How Euler Might Have Constructed $\Gamma(z)$

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Our goal is to construct a "natural" meromorphic function $\Gamma(z)$ satisfying

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(1) = 1. \tag{1}$$

A first attempt is

$$\frac{1}{z}\frac{2}{z+1}\frac{3}{z+2}\cdots . \tag{2}$$

However, this does not converge. To see what is going on, we examine

$$A_n(z) = \frac{1}{z}\frac{2}{z+1}\cdots\frac{n}{z+n-1}. \tag{3}$$

Note that $A_n(1) = 1$ and

$$\begin{aligned} A_n(z+1) &= \frac{2}{z+1}\frac{3}{z+2}\cdots\frac{n+1}{z+n}\frac{1}{n+1} \\ &= \frac{z}{n+1}\,A_{n+1}(z). \end{aligned} \tag{4}$$

This suggests trying

$$\begin{aligned} \Gamma_n(z) &= n^{z-1}\,\frac{1}{z}\frac{2}{z+1}\cdots\frac{n}{z+n-1} \\ &= n^z\,\frac{1}{z}\frac{1}{z+1}\cdots\frac{n-1}{z+n-1} \\ &= n^z\,\frac{1}{z}\frac{1}{1+z}\frac{1}{1+z/2}\cdots\frac{1}{1+z/(n-1)}. \end{aligned} \tag{5}$$

Note that $\Gamma_n(1) = 1$ and

$$\begin{aligned} \Gamma_n(z+1) &= n^z\,\frac{2}{z+1}\frac{3}{z+2}\cdots\frac{n+1}{z+n}\frac{1}{n+1} \\ &= \frac{n}{n+1}\,z\Gamma_{n+1}(z). \end{aligned} \tag{6}$$

Now write

$$n^z = e^{z\ \log n} = e^{z(1+1/2+\cdots+1/(n-1))-\gamma_n z}, \tag{7}$$

so

$$\Gamma_n(z) = \frac{1}{ze^{\gamma_n z}}\prod_{k=1}^{n-1}\frac{e^{z/k}}{1+\frac{z}{k}}, \tag{8}$$

or

$$\frac{1}{\Gamma_n(z)} = ze^{\gamma_n z} \prod_{k=1}^{n-1} \Big(1 + \frac{z}{k}\Big) e^{-z/k}. \tag{9}$$

Simple estimates show that this converges as $n \to \infty$, and we have $\Gamma_n(z) \to \Gamma(z)$, with

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{k=1}^{\infty} \Big(1 + \frac{z}{k}\Big) e^{-z/k}. \tag{10}$$

Here

$$\gamma = \lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{1}{k} - \log n, \tag{11}$$

which is known as Euler's constant. Now the result (1) follows from (6).

The Gamma function $\Gamma(z)$ is related to the sine function, as follows.

$$\begin{aligned} \frac{1}{\Gamma(z)\Gamma(-z)} &= -z^2 \prod_{k=1}^{\infty} \Big(1 - \frac{z^2}{k^2}\Big) \\ &= -z\, \frac{\sin \pi z}{\pi}. \end{aligned} \tag{12}$$

The second identity in (12) is Euler's product formula for the sine. Using this one can compute $\Gamma(1/2)$ as follows. Taking $z = 1/2$ in (12) yields.

$$\Gamma\Big(\frac{1}{2}\Big)\Gamma\Big(-\frac{1}{2}\Big) = -2\pi. \tag{13}$$

Since $\Gamma(1/2) = \Gamma(1-1/2) = -(1/2)\Gamma(-1/2)$, we have $\Gamma(1/2)^2 = \pi$. Since $\Gamma(x) > 0$ for $x > 0$, we deduce that

$$\Gamma\Big(\frac{1}{2}\Big) = \sqrt{\pi}. \tag{14}$$