma 2.1. *If $\mathcal{C} : B^n \to B^n$ is a conformal diffeomorphism, and $\mathrm{PI}_{\mathcal{H}} f$ is defined by (2.3), then*

$$(\mathrm{PI}_{\mathcal{H}} f) \circ \mathcal{C} = \mathrm{PI}_{\mathcal{H}}(f \circ \mathcal{C}). \tag{2.6}$$

Proof. We need to show that, for each $x \in B^n$,

$$\int\limits_{S^{n-1}} f \circ \mathcal{C}_{\mathcal{C}(x)}\, dS = \int\limits_{S^{n-1}} f \circ \mathcal{C} \circ \mathcal{C}_x \, dS. \tag{2.7}$$

By the comment about the invariance of the right side of (2.3) under a change of the form (2.3A), it suffices to note that $\mathcal{C}_{\mathcal{C}(x)}$ and $\mathcal{C} \circ \mathcal{C}_x$ take 0 to the same point. Indeed, both maps take 0 to $\mathcal{C}(x)$, so (2.6) holds.

To proceed, since the hyperbolic ball is a rank-one symmetric space, $u = \mathrm{PI}_{\mathcal{H}} f$ is harmonic (with respect to $\Delta_{\mathcal{H}}$) if and only if it satisfies the mean value property

$$u(x) = \mathrm{Avg}_{\Sigma_r(x)}\, u, \tag{2.8}$$

where $\Sigma_r(x)$ consists of all points of hyperbolic distance r from x, carrying the area element induced by the hyperbolic metric. By (2.6), it suffices to show that, for all $f \in C(S^{n-1})$, (2.8) holds at $x = 0$ for $u = \mathrm{PI}_{\mathcal{H}} f$. Thus we need to show that

$$\mathrm{PI}_{\mathcal{H}} f(0) = \mathrm{Avg}_{y \in \Sigma_r(0)}\, \mathrm{PI}_{\mathcal{H}} f(y), \tag{2.9}$$

or equivalently

$$\int\limits_{S^{n-1}} f\, dS = \mathrm{Avg}_{y \in \Sigma_r(0)} \int\limits_{S^{n-1}} f \circ \mathcal{C}_y \, dS. \tag{2.10}$$

Again exploiting the invariance of the right side of (2.3) under the change (2.3A), we see that it suffices to pick $y_0 \in \Sigma_r(0)$ and show that

$$\int\limits_{S^{n-1}} f\, dS = \mathrm{Avg}_{R \in SO(n)} \int\limits_{S^{n-1}} f \circ R \circ \mathcal{C}_{y_0} \, dS. \tag{2.11}$$

But

$$\mathrm{Avg}_{R \in SO(n)}\, f \circ R \equiv \frac{1}{A_{n-1}} \int\limits_{S^{n-1}} f\, dS, \tag{2.12}$$

so (2.11) holds. Hence $\mathrm{PI}_{\mathcal{H}} f$, given by (2.3), does solve the Dirichlet problem (1.2A).

We can specialize (2.6) to the case $\mathcal{C} = R \in SO(n)$ and differentiate, to obtain

$$X(\mathrm{PI}_{\mathcal{H}} f) = \mathrm{PI}_{\mathcal{H}}(Xf), \quad X \in so(n), \tag{2.13}$$

given $f \in C^1(S^{n-1})$. In order to extend the estimate (1.9) to higher dimensions, we need to analyze the radial derivative of $\mathrm{PI}_{\mathcal{H}} f$, for f in a suitable class of functions on S^{n-1}. To do this, we find it convenient to shift to the upper half space version of hyperbolic space.

3. Moving to the upper half plane

The ball B^n with metric tensor (1.1) is isometric to $\mathbb{R}^n_+$, with metric tensor

$$ds^2 = x_n^{-2} \sum dx_j^2, \tag{3.1}$$

via the conformal map

$$\mathcal{T} : \mathbb{R}^n \setminus \{e_n\} \to \mathbb{R}^n, \quad -\mathcal{T}(x) = 2|x - e_n|^{-2}(x - e_n) + e_n, \tag{3.2}$$

restricted to B^n, yielding

$$\mathcal{T} : B^n \to \mathbb{R}^n_+. \tag{3.2A}$$

The restriction $\mathcal{S} = -\mathcal{T}|_{S^{n-1}}$ is given by

$$\mathcal{S}(x) = (1 - x_n)^{-1}(x', 0), \quad \mathcal{S} : S^{n-1} \setminus \{e_n\} \to \mathbb{R}^{n-1}, \tag{3.3}$$

if $x = (x', x_n)$. This is stereographic projection. A computation (cf. [T2], p. 229) gives for $y = \mathcal{S}(x) = (1 - x_n)^{-1}x'$,

$$\mathcal{S}^* \sum_{j=1}^{n-1} dy_j^2 = (1 - x_n)^{-2} \sum_{j=1}^{n} dx_j^2. \tag{3.5}$$

Hence, for the inverse

$$\psi : \mathbb{R}^{n-1} \longrightarrow S^{n-1}, \tag{3.6}$$

we have

$$\psi^* \sum_{j=1}^{n} dx_j^2 = (1 - x_n)^2 \sum_{j=1}^{n-1} dy_j^2. \tag{3.7}$$

A computation gives

$$x_n = \frac{|y|^2 - 1}{|y|^2 + 1}, \quad 1 - x_n = \frac{2}{|y|^2 + 1}, \tag{3.8}$$

so

$$\psi^* \sum_{j=1}^{n} dx_j^2 = \frac{4}{(|y|^2 + 1)^2} \sum_{j=1}^{n-1} dy_j^2, \tag{3.9}$$

and hence, with dS as in (2.3),

$$\psi^* dS = \frac{C}{(|y|^2+1)^{n-1}}\, dy, \quad y \in \mathbb{R}^{n-1}. \tag{3.10}$$

We have the conjugated operator

$$\mathrm{PI}_{\mathfrak{h}}\, f = (\mathrm{PI}_{\mathcal{H}}(f \circ \mathcal{T}^{-1})) \circ \mathcal{T}, \tag{3.11}$$

giving

$$\mathrm{PI}_{\mathfrak{h}} : C_0(\mathbb{R}^{n-1}) \longrightarrow C(\overline{\mathbb{R}}^n_+), \tag{3.12}$$

for which $u = \mathrm{PI}_{\mathfrak{h}}\, f$ solves

$$\Delta_{\mathfrak{h}} u = 0, \quad u\big|_{R^{n-1}} = f, \tag{3.13}$$

where $\Delta_{\mathfrak{h}}$ is the Laplace-Beltrami operator on $\mathbb{R}^n$, with metric tensor (3.1). From (2.1) and (3.2)–(3.10), we have

$$\begin{aligned} \mathrm{PI}_{\mathfrak{h}}\, f(e_n) &= \frac{1}{A_{n-1}} \int\limits_{\mathbb{R}^{n-1}} f\, \psi^* dS \\ &= C \int\limits_{\mathbb{R}^{n-1}} f(y)(|y|^2+1)^{-(n-1)}\, dy. \end{aligned} \tag{3.14}$$

Then, if we have a have a conformal diffeomorphism of $\mathbb{R}^n_+$ onto itself (which hence is an isometry for the metric (3.1))

$$\mathcal{C}_X : \mathbb{R}^n_+ \to \mathbb{R}^n_+, \quad \mathcal{C}_X(e_n) = X, \tag{3.15}$$

parallel to (2.3) we have

$$\mathrm{PI}_{\mathfrak{h}}\, f(X) = C \int\limits_{\mathbb{R}^{n-1}} f \circ \mathcal{C}_X(y)(|y|^2+1)^{-(n-1)}\, dy. \tag{3.16}$$

In this case, if $X = x + te_n,\ x \in \mathbb{R}^{n-1},\ t > 0$, we can use

$$\mathcal{C}_X(y) = x + ty, \tag{3.17}$$

so

$$\mathrm{PI}_{\mathfrak{h}}\, f(x+te_n) = C \int\limits_{\mathbb{R}^{n-1}} f(x+ty)(|y|^2+1)^{-(n-1)}\, dy. \tag{3.18}$$

The constant C is determined by the identity

$$C \int_{\mathbb{R}^{n-1}} (|y|^2+1)^{-(n-1)}\,dy = 1.$$

It is elementary that (3.18) tends to $f(x)$ as $t \searrow 0$. We make a change of variable and exploit evenness to write this as a convolution:

$$\mathrm{PI}_{\mathfrak{h}}\, f(x+te_n) = Ct^{-(n-1)} \int_{\mathbb{R}^{n-1}} f(x-y)\Bigl(\Bigl|\frac{y}{t}\Bigr|^2+1\Bigr)^{-(n-1)}\,dy. \tag{3.19}$$

Remark. This formula for $\mathrm{PI}_{\mathfrak{h}}$ is a special case of a Poisson integral formula for general rank-one symmetric spaces. See Appendix E for more on this.

Compare (3.19) to the Euclidean Poisson integral

$$\mathrm{PI}_{\mathfrak{e}}\, f(x+te_n) = C't^{-(n-1)} \int_{\mathbb{R}^{n-1}} f(x-y)\Bigl(\Bigl|\frac{y}{t}\Bigr|^2+1\Bigr)^{-n/2}\,dy. \tag{3.20}$$

These coincide when $n=2$, but not when $n>2$. Both (3.19) and (3.20) have the form

$$\mathrm{PI}_a\, f(x+te_n) = \int_{\mathbb{R}^{n-1}} f(x-y)p_a(t,y)\,dy, \tag{3.21}$$

with

$$\begin{aligned} p_{\mathfrak{h}}(t,y) &= Ct^{-(n-1)}\Bigl(\Bigl|\frac{y}{t}\Bigr|^2+1\Bigr)^{-(n-1)} = Ct^{n-1}(|y|^2+t^2)^{-(n-1)}, \\ p_{\mathfrak{e}}(t,y) &= Ct^{-(n-1)}\Bigl(\Bigl|\frac{y}{t}\Bigr|^2+1\Bigr)^{-n/2} = Ct(|y|^2+t^2)^{-n/2}. \end{aligned} \tag{3.22}$$

Note that, for $n \ge 3$, $p_{\mathfrak{h}}(t,y)$ decreases as $|y| \to \infty$ faster than $p_{\mathfrak{e}}(t,y)$ does. This will have significant consequences.

Regarding the relevance of (3.19) to the analysis of $\mathrm{PI}_{\mathcal{H}}\, g$, given $g \in C(S^{n-1})$, the following observation is in order. Using a partition of unity, we can write $g = g_1 + g_2$, where g_1 vanishes in a neighborhood of e_n and g_2 vanishes in a neighborhood of $-e_n$. With $f_1 = g_1 \circ \mathcal{T}$, we have

$$\mathrm{PI}_{\mathcal{H}}\, g_1 = (\mathrm{PI}_{\mathfrak{h}}\, f_1) \circ \mathcal{T}^{-1}, \tag{3.23}$$

and $f_1 \in C(\mathbb{R}^{n-1})$ has compact support. A rotation reduces the study of $\mathrm{PI}_{\mathcal{H}}\, g_2$ to that of $\mathrm{PI}_{\mathcal{H}}\, g_1$, so we can concentrate on (3.19) with f *compactly supported* on $\mathbb{R}^{n-1}$.

Here is a first indication of differences between (3.19) and (3.20) when $n \ge 3$.

Proposition 3.1. *Assume* $f \in C(\mathbb{R}^{n-1})$ *is supported on a compact set* $K \subset \mathbb{R}^{n-1}$. *Let* $\widetilde{K} \subset \mathbb{R}^{n-1}$ *be a compact set disjoint from* K. *Then*

$$|\,\mathrm{PI}_{\mathfrak{h}}\, f(x+te_n)| \le Ct^{n-1} \quad \textit{for} \ \ x \in \widetilde{K}. \tag{3.24}$$

The proof is a simple consequence of (3.19) (or (3.21)–(3.22)). Note by contrast that the hypotheses give

$$|\,\mathrm{PI}_{\mathfrak{e}}\, f(x+te_n)| \le Ct \quad \text{for} \ \ x \in \widetilde{K}. \tag{3.25}$$

For $n = 2$, (3.24) and (3.25) have the same strength (as they must), but for $n \ge 3$, (3.24) is much stronger than (3.25).

Since $\mathrm{PI}_{\mathfrak{h}}$ is a convolution operator for each t, it commutes with ∂_j for $1 \le j \le n-1$. We have

$$\partial_j \,\mathrm{PI}_{\mathfrak{h}}\, f = \mathrm{PI}_{\mathfrak{h}}(\partial_j f), \quad 1 \le j \le n-1. \tag{3.26}$$

Hence, if $f \in C^1(\mathbb{R}^{n-1})$ has compact support, $\partial_j \,\mathrm{PI}_{\mathfrak{h}}\, f$ is bounded and continuous on $\overline{\mathbb{R}}^n_+$, for $1 \le j \le n-1$. We will estimate the t-derivative in §4.

4. Fourier integral representation of $\mathrm{PI}_{\mathfrak{h}} f$

Using the Fourier transform

$$\hat{g}(\xi) = \int\limits_{\mathbb{R}^{n-1}} g(y) e^{-iy\cdot\xi}\, dy, \tag{4.1}$$

we can write $\mathrm{PI}_{\mathfrak{h}} f$ in (3.19) as

$$\mathrm{PI}_{\mathfrak{h}} f(x + te_n) = (2\pi)^{-(n-1)} \int\limits_{\mathbb{R}^{n-1}} \hat{p}_{\mathfrak{h}}(t\xi) \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi, \tag{4.2}$$

where

$$p_{\mathfrak{h}}(y) = p_{\mathfrak{h}}(1, y) = C(|y|^2 + 1)^{-(n-1)}. \tag{4.3}$$

The constant C is the one that makes $\hat{p}_{\mathfrak{h}}(0) = 1$. As is well known, for the Euclidean Poisson integral (3.20), we have

$$\mathrm{PI}_{\mathfrak{e}} f(x + te_n) = (2\pi)^{-(n-1)} \int\limits_{\mathbb{R}^{n-1}} e^{-t|\xi|} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi. \tag{4.4}$$

Here we want to analyze $\hat{p}_{\mathfrak{h}}(t\xi)$. Note that

$$\hat{p}_{\mathfrak{h}}(\xi) = C(1 - \Delta_\xi)^{-\alpha} \delta(\xi), \quad \alpha = n - 1. \tag{4.5}$$

From this we see that, for all $\varepsilon > 0$,

$$\hat{p}_{\mathfrak{h}} \in H^{(3/2)(n-1)-\varepsilon,2}(\mathbb{R}^{n-1}), \ \text{ hence } \ \hat{p}_{\mathfrak{h}} \in C^{n-1-\varepsilon}(\mathbb{R}^{n-1}). \tag{4.6}$$

Also

$$\hat{p}_{\mathfrak{h}} \in C^\infty(\mathbb{R}^{n-1} \setminus 0) \ \text{ and is exponentially decreasing as } \ |\xi| \to \infty. \tag{4.7}$$

Note from (4.6) that

$$n \geq 3 \Longrightarrow \hat{p}_{\mathfrak{h}} \in C^1(\mathbb{R}^{n-1}). \tag{4.8}$$

Also, by radial symmetry, $\hat{p}_{\mathfrak{h}}$ is a radial function. Hence

$$\nabla_\xi \hat{p}_{\mathfrak{h}}(0) = 0. \tag{4.9}$$

It follows that we can write

$$\hat{p}_{\mathfrak{h}}(\xi) = Q_n(|\xi|), \quad Q_n \in C^1([0,\infty)), \quad Q'(0) = 0, \tag{4.10}$$

and also

$$Q_n \in C^\infty((0,\infty)), \quad Q_n(r) \ \text{ exponentially decreasing as } \ r \to \infty. \tag{4.11}$$

Then (4.2) becomes

$$\mathrm{PI}_{\mathfrak{h}}\, f(x+te_n) = (2\pi)^{-(n-1)} \int\limits_{\mathbb{R}^{n-1}} Q_n(t|\xi|)\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi, \tag{4.12}$$

and we get

$$\frac{\partial}{\partial t}\mathrm{PI}_{\mathfrak{h}}\, f(x+te_n) = (2\pi)^{-(n-1)} \int\limits_{\mathbb{R}^{n-1}} Q_n'(t|\xi|)|\xi|\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi. \tag{4.13}$$

The following is an immediate consequence of (4.10)–(4.13), together with the Lebesgue dominated convergence theorem.

Proposition 4.1. *If $f \in \mathcal{S}(\mathbb{R}^{n-1})$ and $n \geq 3$, then*

$$\frac{\partial}{\partial t}\mathrm{PI}_{\mathfrak{h}}\, f \in C(\overline{\mathbb{R}}^n_+), \tag{4.14}$$

and

$$\lim_{t\searrow 0} \frac{\partial}{\partial t}\mathrm{PI}_{\mathfrak{h}}\, f(x+te_n) = 0. \tag{4.15}$$

Recalling the discussion around (3.23), we have the following.

Corollary 4.2. *For $n \geq 2$,*

$$\mathrm{PI}_{\mathcal{H}} : C^\infty(S^{n-1}) \longrightarrow C^1(\overline{B}^n). \tag{4.16}$$

From here, the arguments involving (1.6)–(1.8) give the following partial extension of (1.9).

Theorem 4.3. *Given $f \in C^\infty(S^{n-1})$, $u = \mathrm{PI}_{\mathcal{H}}\, f$ satisfies*

$$|du(x)|_{\mathcal{H}} \leq Ce^{-\rho(x)}. \tag{4.17}$$

We next produce a sharpening of Proposition 4.1, when $n \geq 4$. In such a case, we can use (4.6) to sharpen (4.8) to

$$n \geq 4 \Longrightarrow \hat{p}_{\mathfrak{h}} \in C^2(\mathbb{R}^{n-1}). \tag{4.17}$$

Then we can supplement (4.10) by $Q_n \in C^2([0,\infty))$, and (4.13) by

$$\frac{\partial^2}{\partial t^2}\mathrm{PI}_{\mathfrak{h}}\, f(x+te_n) = (2\pi)^{-(n-1)} \int\limits_{\mathbb{R}^{n-1}} Q_n''(t|\xi|)|\xi|^2\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi. \tag{4.18}$$

Hence, if $f \in \mathcal{S}(\mathbb{R}^{n-1})$ we have $\partial_t^2\,\mathrm{PI}_{\mathfrak{h}}\, f \in C(\overline{\mathbb{R}}^n_+)$ and

$$\frac{\partial^2}{\partial t^2}\mathrm{PI}_{\mathfrak{h}}\, f(x+te_n)\Big|_{t=0} = -Q_n''(0)\Delta f(x), \tag{4.19}$$

where $\Delta = \partial_1^2 + \cdots + \partial_{n-1}^2$. We have the following improvement of (4.15).

Proposition 4.4. *If* $n \geq 4$ *and* $f \in \mathcal{S}(\mathbb{R}^{n-1})$*, then* $u = \mathrm{PI}_{\mathfrak{h}}\, f$ *satisfies*

$$|\partial_t u(x + te_n)| \leq Ct. \tag{4.20}$$

(For the case $n = 3$, see (5.22).)

Corollary 4.5. *For* $n \geq 4$*,*

$$\mathrm{PI}_{\mathcal{H}} : C^\infty(S^{n-1}) \longrightarrow C^2(\overline{B}^n). \tag{4.21}$$

We obtain a finer analysis of $\mathrm{PI}_{\mathfrak{h}}\, f$ in the following section.

5. Finer analysis of $\mathrm{PI}_{\mathfrak{h}}\, f$

Recall that

$$\mathrm{PI}_{\mathfrak{h}}\, f(x+te_n) = (2\pi)^{-(n-1)} \int_{\mathbb{R}^{n-1}} \hat{p}_{\mathfrak{h}}(t\xi)\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi, \tag{5.1}$$

with

$$\begin{aligned} \hat{p}_{\mathfrak{h}}(\xi) &= C(1-\Delta_\xi)^{-(n-1)}\delta(\xi) \\ &= Q_n(|\xi|). \end{aligned} \tag{5.2}$$

We have (4.6)–(4.7) for $\hat{p}_{\mathfrak{h}}$ and (4.10)–(4.11) for Q_n. We now want to go further. It is classical (and most of the supporting calculations are given in [St]) that, for $\xi \in \mathbb{R}^{n-1}$, $\alpha > 0$,

$$(1-\Delta_\xi)^{-\alpha}\delta(\xi) = C|\xi|^{\alpha-(n-1)/2}K_{(n-1)/2-\alpha}(|\xi|), \tag{5.3}$$

where $K_\nu(r)$ is the modified Bessel function called MacDonald's function. Hence (using $K_{-\nu} = K_\nu$) we have

$$Q_n(|\xi|) = C|\xi|^{(n-1)/2}K_{(n-1)/2}(|\xi|). \tag{5.4}$$

If n is an even integer, then $\nu = (n-1)/2$ is a half-integer, while if n is an odd integer, then $\nu = (n-1)/2$ is an integer. We discuss some properties of K_ν for such ν. Details can be found in [Leb]. First,

$$K_{1/2}(r) = \Big(\frac{\pi}{2r}\Big)^{1/2} e^{-r}, \tag{5.5}$$

so for $n=2$ the formula

$$\mathrm{PI}_{\mathfrak{h}}\, f(x+te_n) = (2\pi)^{-(n-1)} \int_{\mathbb{R}^{n-1}} Q_n(t|\xi|)\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi \tag{5.6}$$

agrees with (4.4). We can proceed from here via the recurrence relation

$$K_{\nu+1}(r) = -r^\nu \frac{d}{dr}\Big(r^{-\nu}K_\nu(r)\Big), \tag{5.7}$$

which implies

$$r^{\nu+1}K_{\nu+1}(r) = -r\frac{d}{dr}\Big(r^\nu K_\nu(r)\Big) + 2\nu r^\nu K_\nu(r). \tag{5.8}$$

Starting with (5.5), we get, for example,

$$r^{3/2}K_{3/2}(r) = \sqrt{\frac{\pi}{2}}\big(re^{-r} + e^{-r}\big). \tag{5.9}$$

It follows inductively from (5.8) that if $\ell \in \mathbb{Z}^+$,

$$r^{\ell+1/2}K_{\ell+1/2}(r) \in C^\infty([0,\infty)), \tag{5.10}$$

and hence

$$n \geq 2 \text{ even } \Longrightarrow Q_n \in C^\infty([0,\infty)). \tag{5.11}$$

Consequently, for n even, Propositions 4.1–4.2 readily extend as follows.

Proposition 5.1. *If $n \geq 2$ is even, then*

$$\mathrm{PI}_{\mathfrak{h}} : \mathcal{S}(\mathbb{R}^{n-1}) \longrightarrow C^\infty(\overline{\mathbb{R}}^n_+), \tag{5.12}$$

and, for all $\ell \in \mathbb{Z}^+$,

$$\frac{\partial^\ell}{\partial t^\ell}\,\mathrm{PI}_{\mathfrak{h}}\, f(x + te_n) = (2\pi)^{-(n-1)} \int\limits_{\mathbb{R}^{n-1}} Q^{(\ell)}(t|\xi|)|\xi|^\ell \hat{f}(\xi)e^{ix\cdot\xi}\, d\xi. \tag{5.13}$$

Hence, for such n,

$$\mathrm{PI}_{\mathcal{H}} : C^\infty(S^{n-1}) \longrightarrow C^\infty(\overline{B}^n). \tag{5.14}$$

Next, if $n = 2\ell + 1$ is odd, then $\nu = (n-1)/2 = \ell$ is an integer. In such a case (cf. [Leb]) we have

(5.15)

$$\begin{aligned}\Big(\frac{r}{2}\Big)^\ell & K_\ell(r) \\ &= \frac{1}{2}\sum_{k=0}^{\ell-1} \frac{(-1)^k(\ell-k-1)!}{k!}\Big(\frac{r}{2}\Big)^{2k} \\ &\quad + \frac{(-1)^\ell}{2}\Big(\frac{r}{2}\Big)^{2\ell}\sum_{k=0}^\infty \frac{1}{k!(k+\ell)!}\Big(\frac{r}{2}\Big)^{2k}\Big[2\log\frac{r}{2} - \psi(k+1) - \psi(k+\ell+1)\Big],\end{aligned}$$

where (with γ denoting the Euler constant)

$$\psi(1) = -\gamma, \quad \psi(\ell+1) = -\gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{\ell}, \quad \ell = 1, 2, 3 \ldots. \tag{5.16}$$

For example,

$$Q_3(|\xi|) = C|\xi|K_1(|\xi|) \\ = A_1(|\xi|^2) + B_1(|\xi|^2)|\xi|^2 \log|\xi|, \tag{5.17}$$

with $A_1, B_1 \in C^\infty([0,\infty))$. More generally,

$$Q_{2\ell+1}(|\xi|) = C|\xi|^\ell K_\ell(|\xi|) \\ = A_\ell(|\xi|^2) + B_\ell(|\xi|^2)|\xi|^{2\ell} \log|\xi|, \tag{5.18}$$

with $A_\ell, B_\ell \in C^\infty([0,\infty))$. Here we see that the conclusion for regularity on Q_n on $[0,\infty)$ that follows from (4.6) is fairly sharp:

$$n = 2\ell + 1 \Longrightarrow Q_n \in C^{2\ell-\varepsilon}([0,\infty)), \quad Q_n \notin C^{2\ell}([0,\infty)). \tag{5.19}$$

Note from (5.17) that

$$Q_3'(|\xi|) = \widetilde{A}(|\xi|) + \widetilde{B}(|\xi|)|\xi| \log|\xi|, \tag{5.20}$$

with $\widetilde{A}, \widetilde{B} \in C^\infty([0,\infty))$. Hence, for $f \in \mathcal{S}(\mathbb{R}^2)$,

$$\frac{\partial}{\partial t} \mathrm{PI}_{\mathfrak{h}}\, f(x + te_3) = (2\pi)^{-2} \int_{\mathbb{R}^2} Q_3'(t|\xi|)|\xi|\hat{f}(\xi) e^{ix\cdot\xi}\, d\xi \tag{5.21}$$

yields

$$\Bigl| \frac{\partial}{\partial t} \mathrm{PI}_{\mathfrak{h}}\, f(x + te_3) \Bigr| \le C\, t \log \frac{1}{t}, \tag{5.22}$$

which takes the place of (4.20) when $n = 3$ and improves on (4.15) in that case.

6. Slightly better than C^1 boundary data

Recall that

$$
\begin{aligned}
\mathrm{PI}_{\mathfrak{h}} f(x+te_n) &= t^{-(n-1)} \int\limits_{\mathbb{R}^{n-1}} f(x-y) p_{\mathfrak{h}}\Big(\frac{y}{t}\Big)\, dy \\
&= (2\pi)^{-(n-1)} \int\limits_{\mathbb{R}^{n-1}} Q_n(t|\xi|) \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi,
\end{aligned}
\tag{6.1}
$$

with

$$
p_{\mathfrak{h}}(y) = C(|y|^2+1)^{-(n-1)}, \quad Q_n(|\xi|) = \hat{p}_{\mathfrak{h}}(\xi). \tag{6.2}
$$

Using (5.4), we saw that

$$
n \geq 2 \text{ even } \Longrightarrow Q_n \in C^\infty([0,\infty)), \tag{6.3}
$$

while

$$
n = 2\ell+1 \Longrightarrow Q_n(|\xi|) = A_\ell(|\xi|^2) + B_\ell(|\xi|^2)|\xi|^{2\ell} \log|\xi|, \tag{6.4}
$$

with $A_\ell, B_\ell \in C^\infty([0,\infty))$. Also, in all cases,

$$
Q_n \in C^\infty((0,\infty)), \quad \text{exponentially decreasing at } \infty. \tag{6.5}
$$

We mention that one could avoid (5.4) and special function theory, and deduce (6.3)–(6.4) from asymptotic analysis developed in Chapter 3, §8 of [T3].

It is clear from (6.1)–(6.2) that if f is bounded and continuous on $\mathbb{R}^{n-1}$, then $\mathrm{PI}_{\mathfrak{h}} f \in C(\overline{\mathbb{R}}^n_+)$. Since, for $1 \leq j \leq n-1$, $\partial_j \mathrm{PI}_{\mathfrak{h}} f = \mathrm{PI}_{\mathfrak{h}}(\partial_j f)$, if also $\partial_j f$ is bounded and continuous on $\mathbb{R}^{n-1}$, then $\partial_j \mathrm{PI}_{\mathfrak{h}} f \in C(\overline{\mathbb{R}}^n_+)$. As noted in §5, it follows from (6.4) that $Q_n \in C^1([0,\infty))$ in all cases, and

$$
\frac{\partial}{\partial t} \mathrm{PI}_{\mathfrak{h}} f(x+te_n) = (2\pi)^{-(n-1)} \int\limits_{\mathbb{R}^{n-1}} Q'(t|\xi|)|\xi| \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi. \tag{6.6}
$$

Since Q_n' is bounded and continuous on $[0,\infty)$ it readily follows that $\partial_t \mathrm{PI}_{\mathfrak{h}} f \in C(\overline{\mathbb{R}}^n_+)$ whenever $f \in \mathcal{S}(\mathbb{R}^{n-1})$, and more generally whenever $|\xi|\hat{f}(\xi) \in L^1(\mathbb{R}^{n-1})$. We want to establish the following stronger result.

Proposition 6.1. *Given $f \in C^1(\mathbb{R}^{n-1})$ with compact support, define $g \in \mathcal{S}'(\mathbb{R}^{n-1})$ by*

$$\hat{g}(\xi) = |\xi|\hat{f}(\xi). \tag{6.7}$$

If g is bounded and continuous on $\mathbb{R}^{n-1}$, then

$$\frac{\partial}{\partial t}\,\mathrm{PI}_{\mathfrak{h}}\, f \in C(\overline{\mathbb{R}}^n_+). \tag{6.8}$$

Proof. We have

$$\begin{aligned} \frac{\partial}{\partial t}\,\mathrm{PI}_{\mathfrak{h}}\, f(x+te_n) &= (2\pi)^{-(n-1)} \int\limits_{\mathbb{R}^{n-1}} Q_n'(t|\xi|)\hat{g}(\xi)e^{ix\cdot\xi}\,d\xi \\ &= t^{-(n-1)} \int\limits_{\mathbb{R}^{n-1}} g(x-y) r_n\Big(\frac{y}{t}\Big)\,dy, \end{aligned} \tag{6.9}$$

where

$$\hat{r}_n(\xi) = Q_n'(|\xi|). \tag{6.10}$$

The continuity result (6.8) is an immediate consequence of the result

$$r_n \in L^1(\mathbb{R}^{n-1}), \tag{6.11}$$

so it remains to prove (6.11). Given (6.3)–(6.5), it suffices to prove the following.

Lemma 6.2. *Given $\varphi \in C_0^\infty(\mathbb{R}^{n-1})$, define*

$$\psi_0(\xi) = \varphi(\xi)|\xi|, \quad \psi_1(\xi) = \varphi(\xi)|\xi|\log|\xi|. \tag{6.12}$$

Then

$$\hat{\psi}_0,\ \hat{\psi}_1 \in L^1(\mathbb{R}^{n-1}). \tag{6.13}$$

Proof. It follows from asymptotic analysis of Fourier transforms developed in Chapter 3, §8 of [T3] that

$$|\hat{\psi}_0(y)| \le C(1+|y|)^{-n}, \tag{6.14}$$

which gives (6.13) for $\hat{\psi}_0$. The asymptotic analysis of $\hat{\psi}_1(y)$ will be given in Appendix D.

We discuss when (6.7) defines a bounded continuous g. Note that

$$g = \sum_j R_j \partial_j f, \tag{6.15}$$

where R_j (a Riesz transform) is Fourier multiplication by $-i\xi_j/|\xi|$, hence convolution by a principal value kernel on $\mathbb{R}^{-(n-1)}$ that is homogeneous of degree $-(n-1)$. Given $f \in C_0^1(\mathbb{R}^{n-1})$, $\partial_j f$ is continuous and has compact support. Clearly g is bounded and continuous outside any neighborhood of supp f, so whether g is bounded and continuous on $\mathbb{R}^{n-1}$ is a local question. It is well known that if a function h has compact support in $\mathbb{R}^{n-1}$ and is Hölder continuous with positive exponent, then $R_j h$ is bounded and continuous. Hence Proposition 6.1 applies to compactly supported $f \in C^{1+\varepsilon}(\mathbb{R}^{n-1})$, given $\varepsilon > 0$. More generally, given f compactly supported, if

$$f \in B^1_{\infty,1}(\mathbb{R}^{n-1}), \tag{6.16}$$

where one has

$$f \in B^s_{\infty,1}(\mathbb{R}^{n-1}) \Longleftrightarrow \sum_{\ell \geq 0} 2^{\ell s} \|\psi_\ell(D) f\|_{L^\infty} < \infty, \tag{6.17}$$

where $\{\psi_\ell : \ell \geq 0\}$ is a Littlewood-Paley partition of unity, then $R_j \partial_j f$ is bounded and continuous, so Proposition 6.1 applies to such functions.

Moving over to the setting of B^n, we have the following.

Proposition 6.3. *Given* $\varepsilon > 0$,

$$\mathrm{PI}_{\mathcal{H}} : C^{1+\varepsilon}(S^{n-1}) \longrightarrow C^1(\overline{B}^n). \tag{6.18}$$

More generally,

$$\mathrm{PI}_{\mathcal{H}} : B^1_{\infty,1}(S^{n-1}) \longrightarrow C^1(\overline{B}^n). \tag{6.19}$$

A. C^1 boundary data and beyond

For now, we restrict attention to B^n with $n = 2$, so $\mathrm{PI}_{\mathcal{H}}$ and PI coincide. As noted in the introduction, it is not the case that $\mathrm{PI} : C^1(S^1) \to C^1(\overline{B}^2)$. One does have

$$\mathrm{PI} : C^1(S^1) \longrightarrow C^\alpha(\overline{B}^2), \quad \forall\, \alpha < 1, \tag{A.1}$$

which implies for $u = \mathrm{PI}\, f = \mathrm{PI}_{\mathcal{H}}\, f$,

$$|du(x)|_{\mathcal{E}} \le C_\varepsilon (1 - |x|)^{-\varepsilon}, \quad \forall\, \varepsilon > 0, \tag{A.2}$$

hence, via (1.6) and (1.8),

$$|du(x)|_{\mathcal{H}} \le C_\varepsilon e^{(1-\varepsilon)\rho(x)}, \quad \forall\, \varepsilon > 0, \tag{A.3}$$

a result consistent with (1.9A). Here we go further, and produce a sharper estimate for a broader class of boundary data. Namely, we take

$$f \in C^1_*(S^1) = B^1_{\infty,\infty}(S^1). \tag{A.4}$$

One definition of $C^\alpha_*(S^1) = B^\alpha_{\infty,\infty}(S^1)$ is that a distribution f on S^1 belongs to this space if and only if

$$\|\psi_\ell(D) f\|_{L^\infty} \le C 2^{-\ell\alpha}, \tag{A.5}$$

where $\{\psi_\ell : \ell \ge 0\}$ is a Littlewood-Paley partition of unity and $\psi_\ell(D)$ is Fourier multiplication by ψ_ℓ. Given $f \in C^1_*(S^1)$, we have

$$\frac{\partial}{\partial \theta} \mathrm{PI}\, f = \mathrm{PI}\, \frac{\partial f}{\partial \theta}, \quad r \frac{\partial}{\partial r} \mathrm{PI}\, f = -\,\mathrm{PI}(\Lambda f), \tag{A.6}$$

where

$$\Lambda f(\theta) = \sum_k |k| \hat{f}(k) e^{ik\theta}. \tag{A.7}$$

As is well known,

$$f \in C^1_*(S^1) \Longrightarrow \partial_\theta f,\ \Lambda f \in C^0_*(S^1). \tag{A.8}$$

We have the following.

Lemma A.1. *If* $g \in C^0_*(S^1)$, *then*

$$|\operatorname{PI} g(re^{i\theta})| \le C \log \frac{1}{1-r}. \tag{A.9}$$

Granted this lemma, we have

$$f \in C^1_*(S^1) \Longrightarrow |d \operatorname{PI} f(x)|_{\mathcal{E}} \le C \log \frac{1}{1-|x|} \approx C\rho(x), \tag{A.10}$$

and hence $u = \operatorname{PI}_{\mathcal{H}} f$ satisfies

$$|du(x)|_{\mathcal{H}} \le C(1+\rho(x))e^{-\rho(x)}, \tag{A.11}$$

an estimate that is stronger than (A.3).

To prove (A.9), write

$$\begin{aligned} \operatorname{PI} g(re^{i\theta}) &= \sum_{\ell \ge 0} G_{\ell r}(\theta), \\ G_{\ell r}(\theta) &= \sum_k \psi_\ell(k) r^{|k|} \hat{g}(k) e^{ik\theta}. \end{aligned} \tag{A.12}$$

Then

$$\begin{aligned} \|G_{\ell r}\|_{L^\infty(S^1)} &\le C r^{2^\ell} \|\psi_\ell(D) g\|_{L^\infty} \\ &\le C r^{2^\ell} \|g\|_{C^0_*}, \end{aligned} \tag{A.13}$$

so

$$|\operatorname{PI} g(re^{i\theta})| \le C \|g\|_{C^0_*} \sum_{\ell \ge 0} r^{2^\ell}. \tag{A.14}$$

Now, with $r = e^{-t},\ t \in (0, 1/2]$,

$$\begin{aligned} \sum_{\ell \ge 0} r^{2^\ell} &\le C \int_1^\infty e^{-te^s}\, ds \\ &\le C \int_1^\infty e^{-ty} \frac{dy}{y} \\ &\le C \log \frac{1}{t} \\ &\approx C \log \frac{1}{1-r}, \end{aligned} \tag{A.15}$$

yielding (A.9).

B. Finer results for even n

Recall that

$$\mathrm{PI}_{\mathfrak{h}}\, f(x+te_n)=(2\pi)^{-(n-1)}\int\limits_{\mathbb{R}^{n-1}} Q_n(t|\xi|)\hat{f}(\xi)e^{ix\cdot\xi}\,d\xi, \tag{B.1}$$

with

$$Q_n(r)=C_n r^{(n-1)/2}K_{(n-1)/2}(r), \tag{B.2}$$

so if $n=2\ell+2$ is even,

$$Q_{2\ell+2}(r)=C'_\ell r^{\ell+1/2}K_{\ell+1/2}(r). \tag{B.3}$$

For $\ell=0$ we have (5.5), and for larger $\ell\in\mathbb{Z}^+$ we can apply the recursion (5.8). This can be rewritten as follows. Define $q_\ell(r)$ by

$$Q_{2\ell+2}(r)=q_\ell(r)e^{-r}. \tag{B.4}$$

Then it follows from (5.4)–(5.5) and the fact that

$$Q_n(0)=1,\quad \text{hence}\ \ q_\ell(0)=1, \tag{B.5}$$

that

$$q_0(r)=1. \tag{B.6}$$

Then it follows from the recursion (5.8) together with (B.5) that

$$q_{\ell+1}(r)=q_\ell(r)+\frac{r}{2\ell+1}(q_\ell(r)-q'_\ell(r)). \tag{B.7}$$

In particular,

$$q_1(r)=1+r,\quad q_2(r)=1+r+\frac{r^2}{3}. \tag{B.8}$$

Generally, $q_\ell(r)$ is a polynomial of degree ℓ in r:

$$q_\ell(t)=\sum_{j=0}^{r}\alpha_{\ell j}r^j, \tag{B.9}$$

so, for $n=2\ell+2$,

$$\mathrm{PI}_{\mathfrak{h}}\, f(x+te_n)=(2\pi)^{-(n-1)}\sum_{j=0}^{\ell}\alpha_{\ell j}t^j\int |\xi|^j e^{-t|\xi|}\hat{f}(\xi)e^{ix\cdot\xi}\,d\xi. \tag{B.10}$$

Comparison with (4.4) for the Euclidean Poisson integral gives the following.

Proposition B.1. *If $n = 2\ell + 2$ and f is given on $\mathbb{R}^{n-1}$,*

$$\mathrm{PI}_{\mathfrak{h}}\, f(x + te_n) = \sum_{\ell=0}^{\ell} \alpha_{\ell j} t^j \Big(-\frac{\partial}{\partial t}\Big)^j \mathrm{PI}_{\mathfrak{e}}\, f(x + te_n). \tag{B.11}$$

Note that the right side of (B.11) can be written

$$\sum_{j=0}^{\ell} \beta_{\ell j} \Big(t\frac{\partial}{\partial t}\Big)^j \mathrm{PI}_{\mathfrak{e}}\, f(x + te_n), \tag{B.12}$$

with $\beta_{\ell j} \in \mathbb{R}$. This allows us to apply standard ellpitic regularity results for $\mathrm{PI}_{\mathfrak{e}}$. For example, if f has compact support in $\mathbb{R}^{n-1}$ and $u = \mathrm{PI}_{\mathfrak{e}}\, f$, then

$$f \in C^r(\mathbb{R}^{n-1}) \Rightarrow u,\ (t\partial_t)^j u \in C^r(\overline{\mathbb{R}}^n_+), \quad \forall\, j \in \mathbb{N}, \tag{B.13}$$

if $r > 0,\ r \notin \mathbb{Z}$, with a Zygmund space replacement for $r \in \mathbb{Z}^+$. Hence the same regularity result holds for $u = \mathrm{PI}_{\mathfrak{h}}\, f$. This regularity result carries over to the setting of the ball, yielding the following sharpeninig of (5.14) and (6.18).

Proposition B.2. *If n is even, then, for $r > 0$,*

$$\mathrm{PI}_{\mathcal{H}} : C^r_*(S^{n-1}) \longrightarrow C^r_*(\overline{B}^n). \tag{B.14}$$

Similarly, other standard elliptic regularity results for $u = \mathrm{PI}_{\mathfrak{e}}\, f$ extend. We have

$$\mathrm{PI}_{\mathcal{H}} : H^{s,2}(S^{n-1}) \longrightarrow H^{s+1/2,2}(B^n), \tag{B.15}$$

for $s \geq -1/2$, and, more generally, for $1 < p < \infty$,

$$\mathrm{PI}_{\mathcal{H}} : B^s_{p,p}(S^{n-1}) \longrightarrow H^{s+1/p,p}(B^n), \tag{B.16}$$

provided $n \geq 2$ is even.

Remark 1. Let us be more explicit about (B.12) when $n = 4$. We then have

$$\mathrm{PI}_{\mathfrak{h}}\, f(x + te_n) = \mathrm{PI}_{\mathfrak{e}}\, f(x + te_n) - t\partial_t\, \mathrm{PI}_{\mathfrak{e}}\, f(x + te_n). \tag{B.17}$$

Note that this implies

$$\partial_t\, \mathrm{PI}_{\mathfrak{h}}\, f(x + te_n) = t(\partial_1^2 + \partial_2^2 + \partial_3^2)\, \mathrm{PI}_{\mathfrak{e}}\, f(x + te_n), \tag{B.18}$$

which is consistent with Proposition 3.1 and with Proposition 4.1.

Remark 2. A correspondence between $\mathrm{PI}_{\mathcal{H}}$ and its Euclidean counterpart on S^{n-1} analogous to (B.11) is given in [GJ], which also noted (B.14) as a consequence.

C. Further results for odd n

As seen in §5, if $n = 2\ell + 1$ is odd, we can write

$$Q_n(|\xi|) = A(|\xi|) + B(|\xi|) \log |\xi|, \tag{C.1}$$

with

$$A, B \in C^\infty([0, \infty)), \quad \text{exponentially decreasing,} \tag{C.2}$$

and furthermore,

$$B(|\xi|) = B^\#(|\xi|)|\xi|^{2\ell}, \tag{C.3}$$

where $B^\#$ has property (C.2). Also, by (5.18),

$$A \text{ and } B^\# \text{ are smooth functions of } |\xi|^2. \tag{C.3A}$$

It follows that, for $f \in L^1(\mathbb{R}^{n-1})$,

$$\mathrm{PI}_{\mathfrak{h}} f = \mathrm{PI}_{\mathfrak{a}} f + \mathrm{PI}_{\mathfrak{bl}} f, \tag{C.4}$$

with

$$\begin{aligned} \mathrm{PI}_{\mathfrak{a}} f(x + te_n) &= (2\pi)^{-(n-1)} \int A(t|\xi|) \hat{f}(\xi) e^{ix\cdot\xi} \, d\xi, \\ \mathrm{PI}_{\mathfrak{bl}} f(x + te_n) &= (2\pi)^{-(n-1)} \int B(t|\xi|)(\log |t\xi|) \hat{f}(\xi) e^{ix\cdot\xi} \, d\xi. \end{aligned} \tag{C.5}$$

Note that

$$\mathrm{PI}_{\mathfrak{bl}} f = (\log t) \, \mathrm{PI}_{\mathfrak{b}} f + \mathrm{PI}_{\mathfrak{l}} f, \tag{C.6}$$

where

$$\begin{aligned} \mathrm{PI}_{\mathfrak{b}} f(x + te_n) &= (2\pi)^{-(n-1)} \int B(t|\xi|) \hat{f}(\xi) e^{ix\cdot\xi} \, d\xi, \\ \mathrm{PI}_{\mathfrak{l}} f(x + te_n) &= (2\pi)^{-(n-1)} \int B(t|\xi|)(\log |\xi|) \hat{f}(\xi) e^{ix\cdot\xi} \, d\xi. \end{aligned} \tag{C.7}$$

Also,

$$\begin{aligned} \mathrm{PI}_{\mathfrak{l}} f(x + te_n) &= t^{2\ell} \, \mathrm{PI}_{\mathfrak{l}}^\# f(x + te_n) \\ &= (2\pi)^{-(n-1)} t^{2\ell} \int B^\#(t|\xi|)|\xi|^{2\ell} (\log |\xi|) \hat{f}(\xi) e^{ix\cdot\xi} \, d\xi. \end{aligned} \tag{C.8}$$

It is elementary that

$$\mathrm{PI}_{\mathfrak{a}}, \mathrm{PI}_{\mathfrak{b}}, \mathrm{PI}_{\mathfrak{l}}, \mathrm{PI}_{\mathfrak{l}}^\# : \mathcal{S}(\mathbb{R}^{n-1}) \longrightarrow C^\infty(\overline{\mathbb{R}}^n_+). \tag{C.9}$$

Transforming to the setting of the ball, we have the following.

Proposition C.1. *Assume $n \geq 3$ is odd. Given $f \in C^\infty(S^{n-1})$, we have*

$$\mathrm{PI}_{\mathcal{H}} f(x) = u(x) + v(x)(1-|x|^2)^{n-1}\log(1-|x|^2), \quad u, v \in C^\infty(\overline{B}^n). \tag{C.10}$$

We establish the following Zygmund space mapping properties.

Proposition C.2. *Assume $n \geq 3$ is odd. Take $\varphi \in C_0^\infty(\overline{\mathbb{R}}^n_+)$, and let f be a compactly supported function on $\mathbb{R}^{n-1}$. Then*

$$f \in C^r_*(\mathbb{R}^{n-1}) \Longrightarrow \varphi\, \mathrm{PI}_{\mathfrak{a}} f \in C^r_*(\overline{\mathbb{R}}^n_+), \quad \forall\, r \in (0,\infty), \tag{C.11}$$

and

$$f \in C^r_*(\mathbb{R}^{n-1}) \Longrightarrow \varphi\, \mathrm{PI}_{\mathfrak{bl}} f \in C^r_*(\overline{\mathbb{R}}^n_+), \quad \forall\, r \in (0,n-1). \tag{C.11A}$$

As the stated result for $\mathrm{PI}_{\mathfrak{a}} f$ is relatively straightforward, we concentrate on $\mathrm{PI}_{\mathfrak{bl}} f$. We start with the case $0 < r < 1$. We have

$$\begin{aligned} &t\partial_t\, \mathrm{PI}_{\mathfrak{bl}} f(x+te_n) \\ &= (2\pi)^{(n-1)} \int B'(t|\xi|)|t\xi|(\log|t\xi|)\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi \\ &\quad + (2\pi)^{(n-1)} \int B_1(t|\xi|)\, |t\xi|\, \hat{f}(\xi)e^{ix\cdot\xi}\, d\xi, \end{aligned} \tag{C.12}$$

where

$$B_1(r) = r^{-1}B(r) = O(r^{2\ell-1}), \quad \text{as} \quad r \to 0. \tag{C.13}$$

Using the notation

$$\begin{aligned} \mathrm{PI}^t_{\mathfrak{e}} f(x) &= \mathrm{PI}_{\mathfrak{e}} f(x+te_n) \\ &= (2\pi)^{-(n-1)} \int e^{-t|\xi|}\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi, \end{aligned} \tag{C.14}$$

and

$$P^t_F f(x) = (2\pi)^{-(n-1)} \int F(t\xi)\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi, \tag{C.15}$$

we see that the first term on the right side of (C.12) is equal to

$$P^t_F\, t\partial_t\, \mathrm{PI}^{t/2}_{\mathfrak{e}} f(x), \tag{C.16}$$

where

$$F(\xi) = B'(|\xi|)(\log|\xi|)e^{|\xi|/2}. \tag{C.17}$$

Note that $F(\xi)$ is smooth on $(0,\infty)$ and exponentially decreasing as $|\xi| \to \infty$. Furthermore, since $B'(0)=0$, Lemma 6.2 implies

$$\widehat{F} \in L^1(\mathbb{R}^{n-1}). \tag{C.18}$$

Hence

$$\|P_F^t \, t\partial_t \, \mathrm{PI}_{\mathfrak{e}}^{t/2} f\|_{L^\infty(\mathbb{R}^{n-1})} \le C \|t\partial_t \, \mathrm{PI}_{\mathfrak{e}}^{t/2} f\|_{L^\infty(\mathbb{R}^{n-1})}. \tag{C.19}$$

Now if $f \in C_*^r(\mathbb{R}^{n-1})$ is compactly supported (and $r \in (0,1)$), it is classical that

$$\|t\partial_t \, \mathrm{PI}_{\mathfrak{e}}^{t/2} f\|_{L^\infty} \le Ct^r. \tag{C.20}$$

This, together with (C.19) and a similar (but simpler) analysis of the second term on the right side of (C.12), yields

$$\sup_x |t\partial_t \, \mathrm{PI}_{\mathfrak{bl}} f(x+te_n)| \le Ct^r, \tag{C.21}$$

in this case. A similar argument gives

$$\sup_x |t\partial_j \, \mathrm{PI}_{\mathfrak{bl}} f(x+te_n)| \le Ct^r, \quad 1 \le j \le n-1. \tag{C.22}$$

The result (C.11A), for $r \in (0,1)$, follows from (C.21)–(C.22), by a standard argument.

Next, let us assume $r \in (1,2)$. We have

$$\begin{aligned} &t\partial_t^2 \, \mathrm{PI}_{\mathfrak{bl}} f(x+te_n) \\ &= (2\pi)^{-(n-1)} \int B''(t|\xi|)|t\xi|(\log|t\xi|)|\xi|\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi \\ &\quad + (2\pi)^{-(n-1)} \int B_2(t|\xi|)|t\xi|\,|\xi|\hat{f}(\xi)\, d\xi, \end{aligned} \tag{C.23}$$

where (with B_1 as in (C.13)),

$$B_2(r) = r^{-1}B'(r) + B_1'(r) = O(r^{2\ell-2}), \quad \text{as} \ \ r \to 0. \tag{C.24}$$

Parallel to (C.16), the first term on the right side of (C.23) is equal to

$$P_G^t \, t\partial_t^2 \, \mathrm{PI}_{\mathfrak{e}}^{t/2} f(x), \tag{C.25}$$

where

$$G(\xi) = B''(|\xi|)(\log|\xi|)e^{|\xi|/2}. \tag{C.26}$$

If $n \geq 5$, so $\ell \geq 2$, we have $B''(0) = 0$ and hence $\widehat{G} \in L^1(\mathbb{R}^{n-1})$, and arguments as in (C.19)–(C.21) yield

$$\sup_x |t\partial_t^2 \operatorname{PI}_{\mathfrak{bl}} f(x + te_n)| \leq Ct^{r-1}. \tag{C.27}$$

A similar argument (more parallel to (C.21)-(C.22) actually) gives

$$\sup_x |t\partial_t \partial_j \operatorname{PI}_{\mathfrak{bl}} f(x + te_n)| \leq t^{r-1}, \tag{C.28}$$

and

$$\sup_x |t\partial_j \partial_k \operatorname{PI}_{\mathfrak{bl}} f(x + te_n)| \leq Ct^{r-1}, \tag{C.29}$$

and (C.27)–(C.29) yields $\nabla_{x,t} \operatorname{PI}_{\mathfrak{bl}} f \in C_*^{r-1}(\overline{\mathbb{R}}^n_+)$, thus giving (C.11A) for $r \in (1, 2)$, at least if $\ell \geq 2$. The result for $r = 1$ follows by interpolation.

When $n = 3$ (so $\ell = 1$), $B''(0) \neq 0$, and we do not have $\widehat{G} \in L^1(\mathbb{R}^{n-1})$. To treat this case, we use

$$\begin{aligned} t^2\partial_t^3 &\operatorname{PI}_{\mathfrak{bl}} f(x + te_n) \\ &= (2\pi)^{-(n-1)} \int B'''(t|\xi|)|t\xi|^2(\log|t\xi|)|\xi|\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi \\ &\quad + (2\pi)^{-(n-1)} \int B''(t|\xi|)|t\xi|\,|\xi|\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi \\ &\quad + (2\pi)^{-(n-1)} \int B_2'(t|\xi|)\,|t\xi|^2\,|\xi|\,\hat{f}(\xi)e^{ix\cdot\xi}\, d\xi, \end{aligned} \tag{C.30}$$

with B_2 as in (C.24); in particular, B_2' has properties as in (C.2). As before, the firest term on the right side of (C.30) is the toughest. It is equal to

$$P_H^t\, t^2\partial_t^3 \operatorname{PI}_{\mathfrak{e}}^{t/2} f(x), \tag{C.31}$$

where

$$H(\xi) = B'''(|\xi|)(\log|\xi|)e^{|\xi|/2}. \tag{C.32}$$

This time, by (C.3A), $B'''(0) = 0$, so

$$\widehat{H} \in L^1(\mathbb{R}^{n-1})), \quad (n = 3). \tag{C.33}$$

Hence

$$\|P_H^t\, t^2\partial_t^3 \operatorname{PI}_{\mathfrak{e}}^{t/2} f\|_{L\infty} \leq C\|t^2\partial^3 \operatorname{PI}_e^{t/2} f\|_{L^\infty}, \tag{C.34}$$

which, for compactly supported $f \in C^r_*,\ r \in (1,2)$, is $\le Ct^{r-1}$. This plus simpler analyses of the rest of the terms in the right side of (C.30) gives

$$\sup_x |t^2 \partial_t^3 \operatorname{PI}_{\mathfrak{bl}} f(x+te_n)| \le Ct^{r-1}, \tag{C.35}$$

for $n = 3$, replacing (C.27). This plus analogues of (C.28)–(C.29) give (C.11A), for $n = 3,\ r \in (1,2)$.

For $n = 3$ (or for $r \in (0,2)$), the proof of Proposition C.2 is done. For $n \ge 5$ and $r \in (2, n-1)$, analogous arguments work, to prove Proposition C.2. We omit the details.

From here we have the following, which was proven in [GJ], via different arguments.

Proposition C.3. *Assume $n \ge 3$ is odd. Then*

$$\operatorname{PI}_{\mathcal{H}} : C^r_*(S^{n-1}) \longrightarrow C^r_*(\overline{B}^n), \quad \textit{for} \ \ r \in (0, n-1). \tag{C.36}$$

Here is a result that is true for all $r > 0$. Let $f \in C^r_*(\mathbb{R}^{n-1})$ be compactly supported. Then (C.4)–(C.6) hold, and

$$\begin{aligned} u &= \varphi \operatorname{PI}_{\mathfrak{a}} f \in C^r_*(\overline{\mathbb{R}}^n_+), \\ v &= \varphi \operatorname{PI}_{\mathfrak{b}} f \in C^r_*(\overline{\mathbb{R}}^n_+), \\ w &= \varphi \operatorname{PI}_{\mathfrak{l}} f \in C^s_*(\overline{\mathbb{R}}^n_+), \quad \forall\, s < r. \end{aligned} \tag{C.37}$$

Also

$$\partial_t^j v\big|_{t=0} = \partial_t^j w\big|_{t=0} = 0, \quad \text{for} \ \ 0 \le j \le \min(n-1, r). \tag{C.38}$$

Transfer to the ball gives the following.

Proposition C.4. *Assume $n \ge 3$ is odd. Then, for all $r > 0$,*

$$f \in C^r_*(S^{n-1}) \Longrightarrow \operatorname{PI}_{\mathcal{H}} f(x) = u(x) + w(x) + v(x) \log(1 - |x|^2), \tag{C.39}$$

with

$$u, v \in C^r_*(\overline{B}^n), \quad w \in C^s_*(\overline{B}^n),\ \forall\, s < r, \tag{C.40}$$

and

$$\partial_\nu^j v\big|_{S^{n-1}} = \partial_\nu^j w\big|_{S^{n-1}} = 0, \quad \textit{for} \ \ 0 \le j \le \min(n-1, r). \tag{C.41}$$

According to Proposition C.3, it seems there is some cancellation of singularities in the last two terms of (C.39) if $r < n-1$, but not if $r > n-1$.

D. Asymptotic analysis of $\hat{\psi}_1$ in (6.12)–(6.13)

Here we produce an asymptotic analysis of $\hat{\psi}_1(x)$ that will complete the proof of Lemma 6.2. Recall that

$$\psi_1(\xi) = \varphi(\xi)|\xi| \log |\xi|, \quad \varphi \in C_0^\infty(\mathbb{R}^{n-1}). \tag{D.1}$$

Note that

$$\hat{\psi}_1(x) = \hat{\varphi} * \hat{\psi}_L(x), \quad \psi_L(\xi) = |\xi| \log |\xi|, \tag{D.2}$$

and $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^{n-1})$, so it suffices to identify $\hat{\psi}_L$. We get this from the following well known formula (cf. [T3], Chapter 3, (8.33)).

$$\text{On } \mathbb{R}^{n-1}, \ \ \sigma_s(\xi) = |\xi|^s \Longrightarrow \hat{\sigma}_s(x) = F(s)|x|^{-s-(n-1)}, \tag{D.3}$$

with

$$F(s) = 2^{s+(n-1)/2} \, \frac{\Gamma((s+n-1)/2)}{\Gamma(-s/2)}, \tag{D.4}$$

provided $s \in \mathbb{C}$ satisfies

$$s \notin \{-(n-1) - 2j : j \in \mathbb{Z}^+\} \cup \{2j : j \in \mathbb{Z}^+\}, \tag{D.5}$$

where $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$. Applying $\partial/\partial s$ gives

$$\begin{aligned} &\psi_{s,L}(\xi) = |\xi|^s \log |\xi| \\ &\Longrightarrow \hat{\psi}_{s,L}(x) = F'(s)|x|^{-s-(n-1)} - F(s)|x|^{-s-(n-1)} \log |x|, \end{aligned} \tag{D.6}$$

provided (D.5) holds. Setting $s = 1$ gives

$$\hat{\psi}_L(x) = F'(1)|x|^{-n} - F(1)|x|^{-n} \log |x|. \tag{D.7}$$

This is clearly integrable on $\mathbb{R}^{n-1} \setminus B_1(0)$, so $\hat{\psi}_1 \in L^1(\mathbb{R}^{n-1})$.

E. Complex hyperbolic space and other symmetric spaces

Let $\mathcal{B}^n \subset \mathbb{C}^n$ be the unit ball, endowed with the metric tensor

$$ds^2 = \frac{1}{1-r^2} \sum (dx_j^2 + dy_j^2) + \frac{r^2(dr^2 + (J\,dr)^2)}{(1-r^2)^2}, \quad r = |z|. \tag{E.1}$$

This complete Riemannian manifold has a transitive group G of isometries, which are biholomorphic maps on $\mathcal{B}^n$. Unitary operators on $\mathbb{C}^n$ provide a subgroup of G, preserving 0, making $\mathcal{B}^n$ a rank-one symmetric space. Its Poisson integral

$$\mathrm{PI}_{\mathcal{B}} : C(S^{2n-1}) \longrightarrow C(\overline{\mathcal{B}}^n) \cap C^\infty(\mathcal{B}^n), \tag{E.2}$$

has the property that $u = \mathrm{PI}_{\mathcal{B}}\, f$ solves

$$\Delta_{\mathcal{B}} u = 0 \ \text{ on } \ \mathcal{B}^n, \quad u\big|_{S^{2n-1}} = f, \tag{E.3}$$

where $\Delta_{\mathcal{B}}$ is the Laplace-Beltrami operator on $\mathcal{B}^n$, with metric tensor (E.1). This can be constructed by a process similar to that used in §2. First, parallel to (2.1), we have

$$\mathrm{PI}_{\mathcal{B}}\, f(0) = \frac{1}{A_{2n-1}} \int\limits_{S^{2n-1}} f\, dS. \tag{E.4}$$

Then, given $z \in \mathcal{B}^n$, let

$$\mathcal{C}_z : \mathcal{B}^n \longrightarrow \mathcal{B}^n, \quad \mathcal{C}_z(0) = z, \tag{E.5}$$

be an element of G taking 0 to z. (Such a map dexends to a diffeomorphism of $\overline{\mathcal{B}}^n$ onto itself.) If $v_z = u \circ \mathcal{C}_z$, then $\Delta_{\mathcal{B}} u = 0 \Rightarrow \Delta_{\mathcal{B}} v_z = 0$, so

$$\mathrm{PI}_{\mathcal{B}}\, f(z) = v_z(0) = \frac{1}{A_{2n-1}} \int\limits_{S^{2n-1}} f \circ \mathcal{C}_z\, dS. \tag{E.6}$$

Now $\mathcal{C}_z$ in (E.5) is well defined only up to a factor in $U(n)$:

$$\mathcal{C}'_z = \mathcal{C}_z \circ R, \quad R \in U(n), \tag{E.7}$$

but altering $\mathcal{C}_z$ by such a factor leaves the right side of (E.6) unchanged. A proof that (E.6) gives the solution to (E.3) is analogous to arguments given in §2.

There is a Cayley transform of $\mathcal{B}^n$ onto the Siegel upper half space $\Omega^n \subset \mathbb{C}^n$. and a corresponding analogue of the Poisson integral of §3. The group $G \approx SU(n, 1)$, which acts on $\mathcal{B}^n$, also acts on Ω^n. It has an Iwasawa decomposition $G = NAK$. The nilpotent group N is isomorphic to the Heisenberg group $\mathbb{H}^{n-1}$; this group acts simply transitively on $\partial\Omega^n$, and NA acts simply transitively on Ω^n. The analogue of (3.19) is a map

$$\mathrm{PI}_{\mathfrak{ch}} : C_0^\infty(\mathbb{H}^{n-1}) \longrightarrow C(\overline{\Omega}^n). \tag{E.7A}$$

The formula for $\mathrm{PI}_{\mathfrak{ch}}$ is a special case of a more general class of formulas we briefly describe, in the setting of a rank-one symmetric space $X = G/K$. Again the Iwasawa decomposition $G = NAK$ plays a role, together with the associated Lie algebra decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$. In the rank one setting,

$$\mathfrak{n} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}, \tag{E.8}$$

and if

$$h = \exp(X + Y), \quad X \in \mathfrak{g}_{-\alpha}, \quad Y \in \mathfrak{g}_{-2\alpha}, \tag{E.9}$$

one has (cf. [Hel], pp. 65–67)

$$\mathrm{PI}_N : C_0^\infty(N) \longrightarrow C(\overline{X}), \tag{E.9A}$$

given by

$$\mathrm{PI}_N\, f(gK) = \int_N P(gK, h) f(h)\, dh, \tag{E.10}$$

where dh is Haar measure on N, and with $a_s = \exp(sH) \in A$ (which is one-dimensional here), and $\tilde{h} \in N$,

$$P(\tilde{h} a_s K, h) = P(a_s K, \tilde{h}^{-1} h), \tag{E.11}$$

with

$$P(a_s K, h) = \Bigl(\frac{e^{2s}}{(1 + c\|e^s X\|^2)^2 + 4c\|e^{2s} Y\|^2} \Bigr)^{p/2+q}, \tag{E.12}$$

where

$$p = \dim \mathfrak{g}_{-\alpha}, \quad q = \dim \mathfrak{g}_{-2\alpha}, \quad c = \frac{1}{4(p + 4q)}. \tag{E.13}$$

Taking $t = e^{-s}$ gives

$$P(a_s K, h) = t^{-(p+2q)} \Bigl[\Bigl(1 + c\Bigl\| \frac{X}{t} \Bigr\|^2 \Bigr)^2 + 4c\Bigl\| \frac{Y}{t^2} \Bigr\|^2 \Bigr]^{-(p/2+q)}. \tag{E.14}$$

We see that the Poisson integral of $f \in L^p(N)$ is a one-parameter family of convolution operators on the nilpotent group N. In case $X = B^n$ is the real hyperbolic ball, $N = \mathbb{R}^{n-1}$ is abelian, $p = n - 1$, $q = 0$, and (E.14) reduces to (3.19) (up to a scaling of the norm). In case $X = \mathcal{B}^n$ is the complex hyperbolic ball, $p = 2(n-1)$, $q = 1$, and N is the $(2n-1)$-dimensional Heisenberg group $\mathbb{H}^{n-1}$. In such a case, (E.14) yields

$$P(t,h) = t^{-2n}\Big[\Big(1 + c\Big\|\frac{X}{t}\Big\|^2\Big)^2 + 4c\Big\|\frac{Y}{t^2}\Big\|^2\Big]^{-n}, \tag{E.15}$$

which has the same scaling as the "Heisenberg heat semigroup"

$$e^{t^2\mathcal{L}_0}, \tag{E.16}$$

where $\mathcal{L}_0$ is the "Heisenberg Laplacian," a subelliptic operator on $\mathbb{H}^{n-1}$. (Cf. [T2], Chapter 1, §7.) This suggests the usefulness of the analysis developed in [FS], followed by works such as [RS] and [T1]. Other work on the Dirichlet problem on the complex hyperbolic ball includes [K1], [Fol], and [Gr1]. Further results, including the Dirichlet problem on a strongly pseudoconvex domain with the Bergman metric and related metrics are given in [Gr2] and [LM].

F. Further convergence results

Let us set, for $t > 0$, $f \in L^p(\mathbb{R}^{n-1})$, $x \in \mathbb{R}^{n-1}$,

$$P^t_{\mathfrak{h}} f(x) = \mathrm{PI}_{\mathfrak{h}}\, f(x + te_n), \tag{F.1}$$

and, for $r \in [0,1)$, $f \in L^p(S^{n-1})$, $\omega \in S^{n-1}$,

$$P^r_{\mathcal{H}} f(\omega) = \mathrm{PI}_{\mathcal{H}}\, f(r\omega). \tag{F.2}$$

The explicit formula (3.19) for $\mathrm{PI}_{\mathfrak{h}}$ fits in with standard results to yield the following.

Proposition F.1. *Given $p \in [1,\infty)$,*

$$f \in L^p(\mathbb{R}^{n-1}) \Longrightarrow \|P^t_{\mathfrak{h}} f - f\|_{L^p} \to 0 \quad \textit{as} \quad t \searrow 0. \tag{F.3}$$

Also $P^t_{\mathfrak{h}} f \to f$ a.e. on $\mathbb{R}^{n-1}$, and if $N_\alpha f$ is the nontangential maximal function

$$N_\alpha f(x) = \sup\{|\,\mathrm{PI}_{\mathfrak{h}}\, f(x + y + te_n)| : 0 < t \le 1,\ |y| \le \alpha t\}, \tag{F.4}$$

then

$$\|N_\alpha f\|_{L^p} \le C_{\alpha p} \|f\|_{L^p}, \ \textit{for} \ \ \alpha \in (0,\infty),\ p \in (1,\infty]. \tag{F.5}$$

From here, the relation (3.23) plus the arguments given in that paragraph yield the following.

Proposition F.2. *Given $p \in (1,\infty]$, $f \in L^p(S^{n-1})$, $\beta \in (1,\infty)$, if we set*

$$\mathcal{N}_\beta f(\omega) = \sup\{|\,\mathrm{PI}_{\mathcal{H}}\, f(x)| : |x - \omega| \le \beta\, dist(x, S^{n-1})\}, \tag{F.6}$$

we have

$$\|\mathcal{N}_\beta f\|_{L^p} \le C_{\beta p} \|f\|_{L^p}. \tag{F.7}$$

Since $|P^r_{\mathcal{H}} f(\omega)| \le \mathcal{N}_\beta f(\omega)$, Proposition F.2 implies

$$\|P^r_{\mathcal{H}} f(\omega)\|_{L^p} \le C_p \|f\|_{L^p}, \quad 1 < p \le \infty,\ 0 < r < 1, \tag{F.8}$$

with C_p independent of r.

Remark. A computation of the integral kernel in (2.3), parallel to that in (3.19), would no doubt yield Proposition F.2 directly, and extend it to the case $p = 1$.

Since

$$f \in C(S^{n-1}) \Longrightarrow P^r_{\mathcal{H}} f \to f \ \text{ uniformly as } \ r \nearrow 0, \tag{F.9}$$

(F.8) and the denseness of $C(S^{n-1})$ in $L^p(S^{n-1})$ for $p < \infty$ give the following.

Proposition F.3. *Given* $1 < p < \infty$,

$$f \in L^p(S^{n-1}) \Longrightarrow P_{\mathcal{H}}^r f \to f \quad \text{in} \quad L^p\text{-norm, as} \quad r \nearrow 1. \tag{F.10}$$

Let

$$\Lambda = (-\Delta_S + 1)^{1/2}, \tag{F.11}$$

where Δ_S is the Laplace-Beltrami operator on S^{n-1}. Symmetry implies that $P_{\mathcal{H}}^r$ commutes with the natural action of $SO(n)$ on $L^p(S^{n-1})$, hence with Λ and all its powers. Now we have L^p-Sobolev spaces

$$H^{s,p}(S^{n-1}) = \Lambda^{-s} L^p(S^{n-1}), \quad p \in (1, \infty), \ s \in \mathbb{R}. \tag{F.12}$$

The fact that

$$P_{\mathcal{H}}^r \Lambda^{-s} = \Lambda^{-s} P_{\mathcal{H}}^r, \tag{F.13}$$

plus Proposition F.3, then gives the following.

Proposition F.4. *For* $1 < p < \infty, s \in \mathbb{R}$,

$$f \in H^{s,p}(S^{n-1}) \Longrightarrow P_{\mathcal{H}}^r f \to f \quad \text{in} \quad H^{s,p}\text{-norm, as} \quad r \nearrow 1. \tag{F.14}$$

G. Fatou type theorems

Given $u \in C^\infty(B^n)$, set

$$u_r(\omega) = u(r\omega), \quad r \in [0,1),\ \omega \in S^{n-1}. \tag{G.1}$$

We aim to show that if $\Delta_{\mathcal{H}} u = 0$ on B^n and $\{u_r : 0 \le r < 1\}$ is bounded, in a certain Banach space X of functions (or distributions), then

$$u = \mathrm{PI}_{\mathcal{H}}\, f, \tag{G.2}$$

with $f \in X$ (or occasionally, a larger space). We start with the following.

Lemma G.1. *Take $p \in (1,\infty]$. Assume $\Delta_{\mathcal{H}} u = 0$ on B^n and*

$$\{u_r : 0 \le r < 1\} \ \text{ is relatively compact in } \ L^p(S^{n-1}). \tag{G.3}$$

(For $p = \infty$, we can replace $L^\infty(S^{n-1})$ by $C(S^{n-1})$.) Then there exists $f \in L^p(S^{n-1})$ such that (G.2) holds.

Proof. Set $f_r = u_r \in C^\infty(S^{n-1}) \subset L^p(S^{n-1})$, and

$$v_r = \mathrm{PI}_{\mathcal{H}}\, f_r, \quad v_{rs}(\omega) = v_r(s\omega) = P^s_{\mathcal{H}} f_r(\omega). \tag{G.4}$$

The uniform estimates (F.8), plus (F.10), imply that $P^s_{\mathcal{H}} \to I$ uniformly on compact subsets of $L^p(S^{n-1})$ (resp., $C(S^{n-1})$ if $p = \infty$). Hence, by the compactness hypothesis, given $k \in \mathbb{N}$, there exists $r_k < 1$ such that

$$\|P^s_{\mathcal{H}} f_r - f_r\|_{L^p} \le 2^{-k}, \quad \forall r \in [0,1),\ s \ge r_k. \tag{G.5}$$

(We can assume $r_k \nearrow 1$.) It follows that

$$\|v_{r_k r_k} - u_{r_k}\|_{L^p} \le 2^{-k}, \tag{G.6}$$

and hence, by elliptic regularity, given $\varepsilon > 0$,

$$\|v_{r_k} - u\|_{C(B^n_{(1-\varepsilon)r_k})} \le C_\varepsilon 2^{-k}, \tag{G.7}$$

where $B^n_\rho = \{x \in B^n : |x| < \rho\}$. Consequently, as $k \to \infty$,

$$v_{r_k} \longrightarrow u \ \text{ locally uniformly on } \ B^n. \tag{G.8}$$

Now, the compactness hypothesis (G.3) also implies that, perhaps passing to a further subsequence, we have $f \in L^p(S^{n-1})$ ($f \in C(S^{n-1})$ if $p = \infty$) such that

$$f_{r_k} \longrightarrow f \quad \text{in} \quad L^p\text{-norm.} \tag{G.9}$$

Hence

$$P^s_{\mathcal{H}} f_{r_k} \longrightarrow P^s_{\mathcal{H}} f \quad \text{in} \quad L^p\text{-norm, as} \quad k \to \infty, \tag{G.10}$$

uniformly for $s \in [0, 1)$, and therefore

$$v_{r_k} \longrightarrow \mathrm{PI}_{\mathcal{H}}\, f. \tag{G.11}$$

Comparison with (G.8) gives (G.2).

Remark. In the Euclidean case, a dilation argument gives (G.8) directly, without need for (G.5)–(G.7). This allows one to get (G.2) when (G.3) is weakened to boundedness (with a natural modification when $p = 1$), thus directly yielding the result we will establish in Proposition G.4. In the hyperbolic case (for $n \geq 3$) such a dilation argument is not available.

Lemma G.1 yields the following extension.

Corollary G.2. *Given $p \in (1, \infty)$, $s \in \mathbb{R}$, $\Delta_{\mathcal{H}} u = 0$ on B^n, and*

$$\{u_r : 0 \leq r < 1\} \quad \textit{relatively compact in} \quad H^{s,p}(S^{n-1}), \tag{G.12}$$

there exists $f \in H^{s,p}(S^{n-1})$ such that (G.2) holds.

Proof. With Λ as in (F.11), we have

$$\Lambda^s u_r = w_r, \quad \text{relatively compact in} \quad L^p(S^{n-1}), \tag{G.13}$$

with $\Delta_{\mathcal{H}} w = 0$. Hence, by Lemmma G.1, $w = \mathrm{PI}_{\mathcal{H}}\, g$, $g \in L^p(S^{n-1})$. Then

$$u = \mathrm{PI}_{\mathcal{H}}\, f, \quad f = \Lambda^{-s} g \in H^{s,p}(S^{n-1}). \tag{G.14}$$

Here is a significant improvement of Corollary G.2.

Proposition G.3. *Given $p \in (1, \infty)$, $\sigma \in \mathbb{R}$, $\Delta_{\mathcal{H}} u = 0$ on B^n, and*

$$\{u_r : 0 \leq r < 1\} \quad \textit{bounded in} \quad H^{\sigma,p}(S^{n-1}), \tag{G.15}$$

there exists $f \in H^{\sigma,p}(S^{n-1})$ such that (G.2) holds.

Proof. By Rellich's theorem, (G.15) implies (G.12) for $s < \sigma$, so there exists $f \in H^{s,p}(S^{n-1})$ such that (G.2) holds. Then, with $f_r = u_r$ as in (G.1), we have $f_r \to f$ in $H^{s,p}$-norm, as $r \nearrow 1$, by Proposition F.4. But (G.15) implies that, for some subsequence,

$$f_{r_k} \longrightarrow g, \ \text{weakly, in} \quad H^{\sigma,p}(S^{n-1}). \tag{G.16}$$

Hence $f = g \in H^{\sigma,p}(S^{n-1})$.

Now we can record an improvement of Lemma G.1.

Proposition G.4. *Assume* $p \in [1,\infty]$, $\Delta_{\mathcal{H}} u = 0$ *on* B^n, *and*

$$\{u_r : 0 \le r < 1\} \ \text{ is bounded in } \ L^p(S^{n-1}). \tag{G.17}$$

If $p \in (1,\infty]$, *there exists* $f \in L^p(S^{n-1})$ *such that (G.2) holds. If* $p = 1$, *there exists a finite measure* μ *on* S^{n-1} *such that*

$$u = \mathrm{PI}_{\mathcal{H}}\, \mu. \tag{G.18}$$

Proof. If $1 < p < \infty$, the conclusion is the $\sigma = 0$ case of Proposition G.3. Hence, in case (G.17) holds with $p = \infty$, we have, for each $q < \infty$, $f \in L^q(S^{n-1})$ such that $u = \mathrm{PI}_{\mathcal{H}}\, f$. Then $u_r \to f$ in L^q-norm. But the uniform boundedness of u_r then gives $f \in L^\infty(S^{n-1})$.

If (G.17) holds with $p = 1$, Corollary G.2 gives $u = \mathrm{PI}_{\mathcal{H}}\, f$ for some $f \in H^{-\delta,q}(S^{n-1})$, $\delta > 0$, $q > 1$, and we have

$$u_r \longrightarrow f \ \text{ in } \ H^{-\delta,q}\text{-norm}. \tag{G.19}$$

But the L^1 bound implies for a subsequence

$$u_{r_k} \longrightarrow \mu, \tag{G.20}$$

a finite measure on S^{n-1}, in the weak* topology. We have convergence in $\mathcal{D}'(S^{n-1})$ in both (G.19) and (G.20), so $f = \mu$, and (G.18) holds.

Question. Assume $\Delta_{\mathcal{H}} u = 0$ on B^n and

$$|u(x)| \le C(1 - |x|)^{-N}, \tag{G.21}$$

for some $C, N \in (0,\infty)$. Can one show that (G.15) holds for some $\sigma \in \mathbb{R}$?

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