Elliptic Functions

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Sections 30–34 of "Introduction to Complex Analysis"

(And Appendices: §26 and §K)

Contents

Introduction

We develop the basic theory of elliptic functions, starting in §30 with basic constructions of doubly periodic meromorphic functions on $\mathbb{C}$ as lattice sums. One basic case is the Weierstrass $\wp$-function. Its centrality in the subject is established in §31. Section 32 is devoted to a representation of $\wp$ in terms of theta functions, which among other properties have the advantage of being defined by rapidly convergent series. Section 33 expresses certain classes of integrals, known as elliptic integrals, as inverses of elliptic functions. To treat arbitrary elliptic integrals this way, one needs a result known as the solution to the Abel inversion problem. One convenient route to the solution to this problem involves the construction of the Riemann surface of $\sqrt{q(\zeta)}$, when $q(\zeta)$ is a cubic (or quartic) polynomial, with no repeated roots. This is taken up in §34. Results here are precursors of a general theory of Riemann surfaces (treated in another set of notes of the author).

These notes are excerpted from §§30–34 of our monograph "Introduction to Complex Analysis." They have an appendix, covering some elementary notions of a Riemann surface, taken from §26 of that monograph, and an appendix on the rapid evaluation of the Weierstrass $\wp$-function, taken from §K.

30. Periodic and doubly periodic functions - infinite series representations

We can obtain periodic meromorphic functions by summing translates of z^{-k}. For example,

$$f_1(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} \tag{30.1}$$

is meromorphic on $\mathbb{C}$, with poles in $\mathbb{Z}$, and satisfies $f_1(z+1) = f_1(z)$. In fact, we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \frac{\pi^2}{\sin^2 \pi z}. \tag{30.2}$$

To see this, note that both sides have the same poles, and their difference $g_1(z)$ is seen to be an entire function, satisfying $g_1(z+1) = g_1(z)$. Also it is seen that, for $z = x + iy$, both sides of (30.2) tend to 0 as $|y| \to \infty$. This forces $g_1 \equiv 0$.

A second example is

$$\begin{aligned} f_2(z) = \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{1}{z-n} &= \frac{1}{z} + \sum_{n\neq 0} \Bigl(\frac{1}{z-n} + \frac{1}{n}\Bigr) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}. \end{aligned} \tag{30.3}$$

This is also meromorphic on $\mathbb{C}$, with poles in $\mathbb{Z}$, and it is seen to satisfy $f_2(z+1) = f_2(z)$. We claim that

$$\frac{1}{z} + \sum_{n\neq 0} \Bigl(\frac{1}{z-n} + \frac{1}{n}\Bigr) = \pi \cot \pi z. \tag{30.4}$$

In this case again we see that the difference $g_2(z)$ is entire. Furthermore, applying $-d/dz$ to both sides of (30.4), we get the two sides of (30.2), so g_2 is constant. Looking at the last term in (30.3), we see that the left side of (30.4) is odd in z; so is the right side; hence $g_2 = 0$.

As a third example, we consider

$$\begin{aligned} \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{(-1)^n}{z-n} &= \frac{1}{z} + \sum_{n\neq 0} (-1)^n \Bigl(\frac{1}{z-n} + \frac{1}{n}\Bigr) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \frac{2z}{z^2-n^2} \\ &= \frac{1}{z} - 4 \sum_{k=1}^{\infty} \frac{z(1-2k)}{[z^2-(2k-1)^2][z^2-(2k)^2]}. \end{aligned} \tag{30.5}$$

We claim that

$$\frac{1}{z}+\sum_{n\neq 0}(-1)^n\Bigl(\frac{1}{z-n}+\frac{1}{n}\Bigr)=\frac{\pi}{\sin \pi z}. \tag{30.6}$$

In this case we see that their difference $g_3(z)$ is entire and satisfies $g_3(z+2)=g_3(z)$. Also, for $z=x+iy$, both sides of (30.6) tend to 0 as $|y|\to\infty$, so $g_3\equiv 0$.

We now use a similar device to construct doubly periodic meromorphic functions, following K. Weierstrass. These functions are also called elliptic functions. Further introductory material on this topic can be found in [Ahl] and [Hil]. Pick $\omega_1,\omega_2\in\mathbb{C}$, linearly independent over $\mathbb{R}$, and form the lattice

$$\Lambda=\{j\omega_1+k\omega_2 : j,k\in\mathbb{Z}\}. \tag{30.7}$$

In partial analogy with (30.4), we form the "Weierstrass $\wp$-function,"

$$\wp(z;\Lambda)=\frac{1}{z^2}+\sum_{0\neq\omega\in\Lambda}\Bigl(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\Bigr). \tag{30.8}$$

Convergence on $\mathbb{C}\setminus\Lambda$ is a consequence of the estimate

$$\Bigl|\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\Bigr|\le C\frac{|z|}{|\omega|^3},\quad \text{for}\quad |\omega|\ge 2|z|. \tag{30.9}$$

To verify that

$$\wp(z+\omega;\Lambda)=\wp(z;\Lambda),\quad \forall\,\omega\in\Lambda, \tag{30.10}$$

it is convenient to differentiate both sides of (30.8), obtaining

$$\wp'(z;\Lambda)=-2\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^3}, \tag{30.11}$$

which clearly satisfies

$$\wp'(z+\omega;\Lambda)=\wp'(z;\Lambda),\quad \forall\,\omega\in\Lambda. \tag{30.12}$$

Hence

$$\wp(z+\omega;\Lambda)-\wp(z;\Lambda)=c(\omega),\quad \omega\in\Lambda. \tag{30.13}$$

Now (30.8) implies $\wp(z;\Lambda)=\wp(-z;\Lambda)$. Hence, taking $z=-\omega/2$ in (30.13) gives $c(\omega)=0$ for all $\omega\in\Lambda$, and we have (30.10).

Another analogy with (30.4) leads us to look at the function (not to be confused with the Riemann zeta function)

$$\zeta(z;\Lambda)=\frac{1}{z}+\sum_{0\neq\omega\in\Lambda}\Bigl(\frac{1}{z-\omega}+\frac{1}{\omega}+\frac{z}{\omega^2}\Bigr). \tag{30.14}$$

We note that the sum here is obtained from the sum in (30.8) (up to sign) by integrating from 0 to z along any path that avoids the poles. This is enough to establish convergence of (30.14) in $\mathbb{C} \setminus \Lambda$, and we have

$$\zeta'(z;\Lambda) = -\wp(z;\Lambda). \tag{30.15}$$

In view of (30.10), we hence have

$$\zeta(z+\omega;\Lambda) - \zeta(z;\Lambda) = \alpha_\Lambda(\omega), \quad \forall \omega \in \Lambda. \tag{30.16}$$

In this case $\alpha_\Lambda(\omega) \neq 0$, but we can take $a, b \in \mathbb{C}$ and form

$$\zeta_{a,b}(z;\Lambda) = \zeta(z-a;\Lambda) - \zeta(z-b;\Lambda), \tag{30.17}$$

obtaining a meromorphic function with poles at $(a+\Lambda) \cup (b+\Lambda)$, all simple (if $a-b \notin \Lambda$).

Let us compare the doubly periodic function Φ constructed in (24.8)–(24.11), which maps the rectangle with vertices at $-1, 1, 1+ip, -1+ip$ conformally onto the upper half plane $\mathcal{U}$, with $\Phi(-1) = -1, \Phi(0) = 0, \Phi(1) = 1$. (Here p is a given positive number.) As seen there,

$$\Phi(z+\omega) = \Phi(z), \quad \omega \in \Lambda = \{4k + 2i\ell p : k, \ell \in \mathbb{Z}\}. \tag{30.18}$$

Furthermore, this function has simple poles at $(ip+\Lambda) \cup (ip+2+\Lambda)$, and the residues at ip and at $ip+2$ cancel. Thus there exist constants A and B such that

$$\Phi(z) = A\zeta_{ip,ip+2}(z;\Lambda) + B. \tag{30.19}$$

The constants A and B can be evaluated by taking $z = 0, 1$, though the resulting formulas give A and B in terms of special values of $\zeta(z;\Lambda)$ rather than in elementary terms.

Exercises

1. Setting $z = 1/2$ in (30.2), show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Compare (13.79). Differentiate (30.2) repeatedly and obtain formulas for $\sum_{n\ge 1} n^{-k}$ for even integers k.

Hint. Denoting the right side of (30.2) by $f(z)$, show that

$$f^{(\ell)}(z) = (-1)^\ell (\ell+1)! \sum_{n=-\infty}^{\infty} (z-n)^{-(\ell+2)}.$$

Deduce that, for $k \ge 1$,

$$f^{(2k-2)}\Bigl(\frac{1}{2}\Bigr) = (2k-1)! 2^{2k+1} \sum_{n\ge 1,\text{odd}} n^{-2k}.$$

Meanwhile, use

$$\sum_{n=1}^{\infty} n^{-2k} = \sum_{n\ge 1,\mathrm{odd}} n^{-2k} + 2^{-2k}\sum_{n=1}^{\infty} n^{-2k}$$

to get a formula for $\sum_{n=1}^{\infty} n^{-2k}$, in terms of $f^{(2k-2)}(1/2)$.

1A. Set $F(z) = (\pi\cot\pi z) - 1/z$, and use (30.4) to compute $F^{(\ell)}(0)$. Show that, for $|z| < 1$,

$$\pi\cot\pi z = \frac{1}{z} - 2\sum_{k=1}^{\infty}\zeta(2k)z^{2k-1}, \quad \zeta(2k) = \sum_{n=1}^{\infty} n^{-2k}.$$

1B. Recall from Exercise 6 in §12 that, for $|z|$ sufficiently small,

$$\frac{1}{2}\,\frac{e^z+1}{e^z-1} = \frac{1}{z} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}z^{2k-1},$$

with B_k (called the Bernoulli numbers) *rational* numbers for each k. Note that

$$\frac{e^{2\pi iz}+1}{e^{2\pi iz}-1} = \frac{1}{i}\cot\pi z.$$

Deduce from this and Exercise 1A that, for $k \ge 1$,

$$2\zeta(2k) = (2\pi)^{2k}\frac{B_k}{(2k)!}.$$

Relate this to results of Exercise 1.

1C. For an alternative aproach to the results of Exercise 1B, show that

$$G(z) = \pi\cot\pi z \Longrightarrow G'(z) = -\pi^2 - G(z)^2.$$

Using

$$G(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^{2n-1},$$

compute the Laurent series expansions of $G'(z)$ and $G(z)^2$ and deduce that $a_1 = -\pi^2/3$, while, for $n \ge 2$,

$$a_n = -\frac{1}{2n+1}\sum_{\ell=1}^{n-1} a_{n-\ell}a_\ell.$$

In concert with Exercise 1A, show that $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, and also compute $\zeta(6)$ and $\zeta(8)$.

2. Set

$$F(z) = \pi z\prod_{n=1}^{\infty}\Big(1 - \frac{z^2}{n^2}\Big).$$

Show that

$$\frac{F'(z)}{F(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

Using this and (30.3)–(30.4), deduce that

$$F(z) = \sin \pi z,$$

obtaining another proof of (18.21).
Hint. Show that if F and G are meromorphic and $F'/F \equiv G'/G$, then $F = cG$ for some constant c. To find c in this case, note that $F'(0) = \pi$.

3. Show that if Λ is a lattice of the form (30.7) then a meromorphic function satisfying

$$f(z + \omega) = f(z), \quad \forall\, \omega \in \Lambda \tag{30.20}$$

yields a meromorphic function on the torus $\mathbb{T}_\Lambda$, defined by (26.14). Show that if such f has no poles then it must be constant.

We say a parallelogram $\mathcal{P} \subset \mathbb{C}$ is a period parallelogram for a lattice Λ (of the form (30.7)) provided it has vertices of the form $p, p + \omega_1, p + \omega_2, p + \omega_1 + \omega_2$. Given a meromorphic function f satisfying (30.20), pick a period parallelogram $\mathcal{P}$ whose boundary is disjoint from the set of poles of f.

4. Show that

$$\int_{\partial \mathcal{P}} f(z)\, dz = 0.$$

Deduce that

$$\sum_{p_j \in \mathcal{P}} \operatorname{Res}_{p_j}(f) = 0.$$

Deduce that if f has just one pole in $\mathcal{P}$ then that pole cannot be simple.

5. For ζ defined by (30.14), show that, if $\operatorname{Im}(\omega_2/\omega_1) > 0$,

$$\int_{\partial \mathcal{P}} \zeta(z; \Lambda)\, dz = \alpha_\Lambda(\omega_1)\omega_2 - \alpha_\Lambda(\omega_2)\omega_1 = 2\pi i. \tag{30.21}$$

6. Show that α_Λ in (30.16) satisfies

$$\alpha_\Lambda(\omega + \omega') = \alpha_\Lambda(\omega) + \alpha_\Lambda(\omega'), \quad \omega, \omega' \in \Lambda. \tag{30.22}$$

Show that if $\omega \in \Lambda,\ \omega/2 \notin \Lambda$, then

$$\alpha_\Lambda(\omega) = 2\zeta(\omega/2; \Lambda).$$

7. Apply Green's theorem

$$\iint_{\Omega} \Big(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \Big)\, dx\, dy = \int_{\partial \Omega} (f\, dx + g\, dy)$$

in concert with $\zeta'(z;\Lambda) = -\wp(z;\Lambda)$, written as

$$\frac{1}{2}\Bigl(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\Bigr)\zeta(z;\Lambda) = -\wp(z;\Lambda),$$

and with $\Omega = \mathcal{P}$, as in Exercise 5, to establish that

$$\alpha_\Lambda(\omega_1)\overline{\omega}_2 - \alpha_\Lambda(\omega_2)\overline{\omega}_1 = 2i\mathcal{I}(\Lambda), \tag{30.23}$$

with

$$\mathcal{I}(\Lambda) = \lim_{\varepsilon\to 0} \iint\limits_{\mathcal{P}\setminus D_\varepsilon(0)} \wp(z;\Lambda)\,dx\,dy, \tag{30.24}$$

assuming $\mathcal{P}$ is centered at 0.

8. Solve the pair of equations (30.21) and (30.23) for $\alpha_\Lambda(\omega_1)$ and $\alpha_\Lambda(\omega_2)$. Use this in concert with (30.22) to show that

$$\alpha_\Lambda(\omega) = \frac{1}{A(\mathcal{P})}\Bigl(-\mathcal{I}(\Lambda)\omega + \pi\overline{\omega}\Bigr), \quad \omega \in \Lambda, \tag{30.25}$$

where $\mathcal{I}(\Lambda)$ is as in (30.24) and $A(\mathcal{P})$ is the area of $\mathcal{P}$.

9. Show that the constant A in (30.19) satisfies

$$A = \mathrm{Res}_{ip}(\Phi).$$

10. Show that the constants A and B in (30.19) satisfy

$$[\zeta(1-ip;\Lambda) - \zeta(-1-ip;\Lambda)]A + B = 1,$$

and

$$\alpha_\Lambda(4)A + 2B = 0,$$

with Λ given by (30.18).
Hint. Both $\Phi(z)$ and $\zeta(z;\Lambda)$ are odd in z.

In Exercises 11–12, given $p_j \in \mathbb{T}_\Lambda,\ n_j \in \mathbb{Z}^+$, set $\vartheta = \sum n_j p_j$ and define

$$\mathcal{M}_\vartheta(\mathbb{T}_\Lambda) = \{f \ \text{ meromorphic on } \ \mathbb{T}_\Lambda : \text{poles of } f \text{ are at } p_j \text{ and of order} \le n_j\}. \tag{30.26}$$

Set $|\vartheta| = \sum n_j$.

11. Show that $|\vartheta| = 2 \Rightarrow \dim \mathcal{M}_\vartheta(\mathbb{T}_\Lambda) = 2$, and that this space is spanned by 1 and ζ_{p_1,p_2} if $n_1 = n_2 = 1$, and by 1 and $\wp(z - p_1)$ if $n_1 = 2$.
Hint. Use Exercise 4.

12. Show that

$$|\vartheta| = k \ge 2 \Longrightarrow \dim \mathcal{M}_\vartheta(\mathbb{T}_\Lambda) = k. \tag{30.27}$$

Hint. Argue by induction on k, noting that you can augment $|\vartheta|$ by 1 either by adding another p_j or by increasing some positive n_j by 1.

31. The Weierstrass $\wp$ in elliptic function theory

It turns out that a general elliptic function with period lattice Λ can be expressed in terms of $\wp(z;\Lambda)$ and its first derivative. Before discussing a general result, we illustrate this in the case of the functions $\zeta_{a,b}(z;\Lambda)$, given by (30.17). Henceforth we simply denote these functions by $\wp(z)$ and $\zeta_{a,b}(z)$, respectively.

We claim that, if $2\beta \notin \Lambda$, then

$$\frac{\wp'(\beta)}{\wp(z)-\wp(\beta)} = \zeta_{\beta,-\beta}(z) + 2\zeta(\beta). \tag{31.1}$$

To see this, note that both sides have simple poles at $z=\pm\beta$. (As will be shown below, the zeros α of $\wp'(z)$ satisfy $2\alpha \in \Lambda$.) The factor $\wp'(\beta)$ makes the poles cancel, so the difference is entire, hence constant. Both sides vanish at $z=0$, so this constant is zero. We also note that

$$\zeta_{a,b}(z) = \zeta_{\beta,-\beta}(z-\alpha), \quad \alpha = \frac{a+b}{2}, \ \beta = \frac{a-b}{2}. \tag{31.2}$$

As long as $a-b \notin \Lambda$, (31.1) applies, giving

$$\zeta_{a,b}(z) = \frac{\wp'(\beta)}{\wp(z-\alpha)-\wp(\beta)} - 2\zeta(\beta), \quad \alpha = \frac{a+b}{2}, \ \beta = \frac{a-b}{2}. \tag{31.3}$$

We now prove the result on the zeros of $\wp'(z)$ stated above. Assume Λ has the form (30.7).

Proposition 31.1. *The three points $\omega_1/2$, $\omega_2/2$ and $(\omega_1+\omega_2)/2$ are (mod Λ) all the zeros of $\wp'(z)$.*

Proof. Symmetry considerations (oddness of $\wp'(z)$) imply $\wp'(z)=0$ at each of these three points. Since $\wp'(z)$ has a single pole of order 3 in a period parallelogram, these must be all the zeros. (Cf. Exercise 1 below to justify this last point.)

The general result hinted at above is the following.

Proposition 31.2. *Let f be an elliptic function with period lattice Λ. There exist rational functions Q and R such that*

$$f(z) = Q(\wp(z)) + R(\wp(z))\wp'(z). \tag{31.4}$$

Proof. First assume f is even, i.e., $f(z)=f(-z)$. The product of $f(z)$ with factors of the form $\wp(z)-\wp(a)$ lowers the degree of a pole of f at any point $a \notin \Lambda$, so there exists a polynomial P such that $g(z) = P(\wp(z))f(z)$ has poles only in Λ. Note that $g(z)$ is also even. Then there exists a polynomial P_2 such that $g(z) - P_2(\wp(z))$ has its poles annihilated. This function must hence be constant. Hence any even elliptic f must be a rational function of $\wp(z)$.

On the other hand, if $f(z)$ is odd, then $f(z)/\wp'(z)$ is even, and the previous argument applies, so a general elliptic function must have the form (31.4).

The right side of (31.3) does not have the form (31.4), but we can come closer to this form via the identity

$$\wp(z-\alpha) = -\wp(z) - \wp(\alpha) + \frac{1}{4}\Big(\frac{\wp'(z)+\wp'(\alpha)}{\wp(z)-\wp(\alpha)}\Big)^2. \tag{31.5}$$

This identity can be verified by showing that the difference of the two sides is pole free and vanishes at $z = 0$. The right side of (31.5) has the form (31.4) except for the occurrence of $\wp'(z)^2$, which we will dispose of shortly.

Note that $\wp'(z)^2$ is even, with poles (of order 6) on Λ. We can explicitly write this as $P(\wp(z))$, as follows. Set

$$e_j = \wp\Big(\frac{\omega_j}{2}\Big), \quad j = 1, 2, 3, \tag{31.6}$$

where we set $\omega_3 = \omega_1 + \omega_2$. We claim that

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3). \tag{31.7}$$

In fact, both sides of (31.7) have poles of order 6, precisely at points of Λ. Furthermore, by Proposition 31.1, the zeros of $\wp'(z)^2$ occur precisely at $z = \omega_j$ (mod Λ), each zero having multiplicity 2. We also see that the right side of (31.7) has a double zero at $z = \omega_j,\ j = 1, 2, 3$. So the quotient is entire, hence constant. The factor 4 arises by examining the behavior as $z \to 0$.

The identity (31.7) is a differential equation for $\wp(z)$. Separation of variables yields

$$\frac{1}{2}\int \frac{d\wp}{\sqrt{(\wp - e_1)(\wp - e_2)(\wp - e_3)}} = z + c. \tag{31.8}$$

The left side of (31.8) is known as an elliptic integral.

Any cubic polynomial in u is a constant multiple of $(u - e_1)(u - e_2)(u - e_3)$ for some $e_j \in \mathbb{C}$. However, it is not quite the case that every cubic polynomial fits into the current setting. Here is one constraint; another will be produced in (31.15) below.

Proposition 31.2. *Given a lattice $\Lambda \subset \mathbb{C}$, the quantities e_j in (31.6) are all distinct.*

Proof. Note that $\wp(z) - e_j$ has a double pole at each $z \in \Lambda$, and a double zero at $z = \omega_j/2$. Hence, in an appropriate period parallelogram, it has no other zeros (again cf. Exercise 1 below). Hence $\wp(\omega_k/2) - e_j = e_k - e_j \neq 0$ for $j \neq k$.

We can get more insight into the differential equation (31.7) by comparing Laurent series expansions of the two sides about $z = 0$. First, we can deduce from (30.8) that

$$\wp(z) = \frac{1}{z^2} + az^2 + bz^4 + \cdots . \tag{31.9}$$

Of course, only even powers of z arise. Regarding the absence of the constant term and the values of a and b, see Exercise 3 below. We have

$$a = 3\sum_{0\neq\omega\in\Lambda} \frac{1}{\omega^4}, \quad b = 5\sum_{0\neq\omega\in\Lambda} \frac{1}{\omega^6}. \tag{31.10}$$

Hence

$$\wp'(z) = -\frac{2}{z^3} + 2az + 4bz^3 + \cdots . \tag{31.11}$$

It follows, after some computation, that

$$\frac{1}{4}\wp'(z)^2 = \frac{1}{z^6} - \frac{2a}{z^2} - 4b + \cdots , \tag{31.12}$$

while

$$\begin{aligned} &(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3) \\ &\quad = \wp(z)^3 - \tau_1\wp(z)^2 + \tau_2\wp(z) - \tau_3 \\ &\quad = \frac{1}{z^6} - \frac{\tau_1}{z^4} + \frac{3a + \tau_2}{z^2} + (3b - 2a\tau_1 - \tau_3) + \cdots , \end{aligned} \tag{31.13}$$

where

$$\begin{aligned} \tau_1 &= e_1 + e_2 + e_3, \\ \tau_2 &= e_1e_2 + e_2e_3 + e_3e_1, \\ \tau_3 &= e_1e_2e_3. \end{aligned} \tag{31.14}$$

Comparing coefficients in (31.12)–(31.13) gives the following relation:

$$e_1 + e_2 + e_3 = 0. \tag{31.15}$$

It also gives

$$\begin{aligned} e_1e_2 + e_2e_3 + e_1e_3 &= -5a, \\ e_1e_2e_3 &= 7b, \end{aligned} \tag{31.16}$$

where a and b are given by (31.10). Hence we can rewrite the differential equation (31.7) as

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \tag{31.17}$$

where g_2 and g_3, known as the Weierstrass invariants of the lattice Λ, are given by

$$g_2 = 60 \sum_{0\neq\omega\in\Lambda} \frac{1}{\omega^4}, \quad g_3 = 140 \sum_{0\neq\omega\in\Lambda} \frac{1}{\omega^6}. \tag{31.18}$$

Exercises

1. Assume f is meromorphic (and not identically zero) on $\mathbb{T}_\Lambda = \mathbb{C}/\Lambda$. Show that the number of poles of f is equal to the number of zeros of f, counting multiplicity. *Hint.* Let γ bound a period parallelogram, avoiding the zeros and poles of f, and examine

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz.$$

Recall the argument principle, discussed in §17.

2. Show that, given a lattice $\Lambda \subset \mathbb{C}$, and given $\omega \in \mathbb{C}$,

$$\omega \in \Lambda \Longleftrightarrow \wp\Big(\frac{\omega}{2} + z; \Lambda\Big) = \wp\Big(\frac{\omega}{2} - z; \Lambda\Big), \quad \forall\ z. \tag{31.19}$$

Relate this to the proof of Proposition 31.1.

3. Consider

$$\Phi(z) = \wp(z) - \frac{1}{z^2} = \sum_{\omega \in \Lambda \setminus 0} \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\Big), \tag{31.20}$$

which is holomorphic near $z = 0$. Show that $\Phi(0) = 0$ and that, for $k \geq 1$,

$$\frac{1}{k!}\Phi^{(k)}(0) = (k+1) \sum_{\omega \in \Lambda \setminus 0} \omega^{-(k+2)}. \tag{31.21}$$

(These quantities vanish for k odd.) Relate these results to (31.9)–(31.10).

4. Complete the sketch of the proof of (31.5).
Hint. Use the fact that $\wp(z) - z^{-2}$ is holomorphic near $z = 0$ and vanishes at $z = 0$.

5. Deduce from (31.17) that

$$\wp''(z) = 6\wp(z)^2 - \frac{1}{2}g_2. \tag{31.22}$$

6. Say that, near $z = 0$,

$$\wp(z) = \frac{1}{z^2} + \sum_{n \geq 1} b_n z^{2n}, \tag{31.23}$$

where b_n are given by (31.21), with $k = 2n$. Deduce from Exercise 5 that for $n \geq 3$,

$$b_n = \frac{3}{(2n+3)(n-2)} \sum_{k=1}^{n-2} b_k b_{n-k-1}. \tag{31.24}$$

In particular, we have

$$b_3 = \frac{1}{3}b_1^2, \quad b_4 = \frac{3}{11}b_1 b_2,$$

and

$$b_5 = \frac{1}{13}(b_2^2 + 2b_1 b_3) = \frac{1}{13}\Big(b_2^2 + \frac{2}{3}b_1^3\Big).$$

7. Deduce from Exercise 6 that if

$$\sigma_n = \sum_{\omega \in \Lambda \setminus 0} \frac{1}{\omega^{2n}}, \tag{31.25}$$

then for $n \geq 3$,

$$\sigma_n = P_n(\sigma_2, \sigma_3), \tag{31.26}$$

where $P_n(\sigma_2, \sigma_3)$ is a polynomial in σ_2 and σ_3 with coefficients that are positive, rational numbers. Use (31.16) to show that

$$\sigma_2 = -\frac{1}{15}(e_1e_2 + e_2e_3 + e_1e_3), \quad \sigma_3 = \frac{1}{35}\, e_1e_2e_3. \tag{31.27}$$

Note that $b_n = (2n+1)\sigma_{n+1}$. Note also that g_k in (31.17)–(31.18) satisfy $g_2 = 60\sigma_2$ and $g_3 = 140\sigma_3$.

8. Given f as in Exercise 1, show that

$$\frac{1}{2\pi i} \int_\sigma \frac{f'(z)}{f(z)}\, dz \in \mathbb{Z}, \tag{31.28}$$

whenever σ is a closed curve in $\mathbb{T}_\Lambda$ that avoids the zeros and poles of f.

9. Again take f as in Exercise 1. Assume f has zeros at $p_j \in \mathbb{T}_\Lambda$, of order m_j, and poles at $q_j \in \mathbb{T}_\Lambda$, of order n_j, and no other zeros or poles. Show that

$$\sum m_j p_j - \sum n_j q_j = 0 \pmod{\Lambda}. \tag{31.29}$$

Hint. Take γ as in Exercise 1, and consider

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, z\, dz. \tag{31.30}$$

On the one hand, Cauchy's integral theorem (compare (5.19)) implies (31.30) is equal to the left side of (31.29), provided p_j and q_j are all in the period domain. On the other hand, if γ consists of four consecutive line segments, $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, periodicity of $f'(z)/f(z)$ implies that (31.30) equals

$$-\frac{\omega_2}{2\pi i} \int_{\sigma_1} \frac{f'(z)}{f(z)}\, dz + \frac{\omega_1}{2\pi i} \int_{\sigma_4} \frac{f'(z)}{f(z)}\, dz. \tag{31.31}$$

Use Exercise 8 to deduce that the coefficients of ω_1 and ω_2 in (31.31) are integers.

10. Deduce from (31.5) that

$$u + v + w = 0 \Rightarrow \det \begin{pmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{pmatrix} = 0. \tag{31.32}$$

11. Deduce from (31.5) that

$$\wp(2z) = \frac{1}{4}\Big(\frac{\wp''(z)}{\wp'(z)}\Big)^2 - 2\wp(z). \tag{31.33}$$

Hint. Set $\alpha = -z + h$ in (31.5) and let $h \to 0$.

12. Deduce from (31.33), in concert with (31.17) and (31.22), that

$$\wp(2z) = R(\wp(z)),$$

with

$$R(\zeta) = \frac{\zeta^4 + (g_2/2)\zeta^2 + 2g_3\zeta + (g_2/4)^2}{4\zeta^3 - g_2\zeta - g_3}.$$

13. Use (31.3) and (31.5), plus (31.7), to write $\zeta_{a,b}(z)$ (as in (31.3)) in the form (31.4), i.e.,

$$\zeta_{a,b}(z) = Q(\wp(z)) + R(\wp(z))\wp'(z).$$

32. Theta functions and $\wp$

We begin with the function

$$\theta(x,t)=\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} e^{2\pi i n x}, \tag{32.1}$$

defined for $x\in\mathbb{R},\ t>0$, which solves the "heat equation"

$$\frac{\partial\theta}{\partial t}=\frac{1}{4\pi}\frac{\partial^2\theta}{\partial x^2}. \tag{32.2}$$

Note that θ is actually holomorphic on $\{(x,t)\in\mathbb{C}\times\mathbb{C}:\operatorname{Re} t>0\}$. It is periodic of period 1 in x; $\theta(x+1,t)=\theta(x,t)$. Also one has

$$\theta(x+it,t)=e^{\pi t-2\pi i x}\theta(x,t). \tag{32.3}$$

This identity will ultimately lead us to a connection with $\wp(z)$. In addition, we have

$$\theta\Bigl(x+\frac{1}{2},t\Bigr)=\sum_{n\in\mathbb{Z}}(-1)^n e^{-\pi n^2 t} e^{2\pi i n x}, \tag{32.4}$$

and

$$\theta\Bigl(x+\frac{i}{2}t,t\Bigr)=e^{\pi t/4}\sum_{n\in\mathbb{Z}} e^{-\pi(n+1/2)^2 t} e^{2\pi i n x}, \tag{32.5}$$

which introduces series related to but slightly different from that in (32.1).

Following standard terminology, we set $-t=i\tau$, with $\operatorname{Im}\tau>0$, and denote $\theta(z,-i\tau)$ by

$$\vartheta_3(z,\tau)=\sum_{n\in\mathbb{Z}} e^{n^2\pi i\tau} e^{2n\pi i z}=\sum_{n\in\mathbb{Z}} p^{2n} q^{n^2}, \tag{32.6}$$

where

$$p=e^{\pi i z},\quad q=e^{\pi i\tau}. \tag{32.7}$$

This theta function has three partners, namely

$$\vartheta_4(z,\tau)=\sum_{n\in\mathbb{Z}}(-1)^n e^{n^2\pi i\tau} e^{2\pi i n z}=\sum_{n\in\mathbb{Z}}(-1)^n p^{2n} q^{n^2}, \tag{32.8}$$

and

$$\vartheta_1(z,\tau)=i\sum_{n\in\mathbb{Z}}(-1)^n e^{(n-1/2)^2\pi i\tau} e^{(2n-1)\pi i z}=i\sum_{n\in\mathbb{Z}}(-1)^n p^{2n-1} q^{(n-1/2)^2}, \tag{32.9}$$

and finally

$$\vartheta_2(z,\tau) = \sum_{n\in\mathbb{Z}} e^{(n-1/2)^2\pi i\tau} e^{(2n-1)\pi iz} = \sum_{n\in\mathbb{Z}} p^{2n-1} q^{(n-1/2)^2}. \tag{32.10}$$

We will explore these functions, with the goal of relating them to $\wp(z)$. These results are due to Jacobi; we follow the exposition of [MM].

To begin, we record how $\vartheta_j(z+\alpha)$ is related to $\vartheta_k(z)$ for various values of α. Here and (usually) below we will suppress the τ and denote $\vartheta_j(z,\tau)$ by $\vartheta_j(z)$, for short. In the table below we use

$$a = p^{-1}q^{-1/4} = e^{-\pi iz-\pi i\tau/4}, \quad b = p^{-2}q^{-1} = e^{-2\pi iz-\pi i\tau}.$$

Proofs of the tabulated relations are straightforward analogues of (32.3)–(32.5).

Table of Relations among Various Translations of ϑ_j

	$z+1/2$	$z+\tau/2$	$z+1/2+\tau/2$	$z+1$	$z+\tau$	$z+1+\tau$
ϑ_1	ϑ_2	$ia\vartheta_4$	$a\vartheta_3$	$-\vartheta_1$	$-b\vartheta_1$	$b\vartheta_1$
ϑ_2	$-\vartheta_1$	$a\vartheta_3$	$-ia\vartheta_4$	$-\vartheta_2$	$b\vartheta_2$	$-b\vartheta_2$
ϑ_3	ϑ_4	$a\vartheta_2$	$ia\vartheta_1$	ϑ_3	$b\vartheta_3$	$b\vartheta_3$
ϑ_4	ϑ_3	$ia\vartheta_1$	$a\vartheta_2$	ϑ_4	$-b\vartheta_4$	$-b\vartheta_4$

An inspection shows that the following functions

$$F_{jk}(z) = \Big(\frac{\vartheta_j(z)}{\vartheta_k(z)}\Big)^2 \tag{32.11}$$

satisfy

$$F_{jk}(z+\omega) = F_{jk}(z), \quad \forall\,\omega\in\Lambda, \tag{32.12}$$

where

$$\Lambda = \{k+\ell\tau : k,\ell\in\mathbb{Z}\}. \tag{32.13}$$

Note also that

$$G_j(z) = \frac{\vartheta_j'(z)}{\vartheta_j(z)} \tag{32.14}$$

satisfies

$$G_j(z+1) = G_j(z), \quad G_j(z+\tau) = G_j(z) - 2\pi i. \tag{32.15}$$

To relate the functions F_{jk} to previously studied elliptic functions, we need to know the zeros of $\vartheta_k(z)$. Here is the statement:

Proposition 32.1. *We have*

$$\begin{gathered}\vartheta_1(z)=0\Leftrightarrow z\in\Lambda,\quad \vartheta_2(z)=0\Leftrightarrow z\in\Lambda+\frac{1}{2},\\ \vartheta_3(z)=0\Leftrightarrow z\in\Lambda+\frac{1}{2}+\frac{\tau}{2},\quad \vartheta_4(z)=0\Leftrightarrow z\in\Lambda+\frac{\tau}{2}.\end{gathered}\tag{32.16}$$

Proof. In view of the relations tabulated above, it suffices to treat $\vartheta_1(z)$. We first note that

$$\vartheta_1(-z)=-\vartheta_1(z).\tag{32.17}$$

To see this, replace z by $-z$ in (32.8) and simultaneously replace n by $-m$. Then replace m by $n-1$ and observe that (32.17) pops out. Hence ϑ_1 has a zero at $z=0$. We claim it is simple and that ϑ_1 has no others, mod Λ. To see this, let γ be the boundary of a period parallelogram containing 0 in its interior. Then use of (32.15) with $j=1$ easily gives

$$\frac{1}{2\pi i}\int_\gamma \frac{\vartheta_1'(z)}{\vartheta_1(z)}\,dz=1,$$

completing the proof.

Let us record the following complement to (32.17):

$$2\le j\le 4\Longrightarrow \vartheta_j(-z)=\vartheta_j(z).\tag{32.18}$$

The proof is straightforward from the defining formulas (32.6)–(32.9).

We are now ready for the following important result. For consistency with [MM], we slightly reorder the quantities e_1,e_2,e_3. Instead of using (31.6), we set

$$e_1=\wp\Big(\frac{\omega_1}{2}\Big),\quad e_2=\wp\Big(\frac{\omega_1+\omega_2}{2}\Big),\quad e_3=\wp\Big(\frac{\omega_2}{2}\Big),\tag{32.19}$$

where, in the current setting, with Λ given by (32.13), we take $\omega_1=1$ and $\omega_2=\tau$.

Proposition 32.2. *For $\wp(z)=\wp(z;\Lambda)$, with Λ of the form (32.13) and $\vartheta_j(z)=\vartheta_j(z,\tau)$,*

$$\begin{aligned}\wp(z)&=e_1+\Big(\frac{\vartheta_1'(0)}{\vartheta_1(z)}\cdot\frac{\vartheta_2(z)}{\vartheta_2(0)}\Big)^2\\ &=e_2+\Big(\frac{\vartheta_1'(0)}{\vartheta_1(z)}\cdot\frac{\vartheta_3(z)}{\vartheta_3(0)}\Big)^2\\ &=e_3+\Big(\frac{\vartheta_1'(0)}{\vartheta_1(z)}\cdot\frac{\vartheta_4(z)}{\vartheta_4(0)}\Big)^2.\end{aligned}\tag{32.20}$$

Proof. We have from (32.11)–(32.13) that each function $P_j(z)$ on the right side of (32.20) is Λ-periodic. Also Proposition 32.1 implies each P_j has poles of order 2, precisely on Λ. Furthermore, we have arranged that each such function has leading singularity z^{-2} as $z\to 0$, and each P_j is even, by (32.17) and (32.18), so the difference $\wp(z)-P_j(z)$ is constant for each j. Evaluating at $z=1/2$, $(1+\tau)/2$, and $\tau/2$, respectively, shows that these constants are zero, and completes the proof.

Part of the interest in (32.20) is that the series (32.6)–(32.10) for the theta functions are extremely rapidly convergent. To complete this result, we want to express the quantities e_j in terms of theta functions. The following is a key step.

Proposition 32.3. *In the setting of Proposition 32.2,*

$$\begin{aligned}
e_1 - e_2 &= \Bigl(\frac{\vartheta_1'(0)\vartheta_4(0)}{\vartheta_2(0)\vartheta_3(0)}\Bigr)^2 = \pi^2\vartheta_4(0)^4, \\
e_1 - e_3 &= \Bigl(\frac{\vartheta_1'(0)\vartheta_3(0)}{\vartheta_2(0)\vartheta_4(0)}\Bigr)^2 = \pi^2\vartheta_3(0)^4, \qquad (32.21)\\
e_2 - e_3 &= \Bigl(\frac{\vartheta_1'(0)\vartheta_2(0)}{\vartheta_3(0)\vartheta_4(0)}\Bigr)^2 = \pi^2\vartheta_2(0)^4.
\end{aligned}$$

Proof. To get the first part of the first line, evaluate the second identity in (32.20) at $z = 1/2$, to obtain

$$e_1 - e_2 = \Bigl(\frac{\vartheta_1'(0)}{\vartheta_1(1/2)} \cdot \frac{\vartheta_3(1/2)}{\vartheta_3(0)}\Bigr)^2,$$

and then consult the table to rewrite $\vartheta_3(1/2)/\vartheta_1(1/2)$. Similar arguments give the first identity in the second and third lines of (32.21). The proof of the rest of the identities then follows from the next result.

Proposition 32.4. *We have*

$$\vartheta_1'(0) = \pi\vartheta_2(0)\vartheta_3(0)\vartheta_4(0). \tag{32.22}$$

Proof. To begin, consider

$$\varphi(z) = \vartheta_1(2z)^{-1}\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)\vartheta_4(z).$$

Consultation of the table shows that $\varphi(z+\omega) = \varphi(z)$ for each $\omega \in \Lambda$. Also φ is free of poles, so it is constant. The behavior as $z \to 0$ reveals the constant, and yields the identity

$$\vartheta_1(2z) = 2\,\frac{\vartheta_1(z)\vartheta_2(z)\vartheta_3(z)\vartheta_4(z)}{\vartheta_2(0)\vartheta_3(0)\vartheta_4(0)}. \tag{32.23}$$

Now applying log, taking $(d/dz)^2$, evaluating at $z = 0$, and using

$$\vartheta_1''(0) = \vartheta_2'(0) = \vartheta_3'(0) = \vartheta_4'(0) = 0, \tag{32.24}$$

(a consequence of (32.17)–(32.18)), yields

$$\frac{\vartheta_1'''(0)}{\vartheta_1'(0)} = \frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_3''(0)}{\vartheta_3(0)} + \frac{\vartheta_4''(0)}{\vartheta_4(0)}. \tag{32.25}$$

Now, from (32.2) we have

$$\frac{\partial\vartheta_j}{\partial\tau} = \frac{1}{4\pi i}\,\frac{\partial^2\vartheta_j}{\partial z^2}, \tag{32.26}$$

and computing

$$\frac{\partial}{\partial\tau}\Bigl[\log\vartheta_2(0,\tau) + \log\vartheta_3(0,\tau) + \log\vartheta_4(0,\tau) - \log\vartheta_1'(0,\tau)\Bigr] \tag{32.27}$$

and comparing with (32.25) shows that

$$\vartheta_2(0,\tau)\vartheta_3(0,\tau)\vartheta_4(0,\tau)/\vartheta_1'(0,\tau) \text{ is independent of } \tau. \tag{32.28}$$

Thus

$$\vartheta_1'(0) = A\vartheta_2(0)\vartheta_3(0)\vartheta_4(0), \tag{32.29}$$

with A independent of τ, hence independent of $q = e^{\pi i\tau}$. As $q \to 0$, we have

$$\vartheta_1'(0) \sim 2\pi q^{1/4}, \quad \vartheta_2(0) \sim 2q^{1/4}, \quad \vartheta_3(0) \sim 1, \quad \vartheta_4(0) \sim 1, \tag{32.30}$$

which implies $A = \pi$, proving (32.22).

Now that we have Proposition 32.3, we can use $(e_1 - e_3) - (e_1 - e_2) = e_2 - e_3$ to deduce that

$$\vartheta_3(0)^4 = \vartheta_2(0)^4 + \vartheta_4(0)^4. \tag{32.31}$$

Next, we can combine (32.21) with

$$e_1 + e_2 + e_3 = 0 \tag{32.32}$$

to deduce the following.

Proposition 32.5. *In the setting of Proposition 32.2, we have*

$$\begin{aligned} e_1 &= \frac{\pi^2}{3}\big[\vartheta_3(0)^4 + \vartheta_4(0)^4\big], \\ e_2 &= \frac{\pi^2}{3}\big[\vartheta_2(0)^4 - \vartheta_4(0)^4\big], \\ e_3 &= -\frac{\pi^2}{3}\big[\vartheta_2(0)^4 + \vartheta_3(0)^4\big]. \end{aligned} \tag{32.33}$$

Thus we have an efficient method to compute $\wp(z;\Lambda)$ when Λ has the form (32.13). To pass to the general case, we can use the identity

$$\wp(z; a\Lambda) = \frac{1}{a^2}\,\wp\Big(\frac{z}{a};\Lambda\Big). \tag{32.34}$$

See Appendix K for more on the rapid evaluation of $\wp(z;\Lambda)$.

Exercises

1. Show that

$$\frac{d}{dz}\,\frac{\vartheta_1'(z)}{\vartheta_1(z)} = a\wp(z) + b, \tag{32.35}$$

with $\wp(z) = \wp(z;\Lambda)$, Λ as in (32.13). Show that

$$a = -1, \quad b = e_1 + \frac{\vartheta_1''(\omega_1/2)\vartheta_1(\omega_1/2) - \vartheta_1'(\omega_1/2)^2}{\vartheta_1(\omega_1/2)^2}, \tag{32.36}$$

where $\omega_1 = 1$, $\omega_2 = \tau$.

2. In the setting of Exercise 1, deduce that $\zeta_{a,b}(z;\Lambda)$, given by (30.17), satisfies

$$\begin{aligned} \zeta_{a,b}(z;\Lambda) &= \frac{\vartheta_1'(z-a)}{\vartheta_1(z-a)} - \frac{\vartheta_1'(z-b)}{\vartheta_1(z-b)} \\ &= \frac{d}{dz} \log \frac{\vartheta_1(z-a)}{\vartheta_1(z-b)}. \end{aligned} \tag{32.37}$$

3. Show that, if $a \neq e_j$,

$$\frac{1}{\wp(z) - a} = A\zeta_{\alpha,-\alpha}(z) + B, \tag{32.38}$$

where $\wp(\pm\alpha) = a$. Show that

$$A = \frac{1}{\wp'(\alpha)}. \tag{32.39}$$

Identify B.

4. Give a similar treatment of $1/(\wp(z) - a)$ for $a = e_j$. Relate these functions to $\wp(z - \tilde{\omega}_j)$, with $\tilde{\omega}_j$ found from (32.19).

5. Express g_2 and g_3, given in (31.17)–(31.18), in terms of theta functions.
Hint. Use Exercise 7 of §31, plus Proposition 32.5

33. Elliptic integrals

The integral (31.8) is a special case of a class of integrals known as elliptic integrals, which we explore in this section. Let us set

$$q(\zeta) = (\zeta - e_1)(\zeta - e_2)(\zeta - e_3). \tag{33.1}$$

We assume $e_j \in \mathbb{C}$ are distinct and that (as in (31.15))

$$e_1 + e_2 + e_3 = 0, \tag{33.2}$$

which could be arranged by a coordinate translation. Generally, an elliptic integral is an integral of the form

$$\int_{\zeta_0}^{\zeta_1} R(\zeta, \sqrt{q(\zeta)})\, d\zeta, \tag{33.3}$$

where $R(\zeta, \eta)$ is a rational function of its arguments. The relevance of (31.8) is reinforced by the following result.

Proposition 33.1. *Given distinct e_j satisfying (33.2), there exists a lattice Λ, generated by $\omega_1, \omega_2 \in \mathbb{C}$, linearly independent over $\mathbb{R}$, such that if $\wp(z) = \wp(z; \Lambda)$, then*

$$\wp\Bigl(\frac{\omega_j}{2}\Bigr) = e_j, \quad 1 \le j \le 3, \tag{33.4}$$

where $\omega_3 = \omega_1 + \omega_2$.

Given this result, we have from (31.7) that

$$\wp'(z)^2 = 4q(\wp(z)), \tag{33.5}$$

and hence, as in (31.8),

$$\frac{1}{2}\int_{\wp(z_0)}^{\wp(z)} \frac{d\zeta}{\sqrt{q(\zeta)}} = z - z_0, \quad \text{mod } \Lambda. \tag{33.6}$$

The problem of proving Proposition 33.1 is known as the Abel inversion problem. The proof requires new tools, which will be provided in §34. We point out here that there is no difficulty in identifying what the lattice Λ must be. We have

$$\frac{1}{2}\int_{\infty}^{e_j} \frac{d\zeta}{\sqrt{q(\zeta)}} = \frac{\omega_j}{2}, \quad \text{mod } \Lambda, \tag{33.7}$$

by (33.6). One can also verify directly from (33.7) that if the branches are chosen appropriately then $\omega_3 = \omega_1 + \omega_2$. It is not so clear that if Λ is constructed directly from (33.7) then the values of $\wp(z; \Lambda)$ at $z = \omega_j/2$ are given by (33.4), unless one already knows that Proposition 33.1 is true.

Given Proposition 33.1, we can rewrite the elliptic integral (33.3) as follows. The result depends on the particular path γ_{01} from ζ_0 to ζ_1 and on the particular choice of path σ_{01} in $\mathbb{C}/\Lambda$ such that $\wp$ maps σ_{01} homeomorphically onto γ_{01}. With these choices, (33.3) becomes

$$\int_{\sigma_{01}} R\Bigl(\wp(z), \frac{1}{2}\wp'(z)\Bigr)\wp'(z)\,dz, \tag{33.8}$$

or, as we write more loosely,

$$\int_{z_0}^{z_1} R\Bigl(\wp(z), \frac{1}{2}\wp'(z)\Bigr)\wp'(z)\,dz, \tag{33.9}$$

where z_0 and z_1 are the endpoints of σ_{01}, satisfying $\wp(z_j) = \zeta_j$. It follows from Proposition 31.2 that

$$R\Bigl(\wp(z), \frac{1}{2}\wp'(z)\Bigr)\wp'(z) = R_1(\wp(z)) + R_2(\wp(z))\wp'(z), \tag{33.10}$$

for some rational functions $R_j(\zeta)$. In fact, one can describe computational rules for producing such R_j, by using (33.5). Write $R(\zeta,\eta)$ as a quotient of polynomials in (ζ,η) and use (33.5) to obtain that the left side of (33.10) is equal to

$$\frac{P_1(\wp(z)) + P_2(\wp(z))\wp'(z)}{Q_1(\wp(z)) + Q_2(\wp(z))\wp'(z)}, \tag{33.11}$$

for some polynomials $P_j(\zeta), Q_j(\zeta)$. Then multiply the numerator and denominator of (33.11) by $Q_1(\wp(z)) - Q_2(\wp(z))\wp'(z)$ and use (33.5) again on the new denominator to obtain the right side of (33.10).

The integral of (33.3) is now transformed to the integral of the right side of (33.10). Note that

$$\int_{z_0}^{z_1} R_2(\wp(z))\wp'(z)\,dz = \int_{\zeta_0}^{\zeta_1} R_2(\zeta)\,d\zeta, \quad \zeta_j = \wp(z_j). \tag{33.12}$$

This leaves us with the task of analyzing

$$\int_{z_0}^{z_1} R_1(\wp(z))\,dz, \tag{33.13}$$

when $R_1(\zeta)$ is a rational function.

We first analyze (33.13) when $R_1(\zeta)$ is a polynomial. To begin, we have

$$\int_{z_0}^{z_1} \wp(z)\,dz = \zeta(z_0) - \zeta(z_1), \tag{33.14}$$

by (30.15), where $\zeta(z) = \zeta(z;\Lambda)$ is given by (30.14). See (32.35)–(32.36) for a formula in terms of theta functions. Next, differentiating (33.5) gives (as mentioned in Exercise 5 of §31)

$$\wp''(z) = 2q'(\wp(z)) = 6\wp(z)^2 - \frac{1}{2}g_2, \tag{33.15}$$

so

$$6\int_{z_0}^{z_1} \wp(z)^2\,dz = \wp'(z_1) - \wp'(z_0) + \frac{g_2}{2}(z_1 - z_0). \tag{33.16}$$

We can integrate $\wp(z)^{k+2}$ for $k \in \mathbb{N}$ via the following inductive procedure. We have

$$\frac{d}{dz}\big(\wp'(z)\wp(z)^k\big) = \wp''(z)\wp(z)^k + k\wp'(z)^2\wp(z)^{k-1}. \tag{33.17}$$

Apply (33.15) to $\wp''(z)$ and (33.5) (or equivalently (31.17)) to $\wp'(z)^2$ to obtain

$$\frac{d}{dz}\big(\wp'(z)\wp(z)^k\big) = (6+4k)\wp(z)^{k+2} - (3+k)g_2\wp(z)^k - kg_3\wp(z)^{k-1}. \tag{33.18}$$

From here the inductive evaluation of $\int_{z_0}^{z_1} \wp(z)^{k+2}\,dz$, for $k = 1, 2, 3, \dots$, is straightforward.

To analyze (33.13) for a general rational function $R_1(\zeta)$, we see upon making a partial fraction decomposition that it remains to analyze

$$\int_{z_0}^{z_1} (\wp(z) - a)^{-\ell}\,dz, \tag{33.19}$$

for $\ell = 1, 2, 3, \dots$. One can also obtain inductive formulas here, by replacing $\wp(z)^k$ by $(\wp(z) - a)^k$ in (33.18) and realizing that k need not be positive. We get

$$\frac{d}{dz}\big(\wp'(z)(\wp(z) - a)^k\big) = \wp''(z)(\wp(z) - a)^k + k\wp'(z)^2(\wp(z) - a)^{k-1}. \tag{33.20}$$

Now write

$$\begin{aligned} \wp'(z)^2 &= 4\alpha_3(\wp(z) - a)^3 + 4\alpha_2(\wp(z) - a)^2 + 4\alpha_1(\wp(z) - a) + 4\alpha_0, \\ \wp''(z) &= 2A_2(\wp(z) - a)^2 + 2A_1(\wp(z) - a) + 2A_0, \end{aligned} \tag{33.21}$$

where

$$\alpha_j = \frac{q^{(j)}(a)}{j!}, \quad A_j = \frac{q^{(j+1)}(a)}{j!}, \tag{33.22}$$

to obtain

$$\begin{aligned} &\frac{d}{dz}\big(\wp'(z)(\wp(z) - a)^k\big) \\ &\quad = (2A_2 + 4k\alpha_3)(\wp(z) - a)^{k+2} + (2A_1 + 4k\alpha_2)(\wp(z) - a)^{k+1} \\ &\qquad + (2A_0 + 4k\alpha_1)(\wp(z) - a)^k + 4k\alpha_0(\wp(z) - a)^{k-1}. \end{aligned} \tag{33.23}$$

Note that

$$\alpha_0 = q(a), \quad 2A_0 + 4k\alpha_1 = (2+4k)q'(a). \tag{33.24}$$

Thus, if a is not equal to e_j for any j and if we know the integral (33.19) for integral $\ell \le -k$ (for some negative integer k), we can compute the integral for $\ell = 1-k$, as long as $k \neq 0$. If $a = e_j$ for some j, and if we know (33.19) for integral $\ell \le -k-1$, we can compute it for $\ell = -k$, since then $q'(a) \neq 0$.

At this point, the remaining case of (33.19) to consider is the case $\ell = 1$, i.e.,

$$\int_{z_0}^{z_1} \frac{dz}{\wp(z) - a}. \tag{33.25}$$

See Exercises 2–4 of §32 for expressions of $(\wp(z) - a)^{-1}$ in terms of logarithmic derivatives of quotients of theta functions.

Note that the cases $\ell = 0,\ -1$, and 1 of (33.19) are, under the correspondence of (33.3) with (33.8), respectively equal to

$$\int_{\zeta_0}^{\zeta_1} \frac{d\zeta}{\sqrt{q(\zeta)}}, \quad \int_{\zeta_0}^{\zeta_1} (\zeta - a)\, \frac{d\zeta}{\sqrt{q(\zeta)}}, \quad \int_{\zeta_0}^{\zeta_1} \frac{1}{\zeta - a}\, \frac{d\zeta}{\sqrt{q(\zeta)}}. \tag{33.26}$$

These are called, respectively, elliptic integrals of the first, second, and third kind. The material given above expresses the general elliptic integral (33.3) in terms of these cases.

There is another family of elliptic integrals, namely those of the form

$$I = \int R(\zeta, \sqrt{Q(\zeta)})\, d\zeta, \tag{33.27}$$

where $R(\zeta, \eta)$ is a rational function of its arguments and $Q(\zeta)$ is a fourth degree polynomial:

$$Q(\zeta) = (\zeta - a_0)(\zeta - a_1)(\zeta - a_2)(\zeta - a_3), \tag{33.28}$$

with $a_j \in \mathbb{C}$ distinct. Such integrals can be transformed to integrals of the form (33.3), via the change of variable

$$\tau = \frac{1}{\zeta - a_0}, \quad d\zeta = -\frac{1}{\tau^2}\, d\tau. \tag{33.29}$$

One has

$$\begin{aligned} Q\Big(\frac{1}{\tau} + a_0\Big) &= \frac{1}{\tau}\Big(\frac{1}{\tau} + a_0 - a_1\Big)\Big(\frac{1}{\tau} + a_0 - a_2\Big)\Big(\frac{1}{\tau} + a_0 - a_3\Big) \\ &= -\frac{A}{\tau^4}(\tau - e_1)(\tau - e_2)(\tau - e_3), \end{aligned} \tag{33.30}$$

where

$$A = (a_1 - a_0)(a_2 - a_0)(a_3 - a_0), \quad e_j = \frac{1}{a_j - a_0}. \tag{33.31}$$

Then we have

$$I = -\int R\Big(\frac{1}{\tau} + a_0, \frac{\sqrt{-A}}{\tau^2}\sqrt{q(\tau)}\Big)\frac{1}{\tau^2}\, d\tau, \tag{33.32}$$

with $q(\tau)$ as in (33.1). After a further coordinate translation, one can alter e_j to arrange (33.2).

Elliptic integrals are frequently encountered in many areas of mathematics. Here we give two examples, one from differential equations and one from geometry.

Our first example involves the differential equation for the motion of a simple pendulum, which takes the form

$$\ell \frac{d^2\theta}{dt^2} + g \sin\theta = 0, \tag{33.33}$$

where ℓ is the length of the pendulum g the acceleration of gravity (32 ft./sec.^{2} on the surface of the earth), and θ is the angle the pendulum makes with the downward-pointing vertical axis. The total energy of the pendulum is proportional to

$$E = \frac{1}{2}\Bigl(\frac{d\theta}{dt}\Bigr)^2 - \frac{g}{\ell}\cos\theta. \tag{33.34}$$

Applying d/dt to (33.34) and comparing with (33.33) shows that E is constant for each solution to (33.33), so one has

$$\frac{1}{\sqrt{2}}\frac{d\theta}{dt} = \pm\sqrt{E + \frac{g}{\ell}\cos\theta}, \tag{33.35}$$

or

$$\pm \int \frac{d\theta}{\sqrt{E + a\cos\theta}} = \sqrt{2}t + c, \tag{33.36}$$

with $a = g/\ell$. With $\varphi = \theta/2,\ \cos 2\varphi = 1 - 2\sin^2\varphi$, we have

$$\pm \int \frac{d\varphi}{\sqrt{\alpha - \beta\sin^2\varphi}} = \frac{t}{\sqrt{2}} + c', \tag{33.37}$$

with $\alpha = E + a,\ \beta = 2a$. Then setting $\zeta = \sin\varphi,\ d\zeta = \cos\varphi\, d\varphi$, we have

$$\pm \int \frac{d\zeta}{\sqrt{(\alpha - \beta\zeta^2)(1-\zeta^2)}} = \frac{t}{\sqrt{2}} + c', \tag{33.38}$$

which is an integral of the form (33.27). If instead in (33.36) we set $\zeta = \cos\theta$, so $d\zeta = -\sin\theta\, d\theta$, we obtain

$$\mp \int \frac{d\zeta}{\sqrt{(E + a\zeta)(1-\zeta^2)}} = \sqrt{2}\,t + c, \tag{33.39}$$

which is an integral of the form (33.3).

In our next example we produce a formula for the arc length $L(\theta)$ of the portion of the ellipse

$$z(t) = (a\,\cos t, b\,\sin t), \tag{33.40}$$

from $t=0$ to $t=\theta$. We assume $a>b>0$. Note that

$$
\begin{aligned}
|z'(t)|^2 &= a^2\sin^2 t + b^2\cos^2 t \\
&= b^2 + c^2\sin^2 t,
\end{aligned}
\tag{33.41}
$$

with $c^2 = a^2 - b^2$, so

$$
L(\theta) = \int_0^\theta \sqrt{b^2 + c^2\sin^2 t}\, dt. \tag{33.42}
$$

With $\zeta = \sin t,\ u = \sin\theta$, this becomes

$$
\begin{aligned}
&\int_0^u \sqrt{b^2 + c^2\zeta^2}\, \frac{d\zeta}{\sqrt{1-\zeta^2}} \\
&= \int_0^u \frac{b^2 + c^2\zeta^2}{\sqrt{(1-\zeta^2)(b^2+c^2\zeta^2)}}\, d\zeta,
\end{aligned}
\tag{33.43}
$$

which is an integral of the form (33.27).

Exercises

1. Using (33.7) and the comments that follow it, show that, for $j=1,2$,

$$
\frac{1}{2}\int_{e_j}^{e_3} \frac{d\zeta}{\sqrt{q(\zeta)}} = \frac{\omega_{j'}}{2}, \quad \bmod \Lambda, \tag{33.44}
$$

where $j'=2$ if $j=1,\ j'=1$ if $j=2$.

2. Setting $e_{kj} = e_k - e_j$, show that

$$
\frac{1}{2}\int_{e_1}^{e_1+\eta} \frac{d\zeta}{\sqrt{q(\zeta)}} = \frac{1}{2\sqrt{e_{12}e_{13}}} \sum_{k,\ell=0}^{\infty} \binom{-1/2}{k}\binom{-1/2}{\ell} \frac{1}{e_{12}^k e_{13}^\ell}\, \frac{\eta^{k+\ell+1/2}}{k+\ell+1/2} \tag{33.45}
$$

is a convergent power series provided $|\eta| < \min(|e_1-e_2|, |e_1-e_3|)$. Using this and variants to integrate from e_j to $e_j+\eta$ for $j=2$ and 3, find convergent power series for $\omega_j/2$ (mod Λ).

3. Given $k \neq \pm 1$, show that

$$
\int \frac{d\zeta}{\sqrt{(1-\zeta^2)(k^2-\zeta^2)}} = -\frac{1}{\sqrt{2(1-k^2)}} \int \frac{d\tau}{\sqrt{q(\tau)}}, \tag{33.46}
$$

with

$$
\tau = \frac{1}{\zeta+1}, \quad q(\tau) = \Big(\tau - \frac{1}{2}\Big)\Big(\tau - \frac{1}{1-k}\Big)\Big(\tau - \frac{1}{1+k}\Big).
$$

In Exercises 4–9, we assume e_j are real and $e_1 < e_2 < e_3$. We consider

$$\omega_3 = \int_{e_1}^{e_2} \frac{d\zeta}{\sqrt{(\zeta - e_1)(\zeta - e_2)(\zeta - e_3)}}.$$

4. Show that

$$\begin{aligned} \omega_3 &= 2\int_0^{\pi/2} \frac{d\theta}{\sqrt{(e_3 - e_2)\sin^2\theta + (e_3 - e_1)\cos^2\theta}} \\ &= 2I\big(\sqrt{e_3 - e_2}, \sqrt{e_3 - e_1}\big), \end{aligned} \tag{33.48}$$

where

$$I(r, s) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{r^2\sin^2\theta + s^2\cos^2\theta}}. \tag{33.49}$$

Note that in (33.48), $0 < r < s$.

Exercises 5–7 will be devoted to showing that

$$I(r, s) = \frac{\pi}{2M(s, r)}, \tag{33.50}$$

if $0 < r \le s$, where $M(s, r)$ is the Gauss arithmetic-geometric mean, defined below.

5. Given $0 < b \le a$, define inductively

$$(a_0, b_0) = (a, b), \quad (a_{k+1}, b_{k+1}) = \Big(\frac{a_k + b_k}{2}, \sqrt{a_k b_k}\Big). \tag{33.51}$$

Show that

$$a_0 \ge a_1 \ge a_2 \ge \cdots \ge b_2 \ge b_1 \ge b_0.$$

Show that

$$a_{k+1}^2 - b_{k+1}^2 = (a_{k+1} - a_k)^2.$$

Monotonicity implies $a_k - a_{k+1} \to 0$. Deduce that $a_{k+1} - b_{k+1} \to 0$, and hence

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = M(a, b), \tag{33.52}$$

the latter identity being the definition of $M(a, b)$. Show also that

$$a_{k+1}^2 - b_{k+1}^2 = \frac{1}{4}(a_k - b_k)^2,$$

hence

$$a_{k+1} - b_{k+1} = \frac{(a_k - b_k)^2}{8a_{k+2}}. \tag{33.53}$$

Deduce from (33.53) that convergence in (33.52) is quite rapid.

6. Show that the asserted identity (33.50) holds if it can be demonstrated that, for $0 < r \le s$,

$$I(r,s) = I\Big(\sqrt{rs}, \frac{r+s}{2}\Big). \tag{33.54}$$

Hint. Show that (33.54) $\Rightarrow I(r,s) = I(m,m)$, with $m = M(s,r)$, and evaluate $I(m,m)$.

7. Take the following steps to prove (33.54). Show that you can make the change of variable from θ to φ, with

$$\sin\theta = \frac{2s\sin\varphi}{(s+r)+(s-r)\sin^2\varphi}, \quad 0 \le \varphi \le \frac{\pi}{2}, \tag{33.55}$$

and obtain

$$I(r,s) = \int_0^{\pi/2} \frac{2\,d\varphi}{\sqrt{4rs\sin^2\varphi + (s+r)^2\cos^2\varphi}}. \tag{33.56}$$

Show that this yields (33.54).

8. In the setting of Exercise 4, deduce that

$$\omega_3 = \frac{\pi}{M(\sqrt{e_3-e_1}, \sqrt{e_3-e_2})}. \tag{33.57}$$

9. Similarly, show that

$$\begin{aligned}\omega_1 &= \int_{e_2}^{e_3} \frac{d\zeta}{\sqrt{(\zeta-e_1)(\zeta-e_2)(\zeta-e_3)}} \\ &= 2i\int_0^{\pi/2} \frac{d\theta}{\sqrt{(e_2-e_1)\sin^2\theta + (e_3-e_1)\cos^2\theta}} \\ &= \frac{\pi i}{M(\sqrt{e_3-e_1}, \sqrt{e_2-e_1})}.\end{aligned} \tag{33.58}$$

10. Set $x = \sin\theta$ to get

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\beta^2\sin^2\theta}} = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\beta^2x^2)}}.$$

Write $1-\beta^2\sin^2\theta = (1-\beta^2)\sin^2\theta + \cos^2\theta$ to deduce that

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\beta^2x^2)}} = \frac{\pi}{2M(1,\sqrt{1-\beta^2})}, \tag{3.59}$$

if $\beta \in (-1,1)$.

11. Parallel to Exercise 10, show that

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1+\beta^2 \sin^2\theta}} = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1+\beta^2 x^2)}},$$

and deduce that

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1+\beta^2 x^2)}} = \frac{\pi}{2M(\sqrt{1+\beta^2},1)}, \tag{3.60}$$

if $\beta \in \mathbb{R}$. A special case is

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{2M(\sqrt{2},1)}. \tag{3.61}$$

For more on the arithmetic-geometric mean (AGM), see [BB].

34. The Riemann surface of $\sqrt{q(\zeta)}$

Recall from §33 the cubic polynomial

$$q(\zeta) = (\zeta - e_1)(\zeta - e_2)(\zeta - e_3), \tag{34.1}$$

where $e_1, e_2, e_3 \in \mathbb{C}$ are distinct. Here we will construct a compact Riemann surface M associated with the "double valued" function $\sqrt{q(\zeta)}$, together with a holomorphic map

$$\varphi : M \longrightarrow S^2, \tag{34.2}$$

and discuss some important properties of M and φ. We will then use this construction to prove Proposition 33.1. Material developed below will use some basic results on manifolds, particularly on surfaces, which are generally covered in beginning topology courses. Background may be found in [Mun] and [Sp], among other places. See also Appendix C for some background.

To begin, we set $e_4 = \infty$ in the Riemann sphere $\mathbb{C} \cup \{\infty\}$, identified with S^2 in §26. Reordering if necessary, we arrange that the geodesic γ_{12} from e_1 to e_2 is disjoint from the geodesic γ_{34} from e_3 to e_4. We slit S^2 along γ_{12} and along γ_{34}, obtaining X, a manifold with boundary, as illustrated in the top right portion of Fig. 34.1. Now

$$M = X_1 \cup X_2 / \sim, \tag{34.3}$$

where X_1 and X_2 are two copies of X, and the equivalence relation $\sim$ identifies the upper boundary of X_1 along the slit γ_{12} with the lower boundary of X_2 along this slit and vice-versa, and similarly for γ_{34}. This is illustrated in the middle and bottom parts of Fig. 34.1. The manifold M is seen to be topologically equivalent to a torus.

The map $\varphi : M \to S^2$ in (34.2) is tautological. It is two-to-one except for the four points $p_j = \varphi^{-1}(e_j)$. Recall the definition of a Riemann surface given in §26, in terms of coordinate covers. The space M has a unique Riemann surface structure for which φ is holomorphic. A coordinate taking a neighborhood of p_j in M bijectively onto a neighborhood of the origin in $\mathbb{C}$ is given by $\varphi_j(x) = (\varphi(x) - e_j)^{1/2}$, for $1 \le j \le 3$, with $\varphi(x) \in S^2 \approx \mathbb{C} \cup \{\infty\}$, and a coordinate mapping a neighborhood of p_4 in M bijectively onto a neighborhood of the origin in $\mathbb{C}$ is given by $\varphi_4(x) = \varphi(x)^{-1/2}$.

Now consider the double-valued form $d\zeta/\sqrt{q(\zeta)}$ on S^2, having singularities at $\{e_j\}$. This pulls back to a single-valued 1-form α on M. Noting that if $w^2 = \zeta$ then

$$\frac{d\zeta}{\sqrt{\zeta}} = 2\, dw, \tag{34.4}$$

and that if $w^2 = 1/\zeta$ then

$$\frac{d\zeta}{\sqrt{\zeta^3}} = -2\, dw, \tag{34.5}$$

we see that α is a smooth, holomorphic 1-form on M, with no singularities, and also that α has no zeros on M. Using this, we can prove the following.

Proposition 34.1. *There is a lattice $\Lambda_0 \subset \mathbb{C}$ and a holomorphic diffeomorphism*

$$\psi : M \longrightarrow \mathbb{C}/\Lambda_0. \tag{34.6}$$

Proof. Given M homeomorphic to $S^1 \times S^1$, we have closed curves c_1 and c_2 through p_1 in M such that each closed curve γ in M is homotopic to a curve starting at p_1, winding n_1 times along c_1, then n_2 times along c_2, with $n_j \in \mathbb{Z}$. Say $\omega_j = \int_{c_j} \alpha$. We claim ω_1 and ω_2 are linearly independent over $\mathbb{R}$. First we show that they are not both 0. Indeed, if $\omega_1 = \omega_2 = 0$, then

$$\Psi(z) = \int_{p_0}^{z} \alpha \tag{34.7}$$

would define a non-constant holomorphic map $\Psi : M \to \mathbb{C}$, which would contradict the maximum principle. Let us say $\omega_2 \neq 0$, and set $\beta = \omega_2^{-1}\alpha$. Then $\Psi_1(z) = \int_{p_0}^{z} \beta$ is well defined modulo an additive term of the form $j + k(\omega_1/\omega_2)$, with $j, k \in \mathbb{Z}$. If ω_1/ω_2 were real, then $\operatorname{Im}\Psi_1 : M \to \mathbb{R}$ would be a well defined harmonic function, hence (by the maximum principle) constant, forcing Ψ constant, and contradicting the fact that $\alpha \neq 0$.

Thus we have that $\Lambda_1 = \{n_1\omega_1 + n_2\omega_2 : n_j \in \mathbb{Z}\}$ is a lattice, and that (34.7) yields a well defined holomorphic map

$$\Psi : M \longrightarrow \mathbb{C}/\Lambda_1. \tag{34.8}$$

Since α is nowhere vanishing, Ψ is a local diffeomorphism. Hence it must be a covering map. This gives (34.6), where Λ_0 is perhaps a sublattice of Λ_1.

We now prove Proposition 33.1, which we restate here.

Proposition 34.2. *Let e_1, e_2, e_3 be distinct points in $\mathbb{C}$, satisfying*

$$e_1 + e_2 + e_3 = 0. \tag{34.9}$$

There exists a lattice $\Lambda \subset \mathbb{C}$, generated by ω_1, ω_2, linearly independent over $\mathbb{R}$, such that if $\wp(z) = \wp(z; \Lambda)$, then

$$\wp\Big(\frac{\omega_j}{2}\Big) = e_j, \quad 1 \le j \le 3, \tag{34.10}$$

where $\omega_3 = \omega_1 + \omega_2$.

Proof. We have from (34.2) and (34.6) a holomorphic map

$$\Phi : \mathbb{C}/\Lambda_0 \longrightarrow S^2, \tag{34.11}$$

which is a branched double cover, branching over e_1, e_2, e_3, and ∞. We can regard Φ as a meromorphic function on $\mathbb{C}$, satisfying

$$\Phi(z + \omega) = \Phi(z), \quad \forall\, \omega \in \Lambda_0. \tag{34.12}$$

Furthermore, translating coordinates, we can assume Φ has a double pole, precisely at points in Λ_0. It follows that there are constants a and b such that

$$\Phi(z) = a\wp_0(z) + b, \quad a \in \mathbb{C}^*,\ b \in \mathbb{C}, \tag{34.13}$$

where $\wp_0(z) = \wp(z; \Lambda_0)$. Hence $\Phi'(z) = a\wp_0'(z)$, so by Proposition 31.1 we have

$$\Phi'(z) = 0 \Longleftrightarrow z = \frac{\omega_{0j}}{2}, \quad \text{mod } \Lambda_0, \tag{34.14}$$

where ω_{01}, ω_{02} generate Λ_0 and $\omega_{03} = \omega_{01} + \omega_{02}$. Hence (perhaps after some reordering)

$$e_j = a\wp_0\Bigl(\frac{\omega_{0j}}{2}\Bigr) + b. \tag{34.15}$$

Now if $e_j' = \wp_0(\omega_{0j}/2)$, we have by (31.15) that $e_1' + e_2' + e_3' = 0$, so (34.9) yields

$$b = 0. \tag{34.16}$$

Finally, we set $\Lambda = a^{-1/2}\Lambda_0$ and use (32.34) to get

$$\wp(z; \Lambda) = a\,\wp(a^{1/2}z; \Lambda_0). \tag{34.17}$$

Then (34.10) is achieved.

We mention that a similar construction works to yield a compact Riemann surface $M \to S^2$ on which there is a single valued version of $\sqrt{q(\zeta)}$ when

$$q(\zeta) = (\zeta - e_1) \cdots (\zeta - e_m), \tag{34.18}$$

where $e_j \in \mathbb{C}$ are distinct, and $m \geq 2$. If $m = 2g+1$, one has slits from e_{2j-1} to e_{2j}, for $j = 1, \dots, g$, and a slit from e_{2g+1} to ∞, which we denote e_{2g+2}. If $m = 2g+2$, one has slits from e_{2j-1} to e_{2j}, for $j = 1, \dots, g+1$. Then X is constructed by opening the slits, and M is constructed as in (34.3). The picture looks like that in Fig. 34.1, but instead of two sets of pipes getting attached, one has $g+1$ sets. One gets a Riemann surface M with g holes, called a surface of genus g. Again the double-valued form $d\zeta/\sqrt{q(\zeta)}$ on S^2 pulls back to a single-valued 1-form α on M, with no singularities, except when $m = 2$ (see the exercises). If $m = 4$ (so again $g = 1$), α has no zeros. If $m \geq 5$ (so $g \geq 2$), α has a zero at $\varphi^{-1}(\infty)$. Proposition 34.1 extends to the case $m = 4$. If $m \geq 5$ the situation changes. It is a classical result that M is covered by the disk D rather than by $\mathbb{C}$. The pull-back of α to D is called an automorphic form. For much more on such matters, and on more general constructions of Riemann surfaces, we recommend [FK] and [MM].

We end this section with a brief description of a Riemann surface, conformally equivalent to M in (34.3), appearing as a submanifold of complex projective space $\mathbb{CP}^2$. More details on such a construction can be found in [Cl] and [MM].

To begin, we define complex projective space $\mathbb{CP}^n$ as $(\mathbb{C}^{n+1} \setminus 0)/\sim$, where we say z and $z' \in \mathbb{C}^{n+1} \setminus 0$ satisfy $z \sim z'$ provided $z' = az$ for some $a \in \mathbb{C}^*$. Then $\mathbb{CP}^n$ has the structure of a complex manifold. Denote by $[z]$ the equivalence class in $\mathbb{CP}^n$ of $z \in \mathbb{C}^{n+1} \setminus 0$. We note that the map

$$\kappa : \mathbb{CP}^1 \longrightarrow \mathbb{C} \cup \{\infty\} \tag{34.19}$$

given by

$$\begin{aligned} \kappa([(z_1, z_2)]) &= z_1/z_2, \quad z_2 \neq 0, \\ \kappa([(1, 0)]) &= \infty, \end{aligned} \tag{34.20}$$

is a holomorphic diffeomorphism, so $\mathbb{CP}^1 \approx S^2$.

Now given distinct $e_1, e_2, e_3 \in \mathbb{C}$, we can define $M_e \subset \mathbb{CP}^2$ to consist of elements $[(w, \zeta, t)]$ such that $(w, \zeta, t) \in \mathbb{C}^3 \setminus 0$ satisfies

$$w^2 t = (\zeta - e_1 t)(\zeta - e_2 t)(\zeta - e_3 t). \tag{34.21}$$

One can show that M_e is a smooth complex submanifold of $\mathbb{CP}^2$, possessing then the structure of a compact Riemann surface. An analogue of the map (34.2) is given as follows.

Set $p = [(1, 0, 0)] \in \mathbb{CP}^2$. Then there is a holomorphic map

$$\psi : \mathbb{CP}^2 \setminus p \longrightarrow \mathbb{CP}^1, \tag{34.22}$$

given by

$$\psi([(w, \zeta, t)]) = [(\zeta, t)]. \tag{34.23}$$

This restricts to $M_e \setminus p \to \mathbb{CP}^1$. Note that $p \in M_e$. While ψ in (34.22) is actually singular at p, for the restriction to $M_e \setminus p$ this is a removable singularity, and one has a holomorphic map

$$\varphi_e : M_e \longrightarrow \mathbb{CP}^1 \approx \mathbb{C} \cup \{\infty\} \approx S^2, \tag{34.24}$$

given by (34.22) on $M_e \setminus p$ and taking p to $[(1, 0)] \in \mathbb{CP}^1$, hence to $\infty \in \mathbb{C} \cup \{\infty\}$. This map can be seen to be a 2-to-1 branched covering, branching over $\mathcal{B} = \{e_1, e_2, e_3, \infty\}$. Given $q \in \mathbb{C}$, $q \notin \mathcal{B}$, and a choice $r \in \varphi^{-1}(q) \subset M$ and $r_e \in \varphi_e^{-1}(q) \subset M_e$, there is a unique holomorphic diffeomorphism

$$\Gamma : M \longrightarrow M_e, \tag{34.25}$$

such that $\Gamma(r) = r_e$ and $\varphi = \varphi_e \circ \Gamma$.

Exercises

1. Show that the covering map Ψ in (34.8) is actually a diffeomorphism, and hence $\Lambda_0 = \Lambda_1$.

2. Suppose Λ_0 and Λ_1 are two lattices in $\mathbb{C}$ such that $\mathbb{T}_{\Lambda_0}$ and $\mathbb{T}_{\Lambda_1}$ are conformally equivalent, via a holomorphic diffeomorphism

$$f : \mathbb{C}/\Lambda_0 \longrightarrow \mathbb{C}/\Lambda_1. \tag{34.26}$$

Show that f lifts to a holomorphic diffeomorphism F of $\mathbb{C}$ onto itself, such that $F(0) = 0$, and hence that $F(z) = az$ for some $a \in \mathbb{C}^*$. Deduce that $\Lambda_1 = a\Lambda_0$.

3. Consider the upper half-plane $\mathcal{U} = \{\tau \in \mathbb{C} : \operatorname{Im} \tau > 0\}$. Given $\tau \in \mathcal{U}$, define

$$\Lambda(\tau) = \{m + n\tau : m, n \in \mathbb{Z}\}. \tag{34.27}$$

Show that each lattice $\Lambda \subset \mathbb{C}$ has the form $\Lambda = a\Lambda(\tau)$ for some $a \in \mathbb{C}^*$, $\tau \in \mathcal{U}$.

4. Define the maps $\alpha, \beta : \mathcal{U} \to \mathcal{U}$ by

$$\alpha(\tau) = -\frac{1}{\tau}, \quad \beta(\tau) = \tau + 1. \tag{34.28}$$

Show that, for each $\tau \in \mathcal{U}$,

$$\Lambda(\alpha(\tau)) = \tau^{-1}\,\Lambda(\tau), \quad \Lambda(\beta(\tau)) = \Lambda(\tau). \tag{34.29}$$

5. Let $\mathcal{G}$ be the group of automorphisms of $\mathcal{U}$ generated by α and β, given in (34.28). Show that if $\tau, \tau' \in \mathcal{U}$,

$$\mathbb{C}/\Lambda(\tau) \approx \mathbb{C}/\Lambda(\tau'), \tag{34.30}$$

in the sense of being holomorphically diffeomorphic, if and only if

$$\tau' = \gamma(\tau), \quad \text{for some} \ \ \gamma \in \mathcal{G}. \tag{34.31}$$

6. Show that the group $\mathcal{G}$ consists of linear fractional transformations of the form

$$L_A(\tau) = \frac{a\tau + b}{c\tau + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{34.32}$$

where $a, b, c, d \in \mathbb{Z}$ and $\det A = 1$, i.e., $A \in Sl(2, \mathbb{Z})$. Show that

$$\mathcal{G} \approx Sl(2, \mathbb{Z})/\{\pm I\}.$$

In Exercises 7–8, we make use of the covering map $\Psi : \mathcal{U} \to \mathbb{C} \setminus \{0, 1\}$, given by (25.5), and results of Exercises 1–8 of §25, including (25.10)–(25.11), i.e.,

$$\Psi(\alpha(\tau)) = \frac{1}{\Psi(\tau)}, \quad \Psi(\beta(\tau)) = 1 - \Psi(\tau). \tag{34.33}$$

7. Given $\tau, \tau' \in \mathcal{U}$, we say $\tau \sim \tau'$ if and only if (34.30) holds. Show that, given

$$\tau,\ \tau' \in \mathcal{U}, \quad w = \Psi(\tau),\ w' = \Psi(\tau') \in \mathbb{C} \setminus \{0, 1\}, \tag{34.34}$$

we have

$$\tau \sim \tau' \Longleftrightarrow w' = F(w) \ \text{ for some } \ F \in \mathcal{G}, \tag{34.35}$$

where $\mathcal{G}$ is the group (of order 6) of automorphisms of $\mathbb{C} \setminus \{0, 1\}$ arising in Exercise 6 of §25.

8. Bringing in the map $H : S^2 \to S^2$ arising in Exercise 8 of §25, i.e.,

$$H(w) = \frac{4}{27} \frac{(w^2 - w + 1)^3}{w^2(w-1)^2}, \tag{34.36}$$

satisfying (25.23), i.e.,

$$H\Bigl(\frac{1}{w}\Bigr) = H(w), \quad H(1-w) = H(w), \tag{34.37}$$

show that

$$w' = F(w) \ \text{ for some } \ F \in \mathcal{G} \Longleftrightarrow H(w') = H(w). \tag{34.38}$$

Deduce that, for $\tau, \tau' \in \mathcal{U}$,

$$\tau \sim \tau' \Longleftrightarrow H \circ \Psi(\tau') = H \circ \Psi(\tau). \tag{34.39}$$

Exercises 9–14 deal with the Riemann surface M of $\sqrt{q(\zeta)}$ when

$$q(\zeta) = (\zeta - e_1)(\zeta - e_2), \tag{34.40}$$

and $e_1, e_2 \in \mathbb{C}$ are distinct.

9. Show that the process analogous to that pictured in Fig. 34.1 involves the attachment of one pair of pipes, and M is topologically equivalent to a sphere. One gets a branched covering $\varphi : M \to S^2$, as in (34.2).

10. Show that the double-valued form $d\zeta/\sqrt{q(\zeta)}$ on S^2 pulls back to a single-valued form α on M. Using (34.4), show that α is a smooth nonvanishing form except at $\{p_1, p_2\} = \varphi^{-1}(\infty)$. In a local coordinate system about p_j of the form $\varphi_j(x) = \varphi(x)^{-1}$, use a variant of (34.4)–(34.5) to show that α has the form

$$\alpha = (-1)^j \frac{g(z)}{z}\, dz, \tag{34.41}$$

where $g(z)$ is holomorphic and $g(0) \neq 0$.

11. Let c be a curve in $M \setminus \{p_1, p_2\}$ with winding number 1 about p_1. Set

$$\omega = \int\limits_c \alpha, \quad L = \{k\omega : k \in \mathbb{Z}\} \subset \mathbb{C}. \tag{34.42}$$

Note that Exercise 10 implies $\omega \neq 0$. Pick $q \in M \setminus \{p_1, p_2\}$. Show that

$$\Psi(z) = \int_q^z \alpha \tag{34.43}$$

yields a well defined holomorphic map

$$\Psi : M \setminus \{p_1, p_2\} \longrightarrow \mathbb{C}/L. \tag{34.44}$$

12. Show that Ψ in (34.44) is a holomorphic diffeomorphism of $M \setminus \{p_1, p_2\}$ onto $\mathbb{C}/L$.
Hint. To show Ψ is onto, use (34.41) to examine the behavior of Ψ near p_1 and p_2.

13. Produce a holomorphic diffeomorphism $\mathbb{C}/L \approx \mathbb{C} \setminus \{0\}$, and then use (34.44) to obtain a holomorphic diffeomorphism

$$\Psi_1 : M \setminus \{p_1, p_2\} \longrightarrow S^2 \setminus \{0, \infty\}. \tag{34.45}$$

Show that this extends uniquely to a holomorphic diffeomorphism

$$\Psi_1 : M \longrightarrow S^2. \tag{34.46}$$

14. Note that with a linear change of variable we can arrange $e_j = (-1)^j$ in (34.40). Relate the results of Exercises 9–13 to the identity

$$\int_0^z (1 - \zeta^2)^{-1/2} \, d\zeta = \sin^{-1} z \pmod{2\pi\mathbb{Z}}. \tag{34.47}$$

Appendices

26. The Riemann sphere and other Riemann surfaces

Our main goal here is to describe how the unit sphere $S^2 \subset \mathbb{R}^3$ has a role as a "conformal compactification" of the complex plane $\mathbb{C}$. To begin, we consider a map

$$\mathcal{S} : S^2 \setminus \{e_3\} \longrightarrow \mathbb{R}^2, \tag{26.1}$$

known as stereographic projection; here $e_3 = (0,0,1)$. We define $\mathcal{S}$ as follows:

$$\mathcal{S}(x_1, x_2, x_3) = (1 - x_3)^{-1}(x_1, x_2). \tag{26.2}$$

See Fig. 26.1. A computation shows that $\mathcal{S}^{-1} : \mathbb{R}^2 \to S^2 \setminus \{e_3\}$ is given by

$$\mathcal{S}^{-1}(x, y) = \frac{1}{1 + r^2}\,(2x, 2y, r^2 - 1), \quad r^2 = x^2 + y^2. \tag{26.3}$$

The following is a key geometrical property.

Proposition 26.1. *The map $\mathcal{S}$ is a conformal diffeomorphism of $S^2 \setminus \{e_3\}$ onto $\mathbb{R}^2$.*

In other words, we claim that if two curves γ_1 and γ_2 in S^2 meet at an angle α at $p \neq e_3$, then their images under $\mathcal{S}$ meet at the same angle at $q = \mathcal{S}(p)$. It is equivalent, and slightly more convenient, to show that $F = \mathcal{S}^{-1}$ is conformal. We have

$$DF(q) : \mathbb{R}^2 \longrightarrow T_pS^2 \subset \mathbb{R}^3. \tag{26.4}$$

See Appendix C for more on this. Conformality is equivalent to the statement that there is a positive function $\lambda(p)$ such that, for all $v, w \in \mathbb{R}^2$,

$$DF(q)v \cdot DF(q)w = \lambda(q)\, v \cdot w, \tag{26.5}$$

or in other words,

$$DF(q)^t\, DF(q) = \lambda(q) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{26.6}$$

To check (26.6), we compute DF via (26.3). A calculation gives

$$DF(x, y) = \frac{2}{(1 + r^2)^2} \begin{pmatrix} 1 - x^2 + y^2 & -2xy \\ -2xy & 1 + x^2 - y^2 \\ -2x & -2y \end{pmatrix}, \tag{26.7}$$

and hence

$$DF(x, y)^t\, DF(x, y) = \frac{4}{(1 + r^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{26.8}$$

This gives Proposition 26.1.

Similarly we can define a conformal diffeomorphism

$$\mathcal{S}_- : S^2 \setminus \{-e_3\} \longrightarrow \mathbb{R}^2. \tag{26.9}$$

To do this we take $x_3 \mapsto -x_3$. This reverses orientation, so we also take $x_2 \mapsto -x_2$. Thus we set

$$\mathcal{S}_-(x_1, x_2, x_3) = (1 + x_3)^{-1}(x_1, -x_2). \tag{26.10}$$

Comparing this with (26.3), we see that $\mathcal{S}_- \circ \mathcal{S}^{-1} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ is given by

$$\mathcal{S}_- \circ \mathcal{S}^{-1}(x, y) = \frac{1}{r^2}\,(x, -y). \tag{26.11}$$

Identifying $\mathbb{R}^2$ with $\mathbb{C}$ via $z = x + iy$, we have

$$\mathcal{S}_- \circ \mathcal{S}^{-1}(z) = \frac{\overline{z}}{|z|^2} = \frac{1}{z}. \tag{26.12}$$

Clearly the composition of conformal transformations is conformal, so we could predict in advance that $\mathcal{S}_1 \circ \mathcal{S}^{-1}$ would be conformal and orientation-preserving, hence holomorphic, and (26.12) bears this out.

If we use (26.1) to identify $\mathbb{C}$ with $S^2 \setminus \{e_3\}$, then the one-point compactification $\mathbb{C} \cup \{\infty\}$ is naturally identified with S^2, with ∞ corresponding to the "north pole" e_3. The map (26.12) can be extended from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C} \cup \{\infty\}$, and it switches 0 and ∞.

The concept of a normal family of maps $\Omega \to S$, introduced in §21, is of great interest when $S = S^2 = \mathbb{C} \cup \{\infty\}$. The following result produces a key link with results established in §21.

Proposition 26.2. *Assume $\Omega \subset \mathbb{C}$ is a connected open set. A family $\mathcal{F}$ of holomorphic functions $\Omega \to \mathbb{C}$ is normal with respect to $(\Omega, \mathbb{C} \cup \{\infty\})$ if and only if for each sequence f_ν from $\mathcal{F}$ one of the following happens:*
(a) A subsequence f_{ν_k} converges uniformly on each compact $K \subset \Omega$, as a sequence $f_{\nu_k} : K \to \mathbb{C}$, or
(b) A subsequence f_{ν_k} tends to ∞ uniformly on each compact $K \subset \Omega$.

Proof. Assume $\mathcal{F}$ is a normal family with respect to $(\Omega, \mathbb{C} \cup \{\infty\})$, and f_ν is a sequence of elements of $\mathcal{F}$. Take a subsequence f_{ν_k}, uniformly convegent on each compact K, as a sequence of maps $f_{\nu_k} : K \to S^2$. Say $f_{\nu_k} \to f : \Omega \to S^2$. Pick $p \in \Omega$. We consider two cases.

Case I. First suppose $f(p) = \infty$. Then there exists $N \in \mathbb{Z}^+$ and a neighborhood U of p in Ω such that $|f_{\nu_k}(z)| \geq 1$ for $z \in U, k \geq N$. Set $g_{\nu_k}(z) = 1/f_{\nu_k}(z)$, for $z \in U, k \geq N$. We have $|g_{\nu_k}| \leq 1$ on U, $g_{\nu_k}(z) \neq 0$, and $g_{\nu_k}(z) \to 1/f(z)$, locally uniformly on U (with $1/\infty = 0$), and in particular $g_{\nu_k}(p) \to 0$. By Hurwitz' theorem (Proposition 17.8), this implies $1/f(z) = 0$ on all of U, i.e., $f = \infty$ on U, hence $f = \infty$ on Ω. Hence Case I $\Rightarrow$ Case (b).

Case II. Suppose $f(p) \in \mathbb{C}$, i.e., $f(p) \in S^2 \setminus \{\infty\}$. By the analysis in Case I it follows that $f(z) \in \mathbb{C}$ for all $z \in \Omega$. It is now straightforward to verify Case (a) here.

This gives one implication in Proposition 26.2. The reverse implication is easily established.

The surface S^2 is an example of a Riemann surface, which we define as follows. A Riemann surface is a two-dimensional manifold M covered by open sets $\mathcal{O}_j$ with coordinate charts $\varphi_j : \Omega_j \to \mathcal{O}_j$ having the property that, if $\mathcal{O}_j \cap \mathcal{O}_k \neq \emptyset$, and if $\Omega_{jk} = \varphi_j^{-1}(\mathcal{O}_j \cap \mathcal{O}_k)$, then the diffeomorphism

$$\varphi_k^{-1} \circ \varphi_j : \Omega_{jk} \longrightarrow \Omega_{kj} \tag{26.13}$$

is holomorphic.

Another important class of Riemann surfaces is given as follows. Let $\Lambda \subset \mathbb{R}^2 \approx \mathbb{C}$ be the image of $\mathbb{Z}^2$ under any matrix in $Gl(2,\mathbb{R})$. Then the torus

$$\mathbb{T}_\Lambda = \mathbb{C}/\Lambda \tag{26.14}$$

is a Riemann surface in a natural fashion.

There are many other Riemann surfaces. For example, any oriented two-dimensional Riemannian manifold has a natural structure of a Riemann surface. A proof of this can be found in Chapter 5 of [T2]. An important family of Riemann surfaces holomorphically diffeomorphic to surfaces of the form (26.14) will arise in §34, with implications for the theory of elliptic functions.

Exercises

1. Give an example of a family $\mathcal{F}$ of holomorphic functions $\Omega \to \mathbb{C}$ with the following two properties:
   (a) $\mathcal{F}$ is normal with respect to (Ω, S^2).
   (b) $\{f' : f \in \mathcal{F}\}$ is *not* normal.

Compare Exercise 2 of §21.

2. Given $\Omega \subset \mathbb{C}$ open, let

$$\mathcal{F} = \{f : \Omega \to \mathbb{C} : \mathrm{Re}\, f > 0 \text{ on } \Omega, f \text{ holomorphic}\}.$$

Show that $\mathcal{F}$ is normal with respect to (Ω, S^2). Is $\mathcal{F}$ normal with respect to $(\Omega, \mathbb{C})$?

3. Let $\mathcal{F} = \{z^n : n \in \mathbb{Z}^+\}$. For which regions Ω is $\mathcal{F}$ normal with respect to (Ω, S^2)? Compare Exercise 4 in §21.

4. Show that the set of orientation-preserving conformal diffeomorphisms $\varphi : S^2 \to S^2$ is precisely the set of linear fractional transformations of the form (22.5), with $A \in Gl(2,\mathbb{C})$.
Hint. Given such $\varphi : S^2 \to S^2$, take L_A such that $L_A \circ \varphi$ takes ∞ to ∞, so $\psi = L_A \circ \varphi|_{S^2 \setminus \{\infty\}}$ is a holomorphic diffeomorphism of $\mathbb{C}$ onto itself. What form must ψ have? (Cf. Proposition 11.4.)

5. There is a natural notion of when a map $\varphi : M_1 \to M_2$ between two Riemann surfaces is holomorphic. Write it down. Show that if φ and also $\psi : M_2 \to M_3$ are holomorphic, then so is $\psi \circ \varphi : M_1 \to M_3$.

6. Let $p(z)$ and $q(z)$ be polynomials on $\mathbb{C}$. Assume the roots of $p(z)$ are disjoint from the roots of $q(z)$. Form the meromorphic function

$$R(z) = \frac{p(z)}{q(z)}.$$

Show that $R(z)$ has a unique continuous extension $R : S^2 \to S^2$, and this is holomorphic.

Exercises 7–9 deal with holomorphic maps $F : S^2 \to S^2$. Assume F is not constant.

7. Show that there are only finitely many $p_j \in S^2$ such that $DF(p_j) : T_{p_j}S^2 \to T_{q_j}S^2$ is singular (hence zero), where $q_j = F(p_j)$. The points q_j are called critical values of F.

8. Suppose ∞ is not a critical value of F and that $F^{-1}(\infty) = \{\infty, p_1, \dots, p_k\}$. Show that

$$f(z) = F(z)(z - p_1) \cdots (z - p_k) : \mathbb{C} \longrightarrow \mathbb{C},$$

and $|f(z)| \to \infty$ as $|z| \to \infty$. Deduce that $f(z)$ is a polynomial in z. (Cf. Proposition 11.4.)

9. Show that every holomorphic map $F : S^2 \to S^2$ is of the form treated in Exercise 6 (except for the constant map $F \equiv \infty$).
Hint. Compose with linear fractional transformations and transform F to a map satisfying the conditions of Exercise 8.

10. Given a holomorphic map $f : \Omega \to \mathbb{C}$, set

$$g = \mathcal{S}^{-1} \circ f : \Omega \longrightarrow S^2. \tag{26.13}$$

For $z \in \Omega$, set $q = f(z),\ p = g(z)$, and consider

$$Dg(z) : \mathbb{R}^2 \longrightarrow T_pS^2. \tag{26.14}$$

Using (26.8) (where $F = \mathcal{S}^{-1}$), show that

$$Dg(z)^t\ Dg(z) = 4\Bigl(\frac{|f'(z)|}{1 + |f(z)|^2}\Bigr)^2\ I, \tag{26.15}$$

where I is the identity matrix. The quantity

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2} \tag{26.16}$$

is sometimes called the "spherical derivative" of f.

11. Show that the meromorphic function constructed in Exercises 4–6 of §24 yields a holomorphic map

$$\Phi : \mathbb{T}_\Lambda \longrightarrow S^2, \tag{26.17}$$

where $\Lambda = \{4k + 2i\ell p : k, \ell \in \mathbb{Z}\}$.

K. Rapid evaluation of the Weierstrass $\wp$-function

Given a lattice $\Lambda \subset \mathbb{C}$, the associated Weierstrass $\wp$-function is defined by

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{0\neq\beta\in\Lambda} \Big(\frac{1}{(z-\beta)^2} - \frac{1}{\beta^2} \Big). \tag{K.1}$$

This converges rather slowly, so another method must be used to evaluate $\wp(z;\Lambda)$ rapidly. The classical method, which we describe below, involves a representation of $\wp$ in terms of theta functions. It is most conveniently described in case

$$\Lambda \ \text{ generated by } \ 1 \ \text{ and } \ \tau, \quad \operatorname{Im}\tau > 0. \tag{K.2}$$

To pass from this to the general case, we can use the identity

$$\wp(z;a\Lambda) = \frac{1}{a^2}\wp\Big(\frac{z}{a};\Lambda\Big). \tag{K.3}$$

The material below is basically a summary of material from §32, assembled here to clarify the important application to the task of the rapid evaluation of (K.1).

To evaluate $\wp(z;\Lambda)$, which we henceforth denote $\wp(z)$, we use the following identity:

$$\wp(z) = e_1 + \Big(\frac{\vartheta_1'(0)}{\vartheta_2(0)} \frac{\vartheta_2(z)}{\vartheta_1(z)} \Big)^2. \tag{K.4}$$

See (32.20). Here $e_1 = \wp(\omega_1/2) = \wp(1/2)$, and the theta functions $\vartheta_j(z)$ (which also depend on ω) are defined as follows (cf. (32.6)–(32.10)):

$$\begin{aligned}
\vartheta_1(z) &= i \sum_{n=-\infty}^{\infty} (-1)^n p^{2n-1} q^{(n-1/2)^2}, \\
\vartheta_2(z) &= \sum_{n=-\infty}^{\infty} p^{2n-1} q^{(n-1/2)^2}, \\
\vartheta_3(z) &= \sum_{n=-\infty}^{\infty} p^{2n} q^{n^2}, \\
\vartheta_4(z) &= \sum_{n=-\infty}^{\infty} (-1)^n p^{2n} q^{n^2}.
\end{aligned} \tag{K.5}$$

Here

$$p = e^{\pi i z}, \quad q = e^{\pi i \tau}, \tag{K.6}$$

with τ as in (K.2).

The functions ϑ_1 and ϑ_2 appear in (K.4). Also ϑ_3 and ϑ_4 arise to yield a rapid evaluation of e_1 (cf. (32.33)):

$$e_1 = \frac{\pi^2}{3}\big[\vartheta_3(0)^4 + \vartheta_4(0)^4\big]. \tag{K.7}$$

Note that $(d/dz)p^{2n-1} = \pi i(2n-1)p^{2n-1}$ and hence

$$\vartheta_1'(0) = -\pi \sum_{n=-\infty}^{\infty} (-1)^n(2n-1)q^{(n-1/2)^2}. \tag{K.8}$$

It is convenient to rewrite the formulas for $\vartheta_1(z)$ and $\vartheta_2(z)$ as

$$\begin{aligned} \vartheta_1(z) &= i\sum_{n=1}^{\infty}(-1)^n q^{(n-1/2)^2}(p^{2n-1} - p^{1-2n}), \\ \vartheta_2(z) &= \sum_{n=1}^{\infty} q^{(n-1/2)^2}(p^{2n-1} + p^{1-2n}). \end{aligned} \tag{K.9}$$

also formulas for $\vartheta_1'(0)$ and $\vartheta_j(0)$, which appear in (K.4) and (K.7), can be rewritten:

$$\begin{aligned} \vartheta_1'(0) &= -2\pi\sum_{n=1}^{\infty}(-1)^n(2n-1)q^{(n-1/2)^2}, \\ \vartheta_2(0) &= 2\sum_{n=1}^{\infty} q^{(n-1/2)^2}, \\ \vartheta_3(0) &= 1 + 2\sum_{n=1}^{\infty} q^{n^2}, \\ \vartheta_4(0) &= 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}. \end{aligned} \tag{K.10}$$

Rectangular lattices

We specialize to the case where Λ is a rectangular lattice, of sides 1 and L, more precisely:

$$\Lambda \ \text{ generated by } \ 1 \ \text{ and } \ iL, \quad L > 0. \tag{K.11}$$

Now the formulas established above hold, with $\tau = iL$, hence

$$q = e^{-\pi L}. \tag{K.12}$$

Since q is real, we see that the quantities $\vartheta_1'(0)$ and $\vartheta_j(0)$ in (K.10) are real. It is also convenient to calculate the real and imaginary parts of $\vartheta_j(z)$ in this case. Say

$$z = u + iv, \tag{K.13}$$

with u and v real. Then

$$p^{2n-1} = e^{-(2n-1)\pi v}\big[\cos(2n-1)\pi u + i\sin(2n-1)\pi u\big]. \tag{K.14}$$

We then have

(K.15)

$$\operatorname{Re}(-i\vartheta_1(z)) = -\sum_{n=1}^{\infty}(-1)^n q^{(n-1/2)^2}\big[e^{(2n-1)\pi v} - e^{-(2n-1)\pi v}\big]\cos(2n-1)\pi u,$$

$$\operatorname{Im}(-i\vartheta_1(z)) = \sum_{n=1}^{\infty}(-1)^n q^{(n-1/2)^2}\big[e^{(2n-1)\pi v} + e^{-(2n-1)\pi v}\big]\sin(2n-1)\pi u,$$

and

$$\begin{aligned}\operatorname{Re}\vartheta_2(z) &= \sum_{n=1}^{\infty} q^{(n-1/2)^2}\big[e^{(2n-1)\pi v} + e^{-(2n-1)\pi v}\big]\cos(2n-1)\pi u,\\ \operatorname{Im}\vartheta_2(z) &= -\sum_{n=1}^{\infty} q^{(n-1/2)^2}\big[e^{(2n-1)\pi v} - e^{-(2n-1)\pi v}\big]\sin(2n-1)\pi u.\end{aligned} \tag{K.16}$$

We can calculate these quantities accurately by summing over a small range. Let us insist that

$$-\frac{1}{2} \le u < \frac{1}{2}, \quad -\frac{L}{2} \le v < \frac{L}{2}, \tag{K.17}$$

and assume

$$L \ge 1. \tag{K.18}$$

Then

$$\big|q^{(n-1/2)^2} e^{(2n-1)\pi v}\big| \le e^{-(n^2-3n+5/4)\pi L}, \tag{K.19}$$

and since

$$e^{-\pi} < \frac{1}{20}, \tag{K.20}$$

we see that the quantity in (K.19) is

$$\begin{aligned}&< 0.5\times 10^{-14} \quad \text{for} \quad n=5,\\ &< 2\times 10^{-25} \quad \text{for} \quad n=6,\end{aligned} \tag{K.21}$$

with rapid decrease for $n > 6$. Thus, summing over $1 \le n \le 5$ will give adequate approximations.

For $z = u + iv$ very near 0, where ϑ_1 vanishes and $\wp$ has a pole, the identity

$$\frac{1}{\wp(z) - e_1} = \Big(\frac{\vartheta_2(0)}{\vartheta_1'(0)}\,\frac{\vartheta_1(z)}{\vartheta_2(z)}\Big)^2, \tag{K.22}$$

in concert with (K.10) and (K.15)–(K.16), gives an accurate approximation to $(\wp(z) - e_1)^{-1}$, which in this case is also very small. Note, however, that some care should be taken in evaluating $\mathrm{Re}(-i\vartheta_1(z))$, via the first part of (K.15), when $|z|$ is very small. More precisely, care is needed in evaluating

$$e^{k\pi v} - e^{-k\pi v}, \quad k = 2n - 1 \in \{1, 3, 5, 7, 9\}, \tag{K.23}$$

when v is very small, since then (K.23) is the difference between two quantities close to 1, so evaluating $e^{k\pi v}$ and $e^{-k\pi v}$ separately and subtracting can lead to an undesirable loss of accuracy. In case $k = 1$, one can effect this cancellation at the power series level and write

$$e^{\pi v} - e^{-\pi v} = 2 \sum_{j\geq 1,\mathrm{odd}} \frac{(\pi v)^j}{j!}. \tag{K.24}$$

If $|\pi v| \leq 10^{-2}$, summing over $j \leq 7$ yields substantial accuracy. (If $|\pi v| > 10^{-2}$, separate evaluation of $e^{k\pi v}$ and $e^{-k\pi v}$ should not be a problem.) For other values of k in (K.23), one can derive from

$$(x^k - 1) = (x - 1)(x^{k-1} + \cdots + 1) \tag{K.25}$$

the identity

$$e^{k\pi v} - e^{-k\pi v} = (e^{\pi v} - e^{-\pi v}) \sum_{\ell=0}^{k-1} e^{(2\ell-(k-1))\pi v}, \tag{K.26}$$

which in concert with (K.24) yields an accurate evaluation of each term in (K.23).

Remark. If (K.11) holds with $0 < L < 1$, one can use (K.3), with $a = iL$, to transfer to the case of a lattice generated by 1 and i/L.

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