Euler and Navier-Stokes Equations For Incompressible Fluids

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Introduction

This chapter deals with equations describing motion of an incompressible fluid moving in a fixed compact space M, which it fills completely. We consider two types of fluid motion, with or without viscosity, and two types of compact space, a compact smooth Riemannian manifold with or without boundary. The two types of fluid motion are modeled by the Euler equation

$$\frac{\partial u}{\partial t} + \nabla_u u = -\text{grad } p, \qquad \text{div } u = 0, \tag{0.1}$$

for the velocity field u, in the absence of viscosity, and the Navier-Stokes equation

$$\frac{\partial u}{\partial t} + \nabla_u u = \nu \mathcal{L} u - \text{grad } p, \qquad \text{div } u = 0, \tag{0.2}$$

in the presence of viscosity. In (0.2), ν is a positive constant and $\mathcal{L}$ is the second-order differential operator

$$\mathcal{L} u = \text{div Def } u, \tag{0.3}$$

which on flat Euclidean space is equal to Δu, when div $u = 0$. If there is a boundary, the Euler equation has boundary condition $n \cdot u = 0$, that is, u is tangent to the boundary, while for the Navier-Stokes equation one poses the no-slip boundary condition $u = 0$ on ∂M.

In §1 we derive (0.1) in several forms; we also derive the vorticity equation for the object that is curl u when $\dim M = 3$. We discuss some of the classical physical interpretations of these equations, such as Kelvin's circulation theorem and Helmholtz' theorem on vortex tubes, and include in the exercises other topics, such as steady flows and Bernoulli's law. These phenomena can be compared with analogues for compressible flow, discussed in §5 of Chapter 16.

Sections 2–5 discuss the existence, uniqueness and regularity of solutions to (0.1) and (0.2), on regions with or without boundary. We have devoted separate sections to treatments first without boundary and then with boundary, for these equations, at a cost of a small amount of redundancy. By and large, different analytical problems are emphasized in the separate sections, and their division seems reasonable from a pedagogical point of view.

The treatments in §§2–5 are intended to parallel to a good degree the treatment of nonlinear parabolic and hyperbolic equations in Chapters 15 and 16. Among the significant differences, there is the role of the vorticity equation, which leads to global solutions when $\dim M = 2$. For $\dim M \geq 3$, the question of whether smooth solutions exist for all $t \geq 0$ is still open, with a few exceptions, such as small initial data for (0.2). These problems, as well as variants, such as free boundary problems for fluid flow, remain exciting and perplexing.

In §6 we tackle the question of how solutions to the Navier-Stokes equations on a bounded region behave when the viscosity tends to zero. We stick to two special cases, in which this difficult question turns out to be somewhat tractable. The first is the class of 2D flows on a disk that are circularly symmetric. The second is a class of 3D circular pipe flows, whose detailed description can be found in §6. These cases yield convergence of the velocity fields to the fields solving associated Euler equations, though not in a particularly strong norm, due to boundary layer effects. Section 7 investigates how such velocity convergence yields information on the convergence of the flows generated by such time-varying vector fields.

In Appendix A we discuss boundary regularity for the Stokes operator, needed for the analysis in §5.

1. Euler's equations for ideal incompressible fluid flow

An incompressible fluid flow on a region Ω defines a one-parameter family of volume-preserving diffeomorphisms

$$F(t,\cdot) : \Omega \longrightarrow \Omega, \tag{1.1}$$

where Ω is a Riemannian manifold with boundary; if $\partial\Omega$ is nonempty, we suppose it is preserved under the flow. The flow can be described in terms of its velocity field

$$u(t,y) = F_t(t,x) \in T_y\Omega, \quad y = F(t,x), \tag{1.2}$$

where $F_t(t,x) = (\partial/\partial t)F(t,x)$. If $y \in \partial\Omega$, we assume $u(t,y)$ is tangent to $\partial\Omega$. We want to derive Euler's equation, a nonlinear PDE for u describing the dynamics of fluid flow. We will assume the fluid has uniform density.

If we suppose there are no external forces acting on the fluid, the dynamics are determined by the constraint condition, that $F(t,\cdot)$ preserve volume, or equivalently, that div $u(t,\cdot) = 0$ for all t. The Lagrangian involves the kinetic energy alone, so we seek to find critical points of

$$L(F) = \frac{1}{2}\int_I\int_\Omega \langle F_t(t,x), F_t(t,x)\rangle \, dV\, dt, \tag{1.3}$$

on the space of maps $F : I \times \Omega \to \Omega$ (where $I = [t_0, t_1]$), with the volume-preserving property.

For simplicity, we first treat the case where Ω is a domain in $\mathbb{R}^n$. A variation of F is of the form $F(s,t,x)$, with $\partial F/\partial s = v(t, F(t,x))$, at $s = 0$, where div $v = 0$, v is tangent to $\partial\Omega$, and $v = 0$ for $t = t_0$ and $t = t_1$. We have

$$\begin{aligned} DL(F)v &= \iint \Big\langle F_t(t,x), \frac{d}{dt} v\big(t, F(t,x)\big)\Big\rangle \, dV\, dt \\ &= \iint \Big\langle u\big(t, F(t,x)\big), \frac{d}{dt} v\big(t, F(t,x)\big)\Big\rangle \, dV\, dt \\ &= -\iint \Big\langle \frac{\partial u}{\partial t} + u\cdot\nabla_x u, v\Big\rangle \, dV\, dt. \end{aligned} \tag{1.4}$$

The stationary condition is that this last integral vanish for all such v, and hence, for each t,

$$\int_\Omega \Big\langle \frac{\partial u}{\partial t} + u\cdot\nabla_x u, v\Big\rangle \, dV = 0, \tag{1.5}$$

for all vector fields v on Ω (tangent to $\partial\Omega$), satisfying div $v = 0$.

To restate this as a differential equation, let

$$V_\sigma = \big\{v \in C^\infty(\overline{\Omega}, T\Omega) : \text{div}\ v = 0,\ v \text{ tangent to } \partial\Omega\big\}, \tag{1.6}$$

and let P denote the orthogonal projection of $L^2(\Omega, T\Omega)$ onto the closure of the space V_σ. The operator P is often called the *Leray projection.* The stationary condition becomes

$$\frac{\partial u}{\partial t} + P(u\cdot\nabla_x u) = 0, \tag{1.7}$$

in addition to the conditions

$$\text{div}\ u = 0 \quad \text{on } \Omega \tag{1.8}$$

and

$$u \text{ tangent to } \partial\Omega. \tag{1.9}$$

For a general Riemannian manifold Ω, one has a similar calculation, with $u \cdot \nabla_x u$ in (1.5) generalized simply to $\nabla_u u$, where ∇ is the Riemannian connection on Ω. Thus (1.7) generalizes to

Euler's equation, first form.

$$\frac{\partial u}{\partial t} + P(\nabla_u u) = 0. \tag{1.10}$$

Suppose now $\overline{\Omega}$ is compact. According to the Hodge decomposition, the orthogonal complement in $L^2(\Omega, T)$ of the range of P is equal to the space

$$\big\{\text{grad } p : p \in H^1(\Omega)\big\}.$$

This fact is derived in the problem set following §9 in Chapter 5, entitled "Exercises on spaces of gradient and divergence-free vector fields"; see (9.79)–(9.80). Thus we can rewrite (1.10) as

Euler's equation, second form.

$$\frac{\partial u}{\partial t} + \nabla_u u = -grad\ p. \tag{1.11}$$

Here, p is a scalar function, determined uniquely up to an additive constant (assuming Ω is connected). The function p is identified as "pressure."

It is useful to derive some other forms of Euler's equation. In particular, let $\tilde{u}$ denote the 1-form corresponding to the vector field u via the Riemannian metric on Ω. Then (1.11) is equivalent to

$$\frac{\partial \tilde{u}}{\partial t} + \nabla_u \tilde{u} = -dp. \tag{1.12}$$

We will rewrite this using the Lie derivative. Recall that, for any vector field X,

$$\nabla_u X = \mathcal{L}_u X + \nabla_X u,$$

by the zero-torsion condition on ∇. Using this, we deduce that

$$\langle \mathcal{L}_u \tilde{u} - \nabla_u \tilde{u}, X\rangle = \langle \tilde{u}, \nabla_X u\rangle. \tag{1.13}$$

In case $\langle \tilde{u}, v\rangle = \langle u, v\rangle$ (the Riemannian inner product), we have

$$\langle \tilde{u}, \nabla_X u\rangle = \frac{1}{2}\langle d\,|u|^2, X\rangle, \tag{1.14}$$

using the notation $|u|^2 = \langle u, u\rangle$, so (1.12) is equivalent to

Euler's equation, third form.

$$\frac{\partial \tilde{u}}{\partial t} + \mathcal{L}_u \tilde{u} = d\Big(\frac{1}{2}|u|^2 - p\Big). \tag{1.15}$$

Writing the Lie derivative in terms of exterior derivatives, we obtain

$$\frac{\partial \tilde{u}}{\partial t} + (d\tilde{u})\rfloor u = -d\Big(\frac{1}{2}|u|^2 + p\Big). \tag{1.16}$$

Note also that the condition div $u = 0$ can be rewritten as

$$\delta\tilde{u} = 0. \tag{1.17}$$

In the study of Euler's equation, a major role is played by the *vorticity*, which we proceed to define. In its first form, the vorticity will be taken to be

$$\tilde{w} = d\tilde{u}, \tag{1.18}$$

for each t a 2-form on Ω. The Euler equation leads to a PDE for vorticity; indeed, applying the exterior derivative to (1.15) gives immediately the

Vorticity equation, first form.

$$\frac{\partial \tilde{w}}{\partial t} + \mathcal{L}_u \tilde{w} = 0, \tag{1.19}$$

or equivalently, from (1.16),

$$\frac{\partial \tilde{w}}{\partial t} + d\big(\tilde{w}\rfloor u\big) = 0. \tag{1.20}$$

It is convenient to express this in terms of the covariant derivative. In analogy to (1.13), for any 2-form β and vector fields X and Y, we have

$$\begin{aligned}(\nabla_u\beta - \mathcal{L}_u\beta)(X,Y) &= \beta(\nabla_X u, Y) + \beta(X, \nabla_Y u) \\ &= (\beta\#\nabla u)(X,Y),\end{aligned} \tag{1.21}$$

where the last identity defines $\beta\#\nabla u$. Thus we can rewrite (1.19) as

Vorticity equation, second form.

$$\frac{\partial \tilde{w}}{\partial t} + \nabla_u \tilde{w} - \tilde{w}\#\nabla u = 0. \tag{1.22}$$

It is also useful to consider vorticity in another form. Namely, to $\tilde{w}$ we associate a section w of $\Lambda^{n-2}T$ ($n = \dim\,\Omega$), so that the identity

$$\tilde{w}\wedge\alpha = \langle w, \alpha\rangle\omega \tag{1.23}$$

holds, for every $(n-2)$-form α, where ω is the volume form on Ω. (We assume Ω is oriented.) The correspondence $\tilde{w} \leftrightarrow w$ given by (1.23) depends only on the volume element ω. Hence

$$\text{div } u = 0 \Longrightarrow \mathcal{L}_u \tilde{w} = \widetilde{\mathcal{L}_u w}, \tag{1.24}$$

so (1.19) yields the

Vorticity equation, third form.

$$\frac{\partial w}{\partial t} + \mathcal{L}_u w = 0. \tag{1.25}$$

This vorticity equation takes special forms in two and three dimensions, respectively. When $\dim \Omega = n = 2,\ w$ is a scalar field, often denoted as

$$w = \text{ rot } u, \tag{1.26}$$

and (1.25) becomes the

2-D vorticity equation.

$$\frac{\partial w}{\partial t} + u \cdot \textit{grad } w = 0. \tag{1.27}$$

This is a conservation law. As we will see, this has special implications for two-dimensional incompressible fluid flow. If $n = 3,\ w$ is a vector field, denoted as

$$w = \text{ curl } u, \tag{1.28}$$

and (1.25) becomes the

3-D vorticity equation.

$$\frac{\partial w}{\partial t} + [u, w] = 0, \tag{1.29}$$

or equivalently,

$$\frac{\partial w}{\partial t} + \nabla_u w - \nabla_w u = 0. \tag{1.30}$$

Note that (1.28) is a generalization of the notion of the curl of a vector field on flat $\mathbb{R}^3$. Compare with material in the second exercise set following §8 in Chapter 5.

The first form of the vorticity equation, (1.19), implies

$$\tilde{w}(0) = \left(F^t\right)^* \tilde{w}(t), \tag{1.31}$$

where $F^t(x) = F(t,x)$, $\tilde{w}(t)(x) = \tilde{w}(t,x)$. Similarly, the third form, (1.25), yields

$$w(t,y) = \Lambda^{n-2} DF^t(x)\ w(0,x), \quad y = F(t,x), \tag{1.32}$$

where $DF^t(x) : T_x\Omega \to T_y\Omega$ is the derivative. In case $n = 2$, this last identity is simply $w(t,y) = w(0,x)$, the conservation law mentioned after (1.27).

One implication of (1.31) is the following. Let S be an oriented surface in Ω, with boundary C; let $S(t)$ be the image of S under F^t, and $C(t)$ the image of C; then (1.31) yields

$$\int_{S(t)} \tilde{w}(t) = \int_S \tilde{w}(0). \tag{1.33}$$

Since $\tilde{w} = d\tilde{u}$, this implies the following:

Kelvin's circulation theorem.

$$\int_{C(t)} \tilde{u}(t) = \int_C \tilde{u}(0). \tag{1.34}$$

We take a look at some phenomena special to the case $\dim \Omega = n = 3$, where the vorticity w is a vector field on Ω, for each t. Fix t_0, and consider $w = w(t_0)$. Let S be an oriented surface in Ω, transversal to w. A vortex tube $\mathcal{T}$ is defined to be the union of orbits of w through S, to a second transversal surface S_2 (see Fig. 1.1). For simplicity we will assume that none of these orbits ends at a zero of the vorticity field, though more general cases can be handled by a limiting argument.

FIGURE 1.1

Since $d\tilde{w} = d^2\tilde{u} = 0$, we can use Stokes' theorem to write

$$0 = \int_{\mathcal{T}} d\tilde{w} = \int_{\partial\mathcal{T}} \tilde{w}. \tag{1.35}$$

Now $\partial\mathcal{T}$ consists of three pieces, S and S_2 (with opposite orientations) and the lateral boundary $\mathcal{L}$, the union of the orbits of w from ∂S to ∂S_2. Clearly, the pull-back of $\tilde{w}$ to $\mathcal{L}$ is 0, so (1.35) implies

$$\int_S \tilde{w} = \int_{S_2} \tilde{w}. \tag{1.36}$$

Applying Stokes' theorem again, for $\tilde{w} = d\tilde{u}$, we have

Helmholtz' theorem. *For any two curves C, C_2 enclosing a vortex tube,*

$$\int_C \tilde{u} = \int_{C_2} \tilde{u}. \tag{1.37}$$

This common value is called the strength of the vortex tube $\mathcal{T}$.

Also note that if $\mathcal{T}$ is a vortex tube at $t_0 = 0$, then, for each t, $\mathcal{T}(t)$, the image of $\mathcal{T}$ under F^t, is a vortex tube, as a consequence of (1.32) (with $n = 3$), and furthermore (1.34) implies that the strength of $\mathcal{T}(t)$ is independent of t. This conclusion is also part of Helmholtz' theorem.

To close this section, we note that the Euler equation for an ideal incompressible fluid flow with an external force f is

$$\frac{\partial u}{\partial t} + \nabla_u u = -\text{grad } p + f,$$

in place of (1.11). If f is conservative, of the form $f = -\text{grad } \varphi$, then (1.12) is replaced by

$$\frac{\partial \tilde{u}}{\partial t} + \nabla_u \tilde{u} = -d(p + \varphi).$$

Thus the vorticity $\tilde{w} = d\tilde{u}$ continues to satisfy (1.19), and other phenomena discussed above can be treated in this extra generality.

Indeed, in the case we have considered, of a completely confined, incompressible flow of a fluid of uniform density, adding such a conservative force field has no effect on the velocity field u, just on the pressure, though in other situations such a force field could have more pronounced effects.

Exercises

1. Using the divergence theorem, show that whenever $\text{div } u = 0$, u tangent to $\partial\Omega$, $\overline{\Omega}$ compact, and $f \in C^\infty(\overline{\Omega})$, we have

$$\int_\Omega \mathcal{L}_u f \; dV = 0.$$

Hence show that, for any smooth vector field X on $\overline{\Omega}$,

$$\int_{\Omega} \langle \nabla_u X, X \rangle \, dV = 0.$$

From this, conclude that any (sufficiently smooth) u solving (1.7)–(1.9) satisfies the conservation of energy law

$$\frac{d}{dt} \|u(t)\|^2_{L^2(\Omega)} = 0. \tag{1.38}$$

2. When $\dim \Omega = n = 3$, show that the vorticity field w is divergence free. (*Hint*: div curl.)
3. If u, v are vector fields, $\tilde{u}$ the 1-form associated to u, it is generally true that $\nabla_v \tilde{u} = \widetilde{\nabla_v u}$, but not that $\mathcal{L}_v \tilde{u} = \widetilde{\mathcal{L}_v u}$. Why is that?
4. A fluid flow is called *stationary* provided u is independent of t. Establish *Bernoulli's law*, that for a stationary solution of Euler's equations (1.7)–(1.9), the function $(1/2)|u|^2 + p$ is constant along any streamline (i.e., an integral curve of u).
   In the nonstationary case, show that

$$\frac{1}{2}\left(\frac{\partial}{\partial t} - \mathcal{L}_u\right)|u|^2 = -\mathcal{L}_u p.$$

   (*Hint*: Use Euler's equation in the form (1.16); take the inner product of both sides with u.)
5. Suppose $\dim \Omega = 3$. Recall from the auxiliary exercise set after §8 in Chapter 5 the characterization

$$u \times v = X \Longleftrightarrow \tilde{X} = *(\tilde{u} \wedge \tilde{v}).$$

   Show that the form (1.16) of Euler's equation is equivalent to

$$\frac{\partial u}{\partial t} + (\text{curl } u) \times u = -\text{grad}\left(\frac{1}{2}|u|^2 + p\right). \tag{1.39}$$

   Also, if $\Omega \subset \mathbb{R}^3$, deduce this from (1.11) together with the identity

$$\text{grad}(u \cdot v) = u \cdot \nabla v + v \cdot \nabla u + u \times \ \text{curl } v + v \times \ \text{curl } u,$$

   which is derived in (8.63) of Chapter 5.
6. Deduce the 3-D vorticity equation (1.30) by applying curl to both sides of (1.39) and using the identity

$$\text{curl}(u \times v) = v \cdot \nabla u - u \cdot \nabla v + (\text{div } v)u - (\text{div } u)v,$$

   which is derived in (8.62) of Chapter 5. Also show that the vorticity equation can be written as

$$w_t + \nabla_u w = (\text{Def } u)w, \qquad \text{Def } v = \frac{1}{2}(\nabla v + \nabla v^t). \tag{1.40}$$

   (*Hint*: $w \times w = 0$.)
7. In the setting of Exercise 5, show that, for a stationary flow, $(1/2)|u|^2 + p$ is constant along both any streamline and any vortex line (i.e., an integral curve of $w = \text{curl } u$).
8. For $\dim \Omega = 3$, note that (1.29) implies $[u, w] = 0$ for a stationary flow, with $w = \text{curl } u$. What does Frobenius's theorem imply about this?

9. Suppose u is a (sufficiently smooth) solution to the Euler equation (1.11), also satisfying (1.9), namely, u is tangent to $\partial\Omega$. Show that if $u(0)$ has vanishing divergence, then $u(t)$ has vanishing divergence for all t. (*Hint*: Use the Hodge decomposition discussed between (1.10) and (1.11).)
10. Suppose $\tilde{u}$, the 1-form associated to u, and a 2-form $\tilde{w}$ satisfy the coupled system

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} + \tilde{w}\rfloor u &= -d\Phi, \\ \frac{\partial \tilde{w}}{\partial t} + d(\tilde{w}\rfloor u) &= 0. \end{aligned} \tag{1.41}$$

Show that if $\tilde{w}(0) = d\tilde{u}(0)$, then $\tilde{w}(t) = d\tilde{u}(t)$ for all t. (*Hint*: Set $\tilde{W}(t) = d\tilde{u}(t)$, so by the first half of (1.41), $\partial\tilde{W}/\partial t + d(\tilde{w}\rfloor u) = 0$. Subtract this from the second equation of the pair (1.41).)
11. If u generates a 1-parameter group of isometries of Ω, show that u provides a stationary solution to the Euler equations. (*Hint*: Show that $\text{Def}\, u = 0 \Rightarrow \nabla_u u = (1/2)\, \text{grad}\, |u|^2$.)
12. A flow is called *irrotational* if the vorticity $\tilde{w}$ vanishes. Show that if $\tilde{w}(0) = 0$, then $\tilde{w}(t) = 0$ for all t, under the hypotheses of this section.
13. If a flow is both stationary and irrotational, show that Bernoulli's law can be strengthened to

$$\frac{1}{2}|u|^2 + p \text{ is constant on } \Omega.$$

The common statement of this is that higher fluid velocity means lower pressure (within the limited set of circumstances for which this law holds). (*Hint*: Use (1.16).)
14. Suppose $\overline{\Omega}$ is compact. Show that the space of 1-forms $\tilde{u}$ on $\overline{\Omega}$ satisfying

$$\delta\tilde{u} = 0, \quad d\tilde{u} = 0 \text{ on } \Omega, \quad \langle \nu, u\rangle = 0 \text{ on } \partial\Omega,$$

is the finite-dimensional space of harmonic 1-forms $\mathcal{H}_1^A$, with absolute boundary conditions, figuring into the Hodge decomposition, introduced in (9.36) of Chapter 5. Show that, for $\tilde{u}(0) \in \mathcal{H}_1^A$, Euler's equation becomes the finite-dimensional system

$$\frac{\partial \tilde{u}}{\partial t} + P_h^A(\nabla_u \tilde{u}) = 0, \tag{1.42}$$

where P_h^A is the orthogonal projection of $L^2(\Omega, \Lambda^1)$ onto $\mathcal{H}_1^A$.
15. In the context of Exercise 14, show that an irrotational Euler flow must be stationary, that is, the flow described by (1.42) is trivial. (*Hint*: By (1.16), $\partial\tilde{u}/\partial t = -d\Big((1/2)|u|^2 + p\Big)$, which is orthogonal to $\mathcal{H}_1^A$.)
16. Suppose Ω is a bounded region in $\mathbb{R}^2$, with $k+1$ (smooth) boundary components γ_j. Show that $\mathcal{H}_1^A$ is the k-dimensional space

$$\mathcal{H}_1^A = \{*df : f \in C^\infty(\overline{\Omega}), \Delta f = 0 \text{ on } \Omega, f = c_j \text{ on } \gamma_j\},$$

where the c_j are arbitrary constants. Show that a holomorphic diffeomorphism $F : \overline{\Omega} \to \overline{\mathcal{O}}$ takes $\mathcal{H}_1^A(\Omega)$ to $\mathcal{H}_1^A(\mathcal{O})$.

17. If Ω is a planar region as in Exercise 16, show that the space V_σ of velocity fields for Euler flows defined by (1.6) can be characterized as

$$V_\sigma = \{u : \tilde{u} = *df, f \in C^\infty(\overline{\Omega}), f = c_j \text{ on } \gamma_j\}.$$

Given u in this space, an associated f is called a *stream function*. Show that it is constant along each streamline of u.

18. In the context of Exercise 17, note that $w = \text{rot } u = -\Delta f$. Show that $u \cdot \nabla w = 0$, hence $\partial w/\partial t = 0$, whenever f satisfies a PDE of the form

$$\Delta f = \Phi(f) \text{ on } \Omega, \quad f = c_j \text{ on } \gamma_j. \tag{1.43}$$

19. When $\tilde{u}(0) = *df$ for f satisfying (1.43), show that the resulting flow is stationary, that is, $\partial \tilde{u}/\partial t = 0$, not merely $\partial w/\partial t = 0$. (*Hint*: In this case, $\tilde{u}$ satisfies the linear evolution equation

$$\frac{\partial \tilde{u}}{\partial t} + P_h^A(w \ * \tilde{u}) = 0,$$

as a consequence of (1.16). It suffices to show that $P_h^A(w * \tilde{u}(0)) = 0$, but indeed $w * \tilde{u}(0) = -\Phi(f)df = d\psi(f)$.)

Note. When Ω is simply connected, the argument simplifies.

20. Let $\overline{\Omega}$ be a compact Riemannian manifold, u a solution to (1.7)–(1.9), with associated vorticity $\tilde{w}$. When does

$$\frac{\partial \tilde{w}}{\partial t} = 0, \quad \text{for all } t \Longrightarrow \frac{\partial u}{\partial t} = 0?$$

Begin by considering the following cases:
a) $\mathcal{H}^1(\overline{\Omega}) = 0$. (*Hint*: Use Hodge theory.)
b) $\dim \mathcal{H}^1(\overline{\Omega}) = 1$. (*Hint*: Use conservation of energy.)
c) $\overline{\Omega} \subset\subset \mathbb{R}^2$. (*Hint*: Generalize Exercise 19.)

21. Using the exercises on "spaces of gradient and divergence-free vector fields" in §9 of Chapter 5, show that if we identify vector fields and 1-forms, the Leray projection P is given by

$$Pu = P_\delta^A u + P_h^A u, \quad \text{i.e.,} \quad (I - P)u = P_d^A u = d\delta G^A u. \tag{1.44}$$

22. Let Ω be a smooth, bounded region in $\mathbb{R}^3$ and u a solution to the Euler equation on $I \times \Omega$, where I is a t-interval containing 0. Assume the vorticity w vanishes on $\partial\Omega$ at $t = 0$.
a) Show that $w = 0$ on $\partial\Omega$, for all $t \in I$.
b) Show that the quantity

$$h(t) = \int_\Omega u(t,x) \cdot w(t,x) \; dx, \tag{1.45}$$

is independent of t. This is called the *helicity*. (*Hint*: Use formulas for the adjoint of ∇_u when $\text{div } u = 0$; ditto for ∇_w; recall Exercise 2.)
c) Show that the quantities

$$I(t) = \int_\Omega x \times w(t,x) \; dx, \quad A(t) = \int_\Omega |x|^2 w(t,x) \; dx \tag{1.46}$$

are independent of t. These are called the *impulse* and the *angular impulse*, respectively.

Consider these questions when the hypothesis on w is relaxed to w tangent to $\partial\Omega$ at $t = 0$.

23. Extend results on the conservation of helicity to other 3-manifolds Ω, via a computation of

$$(\partial_t + \mathcal{L}_u)(\tilde{u} \wedge \tilde{w}). \tag{1.47}$$

24. If we consider the motion of an incompressible fluid of *variable* density $\rho(t, x)$, the Euler equations are modified to

$$\rho(u_t + \nabla_u u) = -\text{grad } p, \quad \rho_t + \nabla_u \rho = 0, \tag{1.48}$$

and, as before, div $u = 0$, u tangent to $\partial\Omega$. Show that, in this case, the vorticity $\tilde{w} = d\tilde{u}$ satisfies

$$\partial_t \tilde{w} + \mathcal{L}_u \tilde{w} = \rho^{-2} d\rho \wedge dp. \tag{1.49}$$

(Results in subsequent sections will not apply to this case.)

2. Existence of solutions to the Euler equations

In this section we will examine the existence of solutions to the initial value problem for the Euler equation:

$$\frac{\partial u}{\partial t} + P\nabla_u u = 0, \qquad u(0) = u_0, \tag{2.1}$$

given div $u_0 = 0$, where P is the orthogonal projection of $L^2(M, TM)$ onto the space V_σ of divergence-free vector fields. We suppose M is compact without boundary; regions with boundary will be treated in the next section.

We take an approach very similar to that used for symmetric hyperbolic equations in §2 of Chapter 16. Thus, with J_ε a Friedrichs mollifier such as used there, we consider the approximating equations

$$\frac{\partial u_\varepsilon}{\partial t} + PJ_\varepsilon \nabla_{u_\varepsilon} J_\varepsilon u_\varepsilon = 0, \qquad u_\varepsilon(0) = u_0. \tag{2.2}$$

As in that case, we want to show that u_ε exists on an interval independent of ε, and we want to obtain uniform estimates that allow us to pass to the limit $\varepsilon \to 0$. We begin by estimating the L^2-norm. Noting that $u_\varepsilon(t) = Pu_\varepsilon(t)$, we have

$$\begin{aligned} \frac{d}{dt} \|u_\varepsilon(t)\|_{L^2}^2 &= -2(PJ_\varepsilon \nabla_{u_\varepsilon} J_\varepsilon u_\varepsilon, u_\varepsilon) \\ &= -2(\nabla_{u_\varepsilon} J_\varepsilon u_\varepsilon, J_\varepsilon u_\varepsilon). \end{aligned} \tag{2.3}$$

Now, generally, we have

$$\nabla_v^* w = -\nabla_v w - (\text{div } v)w, \tag{2.4}$$

as shown in §3 of Chapter 2. Consequently, when div $v = 0$, we have

$$(\nabla_v w, w) = -(\nabla_v w, w) = 0. \tag{2.5}$$

Thus (2.3) yields $(d/dt)\|u_\varepsilon(t)\|_{L^2}^2 = 0$, or

$$\|u_\varepsilon(t)\|_{L^2} = \|u_0\|_{L^2}. \tag{2.6}$$

It follows that (2.2) is solvable for all $t \in \mathbb{R}$, when $\varepsilon > 0$.

The next step, to estimate higher-order derivatives of u_ε, is accomplished in almost exact parallel with the analysis (1.8) of Chapter 16, for symmetric hyperbolic systems. Again, to make things simple, let us suppose $M = \mathbb{T}^n$; modifications for the more general case will be sketched below. Then P and J_ε can be taken to be convolution operators, so P, J_ε, and D^α all commute. Then

$$\begin{aligned} \frac{d}{dt}\,\|D^\alpha u_\varepsilon(t)\|_{L^2}^2 &= -2(D^\alpha P J_\varepsilon \nabla_{u_\varepsilon} J_\varepsilon u_\varepsilon, D^\alpha u_\varepsilon) \\ &= -2(D^\alpha L_\varepsilon J_\varepsilon u_\varepsilon, D^\alpha J_\varepsilon u_\varepsilon), \end{aligned} \tag{2.7}$$

where we have set

$$L_\varepsilon w = L(u_\varepsilon, D)w, \tag{2.8}$$

with

$$L(v, D)w = \nabla_v w, \tag{2.9}$$

a first-order differential operator on w whose coefficients $L_j(v)$ depend linearly on v. By (2.4),

$$L_\varepsilon + L_\varepsilon^* = 0, \tag{2.10}$$

since div $u_\varepsilon = 0$, so (2.7) yields

$$\frac{d}{dt}\,\|D^\alpha u_\varepsilon(t)\|_{L^2}^2 = 2([L_\varepsilon, D^\alpha] J_\varepsilon u_\varepsilon, D^\alpha J_\varepsilon u_\varepsilon). \tag{2.11}$$

Now, just as in (1.13) of Chapter 16, the Moser estimates from §3 of Chapter 13 yield

$$\begin{aligned} &\|[L_\varepsilon, D^\alpha]w\|_{L^2} \\ &\le C \sum_j \Big(\|L_j(u_\varepsilon)\|_{H^k} \|\partial_j w\|_{L^\infty} + \|\nabla L_j(u_\varepsilon)\|_{L^\infty} \|\partial_j w\|_{H^{k-1}} \Big). \end{aligned} \tag{2.12}$$

Keep in mind that $L_j(u_\varepsilon)$ is linear in u_ε. Applying this with $w = J_\varepsilon u_\varepsilon$, and summing over $|\alpha| \le k$, we have the basic estimate

$$\frac{d}{dt}\,\|u_\varepsilon(t)\|_{H^k}^2 \le C \|u_\varepsilon(t)\|_{C^1} \|u_\varepsilon(t)\|_{H^k}^2, \tag{2.13}$$

parallel to the estimate (1.15) in Chapter 16, but with a more precise dependence on $\|u_\varepsilon(t)\|_{C^1}$, which will be useful later on. From here, the elementary arguments used to prove Theorem 1.2 in Chapter 16 extend without change to yield the following:

Theorem 2.1. *Given $u_0 \in H^k(M),\ k > n/2+1$, with div $u_0 = 0$, there is a solution u to (2.1) on an interval I about 0, with*

$$u \in L^\infty(I, H^k(M)) \cap \ Lip(I, H^{k-1}(M)). \tag{2.14}$$

We can also establish the uniqueness, and treat the stability and rate of convergence of u_ε to u, just as was done in Chapter 16, §1. Thus, with $\varepsilon \in [0,1]$, we compare a solution u to (2.1) to a solution u_ε to

$$\frac{\partial u_\varepsilon}{\partial t} + PJ_\varepsilon \nabla_{u_\varepsilon} J_\varepsilon u_\varepsilon = 0, \quad u_\varepsilon(0) = v_0. \tag{2.15}$$

Setting $v = u - u_\varepsilon$, we can form an equation for v analogous to equation (1.25) in Chapter 16, and the analysis (1.25)–(1.36) there goes through without change, to give

$$\|v(t)\|_{L^2}^2 \le K_0(t)\Big(\|u_0 - v_0\|_{L^2}^2 + K_2(t)\|I - J_\varepsilon\|_{\mathcal{L}(H^{k-1},L^2)}^2\Big). \tag{2.16}$$

Thus we have

Proposition 2.2. *Given $k > n/2+1$, solutions to (2.1) satisfying (2.14) are unique. They are limits of solutions u_ε to (2.2), and for $t \in I$,*

$$\|u(t) - u_\varepsilon(t)\|_{L^2} \le K_1(t)\|I - J_\varepsilon\|_{\mathcal{L}(H^{k-1},L^2)}. \tag{2.17}$$

Continuing to follow Chapter 16, we can next look at

$$\begin{aligned} \frac{d}{dt}\|D^\alpha J_\varepsilon u(t)\|_{L^2}^2 &= -2(D^\alpha J_\varepsilon P \nabla_u u, D^\alpha J_\varepsilon u) \\ &= -2(D^\alpha J_\varepsilon L(u,D)u, D^\alpha J_\varepsilon u), \end{aligned} \tag{2.18}$$

given the commutativity of P with $D^\alpha J_\varepsilon$, and then we can follow the analysis of (1.40)–(1.45) given there without any change, to get

$$\frac{d}{dt}\ \|J_\varepsilon u(t)\|_{H^k}^2 \le C\big(1 + \|u(t)\|_{C^1}\big)\|u(t)\|_{H^k}^2, \tag{2.19}$$

for the solution u to (2.1) constructed above. Now, as in the proof of Proposition 5.1 in Chapter 16, we can note that (2.19) is equivalent to an integral inequality, and pass to the limit $\varepsilon \to 0$, to deduce

$$\frac{d}{dt}\ \|u(t)\|_{H^k}^2 \le C\big(1 + \|u(t)\|_{C^1}\big)\|u(t)\|_{H^k}^2, \tag{2.20}$$

parallel to (1.46) of Chapter 16, but with a significantly more precise dependence on $\|u(t)\|_{C^1}$. Consequently, as in Proposition 1.4 of Chapter 16, we can sharpen the first part of (2.14) to

$$u \in C(I, H^k(M)).$$

Furthermore, we can deduce that if $u \in C(I, H^k(M))$ solves the Euler equation, $I = (-a, b)$, then u continues beyond the endpoints unless $\|u(t)\|_{C^1}$

blows up at an endpoint. However, for the Euler equations, there is the following important sharpening, due to Beale-Kato-Majda [BKM]:

Proposition 2.3. *If $u \in C(I, H^k(M))$ solves the Euler equations, $k > n/2 + 1$, and if*

$$\sup_{t \in I} \|w(t)\|_{L^\infty} \le K < \infty, \tag{2.21}$$

where w is the vorticity, then the solution u continues to an interval I', containing $\overline{I}$ in its interior, $u \in C(I', H^k(M))$.

For the proof, recall that if $\tilde{u}(t)$ and $\tilde{w}(t)$ are the 1-form and 2-form on M, associated to u and w, then

$$\tilde{w} = d\tilde{u}, \qquad \delta \tilde{u} = 0. \tag{2.22}$$

Hence $\delta \tilde{w} = \delta d \tilde{u} + d \delta \tilde{u} = \Delta \tilde{u}$, where Δ is the Hodge Laplacian, so

$$\tilde{u} = G \delta \tilde{w} + P_0 \tilde{u}, \tag{2.23}$$

where P_0 is a projection onto the space of harmonic 1-forms on M, which is a finite-dimensional space of C^∞-forms. Now $G\delta$ is a pseudodifferential operator of order -1:

$$G\delta = A \in OPS^{-1}(M). \tag{2.24}$$

Consequently, $\|G\delta\tilde{w}\|_{H^{1,p}} \le C_p \|w\|_{L^p}$ for any $p \in (1, \infty)$. This breaks down for $p = \infty$, but, as we show below, we have, for any $s > n/2$,

$$\|A\tilde{w}\|_{C^1} \le C\big(1 + \log^+ \|\tilde{w}\|_{H^s}\big) \|\tilde{w}\|_{L^\infty} + C. \tag{2.25}$$

Therefore, under the hypothesis (2.21), we obtain an estimate

$$\|u(t)\|_{C^1} \le C\big(1 + \log^+ \|u\|_{H^k}^2\big), \tag{2.26}$$

provided $k > n/2 + 1$, using (2.23) and the facts that $\|\tilde{w}\|_{H^{k-1}} \le c\|u\|_{H^k}$ and that $\|u(t)\|_{L^2}$ is constant. Thus (2.20) yields the differential inequality

$$\frac{dy}{dt} \le C(1 + \log^+ y) y, \quad y(t) = \|u(t)\|_{H^k}^2. \tag{2.27}$$

Now one form of Gronwall's inequality (cf. Chapter 1, (5.19)–(5.21)) states that if $Y(t)$ solves

$$\frac{dY}{dt} = F(t, Y), \quad Y(0) = y(0), \tag{2.28}$$

while $dy/dt \le F(t, y)$, and if $\partial F / \partial y \ge 0$, then $y(t) \le Y(t)$ for $t \ge 0$. We apply this to $F(t, Y) = C(1 + \log^+ Y)Y$, so (2.28) gives

$$\int \frac{dY}{(1 + \log^+ Y)Y} = Ct + C_1. \tag{2.29}$$

Since

$$\int_1^\infty \frac{dY}{(1+\log^+ Y)Y} = \infty, \tag{2.30}$$

we see that $Y(t)$ exists for all $t \in [0,\infty)$ in this case. This provides an upper bound

$$\|u(t)\|_{H^k}^2 \le Y(t), \tag{2.31}$$

as long as (2.21) holds. Thus Proposition 2.3 will be proved once we establish the estimate (2.25). We will establish a general result, which contains (2.25).

Lemma 2.4. *If $P \in OPS^0_{1,0}$, $s > n/2$, then*

$$\|Pu\|_{L^\infty} \le C\|u\|_{L^\infty} \cdot \Bigl[1 + \log\Bigl(\frac{\|u\|_{H^s}}{\|u\|_{L^\infty}}\Bigr)\Bigr]. \tag{2.32}$$

We suppose the norms are arranged to satisfy $\|u\|_{L^\infty} \le \|u\|_{H^s}$. Another way to write the result is in the form

$$\|Pu\|_{L^\infty} \le C\varepsilon^\delta \|u\|_{H^s} + C\Bigl(\log \frac{1}{\varepsilon}\Bigr)\|u\|_{L^\infty}, \tag{2.33}$$

for $0 < \varepsilon \le 1$, with C independent of ε. Then, letting $\varepsilon^\delta = \|u\|_{L^\infty}/\|u\|_{H^s}$ yields (2.32). The estimate (2.33) is valid when $s > n/2 + \delta$. We will derive (2.33) from an estimate relating the L^∞-, H^s-, and C^0_*-norms. The Zygmund spaces C^r_* are defined in §8 of Chapter 13.

It suffices to prove (2.33) with P replaced by $P + cI$, where c is greater than the L^2-operator norm of P; hence we can assume $P \in OPS^0_{1,0}$ is elliptic and invertible, with inverse $Q \in OPS^0_{1,0}$. Then (2.33) is equivalent to

$$\|u\|_{L^\infty} \le C\varepsilon^\delta \|u\|_{H^s} + C\Bigl(\log \frac{1}{\varepsilon}\Bigr)\|Qu\|_{L^\infty}. \tag{2.34}$$

Now since $Q : C^0_* \to C^0_*$, with inverse P, and the C^0_*-norm is weaker than the L^∞-norm, this estimate is a consequence of

$$\|u\|_{L^\infty} \le C\varepsilon^\delta \|u\|_{H^s} + C\Bigl(\log \frac{1}{\varepsilon}\Bigr)\|u\|_{C^0_*}, \tag{2.35}$$

for $s > n/2 + \delta$. This result is proved in Chapter 13, §8; see Proposition 8.11 there.

We now have (2.25), so the proof of Proposition 2.3 is complete. One consequence of Proposition 2.3 is the following classical result.

Proposition 2.5. *If dim $M = 2$, $u_0 \in H^k(M)$, $k > 2$, and div $u_0 = 0$, then the solution to the Euler equation (2.1) exists for all $t \in \mathbb{R}$; $u \in C(\mathbb{R}, H^k(M))$.*

Proof. Recall that in this case w is a scalar field and the vorticity equation is

$$\frac{\partial w}{\partial t} + \nabla_u w = 0, \tag{2.36}$$

which implies that, as long as $u \in C(I, H^k(M)),\ t \in I$,

$$\|w(t)\|_{L^\infty} = \|w(0)\|_{L^\infty}. \tag{2.37}$$

Thus the hypothesis (2.21) is fulfilled.

When $\dim M \geq 3$, the vorticity equation takes a more complicated form, which does not lead to (2.37). It remains a major outstanding problem to decide whether smooth solutions to the Euler equation (2.1) persist in this case. There are numerical studies of three-dimensional Euler flows, with particular attention to the evolution of the vorticity, such as [BM].

Having discussed details in the case $M = \mathbb{T}^n$, we now describe modifications when M is a more general compact Riemannian manifold without boundary. One modification is to estimate, instead of (2.7),

$$\begin{aligned} \frac{d}{dt}\,\|\Delta^\ell u_\varepsilon(t)\|_{L^2}^2 &= -2(\Delta^\ell P J_\varepsilon \nabla_{u_\varepsilon} J_\varepsilon u_\varepsilon, \Delta^\ell u_\varepsilon) \\ &= -2(\Delta^\ell P L_\varepsilon J_\varepsilon u_\varepsilon, \Delta^\ell J_\varepsilon u_\varepsilon), \end{aligned} \tag{2.38}$$

the latter identity holding provided $\Delta,\ P$, and J_ε all commute. This can be arranged by taking $J_\varepsilon = e^{\varepsilon\Delta}$; P and Δ automatically commute here. In this case, with D^α replaced by Δ^ℓ, (2.11)–(2.12) go through, to yield the basic estimate (2.13), provided $k = 2\ell > n/2 + 1$. When $[n/2]$ is even, this gives again the results of Theorem 2.1–Proposition 2.5. When $[n/2]$ is odd, the results obtained this way are slightly weaker, if ℓ is restricted to be an integer.

An alternative approach, which fully recovers Theorem 2.1–Proposition 2.5, is the following. Let $\{X_j\}$ be a finite collection of vector fields on M, spanning T_xM at each x, and for $J = (j_1, \dots, j_k)$, let $X^J = \nabla_{X_{j_1}} \cdots \nabla_{X_{j_k}}$, a differential operator of order $k = |J|$. We estimate

$$\frac{d}{dt}\,\|X^J u_\varepsilon(t)\|_{L^2}^2 = -2(X^J P J_\varepsilon L_\varepsilon J_\varepsilon u_\varepsilon, X^J u_\varepsilon). \tag{2.39}$$

We can still arrange that P and J_ε commute, and write this as

$$\begin{aligned} &-2(L_\varepsilon X^J J_\varepsilon u_\varepsilon, X^J J_\varepsilon u_\varepsilon) - 2([X^J, L_\varepsilon] J_\varepsilon u_\varepsilon, X^J J_\varepsilon u_\varepsilon) \\ &\quad - 2(X^J L_\varepsilon J_\varepsilon u_\varepsilon, [X^J, P J_\varepsilon] u_\varepsilon) - 2([X^J, P J_\varepsilon] L_\varepsilon J_\varepsilon u_\varepsilon, X^J u_\varepsilon). \end{aligned} \tag{2.40}$$

Of these four terms, the first is analyzed as before, due to (2.10). For the second term we have the same type of Moser estimate as in (2.12). The new terms to analyze are the last two terms in (2.40). In both cases the key is to see that, for $\varepsilon \in (0, 1]$,

$$[X^J, P J_\varepsilon] \text{ is bounded in } OPS^{k-1}_{1,0}(M) \quad \text{if } |J| = k, \tag{2.41}$$

which follows from the containment $P \in OPS^0_{1,0}(M)$ and the boundedness of J_ε in $OPS^0_{1,0}(M)$. If we push one factor X_{j_1} in X^J from the left side to the right side of the third inner product in (2.40), we dominate each of the last two terms by

$$C\|L_\varepsilon J_\varepsilon u_\varepsilon\|_{H^{k-1}} \cdot \|u_\varepsilon\|_{H^k} \tag{2.42}$$

if $|J| = k$. To complete the estimate, we use the identity

$$\operatorname{div}(u \otimes v) = (\operatorname{div} v)u + \nabla_v u, \tag{2.43}$$

which yields

$$L_\varepsilon J_\varepsilon u_\varepsilon = \operatorname{div}(J_\varepsilon u_\varepsilon \otimes u_\varepsilon). \tag{2.44}$$

Now, by the Moser estimates, we have

$$\|L_\varepsilon J_\varepsilon u_\varepsilon\|_{H^{k-1}} \le C\|J_\varepsilon u_\varepsilon \otimes u_\varepsilon\|_{H^k} \le C\|u_\varepsilon\|_{L^\infty}\|u_\varepsilon\|_{H^k}. \tag{2.45}$$

Consequently, we again obtain the estimate (2.13), and hence the proofs of Theorem 2.1–Proposition 2.5 again go through.

So far in this section we have discussed strong solutions to the Euler equations, for which there is a uniqueness result known. We now give a result of [DM], on the existence of weak solutions to the two-dimensional Euler equations, with initial data less regular than in Proposition 2.5.

Proposition 2.6. *If dim $M = 2$ and $u_0 \in H^{1,p}(M)$, for some $p > 1$, then there exists a weak solution to (2.1):*

$$u \in L^\infty\big(\mathbb{R}^+, H^{1,p}(M)\big) \cap C\big(\mathbb{R}^+, L^2(M)\big). \tag{2.46}$$

Proof. Take $f_j \in C^\infty(M)$, $f_j \to u_0$ in $H^{1,p}(M)$, and let $v_j \in C^\infty(\mathbb{R}^+ \times M)$ solve

$$\frac{\partial v_j}{\partial t} + P \operatorname{div}(v_j \otimes v_j) = 0, \quad \operatorname{div} v_j = 0, \quad v_j(0) = f_j. \tag{2.47}$$

Here we have used (2.43) to write $\nabla_{v_j} v_j = \operatorname{div}(v_j \otimes v_j)$. Let $w_j = \operatorname{rot} v_j$, so $w_j(0) \to \operatorname{rot} u_0$ in $L^p(M)$. Hence $\|w_j(0)\|_{L^p}$ is bounded in j, and the vorticity equation implies

$$\|w_j(t)\|_{L^p} \le C, \quad \forall\, t, j. \tag{2.48}$$

Also $\|v_j(0)\|_{L^2}$ is bounded and hence $\|v_j(t)\|_{L^2}$ is bounded, so

$$\|v_j(t)\|_{H^{1,p}} \le C. \tag{2.49}$$

The Sobolev imbedding theorem gives $H^{1,p}(M) \subset L^{2+2\delta}(M),\ \delta > 0$, when dim $M = 2$, so

$$\|v_j(t) \otimes v_j(t)\|_{L^{1+\delta}} \le C. \tag{2.50}$$

Hence, by (2.47),

$$\|\partial_t v_j(t)\|_{H^{-1,1+\delta}} \le C. \tag{2.51}$$

An interpolation of (2.49) and (2.51) gives

$$v_j \ \text{ bounded in } \ C^r\big([0,\infty), L^s(M)\big), \tag{2.52}$$

for some $r > 0,\ s > 2$. Together with (2.49), this implies

$$\|v_j\| \ \text{ compact in } \ C\big([0,T], L^2(M)\big), \tag{2.53}$$

for any $T < \infty$. Thus we can choose a subsequence v_{j_ν} such that

$$v_{j_\nu} \longrightarrow u \ \text{ in } \ C\big([0,T], L^2(M)\big), \quad \forall\, T < \infty, \tag{2.54}$$

the convergence being in norm. Hence

$$v_{j_\nu} \otimes v_{j_\nu} \longrightarrow u \otimes u \quad \text{in } \ C\big(\mathbb{R}^+, L^1(M)\big), \tag{2.55}$$

so

$$P \ \text{div}(v_{j_\nu} \otimes v_{j_\nu}) \longrightarrow P \ \text{div}(u \otimes u) \quad \text{in } \ C\big(\mathbb{R}^+, \mathcal{D}'(M)\big), \tag{2.56}$$

so the limit satisfies (2.1).

The question of the uniqueness of a weak solution obtained in Proposition 2.6 is open.

It is of interest to consider the case when rot $u_0 = w_0$ is not in $L^p(M)$ for some $p > 1$, but just in $L^1(M)$, or more generally, let w_0 be a finite measure on M. This problem was addressed in [DM], which produced a "measure-valued solution" (i.e., a "fuzzy solution," in the terminology used in Chapter 13, §11). In [Del] it was shown that if w_0 is a *positive* measure (and $M = \mathbb{R}^2$), then there is a global weak solution; see also [ES] and [Mj5]. Other work, with particular attention to cases where rot u_0 is a linear combination of delta functions, is discussed in [MaP]; see also [Cho].

We also mention the extension of Proposition 2.6 in [Cha], to the case $w_0 \in L(\log L)$.

The following provides extra information on the limiting case $p = \infty$ of Proposition 2.6:

Proposition 2.7. *If dim $M = 2$, rot $u_0 \in L^\infty(M)$, and u is a weak solution to (2.1) given by Proposition 2.6, then*

$$u \in C(\mathbb{R}^+ \times M), \tag{2.57}$$

and, for each $t \in \mathbb{R}^+$, in any local coordinate chart on M, if $|x - y| \le 1/2$,

$$|u(t,x) - u(t,y)| \le C|x-y| \ \log \frac{1}{|x-y|} \ \|rot\ u_0\|_{L^\infty}. \tag{2.58}$$

Furthermore, u generates a flow, consisting of homeomorphisms $\mathcal{F}^t : M \to M$.

Proof. The continuity in (2.57) holds whenever $u_0 \in H^{1,p}(M)$ with $p > 2$, as can be deduced from (2.46), its corollary

$$\partial_t u \in L^\infty\big(\mathbb{R}^+, L^p(M)\big), \quad p > 2, \tag{2.59}$$

and interpolation. In fact, this gives a Hölder estimate on u. Next, we have

$$\|\text{rot } u(t)\|_{L^\infty} \le \|\text{rot } u_0\|_{L^\infty}, \quad \forall\, t \ge 0. \tag{2.60}$$

Since $u(t)$ is obtained from rot $u(t)$ via (2.23), the estimate (2.58) is a consequence of the fact that

$$A \in OPS^{-1}(M) \Longrightarrow A : L^\infty(M) \to \text{LLip}(M), \tag{2.61}$$

where, with $\delta(x,y) = \text{dist}(x,y),\ \lambda(\delta) = \delta \log(1/\delta)$,

$$\text{LLip}(M) = \bigl\{ f \in C(M) : |f(x) - f(y)| \le C\lambda\bigl(\delta(x,y)\bigr) \bigr\}. \tag{2.62}$$

The result (2.61) can be established directly from integral kernel estimates. Alternatively, (2.61) follows from the inclusion

$$C^1_*(M) \subset \text{LLip}(M), \tag{2.63}$$

since we know that $A \in OPS^{-1}(M) \Rightarrow A : L^\infty(M) \to C^1_*(M)$. In turn, the inclusion (2.63) is a consequence of the following characterization of LLip, due to [BaC]:

Let $\Psi_0 \in C_0^\infty(\mathbb{R}^n)$ satisfy $\Psi_0(\xi) = 1$ for $|\xi| \le 1$, and set $\Psi_k(\xi) = \Psi_0(2^{-k}\xi)$. Recall that, with $\psi_0 = \Psi_0,\ \psi_k = \Psi_k - \Psi_{k-1}$ for $k \ge 1$,

$$f \in C^0_*(\mathbb{R}^n) \Longleftrightarrow \|\psi_k(D) f\|_{L^\infty} \le C.$$

It follows that, for any $u \in \mathcal{E}'(\mathbb{R}^n)$,

$$u \in C^1_*(\mathbb{R}^n) \Longleftrightarrow \|\nabla \psi_k(D) u\|_{L^\infty} \le C. \tag{2.64}$$

By comparison, we have the following:

Lemma 2.8. *Given $u \in \mathcal{E}'(\mathbb{R}^n)$, we have*

$$u \in LLip(\mathbb{R}^n) \Longleftrightarrow \|\nabla \Psi_k(D) u\|_{L^\infty} \le C(k+1). \tag{2.65}$$

We leave the details of either of these approaches to (2.61) as an exercise. Now, for t-dependent vector fields satisfying (2.57)–(2.58), the existence and uniqueness of solutions of the associated ODEs, and continuous dependence on initial data, are established in Appendix A of Chapter 1, and the rest of Proposition 2.7 follows.

We mention that uniqueness has been established for solutions to (2.1) described by Proposition 2.7; see [Kt1] and [Yud]. A special case of Proposition 2.7 is that for which rot u_0 is piecewise constant. One says these are "vortex patches." There has been considerable interest in properties of the evolution of such vortex patches; see [Che3] and also [BeC].

Exercises

1. Refine the estimate (2.13) to

$$\frac{d}{dt}\,\|u_\varepsilon(t)\|^2_{H^k}\le C\|\nabla u_\varepsilon\|_{L^\infty}\|u_\varepsilon(t)\|^2_{H^k}, \tag{2.66}$$

for $k>n/2+1$.

2. Using interpolation inequalities, show that if $k=s+r,\ s=n/2+1+\delta$, then

$$\frac{d}{dt}\,\|u_\varepsilon(t)\|^2_{H^k}\le C\|u_\varepsilon(t)\|^{2(1+\gamma)}_{H^k},\quad \gamma=\frac{s}{2k}.$$

3. Give a treatment of the Euler equation with an external force term:

$$\frac{\partial u}{\partial t}+\nabla_u u=-\text{grad}\ p+f,\quad \text{div}\ u=0. \tag{2.67}$$

4. The enstrophy of an smooth Euler flow is defined by

$$\text{Ens}(t)=\|w(t)\|^2_{L^2(M)},\quad w=\ \text{vorticity}. \tag{2.68}$$

If u is a smooth solution to (2.1) on $I\times M,\ t\in I$, and dim $M=3$, show that

$$\frac{d}{dt}\|w(t)\|^2_{L^2}=2\Big(\nabla_w u,w\Big)_{L^2}. \tag{2.69}$$

5. Recall the deformation tensor associated to a vector field u,

$$\text{Def}(u)=\frac{1}{2}\Big(\nabla u+\nabla u^t\Big), \tag{2.70}$$

which measures the degree to which the flow of u distorts the metric tensor g. Denote by ϑ_u the associated second-order, symmetric covariant tensor field (i.e., $\vartheta_u=(1/2)\mathcal{L}_u g$). Show that when dim $M=3$, (2.69) is equivalent to

$$\frac{d}{dt}\|w(t)\|^2_{L^2}=2\int\limits_M \vartheta_u(w,w)\ dV. \tag{2.71}$$

6. Show that the estimate (2.32) can be generalized and sharpened to

$$\|Pu\|_{L^\infty}\le C\|u\|_{C^0_*}\cdot\left[1+\log\left(\frac{\|u\|_{H^{s,p}}}{\|u\|_{C^0_*}}\right)\right],\quad P\in OPS^0_{1,\delta}, \tag{2.72}$$

given $\delta\in[0,1),\ p\in(1,\infty)$, and $s>n/p$.

7. Prove Lemma 2.8, and hence deduce (2.61).

3. Euler flows on bounded regions

Having discussed the existence of solutions to the Euler equations for flows on a compact manifold without boundary in §2, we now consider the case of a compact manifold $\overline{M}$ with boundary ∂M (and interior M). We want to solve the PDE

$$\frac{\partial u}{\partial t}+P\nabla_u u=0,\quad \text{div}\ u=0, \tag{3.1}$$

with boundary condition

$$\nu \cdot u = 0 \quad \text{on } \partial M, \tag{3.2}$$

where ν is the normal to ∂M, and initial condition

$$u(0) = u_0. \tag{3.3}$$

We work on the spaces

$$V^k = \{u \in H^k(M, TM) : \text{div } u = 0,\ \nu \cdot u\big|_{\partial M} = 0\}. \tag{3.4}$$

As shown in the third problem set in §9 of Chapter 5 (see (9.79)), V^0 is the closure of V_σ (given by (1.6)) in $L^2(M, TM)$. Hence the Leray projection P is the orthogonal projection of $L^2(M, TM)$ onto V^0. This result uses the Hodge decomposition, and results on the Hodge Laplacian with absolute boundary conditions, which also imply that

$$P : H^k(M, TM) \longrightarrow V^k. \tag{3.5}$$

Furthermore, the Hodge decomposition yields the characterization

$$(I - P)v = -\text{grad } p, \tag{3.6}$$

where p is uniquely defined up to an additive constant by

$$-\Delta p = \text{ div } v \text{ on } M, \quad -\frac{\partial p}{\partial \nu} = \nu \cdot v \text{ on } \partial M. \tag{3.7}$$

See also Exercises 1–2 at the end of this section.

The following estimates will play a central role in our analysis of the Euler equations.

Proposition 3.1. *Let u and v be C^1-vector fields in $\overline{M}$. Assume $u \in V^k$. If $v \in H^{k+1}(M)$, then*

$$\big|(\nabla_u v, v)_{H^k}\big| \le C\Big(\|u\|_{C^1}\|v\|_{H^k} + \|u\|_{H^k}\|v\|_{C^1}\Big)\|v\|_{H^k}, \tag{3.8}$$

while if $v \in V^k$, then

$$\big\|(1 - P)\nabla_u v\big\|_{H^k} \le C\Big(\|u\|_{C^1}\|v\|_{H^k} + \|u\|_{H^k}\|v\|_{C^1}\Big). \tag{3.9}$$

Proof. We begin with the $k = 0$ case of (3.8). Indeed, Green's formula gives

$$(\nabla_u v, w)_{L^2} = -(v, \nabla_u w)_{L^2} - \big(v, (\text{div } u)w\big)_{L^2} + \int\limits_{\partial M} \langle \nu, u\rangle \ \langle v, w\rangle \ dS. \tag{3.10}$$

If div $u = 0$ and $\nu \cdot u\big|_{\partial M} = 0$, the last two terms vanish, so the $k = 0$ case of (3.8) is sharpened to

$$(\nabla_u v, v)_{L^2} = 0 \quad \text{if } u \in V^0 \tag{3.11}$$

and v is C^1 on $\overline{M}$. This also holds if $u \in V^0 \cap C(\overline{M}, T)$ and $v \in H^1$.

To treat (3.8) for $k \geq 1$, we use the following inner product on $H^k(M, T)$. Pick a finite set of smooth vector fields $\{X_j\}$, spanning $T_x\overline{M}$ for each $x \in \overline{M}$, and set

$$(u, v)_{H^k} = \sum_{|J| \leq k} (X^J u, X^J v)_{L^2}, \tag{3.12}$$

where $X^J = \nabla_{X_{j_1}} \cdots \nabla_{X_{j_\ell}}$ are as in (2.39), $|J| = \ell$. Now, we have

$$(X^J \nabla_u v, X^J v)_{L^2} = (\nabla_u X^J v, X^J v)_{L^2} + ([X^J, \nabla_u]v, X^J v)_{L^2}. \tag{3.13}$$

The first term on the right vanishes, by (3.11). As for the second, as in (2.12) we have the Moser estimate

$$\big\|[X^J, \nabla_u]v\big\|_{L^2} \leq C\Big(\|u\|_{C^1}\|v\|_{H^k} + \|u\|_{H^k}\|v\|_{C^1}\Big). \tag{3.14}$$

This proves (3.8).

In order to establish (3.9), it is useful to calculate div $\nabla_u v$. In index notation $X = \nabla_u v$ is given by $X^j = v^j{}_{;k}u^k$, so div $X = X^j{}_{;j}$ yields

$$\text{div } \nabla_u v = v^j{}_{;k;j}u^k + v^j{}_{;k}u^k{}_{;j}. \tag{3.15}$$

If M is flat, we can simply change the order of derivatives of v; more generally, using the Riemann curvature tensor R,

$$v^j{}_{;k;j} = v^j{}_{;j;k} + R^j{}_{\ell jk}v^\ell. \tag{3.16}$$

Noting that $R^j{}_{\ell jk} = \text{Ric}_{\ell k}$ is the Ricci tensor, we have

$$\text{div } \nabla_u v = \nabla_u(\text{div } v) + \text{Ric}(u, v) + \text{Tr}\big((\nabla u)(\nabla v)\big), \tag{3.17}$$

where ∇u and ∇v are regarded as tensor fields of type $(1, 1)$. When div $v = 0$, of course the first term on the right side of (3.17) disappears, so

$$\text{div } v = 0 \Longrightarrow \text{ div } \nabla_u v = \text{Tr}\big((\nabla u)(\nabla v)\big) + \text{Ric}(u, v). \tag{3.18}$$

Note that only first-order derivatives of v appear on the right. Thus P acts on $\nabla_u v$ more like the identity than it might at first appear.

To proceed further, we use (3.6) to write

$$(1 - P)\nabla_u v = -\text{grad } \varphi, \tag{3.19}$$

where, parallel to (3.7), φ satisfies

$$-\Delta\varphi = \text{ div } \nabla_u v \text{ on } M, \quad -\frac{\partial\varphi}{\partial\nu} = \nu \cdot \big(\nabla_u v\big) \text{ on } \partial M. \tag{3.20}$$

The computation of div $\nabla_u v$ follows from (3.18). To analyze the boundary value in (3.20), we use the identity $\langle \nu, \nabla_u v\rangle = \nabla_u\langle \nu, v\rangle - \langle \nabla_u \nu, v\rangle$, and note that when u and v are tangent to ∂M, the first term on the right vanishes. Hence,

$$\langle \nu, \nabla_u v\rangle = -\langle \nabla_u \nu, v\rangle = \widetilde{II}(u, v), \tag{3.21}$$

where $\widetilde{II}$ is the second fundamental form of ∂M. Thus (3.20) can be rewritten as

$$-\Delta\varphi = \operatorname{Tr}\big((\nabla u)(\nabla v)\big) + \operatorname{Ric}(u,v) \text{ on } M, \quad \frac{\partial\varphi}{\partial\nu} = -\widetilde{II}(u,v). \tag{3.22}$$

Note that in the last expression for $\partial\varphi/\partial\nu$ there are no derivatives of v. Now, by (3.22) and the estimates for the Neumann problem derived in Chapter 5, we have

$$\|\nabla\varphi\|_{H^k} \le C\Big(\|u\|_{C^1}\|v\|_{H^k} + \|u\|_{H^k}\|v\|_{C^1}\Big), \tag{3.23}$$

which proves (3.9).

Note that (3.8)–(3.9) yield the estimate

$$\big|(P\nabla_u v, v)_{H^k}\big| \le C\Big(\|u\|_{C^1}\|v\|_{H^k} + \|u\|_{H^k}\|v\|_{C^1}\Big)\|v\|_{H^k}, \tag{3.24}$$

given $u \in V^k, v \in V^{k+1}$.

In order to solve (3.1)–(3.3), we use a Galerkin-type method, following [Tem2]. Fix $k > n/2+1$, where $n = \dim M$, and take $u_0 \in V^k$. We use the inner product on V^k, derived from (3.12). Now there is an isomorphism $B_0 : V^k \to (V^k)'$, defined by $\langle B_0 v, w\rangle = (v,w)_{V^k}$. Using $V^k \subset V^0 \subset (V^k)'$, we define an unbounded, self-adjoint operator B on V^0 by

$$\mathcal{D}(B) = \{v \in V^k : B_0 v \in V^0\}, \quad B = B_0\big|_{\mathcal{D}(B)}. \tag{3.25}$$

This is a special case of the Friedrichs extension method, discussed in general in Appendix A, §8. It follows from the compactness of the inclusion $V^k \hookrightarrow V^0$ that B^{-1} is compact, so V^0 has an orthonormal basis $\{w_j : j = 1,2,\dots\}$ such that $Bw_j = \lambda_j w_j,\ \lambda_j \nearrow \infty$. Let P_j be the orthogonal projection of V^0 onto the span of $\{w_1,\dots,w_j\}$. It is useful to note that

$$(P_j u, v)_{V^0} = (u, P_j v)_{V^0} \quad \textit{and} \quad (P_j u, v)_{V^k} = (u, P_j v)_{V^k}. \tag{3.26}$$

Our approximating equation will be

$$\frac{\partial u_j}{\partial t} + P_j \nabla_{u_j} u_j = 0, \quad u_j(0) = P_j u_0. \tag{3.27}$$

Here, we extend P_j to be the orthogonal projection of $L^2(M,TM)$ onto the span of $\{w_1,\dots,w_j\}$.

We first estimate the V^0-norm (i.e., the L^2-norm) of u_j, using

$$\begin{aligned}\frac{d}{dt}\,\|u_j(t)\|^2_{V^0} &= -2(P_j\nabla_{u_j}u_j, u_j)_{V^0} \\ &= -2(\nabla_{u_j}u_j, u_j)_{L^2}.\end{aligned} \tag{3.28}$$

By (3.11), $(\nabla_{u_j}u_j, u_j)_{L^2} = 0$, so

$$\|u_j(t)\|_{V^0} = \|P_j u_0\|_{L^2}. \tag{3.29}$$

Hence solutions to (3.27) exist for all $t \in \mathbb{R}$, for each j.

Our next goal is to estimate higher-order derivatives of u_j, so that we can pass to the limit $j \to \infty$. We have

$$\frac{d}{dt} \|u_j(t)\|^2_{V^k} = -2(P_j \nabla_{u_j} u_j, u_j)_{V^k} = -2(P\nabla_{u_j} u_j, u_j)_{V^k}, \tag{3.30}$$

using (3.26). We can estimate this by (3.24), so we obtain the basic estimate:

$$\frac{d}{dt} \|u_j(t)\|^2_{V^k} \le C\|u_j\|_{C^1}\|u_j\|^2_{V^k}. \tag{3.31}$$

This is parallel to (2.13), so what is by now a familiar argument yields our existence result:

Theorem 3.2. *Given $u_0 \in V^k,\ k > n/2+1$, there is a solution to (3.1)–(3.3) for t in an interval I about 0, with*

$$u \in L^\infty(I, V^k) \cap \ Lip(I, V^{k-1}). \tag{3.32}$$

The solution is unique, in this class of functions.

The last statement, about uniqueness, as well as results on stability and rate of convergence as $j \to \infty$, follow as in Proposition 2.2.

If u is a solution to (3.1)–(3.3) satisfying (3.32) with initial data $u_0 \in V^k$, we want to estimate the rate of change of $\|u(t)\|^2_{H^k}$, as was done in (2.18)–(2.20). Things will be a little more complicated, due to the presence of a boundary ∂M. Following [KL], we define the smoothing operators J_ε on $H^k(M, TM)$ as follows. Assume M is an open subset (with closure $\overline{M}$) of the compact Riemannian manifold $\widetilde{M}$ without boundary, and let

$$E : H^\ell(M, T) \longrightarrow H^\ell(\widetilde{M}, T), \quad 0 \le \ell \le k+1,$$

be an extension operator, such as we constructed in Chapter 4. Let $R : H^\ell(\widetilde{M}, T) \to H^\ell(M, T)$ be the restriction operator, and set

$$J_\varepsilon u = R\widetilde{J}_\varepsilon E u, \tag{3.33}$$

where $\widetilde{J}_\varepsilon$ is a Friedrichs mollifier on $\widetilde{M}$. If we apply J_ε to the solution $u(t)$ of current interest, we have

$$\begin{aligned} \frac{d}{dt}\|J_\varepsilon u(t)\|^2_{H^k} &= -2(J_\varepsilon P \nabla_u u, J_\varepsilon u)_{H^k} \\ &= -2(J_\varepsilon \nabla_u u, J_\varepsilon u)_{H^k} + 2\big(J_\varepsilon(1-P)\nabla_u u, J_\varepsilon u\big)_{H^k}. \end{aligned} \tag{3.34}$$

Using (3.9), we estimate the last term by

$$\begin{aligned} &2\big|\big(J_\varepsilon(1-P)\nabla_u u, J_\varepsilon u\big)_{H^k}\big| \\ &\quad \le C\big\|(1-P)\nabla_u u\big\|_{H^k} \cdot \|u\|_{H^k} \le C\|u(t)\|_{C^1}\|u(t)\|^2_{H^k}. \end{aligned} \tag{3.35}$$

To analyze the rest of the right side of (3.34), write

$$(J_\varepsilon \nabla_u u, J_\varepsilon u)_{H^k} = \sum_{|J|\le k} \big(X^J J_\varepsilon \nabla_u u, X^J J_\varepsilon u\big)_{L^2}, \tag{3.36}$$

using (3.12). Now we have

$$X^J J_\varepsilon \nabla_u u = X^J [J_\varepsilon, \nabla_u] u + [X^J, \nabla_u] J_\varepsilon u + \nabla_u (X^J J_\varepsilon u). \tag{3.37}$$

We look at these three terms successively. First, by (3.14),

$$\big\|[X^J, \nabla_u] J_\varepsilon u\big\|_{L^2} \le C \|u(t)\|_{C^1(\overline{M})} \|u(t)\|_{H^k(M)}. \tag{3.38}$$

Next, as in (1.44)–(1.45) of Chapter 16 on hyperbolic PDE, we claim to have an estimate

$$\big\|[J_\varepsilon, \nabla_u] u\big\|_{H^k(M)} \le C \|u(t)\|_{C^1(\overline{M})} \|u\|_{H^k(M)}. \tag{3.39}$$

To obtain this, we can use a Friedrichs mollifier $\widetilde{J}_\varepsilon$ on $\widetilde{M}$ with the property that

$$\text{supp } w \subset K \Rightarrow \text{ supp } \widetilde{J}_\varepsilon w \subset K, \quad K = \widetilde{M} \setminus M. \tag{3.40}$$

In that case, if $\widetilde{u} = Eu$ and $\widetilde{w} = Ew$, then

$$[J_\varepsilon, \nabla_u] w = R[\widetilde{J}_\varepsilon, \nabla_{\widetilde{u}}] \widetilde{w}. \tag{3.41}$$

Thus (3.39) follows from known estimates for $\widetilde{J}_\varepsilon$.

Finally, the $L^2(M)$-inner product of the last term in (3.37) with $X^J J_\varepsilon u$ is zero. Thus we have a bound

$$\big|(J_\varepsilon \nabla_u u, J_\varepsilon u)_{H^k}\big| \le C \|u(t)\|_{C^1} \|u(t)\|^2_{H^k}, \tag{3.42}$$

and hence

$$\frac{d}{dt} \big\|J_\varepsilon u(t)\big\|^2_{H^k} \le C \|u(t)\|_{C^1} \|u(t)\|^2_{H^k}. \tag{3.43}$$

As before, we can convert this to an integral inequality and take $\varepsilon \to 0$, obtaining

$$\|u(t)\|^2_{H^k} \le \|u_0\|^2_{H^k} + C \int_0^t \|u(s)\|_{C^1(\overline{M})} \|u(s)\|^2_{H^k} \, ds. \tag{3.44}$$

As with the exploitation of (2.19)–(2.20), we have

Proposition 3.3. *If $k > n/2 + 1$, $u_0 \in V^k$, the solution u to (3.1)–(3.3) given by Theorem 3.2 satisfies*

$$u \in C(I, V^k). \tag{3.45}$$

Furthermore, if I is an open interval on which (3.45) holds, u solving (3.1)–(3.3), and if

$$\sup_{t\in I} \|u(t)\|_{C^1(\overline{M})} \le K < \infty, \tag{3.46}$$

then the solution u continues to an interval I', containing $\overline{I}$ in its interior, $u \in C(I', V^k)$.

We will now extend the result of [BKM], Proposition 2.3, to the Euler flow on a region with boundary. Our analysis follows [Fer] in outline, except that, as in §2, we make use of some of the Zygmund space analysis developed in §8 of Chapter 13.

Proposition 3.4. *If $u \in C(I, V^k)$ solves the Euler equation, with $k > n/2 + 1$, $I = (-a, b)$, and if the vorticity w satisfies*

$$\sup_{t \in I} \|w(t)\|_{L^\infty} \le K < \infty, \tag{3.47}$$

then the solution u continues to an interval I', containing $\overline{I}$ in its interior, $u \in C(I', V^k)$.

To start the proof, we need a result parallel to (2.23), relating u to w.

Lemma 3.5. *If $\tilde{u}$ and $\tilde{w}$ are the 1-form and 2-form on $\overline{M}$, associated to u and w, then*

$$\tilde{u} = \delta G^A \tilde{w} + P_h^A \tilde{u}, \tag{3.48}$$

where G^A is the Green operator for Δ, with absolute boundary conditions, and P_h^A the orthogonal projection onto the space of harmonic 1-forms with absolute boundary conditions.

Proof. We know that

$$d\tilde{u} = \tilde{w}, \quad \delta \tilde{u} = 0, \quad \iota_n \tilde{u} = 0. \tag{3.49}$$

In particular, $\tilde{u} \in H_A^1(M, \Lambda^1)$, defined by (9.11) of Chapter 5. Thus we can write the Hodge decomposition of $\tilde{u}$ as

$$\tilde{u} = (d + \delta) G^A (d + \delta) \tilde{u} + P_h^A \tilde{u}. \tag{3.50}$$

See Exercise 2 in the first exercise set of §9, Chapter 5. By (3.49), this gives (3.48).

Now since G^A is the solution operator to a regular elliptic boundary problem, it follows from Theorem 8.9 (complemented by (8.54)–(8.55)) of Chapter 13 that

$$G^A : C^0(\overline{M}, \Lambda^2) \longrightarrow C_*^2(\overline{M}, \Lambda^2), \tag{3.51}$$

where $C_*^2(\overline{M})$ is a Zygmund space, defined by (8.37)–(8.41) of Chapter 13. Hence, from (3.48), we have

$$\|\tilde{u}(t)\|_{C_*^1} \le C \|\tilde{w}(t)\|_{L^\infty} + C \|\tilde{u}(t)\|_{L^2}. \tag{3.52}$$

Of course, the last term is equal to $C\|\tilde{u}(0)\|_{L^2}$. Thus, under the hypothesis (3.47), we have

$$\|u(t)\|_{C^1_*} \le K' < \infty, \quad t \in I. \tag{3.53}$$

Now the estimate (8.53) of Chapter 13 gives

$$\|u(t)\|_{C^1} \le C\Big[1 + \log^+\big(\|u(t)\|_{H^k}\big)\Big], \tag{3.54}$$

for any $k > n/2 + 1$, parallel to (2.26).

To prove Proposition 3.4, we can exploit (3.43) in the same way we did (2.19), to obtain, via (3.54), the estimate

$$\frac{dy}{dt} \le C\big(1 + \log^+ y\big)y, \qquad y(t) = \|u(t)\|^2_{H^k}. \tag{3.55}$$

A use of Gronwall's inequality exactly as in (2.27)–(2.31) finishes the proof.

As in §2, one consequence of Proposition 2.4 is the classical global existence result when $\dim M = 2$.

Proposition 3.6. *If* $\dim M = 2$ *and* $u_0 \in V^k$, $k > 2$, *then the solution to the Euler equations (3.1)–(3.3) exists for all* $t \in \mathbb{R}$; $u \in C(\mathbb{R}, V^k)$.

Proof. As in (2.36), the vorticity w is a scalar field, satisfying

$$\frac{\partial w}{\partial t} + \nabla_u w = 0.$$

Since u is tangent to ∂M, this again yields

$$\|w(t)\|_{L^\infty} = \|w(0)\|_{L^\infty}.$$

Exercises

1. Show that if $u \in L^2(M, TM)$ and $\operatorname{div} u = 0$, then $\nu \cdot u\big|_{\partial M}$ is well defined in $H^{-1}(\partial M)$. Hence (3.4) is well defined for $k = 0$.
2. Show that the result (3.6)–(3.7) specifying $(I - P)v$ follows from (1.44). (*Hint*: Take $p = -\delta G^A \tilde{v}$.)
3. Show that the result (3.5) that $P : H^k(M, TM) \to V^k$ follows from (1.44). Show that V^k is dense in V^ℓ, for $0 \le \ell < k$.
4. For $s \in [0, \infty)$, define V^s by (3.4) with $s = k$, not necessarily an integer. Equivalently,

$$V^s = V^0 \cap H^s(M, TM).$$

Demonstrate the interpolation property

$$[V^0, V^k]_\theta = V^{k\theta}, \quad 0 < \theta < 1.$$

(*Hint*: Show that $P : H^s(M, TM) \to V^s$, and make use of this fact.)

5. Let u be a 1-form on M. Show that $d^*du = v$, where, in index notation,

$$-v_j = u_{j;k}{}^{;k} - u_{k;j}{}^{;k}.$$

In analogy with (3.15)–(3.16), reorder the derivatives in the last term to deduce that $d^*du = \nabla^*\nabla u - dd^*u + \mathrm{Ric}(u)$, or equivalently,

$$(d^*d + dd^*)u = \nabla^*\nabla u + \mathrm{Ric}(u), \tag{3.56}$$

which is a special case of the Weitzenbock formula. Compare with (4.16) of Chapter 10.

6. Construct a Friedrichs mollifier on $\widetilde{M}$, a compact manifold without boundary, having the property (3.40). (*Hint*: In the model case $\mathbb{R}^n$, consider convolution by $\varepsilon^{-n}\varphi(x/\varepsilon)$, where we require $\int \varphi(x)dx = 1$, and $\varphi \in C_0^\infty(\mathbb{R}^n)$ is supported on $|x - e_1| \le 1/2,\ e_1 = (1, 0, \dots, 0)$.)

4. Navier-Stokes equations

We study here the Navier-Stokes equations for the viscous incompressible flow of a fluid on a compact Riemannian manifold M. The equations take the form

$$\frac{\partial u}{\partial t} + \nabla_u u = \nu\mathcal{L}u - \ \mathrm{grad}\ p, \qquad \mathrm{div}\ u = 0, \quad u(0) = u_0. \tag{4.1}$$

for the velocity field u, where p is the pressure, which is eliminated from (4.1) by applying P, the orthogonal projection of $L^2(M, TM)$ onto the kernel of the divergence operator. In (4.1), ∇ is the covariant derivative. For divergence-free fields u, one has the identity

$$\nabla_u u = \mathrm{div}(u \otimes u), \tag{4.2}$$

the right side being the divergence of a second-order tensor field. This is a special case of the general identity $\mathrm{div}(u \otimes v) = \nabla_v u + (\mathrm{div}\ v)u$, which arose in (2.43). The quantity ν in (4.1) is a positive constant. If $M = \mathbb{R}^n$, $\mathcal{L}$ is the Laplace operator Δ, acting on the separate components of the velocity field u.

Now, if M is not flat, there are at least two candidates for the role of the Laplace operator, the Hodge Laplacian

$$\Delta = -(d^*d + dd^*),$$

or rather its conjugate upon identifying vector fields and 1-forms via the Riemannian metric ("lowering indices"), and the Bochner Laplacian

$$\mathcal{L}_B = -\nabla^*\nabla,$$

where $\nabla : C^\infty(M, TM) \to C^\infty(M, T^* \otimes T)$ arises from the covariant derivative. In order to see what $\mathcal{L}$ is in (4.1), we record another form of (4.1), namely

$$\frac{\partial u}{\partial t} + \nabla_u u = \nu\ \mathrm{div}\ S - \ \mathrm{grad}\ p, \qquad \mathrm{div}\ u = 0, \tag{4.3}$$

where S is the "stress tensor"

$$S = \nabla u + \nabla u^t = 2 \text{ Def } u,$$

also called the "deformation tensor." This tensor was introduced in Chapter 2, §3; cf. (3.35). In index notation, $S^{jk} = u^{j;k} + u^{k;j}$, and the vector field div S is given by

$$S^{jk}{}_{;k} = u^{j;k}{}_{;k} + u^{k;j}{}_{;k}.$$

The first term on the right is $-\nabla^*\nabla u$. The second term can be written (as in (3.16)) as

$$u^k{}_{;k}{}^{;j} + R^k{}_{\ell k}{}^j u^\ell = \big(\text{grad div } u + \text{Ric}(u)\big)^j.$$

Thus, as long as div $u = 0$,

$$\text{div } S = -\nabla^*\nabla u + \text{Ric}(u).$$

By comparison, a special case of the Weitzenbock formula, derivable in a similar fashion (see Exercise 5 in the previous section), is

$$\Delta u = -\nabla^*\nabla u - \text{Ric}(u)$$

when u is a 1-form. In other words, on ker div,

$$\mathcal{L}u = \Delta u + 2 \text{ Ric}(u). \tag{4.4}$$

The Hodge Laplacian Δ has the property of commuting with the projection P onto ker div, as long as M has no boundary. For simplicity of exposition, we will restrict attention throughout the rest of this section to the case of Riemannian manifolds M for which Ric is a constant scalar multiple c_0 of the identity, so

$$\mathcal{L} = \Delta + 2c_0 \quad \text{on ker div}, \tag{4.5}$$

and the right side also commutes with P. Then we can rewrite (4.1) as

$$\frac{\partial u}{\partial t} = \nu\mathcal{L}u - P\nabla_u u, \quad u(0) = u_0, \tag{4.6}$$

where, as above, the vector field u_0 is assumed to have divergence zero. Let us note that, in any case,

$$\mathcal{L} = -2 \,\text{Def}^*\text{Def}$$

is a negative-semidefinite operator.

We will perform an analysis similar to that of §2; in this situation we will obtain estimates independent of ν, and we will be in a position to pass to the limit $\nu \to 0$. We begin with the approximating equation

$$\frac{\partial u_\varepsilon}{\partial t} + PJ_\varepsilon\nabla_{u_\varepsilon}J_\varepsilon u_\varepsilon = \nu J_\varepsilon\mathcal{L}J_\varepsilon u_\varepsilon, \quad u_\varepsilon(0) = u_0, \tag{4.7}$$

parallel to (2.2), using a Friedrichs mollifier J_ε. Arguing as in (2.3)–(2.6), we obtain

$$\frac{d}{dt}\,\|u_\varepsilon(t)\|_{L^2}^2 = -4\nu\|\mathrm{Def}\ J_\varepsilon u_\varepsilon(t)\|_{L^2}^2 \le 0, \tag{4.8}$$

hence

$$\|u_\varepsilon(t)\|_{L^2} \le \|u_0\|_{L^2}. \tag{4.9}$$

Thus it follows that (4.7) is solvable for all $t \in \mathbb{R}$ whenever $\nu \ge 0$ and $\varepsilon > 0$.

We next estimate higher-order derivatives of u_ε, as in §2. For example, if $M = \mathbb{T}^n$, following (2.7)–(2.13), we obtain now

$$\begin{aligned}\frac{d}{dt}\,\|u_\varepsilon(t)\|_{H^k}^2 &\le C\|u_\varepsilon(t)\|_{C^1}\|u_\varepsilon(t)\|_{H^k}^2 - 4\nu\|\mathrm{Def}\ J_\varepsilon u_\varepsilon(t)\|_{H^k}^2 \\ &\le C\|u_\varepsilon(t)\|_{C^1}\|u_\varepsilon(t)\|_{H^k}^2,\end{aligned} \tag{4.10}$$

for $\nu \ge 0$. For more general M, one has similar results parallel to analyses of (2.34) and (2.35). Note that the factor C is independent of ν. As in Theorem 2.1 (see also Theorem 1.2 of Chapter 16), these estimates are sufficient to establish a local existence result, for a limit point of u_ε as $\varepsilon \to 0$, which we denote by u_ν.

Theorem 4.1. *Given $u_0 \in H^k(M)$, $k > n/2 + 1$, with div $u_0 = 0$, there is a solution u_ν on an interval $I = [0, A)$ to (4.6), satisfying*

$$u_\nu \in L^\infty(I, H^k(M)) \cap Lip(I, H^{k-2}(M)). \tag{4.11}$$

The interval I and the estimate of u_ν in $L^\infty(I, H^k(M))$ can be taken independent of $\nu \ge 0$.

We can also establish the uniqueness, and treat the stability and rate of convergence of u_ε to $u = u_\nu$ as before. Thus, with $\varepsilon \in [0, 1]$, we compare a solution $u = u_\nu$ to (4.6) to a solution $u_{\nu\varepsilon} = w$ to

$$\frac{\partial w}{\partial t} + PJ_\varepsilon \nabla_w J_\varepsilon w = \nu J_\varepsilon \mathcal{L} J_\varepsilon w, \quad w(0) = w_0. \tag{4.12}$$

Setting $v = u_\nu - u_{\nu\varepsilon}$, we have again an estimate of the form (2.16), hence:

Proposition 4.2. *Given $k > n/2 + 1$, solutions to (4.6) satisfying (4.11) are unique. They are limits of solutions $u_{\nu\varepsilon}$ to (4.7), and, for $t \in I$,*

$$\|u_\nu(t) - u_{\nu\varepsilon}(t)\|_{L^2} \le K_1(t)\|I - J_\varepsilon\|_{\mathcal{L}(H^{k-1}, L^2)}, \tag{4.13}$$

the quantity on the right being independent of $\nu \in [0, \infty)$.

Continuing to follow §2, we can next look at

$$\begin{aligned}\frac{d}{dt}\,\|D^\alpha J_\varepsilon u_\nu(t)\|_{L^2}^2 = &-2\big(D^\alpha J_\varepsilon L(u_\nu, D)u_\nu, D^\alpha J_\varepsilon u_\nu\big) \\ &-2\nu\|\mathrm{Def}\ D^\alpha J_\varepsilon u_\nu(t)\|_{L^2}^2,\end{aligned} \tag{4.14}$$

parallel to (2.18), and as in (2.19)–(2.20) deduce

$$\begin{aligned}\frac{d}{dt}\,\|u_\nu(t)\|_{H^k}^2 &\le C\|u_\nu(t)\|_{C^1}\|u_\nu(t)\|_{H^k}^2 - 4\nu\|\mathrm{Def}\ u_\nu(t)\|_{H^k}^2 \\ &\le C\|u_\nu(t)\|_{C^1}\|u_\nu(t)\|_{H^k}^2.\end{aligned} \tag{4.15}$$

This time, the argument leading to $u \in C(I, H^k(M))$, in the case of the solution to a hyperbolic equation or the Euler equation (2.1), gives for u_ν solving (4.6) with $u_0 \in H^k(M)$,

$$u_\nu \text{ is continuous in } t \text{ with values in } H^k(M), \quad \text{at } t = 0, \tag{4.16}$$

provided $k > n/2 + 1$. At other points $t \in I$, one has right continuity in t. This argument does not give left continuity since the evolution equation (4.6) is not well posed backward in time. However, a much stronger result holds for positive $t \in I$, as will be seen in (4.17) below.

Having considered results with estimates independent of $\nu \ge 0$, we now look at results for fixed $\nu > 0$ (or which at least require ν to be bounded away from 0). Then (4.6) behaves like a semilinear parabolic equation, and we will establish the following analogue of Proposition 1.3 of Chapter 15. We assume $n \ge 2$.

Proposition 4.3. *If div $u_0 = 0$ and $u_0 \in L^p(M)$, with $p > n =$ dim M, and if $\nu > 0$, then (4.6) has a unique short-time solution on an interval $I = [0, T]$:*

$$u = u_\nu \in C(I, L^p(M)) \cap C^\infty((0, T) \times M). \tag{4.17}$$

Proof. It is useful to rewrite (4.6) as

$$\frac{\partial u}{\partial t} + P\ \mathrm{div}(u \otimes u) = \nu \mathcal{L} u, \quad u(0) = u_0, \tag{4.18}$$

using the identity (4.2). In this form, the parallel with (1.16) of Chapter 15, namely,

$$\frac{\partial u}{\partial t} = \nu \Delta u + \sum \partial_j F_j(u),$$

is evident. The proof is done in the same way as the results on semilinear parabolic equations there. We write (4.18) as an integral equation

$$u(t) = e^{t\nu\mathcal{L}} u_0 - \int_0^t e^{(t-s)\nu\mathcal{L}} P\ \mathrm{div}\big(u(s) \otimes u(s)\big)\, ds = \Psi u(t), \tag{4.19}$$

and look for a fixed point of

$$\Psi : C(I, X) \to C(I, X), \quad X = L^p(M) \cap \ \ker \mathrm{div}\ . \tag{4.20}$$

As in the proof of Propositions 1.1 and 1.3 in Chapter 15, we fix $\alpha > 0$, set

$$Z = \{u \in C([0, T], X) : u(0) = u_0,\ \|u(t) - u_0\|_X \le \alpha\}, \tag{4.21}$$

and show that if $T > 0$ is small enough, then $\Psi : Z \to Z$ is a contraction map. For that, we need a Banach space Y such that

$$\Phi : X \to Y \text{ is Lipschitz, uniformly on bounded sets,} \tag{4.22}$$

$$e^{t\mathcal{L}} : Y \to X, \quad \text{for } t > 0, \tag{4.23}$$

and, for some $\gamma < 1$,

$$\|e^{t\mathcal{L}}\|_{\mathcal{L}(Y,X)} \le Ct^{-\gamma}, \quad \text{for } t \in (0, 1]. \tag{4.24}$$

The map Φ in (4.22) is

$$\Phi(u) = P \operatorname{div}(u \otimes u). \tag{4.25}$$

We set

$$Y = H^{-1,p/2}(M) \cap \ker \operatorname{div},$$

and these conditions are all seen to hold, as long as $p > n$; to check (4.24), use (1.15) of Chapter 15. Thus we have the solution u_ν to (4.6), belonging to $C([0,T], L^p(M))$. To obtain the smoothness stated in (4.17), the proof of smoothness in Proposition 1.3 of Chapter 15 applies essentially verbatim.

Local existence with initial data $u_0 \in L^n(M)$ was established in [Kt4]. We also mention results on local existence when u_0 belongs to certain Morrey spaces, given in [Fed], [Kt5], and [T2].

Note that the length of the interval I on which u_ν is produced in Proposition 4.3 depends only on $\|u_0\|_{L^p}$ (given M and ν). Hence one can get global existence provided one can bound $\|u(t)\|_{L^p(M)}$, for some $p > n$. In view of this we have the following variant of Proposition 2.3 (with a much simpler proof):

Proposition 4.4. *Given $\nu > 0$, $p > n$, if $u \in C([0,T), L^p(M))$ solves (4.6), and if the vorticity w satisfies*

$$\sup_{t\in[0,T)} \|w(t)\|_{L^q} \le K < \infty, \quad q = \frac{np}{n+p}, \tag{4.26}$$

then the solution u continues to an interval $[0, T')$, for some $T' > T$,

$$u \in C([0,T'), L^p(M)) \cap C^\infty((0,T') \times M),$$

solving (4.6).

Proof. As in the proof of Proposition 2.3, we have

$$u = Aw + P_0 u,$$

where P_0 is a projection onto a finite-dimensional space of smooth fields, $A \in OPS^{-1}(M)$. Since we know that $\|u(t)\|_{L^2} \le \|u_0\|_{L^2}$ and since $A : L^q \to H^{1,q} \subset L^p$, we have an L^p-bound on $u(t)$ as $t \nearrow T$, as needed to prove the proposition.

Note that we require on q precisely that $q > n/2$, in order for the corresponding p to exceed n.

Note also that when $\dim M = 2$, the vorticity w is scalar and satisfies the PDE

$$\frac{\partial w}{\partial t} + \nabla_u w = \nu(\Delta + 2c_0)w; \tag{4.27}$$

as long as (4.5) holds, generalizing the $\nu = 0$ case, we have $\|w(t)\|_{L^\infty} \le e^{2\nu c_0 t}\|w(0)\|_{L^\infty}$ (this time by the maximum principle), and consequently global existence.

When $\dim M = 3$, w is a vector field and (as long as (4.5) holds) the vorticity equation is

$$\frac{\partial w}{\partial t} + \nabla_u w - \nabla_w u = \nu\mathcal{L}w. \tag{4.28}$$

It remains an open problem whether (4.1) has global solutions in the space $C^\infty((0,\infty)\times M)$ when $\dim M \ge 3$, despite the fact that one thinks this should be easier for $\nu > 0$ than in the case of the Euler equation. We describe here a couple of results that are known in the case $\nu > 0$.

Proposition 4.5. *Let $k > n/2 + 1$, $\nu > 0$. If $\|u_0\|_{H^k}$ is small enough, then (4.6) has a global solution in $C([0,\infty), H^k) \cap C^\infty((0,\infty)\times M)$.*

What "small enough" means will arise in the course of the proof, which will be a consequence of the first part of the estimate (4.15). To proceed from this, we can pick positive constants A and B such that

$$\|\mathrm{Def}\ u\|_{H^k}^2 \ge A\|u\|_{H^k}^2 - B\|u\|_{L^2}^2,$$

so (4.15) yields

$$\frac{d}{dt}\,\|u(t)\|_{H^k}^2 \le \bigl\{C\|u(t)\|_{C^1} - 2\nu A\bigr\}\|u\|_{H^k}^2 + 2\nu B\|u(t)\|_{L^2}^2.$$

Now suppose

$$\|u_0\|_{L^2}^2 \le \delta \quad \text{and} \quad \|u_0\|_{H^k}^2 \le L\delta;$$

L will be specified below. We require $L\delta$ to be so small that

$$\|v\|_{H^k}^2 \le 2L\delta \Longrightarrow \|v\|_{C^1} \le \frac{\nu A}{C}. \tag{4.29}$$

Recall that $\|u(t)\|_{L^2} \le \|u_0\|_{L^2}$. Consequently, as long as $\|u(t)\|_{H^k}^2 \le 2L\delta$, we have

$$\frac{dy}{dt} \le -\nu Ay + 2\nu B\delta, \quad y(t) = \|u(t)\|_{H^k}^2.$$

Such a differential inequality implies

$$y(t) \le \max\bigl\{y(t_0), 2BA^{-1}\delta\bigr\}, \quad \text{for } t \ge t_0. \tag{4.30}$$

Consequently, if we take $L = 2B/A$ and pick δ so small that (4.29) holds, we have a global bound $\|u(t)\|^2_{H^k} \le L\delta$, and corresponding global existence.

A substantially sharper result of this nature is given in Exercises 4–9 at the end of this section.

We next prove the famous Hopf theorem, on the existence of global *weak* solutions to (4.6), given $\nu > 0$, for initial data $u_0 \in L^2(M)$. The proof is parallel to that of Proposition 1.7 in Chapter 15. In order to make the arguments given here resemble those for viscous flow on Euclidean space most closely, we will assume throughout the rest of this section that (4.5) holds with $c_0 = 0$ (i.e., that $\mathrm{Ric} = 0$).

Theorem 4.6. *Given $u_0 \in L^2(M)$, div $u_0 = 0$, $\nu > 0$, the equation (4.6) has a weak solution for $t \in (0,\infty)$,*

$$\begin{aligned} u \in L^\infty\big(\mathbb{R}^+, L^2(M)\big) \cap L^2_{\mathrm{loc}}\big(\mathbb{R}^+, H^1(M)\big) \\ \cap\ \mathit{Lip}_{\mathrm{loc}}\big(\mathbb{R}^+, H^{-2}(M) + H^{-1,1}(M)\big). \end{aligned} \tag{4.31}$$

We will produce u as a limit point of solutions u_ε to a slight modification of (4.7), namely we require each J_ε to be a projection; for example, take $J_\varepsilon = \chi(\varepsilon\Delta)$, where $\chi(\lambda)$ is the characteristic function of $[-1,1]$. Then J_ε commutes with Δ and with P. We also require $u_\varepsilon(0) = J_\varepsilon u_0$; then $u_\varepsilon(t) = J_\varepsilon u_\varepsilon(t)$. Now from (4.9), which holds here also, we have

$$\{u_\varepsilon : \varepsilon \in (0,1]\} \text{ is bounded in } L^\infty(\mathbb{R}^+, L^2). \tag{4.32}$$

This follows from (4.8), further use of which yields

$$4\nu \int_0^T \|\mathrm{Def}\ u_\varepsilon(t)\|^2_{L^2}\, dt = \|J_\varepsilon u_0\|^2_{L^2} - \|u_\varepsilon(T)\|^2_{L^2}, \tag{4.33}$$

as in (1.39) of Chapter 15. Hence, for each bounded interval $I = [0,T]$,

$$\{u_\varepsilon\} \text{ is bounded in } L^2(I, H^1(M)). \tag{4.34}$$

Now, as in (4.18), we write our PDE for u_ε as

$$\frac{\partial u_\varepsilon}{\partial t} + PJ_\varepsilon \,\mathrm{div}(u_\varepsilon \otimes u_\varepsilon) = \nu\Delta u_\varepsilon, \tag{4.35}$$

since $J_\varepsilon \Delta J_\varepsilon u_\varepsilon = \Delta u_\varepsilon$. From (4.32) we see that

$$\{u_\varepsilon \otimes u_\varepsilon : \varepsilon \in (0,1]\} \text{ is bounded in } L^\infty(\mathbb{R}^+, L^1(M)). \tag{4.36}$$

We use the inclusion $L^1(M) \subset H^{-n/2-\delta}(M)$. Hence, by (4.35), for each $\delta > 0$,

$$\{\partial_t u_\varepsilon\} \text{ is bounded in } L^2(I, H^{-n/2-1-\delta}(M)), \tag{4.37}$$

so

$$\{u_\varepsilon\} \text{ is bounded in } H^1(I, H^{-n/2-1-\delta}(M)). \tag{4.38}$$

As in the proof of Proposition 1.7 in Chapter 15, we now interpolate between (4.34) and (4.38), to obtain

$$\{u_\varepsilon\} \text{ is bounded in } H^s(I, H^{1-s-s(n/2+1+\delta)}(M)), \tag{4.39}$$

and hence, as in (1.45) there,

$$\{u_\varepsilon\} \ \text{ is compact in } \ L^2(I, H^{1-\gamma}(M)), \tag{4.40}$$

for all $\gamma > 0$.

Now the rest of the argument is easy. We can pick a sequence $u_k = u_{\varepsilon_k}$ ($\varepsilon_k \to 0$) such that

$$u_k \to u \quad \text{in } L^2([0,T], H^{1-\gamma}(M)), \text{ in norm}, \tag{4.41}$$

arranging that this hold for all $T < \infty$, and from this it is easy to deduce that u is a desired weak solution to (4.6).

Solutions of (4.6) obtained as limits of u_ε as in the proof of Theorem 4.6 are called Leray-Hopf solutions to the Navier-Stokes equations. The uniqueness and smothness of a Leray-Hopf solution so constructed remain open problems if dim $M \geq 3$. We next show that when dim $M = 3$, such a solution is smooth except for at most a fairly small exceptional set.

Proposition 4.7. *If dim $M = 3$ and u is a Leray-Hopf solution of (4.6), then there is an open dense subset $\mathcal{J}$ of $(0, \infty)$ such that $\mathbb{R}^+ \setminus \mathcal{J}$ has Lebesgue measure zero and*

$$u \in C^\infty(\mathcal{J} \times M). \tag{4.42}$$

Proof. For $T > 0$ arbitrary, $I = [0, T]$, use (4.40). With $u_k = u_{\varepsilon_k}$, passing to a subsequence, we can suppose

$$\|u_{k+1} - u_k\|_E \leq 2^{-k}, \quad E = L^2(I, H^{1-\gamma}(M)). \tag{4.43}$$

Now if we set

$$\Gamma(t) = \sup_k \ \|u_k(t)\|_{H^{1-\gamma}}, \tag{4.44}$$

we have

$$\Gamma(t) \leq \|u_1(t)\|_{H^{1-\gamma}} + \sum_{k=1}^{\infty} \|u_{k+1}(t) - u_k(t)\|_{H^{1-\gamma}}, \tag{4.45}$$

hence

$$\Gamma \in L^2(I). \tag{4.46}$$

In particular, $\Gamma(t)$ is finite *almost everywhere*. Let

$$S = \{t \in I : \Gamma(t) < \infty\}. \tag{4.47}$$

For small $\gamma > 0$, $H^{1-\gamma}(M) \subset L^p(M)$ with p close to 6 when dim $M = 3$, and products of two elements in $H^{1-\gamma}(M)$ belong to $H^{1/2-\gamma'}(M)$, with

$\gamma' > 0$ small. Recalling that u_ε satisfies (4.35), we now apply the analysis used in the proof of Proposition 4.3 to u_k, concluding that, for each $t_0 \in S$, there exists $T(t_0) > 0$, depending only on $\Gamma(t_0)$, such that, for small $\gamma' > 0$, we have

$$\{u_k\} \text{ bounded in } C\big([t_0, t_0 + T(t_0)], H^{1-\gamma}(M)\big) \cap C^\infty\big((t_0, t_0 + T(t_0)) \times M\big).$$

Consequently, if we form the open set

$$\mathcal{J}_T = \bigcup_{t_0 \in S} \big(t_0, t_0 + T(t_0)\big), \tag{4.48}$$

then any weak limit u of $\{u_k\}$ has the property that $u \in C^\infty(\mathcal{J}_T \times M)$. It remains only to show that $I \setminus \mathcal{J}_T$ has Lebesgue measure zero; the denseness of $\mathcal{J}_T$ in I will automatically follow. To see this, fix $\delta_1 > 0$. Since $\text{meas}(I \setminus S) = 0$, there exists $\delta_2 > 0$ such that if $S_{\delta_2} = \{t \in S : T(t) \geq \delta_2\}$, then $\text{meas}(I \setminus S_{\delta_2}) < \delta_1$. But $\mathcal{J}_T$ contains the translate of S_{δ_2} by $\delta_2/2$, so $\text{meas}(I \setminus \mathcal{J}_T) \leq \delta_1 + \delta_2/2$. This completes the proof.

There are more precise results than this. As shown in [CKN], when $M = \mathbb{R}^3$, the subset of $\mathbb{R}^+ \times M$ on which a certain type of Leray-Hopf solution, called "admissible," is not smooth, must have vanishing one-dimensional Hausdorff measure. In [CKN] it is shown that admissible Leray-Hopf solutions exist.

We now discuss some results regarding the uniqueness of weak solutions to the Navier-Stokes equations (4.6). Thus, let $I = [0, T]$, and suppose

$$u_j \in L^\infty\big(I, L^2(M)\big) \cap L^2\big(I, H^1(M)\big), \quad j = 1, 2, \tag{4.49}$$

are two weak solutions to

$$\frac{\partial u_j}{\partial t} + P \operatorname{div}(u_j \otimes u_j) = \nu \Delta u_j, \qquad u_j(0) = u_0, \tag{4.50}$$

where $u_0 \in L^2(M)$, $\operatorname{div} u_0 = 0$. Then $v = u_1 - u_2$ satisfies

$$\frac{\partial v}{\partial t} + P \operatorname{div}(u_1 \otimes v + v \otimes u_2) = \nu \Delta v, \quad v(0) = 0. \tag{4.51}$$

We will estimate the rate of change of $\|v(t)\|^2_{L^2}$, using the following:

Lemma 4.8. *Provided*

$$v \in L^2(I, H^1(M)) \quad \text{and} \quad \frac{\partial v}{\partial t} \in L^2(I, H^{-1}(M)), \tag{4.52}$$

then $\|v(t)\|^2_{L^2}$ is absolutely continuous and

$$\frac{d}{dt} \|v(t)\|^2_{L^2} = 2(v_t, v)_{L^2} \in L^1.$$

Furthermore, $v \in C(I, L^2)$.

Proof. The identity is clear for smooth v, and the rest follows by approximation.

By hypothesis (4.49), the functions u_j satisfy the first part of (4.52). By (4.50), the second part of (4.52) is satisfied provided $u_j \otimes u_j \in L^2(I \times M)$, that is, provided

$$u_j \in L^4(I \times M). \tag{4.53}$$

We now proceed to investigate the L^2-norm of v, solving (4.51). If u_j satisfy both (4.49) and (4.53), we have

$$\begin{aligned} \frac{d}{dt}\,\|v(t)\|_{L^2}^2 &= -2(\nabla_v u_1, v) - 2(\nabla_{u_2} v, v) - 2\nu\|\nabla v\|_{L^2}^2 \\ &= 2(u_1, \nabla_v v) - 2\nu\|\nabla v\|_{L^2}^2, \end{aligned} \tag{4.54}$$

since $\nabla_v^* = -\nabla_v$ and $\nabla_{u_2}^* = -\nabla_{u_2}$ for these two divergence-free vector fields. Consequently, we have

$$\frac{d}{dt}\,\|v(t)\|_{L^2}^2 \le 2\|u_1\|_{L^4} \cdot \|v\|_{L^4} \cdot \|\nabla v\|_{L^2} - 2\nu\|\nabla v\|_{L^2}^2. \tag{4.55}$$

Our goal is to get a differential inequality implying $\|v(t)\|_{L^2} = 0$; this requires estimating $\|v(t)\|_{L^4}$ in terms of $\|v(t)\|_{L^2}$ and $\|\nabla v\|_{L^2}$. Since $H^{1/2}(M^2) \subset L^4(M^2)$ and $H^1(M^3) \subset L^6(M^3)$, we can use the following estimates when dim $M = 2$ or 3:

$$\begin{aligned} \|v\|_{L^4} &\le C\|v\|_{L^2}^{1/2} \cdot \|\nabla v\|_{L^2}^{1/2} + C\|v\|_{L^2}, \quad \dim M = 2, \\ \|v\|_{L^4} &\le C\|v\|_{L^2}^{1/4} \cdot \|\nabla v\|_{L^2}^{3/4} + C\|v\|_{L^2}, \quad \dim M = 3. \end{aligned} \tag{4.56}$$

With these estimates, we are prepared to prove the following uniqueness result:

Proposition 4.9. *Let u_1 and u_2 be weak solutions to (4.6), satisfying (4.49) and (4.53). Suppose dim $M = 2$ or 3; if dim $M = 3$, suppose furthermore that*

$$u_1 \in L^8\big(I, L^4(M)\big). \tag{4.57}$$

If $u_1(0) = u_2(0)$, then $u_1 = u_2$ on $I \times M$.

Proof. For $v = u_1 - u_2$, we have the estimate (4.55). Using (4.56), we have

$$\begin{aligned} 2\|u_1\|_{L^4}\|v\|_{L^4}\|\nabla v\|_{L^2} \le \nu\|\nabla v\|_{L^2}^2 &+ C\nu^{-3}\|v\|_{L^2}^2 \cdot \|u_1\|_{L^4}^4 \\ &+ C\nu^{-1}\|v\|_{L^2}^2 \cdot \|u_1\|_{L^4}^2 \end{aligned} \tag{4.58}$$

when dim $M = 2$, and

$$\begin{aligned} 2\|u_1\|_{L^4}\|v\|_{L^4}\|\nabla v\|_{L^2} \le \nu\|\nabla v\|_{L^2}^2 &+ C\nu^{-7}\|v\|_{L^2}^2 \cdot \|u_1\|_{L^4}^8 \\ &+ C\nu^{-1}\|v\|_{L^2}^2 \cdot \|u_1\|_{L^4}^2 \end{aligned} \tag{4.59}$$

when dim $M = 3$. Consequently,

$$\frac{d}{dt}\,\|v(t)\|_{L^2}^2 \le C_n(\nu)\|v(t)\|_{L^2}^2\Big(\|u_1\|_{L^4}^p + \|u_1\|_{L^4}^2\Big), \tag{4.60}$$

where $p = 4$ if dim $M = 2$ and $p = 8$ if dim $M = 3$. Then Gronwall's inequality gives

$$\|v(t)\|_{L^2}^2 \le \|u_1(0) - u_2(0)\|_{L^2}^2 \,\exp\Big\{C_n(\nu)\int_0^t \Big(\|u_1(s)\|_{L^4}^p + \|u_1(s)\|_{L^4}^2\Big)\,ds\Big\},$$

proving the proposition.

We compare the properties of the last proposition with properties that Leray-Hopf solutions can be shown to have:

Proposition 4.10. *If u is a Leray-Hopf solution to (4.1) and $I = [0,T]$, then*

$$u \in L^4(I\times M) \quad \text{if dim } M = 2, \tag{4.61}$$

and

$$u \in L^{8/3}\big(I, L^4(M)\big) \quad \text{if dim } M = 3. \tag{4.62}$$

Also,

$$u \in L^2\big(I, L^4(M)\big) \quad \text{if dim } M = 4. \tag{4.63}$$

Proof. Since $u \in L^\infty(I,L^2)\cap L^2(I,H^1)$, (4.61) follows from the first part of (4.56), and (4.62) follows from the second part. Similarly, (4.63) follows from the inclusion

$$H^1(M^4) \subset L^4(M^4).$$

In particular, the hypotheses of Proposition 4.9 are seen to hold for Leray-Hopf solutions when dim $M = 2$, so there is a uniqueness result in that case. On the other hand, there is a gap between the conclusion (4.62) and the hypothesis (4.57) when dim $M = 3$.

Exercises

In the exercises below, assume for simplicity that $\mathrm{Ric} = 0$, so (4.5) holds with $c_0 = 0$.

1. One place dissipated energy can go is into heat. Suppose a "temperature" function $T = T(t,x)$ satisfies a PDE

$$\frac{\partial T}{\partial t} + \nabla_u T = \alpha\Delta T + 4\nu|\mathrm{Def}\ u|^2, \tag{4.64}$$

coupled to (4.6), where α is a positive constant. Show that the total energy

$$E(t) = \int_M \Big\{ |u(t,x)|^2 + T(t,x) \Big\} \, dx$$

is conserved, provided u and T possess sufficient smoothness. Discuss local existence of solutions to the coupled equations (4.1) and (4.64).

2. Show that under the hypotheses of Theorem 4.1,

$$u_\nu \to v, \quad \text{as } \nu \to 0,$$

v being the solution to the Euler equation (i.e., the solution to the $\nu = 0$ case of (4.6)). In what topology can you demonstrate this convergence?

3. Give the details of the interpolation argument yielding (4.39).
4. Combining Propositions 4.3 and 4.5, show that if div $u_0 = 0$, $p > n$, and $\|u_0\|_{L^p}$ is small enough, then (4.6) has a global solution

$$u \in C\Big([0,\infty), L^p\Big) \cap C^\infty\Big((0,\infty) \times M\Big).$$

In Exercises 5–10, suppose dim $M = 3$. Let u solve (4.6), with vorticity w.

5. Show that the vorticity satisfies

$$\frac{d}{dt} \, \|w(t)\|_{L^2}^2 = 2(\nabla_w u, w) - 2\nu \|\nabla w\|_{L^2}^2. \tag{4.65}$$

6. Using $(\nabla_w u, w) = -(u, \nabla_w w) - \Big(u, (\text{div } w) w\Big)$, deduce that

$$\frac{d}{dt} \, \|w(t)\|_{L^2}^2 \le C \|u\|_{L^3} \cdot \|w\|_{L^6} \cdot \|\nabla w\|_{L^2} - 2\nu \|\nabla w\|_{L^2}^2.$$

Show that

$$\|w\|_{L^6} \le C \|\nabla w\|_{L^2} + C \|u\|_{L^2}, \tag{4.66}$$

and hence

$$\frac{d}{dt} \, \|w(t)\|_{L^2}^2 \le C \|u\|_{L^3} \Big(\|\nabla w\|_{L^2}^2 + \|u\|_{L^2}^2 \Big) - 2\nu \|\nabla w\|_{L^2}^2.$$

7. Show that

$$\|u\|_{L^3} \le C \|u\|_{L^2}^{1/2} \cdot \|w\|_{L^2}^{1/2} + C \|u\|_{L^2},$$

and hence, if $\|u_0\|_{L^2} = \beta$,

$$\frac{d}{dt} \, \|w(t)\|_{L^2}^2 \le C \Big(\beta^{1/2} \|w\|_{L^2}^{1/2} + \beta \Big) \, \Big(\|\nabla w\|_{L^2}^2 + \beta^2 \Big) - 2\nu \|\nabla w\|_{L^2}^2.$$

8. Show that there exist constants $A, B \in (0,\infty)$, depending on M, such that

$$\|\nabla w\|_{L^2}^2 \ge A \|w\|_{L^2}^2 - B\beta^2, \tag{4.67}$$

and hence that $y(t) = \|w(t)\|_{L^2}^2$ satisfies

$$\frac{dy}{dt} \le C\Big(\beta^{1/2} y^{1/4} + \beta\Big) \beta^2 - \nu A y + \nu B \beta^2$$

as long as

$$C\Big(\beta^{1/2} y^{1/4} + \beta\Big) < \nu. \tag{4.68}$$

9. As long as (4.68) holds, $dy/dt \le \nu\beta^2(1+B) - \nu Ay$. As in (4.30), this gives

$$y(t) \le \max\Big\{y(t_0), \beta^2(1+B)A^{-1}\Big\},, \quad \text{for } t \ge t_0.$$

Thus (4.68) persists as long as $C\Big(\beta(1+B)^{1/4}A^{-1/4} + \beta\Big) < \nu$. Deduce a global existence result for the Navier-Stokes equations (4.1) when dim $M = 3$ and

$$\tag{4.69} \begin{gathered} C\Big(\|u_0\|_{L^2}^{1/2}\|w(0)\|_{L^2}^{1/2} + \|u_0\|_{L^2}\Big) < \nu, \\ C\|u_0\|_{L^2}\Big(1 + (1+B)^{1/4}A^{-1/4}\Big) < \nu. \end{gathered}$$

For other global existence results, see [Bon] and [Che1].

10. Deduce from (4.65) that

$$\frac{d}{dt}\,\|w(t)\|_{L^2}^2 \le C\Big(\|w\|_{L^3}^3 + \|w\|_{L^3}^2\|u\|_{L^2}\Big) - 2\nu\|\nabla w\|_{L^2}^2.$$

Work on this, applying

$$\|w\|_{L^3} \le C\|w\|_{L^2}^{1/2}\cdot\|w\|_{L^6}^{1/2},$$

in concert with (4.66).

11. Generalize results of this section to the case where no extra hypotheses are made on Ric. Consider also cases where *some* assumptions are made (e.g. Ric ≥ 0, or Ric ≤ 0). (*Hint*: Instead of (4.6) or (4.18), we have

$$\frac{\partial u}{\partial t} = \nu\Delta u - P\ \mathrm{div}(u\otimes u) + PBu, \quad Bu = 2\nu\ \mathrm{Ric}(u).)$$

12. Assume u is a Killing field on M, that is, u generates a group of isometries of M. According to Exercise 11 of §1, u provides a steady solution to the Euler equation (1.11). Show that u also provides a steady solution to the Navier-Stokes equation (4.1), provided $\mathcal{L}$ is given by (4.4). If $M = S^2$ or S^3, with its standard metric, show that such u (if not zero) does not give a steady solution to (4.1) if $\mathcal{L}$ is taken to be either the Hodge Laplacian Δ or the Bochner Laplacian $\nabla^*\nabla$. Physically, would you expect such a vector field u to give rise to a viscous force?

13. Show that a t-dependent vector field $u(t)$ on $[0,T)\times M$ satisfying

$$u \in L^1\Big([0,T), \mathrm{Lip}^1(M)\Big)$$

generates a well-defined flow consisting of homeomorphisms.

14. Let u be a solution to (4.1) with $u_0 \in L^p(M),\ p > n$, as in Proposition 4.3. Show that, given $s \in (0,2]$,

$$\|u(t)\|_{H^{s,p}} \le Ct^{-s/2}, \quad 0 < t < T.$$

Taking $s \in (1+n/p, 2)$, deduce from Exercise 13 that u generates a well-defined flow consisting of homeomorphisms.
For further results on flows generated by solutions to the Navier-Stokes equations, see [ChL] and [FGT].

5. Viscous flows on bounded regions

In this section we let $\overline{\Omega}$ be a compact manifold with boundary and consider the Navier-Stokes equations on $\mathbb{R}^+ \times \Omega$,

$$\frac{\partial u}{\partial t} + \nabla_u u = \nu \mathcal{L} u - \text{ grad } p, \quad \text{div } u = 0. \tag{5.1}$$

We will assume for simplicity that Ω is flat, or more generally, $\text{Ric} = 0$ on Ω, so, by (4.4), $\mathcal{L} = \Delta$. When $\partial\Omega \neq \emptyset$, we impose the "no-slip" boundary condition

$$u = 0, \quad \text{for } x \in \partial\Omega. \tag{5.2}$$

We also set an initial condition

$$u(0) = u_0. \tag{5.3}$$

We consider the following spaces of vector fields on Ω, which should be compared to the spaces V_σ of (1.6) and V^k of (3.4). First, set

$$\mathcal{V} = \{u \in C_0^\infty(\Omega, T\Omega) : \text{div } u = 0\}. \tag{5.4}$$

Then set

$$W^k = \text{ closure of } \mathcal{V} \text{ in } H^k(\Omega, T), \quad k = 0, 1. \tag{5.5}$$

Lemma 5.1. *We have $W^0 = V^0$ and*

$$W^1 = \{u \in H_0^1(\Omega, T) : \textit{div } u = 0\}. \tag{5.6}$$

Proof. Clearly, $W^0 \subset V^0$. As noted in §1, it follows from (9.79)–(9.80) of Chapter 5 that

$$(V^0)^\perp = \{\nabla p : p \in H^1(\Omega)\}, \tag{5.7}$$

the orthogonal complement taken in $L^2(\Omega, T)$. To show that $\mathcal{V}$ is dense in V^0, suppose $u \in L^2(\Omega, T)$ and $(u, v) = 0$ for all $v \in \mathcal{V}$. We need to conclude that $u = \nabla p$ for some $p \in H^1(\Omega)$. To accomplish this, let us make note of the following simple facts. First,

(5.8)
$$\nabla : H^1(\Omega) \to L^2(\Omega, T) \text{ has closed range } \mathcal{R}_0; \quad \mathcal{R}_0^\perp = \text{ ker } \nabla^* = V^0.$$

The last identity follows from (5.7). Second, and more directly useful,

$$\begin{aligned} &\nabla : L^2(\Omega) \to H^{-1}(\Omega, T) \text{ has closed range } \mathcal{R}_1, \\ &\mathcal{R}_1^\perp = \text{ ker } \nabla^* = \{u \in H_0^1(\Omega, T) : \text{div } u = 0\} = W_\Omega^{(1)}, \end{aligned} \tag{5.9}$$

the last identity defining $W_\Omega^{(1)}$.

Now write Ω as an increasing union $\Omega_1 \subset\subset \Omega_2 \subset\subset \cdots \nearrow \Omega$, each Ω_j having smooth boundary. We claim $u_j = u|_{\Omega_j}$ is orthogonal to $W_{\Omega_j}^{(1)}$, defined as in (5.9). Indeed, if $v \in W^{(1)}_{\Omega_j}$ (and you extend v to be 0 on $\Omega \setminus \Omega_j$), then $\rho(\varepsilon\sqrt{-\Delta})v = v_\varepsilon$ belongs to $\mathcal{V}$ if $\hat{\rho} \in C_0^\infty(\mathbb{R})$ and ε is small, and $v_\varepsilon \to v$ in H^1-norm if $\rho(0) = 1$, so $(u, v) = \lim(u, v_\varepsilon) = 0$. From (5.9) it follows that there exist $p_j \in L^2(\Omega_j)$ such that $u = \nabla p_j$ on Ω_j; p_j is uniquely determined up to an additive constant (if Ω_j is connected) so we can make all the p_j fit together, giving $u = \nabla p$. If $u \in L^2(\Omega, T)$, p must belong to $H^1(\Omega)$.

The same argument works if $u \in H^{-1}(\Omega, T)$ is orthogonal to $\mathcal{V}$; we obtain $u = \nabla p$ with $p \in L^2(\Omega)$; one final application of (5.9) then yields (5.6), finishing off the lemma.

Thus, if $u_0 \in W^1$, we can rephrase (5.1), demanding that

$$\frac{d}{dt}\,(u, v)_{W^0} + (\nabla_u u, v)_{W^0} = -\nu(u, v)_{W^1}, \quad \text{for all } v \in \mathcal{V}. \tag{5.10}$$

Alternatively, we can rewrite the PDE as

$$\frac{\partial u}{\partial t} + P\ \nabla_u u = -\nu A u. \tag{5.11}$$

Here, P is the orthogonal projection of $L^2(\Omega, T)$ onto $W^0 = V^0$, namely, the same P as in (1.10) and (3.1), hence described by (3.5)–(3.6). The operator A is an unbounded, positive, self-adjoint operator on W^0, defined via the Friedrichs extension method, as follows. We have $A_0 : W^1 \to (W^1)^*$ given by

$$\langle A_0 u, v\rangle = (u, v)_{W^1} = (du, dv)_{L^2}, \tag{5.12}$$

the last identity holding because div $u =$ div $v = 0$. Then set

$$\mathcal{D}(A) = \{u \in W^1 : A_0 u \in W^0\}, \quad A = A_0\big|_{\mathcal{D}(A)}, \tag{5.13}$$

using $W^1 \subset W^0 \subset (W^1)^*$. Automatically, $\mathcal{D}(A^{1/2}) = W^1$. The operator A is called the Stokes operator. The following result is fundamental to the analysis of (5.1)–(5.2):

Proposition 5.2. *$\mathcal{D}(A) \subset H^2(\Omega, T)$. In fact, $\mathcal{D}(A) = H^2(\Omega, T) \cap W^1$.*

In fact, if $u \in \mathcal{D}(A)$ and $Au = f \in W^0$, then $(f, v)_{L^2} = (-\Delta u, v)_{L^2}$, for all $v \in \mathcal{V}$. We know $\Delta u \in H^{-1}$, so from Lemma 5.1 and (5.9) we conclude that there exists $p \in L^2(\Omega)$ such that

$$-\Delta u = f + \nabla p. \tag{5.14}$$

Also we know that div $u = 0$ and $u \in H_0^1(\Omega, T)$. We want to conclude that $u \in H^2$ and $p \in H^1$. Let us identify vector fields and 1-forms, so

$$-\Delta u = f + dp, \quad \delta u = 0, \quad u\big|_{\partial\Omega} = 0. \tag{5.15}$$

In order not to interrupt the flow of the analysis of (5.1)–(5.2), we will show in Appendix A at the end of this chapter that solutions to (5.15) possess appropriate regularity.

We will define

$$W^s = \mathcal{D}(A^{s/2}), \quad s \geq 0. \tag{5.16}$$

Note that this is consistent with (5.5), for $s = k = 0$ or 1.

We now construct a local solution to the initial-value problem for the Navier-Stokes equation, by converting (5.11) into an integral equation:

$$u(t) = e^{-t\nu A}\, u_0 - \int_0^t e^{(s-t)\nu A}\, P\, \operatorname{div}\bigl(u(s) \otimes u(s)\bigr)\, ds = \Psi u(t). \tag{5.17}$$

We want to find a fixed point of Ψ on $C(I, X)$, for $I = [0, T]$, with some $T > 0$, and X an appropriate Banach space. We take X to be of the form

$$X = W^s = \mathcal{D}(A^{s/2}), \tag{5.18}$$

for a value of s to be specified below. As in the construction in §4, we need a Banach space Y such that

$$\Phi : X \to Y \text{ is Lipschitz, uniformly on bounded sets,} \tag{5.19}$$

where

$$\Phi(u) = P\, \operatorname{div}(u \otimes u), \tag{5.20}$$

and such that, for some $\gamma < 1$,

$$\|e^{-tA}\|_{\mathcal{L}(Y,X)} \leq C t^{-\gamma}, \tag{5.21}$$

for $t \in (0, 1]$. We take

$$Y = W^0. \tag{5.22}$$

As $\|e^{-tA}\|_{\mathcal{L}(W^0,W^s)} \sim C t^{-s/2}$ for $t \leq 1$, the condition (5.21) requires $s \in (0, 2)$, in (5.18). We need to verify (5.19). Note that, by Proposition 5.2 and interpolation,

$$W^s \subset H^s(\Omega, T), \quad \text{for } 0 \leq s \leq 2. \tag{5.23}$$

Thus (5.19) will hold provided

$$M : H^s(\Omega, T) \to H^1(\Omega, T \otimes T), \quad \text{with } M(u) = u \otimes u. \tag{5.24}$$

Lemma 5.3. *Provided $\dim \Omega \leq 5$, there exists $s_0 < 2$ such that (5.24) holds for all $s > s_0$.*

Proof. If $\dim M = n$, one has

$$H^{n/2+\varepsilon} \cdot H^{n/2+\varepsilon} \subset H^{n/2+\varepsilon} \quad \text{and } H^{n/4} \cdot H^{n/4} \subset H^0 = L^2,$$

the latter because $H^{n/4} \subset L^4$. Other inclusions

$$H^r \cdot H^r \subset H^\sigma, \quad r = \frac{n}{4} + \frac{\sigma}{2} + \varepsilon\theta, \quad \sigma = \theta\Big(\frac{1}{2}n + \varepsilon\Big), \tag{5.25}$$

follow by a straightforward interpolation. One sees that (5.24) holds for $s > s_0$ with

$$s_0 = \frac{n}{4} + \frac{1}{2} \quad (\text{ if } n \geq 2). \tag{5.26}$$

For $2 \leq n \leq 5$, s_0 increases from 1 to 7/4; for $n = 6$, $s_0 = 2$.

Thus we have an existence result:

Proposition 5.4. *Suppose dim $\Omega \leq 5$. If s_0 is given by (5.26) and $u_0 \in W^s$ for some $s \in (s_0, 2)$, then there exists $T > 0$ such that (5.17) has a unique solution*

$$u \in C\big([0,T], W^s\big). \tag{5.27}$$

We can extend the last result a bit once the following is established:

Proposition 5.5. *Set $V^s = V^0 \cap H^s(\Omega, T)$, for $0 \leq s \leq 1$. We have*

$$W^s = V^s, \quad \textit{for } 0 \leq s < \frac{1}{2}, \tag{5.28}$$

and hence

$$P : H^s(\Omega, T) \longrightarrow W^s, \tag{5.29}$$

for such s.

Proof. To deduce (5.29) from (5.28), note that, by (3.5), $P : H^s(\Omega, T) \to H^s(\Omega, T)$ for $s = 0, 1$, hence, by interpolation, for all $s \in [0, 1]$, so $P : H^s(\Omega, T) \to V^s$, for $s \in [0, 1]$.

To establish (5.28), recall that $W^1 = \mathcal{D}(A^{1/2}) = V^0 \cap H^1_0(\Omega, T)$. We hence have

$$W^s = [V^0, V^0 \cap H^1_0(\Omega, T)]_s, \quad \text{for } 0 \leq s \leq 1.$$

Thus (5.28) will follow from the identity

$$[V^0, V^0 \cap H^1_0(\Omega, T)]_s = V^0 \cap [L^2(\Omega, T), H^1_0(\Omega, T)]_s, \quad 0 \leq s \leq 1, \tag{5.30}$$

since, as seen in (5.37) of Chapter 4,

$$[L^2(\Omega), H^1_0(\Omega)]_s = H^s(\Omega), \quad \text{for } 0 \leq s < \frac{1}{2}. \tag{5.31}$$

Following [FM], we make use of the following result to establish (5.30):

Lemma 5.6. *There is a continuous projection Q from $L^2(\Omega, T)$ onto V^0 such that Q maps $H^2(\Omega,T)\cap H^1_0(\Omega,T) = \mathcal{D}(\Delta)$ to $H^2(\Omega,T)\cap W^1 = \mathcal{D}(A)$.*

Here Δ is the Laplace operator on Ω, with Dirichlet boundary condition. We know that

$$[L^2(\Omega,T), H^2(\Omega,T)\cap H^1_0(\Omega,T)]_{1/2} = \mathcal{D}\big((-\Delta)^{1/2}\big) = H^1_0(\Omega,T), \tag{5.32}$$

so the lemma implies that the projection Q has the property

$$Q : H^1_0(\Omega,T) \longrightarrow W^1 = V^0 \cap H^1_0(\Omega,T), \tag{5.33}$$

and (5.30) is a straightforward consequence of this result.

Proof of lemma. We define the continuous operator $Q_0 : \mathcal{D}(\Delta) \to \mathcal{D}(A)$ by

$$Q_0 u = -A^{-1} P \Delta u, \quad u \in \mathcal{D}(\Delta). \tag{5.34}$$

Since $Q_0 u = u$ for $u \in \mathcal{D}(A) = \mathcal{D}(\Delta)\cap V^0$ and since $\mathcal{D}(A)$ is dense in V^0, it suffices to show that Q_0 can be extended to a bounded operator from $L^2(\Omega,T)$ to V^0. Indeed, by the self-adjointness of A and Δ, we have, for the adjoint, mapping V^0 to $L^2(\Omega,T)$,

$$Q_0^* = -\Delta \iota A^{-1}, \quad \iota : V^0 \hookrightarrow L^2(\Omega,T), \tag{5.35}$$

which is a bounded operator from V^0 to $L^2(\Omega,T)$, since the inclusion ι maps $\mathcal{D}(A)$ into $\mathcal{D}(\Delta)$. This proves the lemma, so Proposition 5.5 is established.

We now return to the integral equation (5.17), replacing $Y = W^0$ in (5.22) by

$$Y = W^\sigma = V^\sigma, \quad \sigma \in \Big[0, \frac{1}{2}\Big). \tag{5.36}$$

We take $X = W^s$, as in (5.18), and this time we need $s - \sigma \in (0,2)$ in order for (5.21) to hold with $\gamma < 1$. Higher regularity for the Stokes operator gives

$$W^s \subset H^s(\Omega,T), \quad \text{for } s \in \mathbb{R}^+, \tag{5.37}$$

extending (5.23). Thus (5.19) will hold provided we extend (5.24) to

$$M : H^s(\Omega,T) \longrightarrow H^{1+\sigma}(\Omega, T\otimes T), \quad M(u) = u \otimes u. \tag{5.38}$$

Let us write

$$\sigma = \frac{1}{2} - \delta, \quad s = 2 + \sigma - \delta = \frac{5}{2} - 2\delta.$$

By the arguments used in Lemma 5.3, we have the following:

Lemma 5.7. *Provided $\dim \Omega \le 6$, if $\delta \in (0, 1/2)$ is small enough, and $\sigma = 1/2 - \delta$, there exists $s_0 \in (\sigma, 2+\sigma)$ such that (5.38) holds for all $s > s_0$.*

Proof. If $n \le 4$, then $H^s(\Omega)$ is an algebra for $s = 5/2 - 2\delta$ if δ is small enough. If $n \ge 5$, we can take $s_0 = (n+3)/4$.

Thus we have the following complement to Proposition 5.4:

Proposition 5.8. *Suppose dim $\Omega \le 6$. If $\delta > 0$ is small enough, $s = 5/2 - 2\delta$, and $u_0 \in W^s$, then there exists $T > 0$ such that (5.17) has a unique solution in $C\big([0,T], W^s\big)$.*

There are results on higher regularity of strong solutions, for $0 < t < T$. We refer to [Tem3] for a discussion of this.

Having treated strong solutions, we next establish the Hopf theorem on the global existence of weak solutions to the Navier-Stokes equations, in the case of domains with boundary.

Theorem 5.9. *Assume dim $\Omega \le 3$. Given $u_0 \in W^0, \nu > 0$, the system (5.1)–(5.3) has a weak solution for $t \in (0,\infty)$,*

$$u \in L^\infty(\mathbb{R}^+, W^0) \cap L^2_{\text{loc}}(\mathbb{R}^+, W^1). \tag{5.39}$$

The proof is basically parallel to that of Theorem 4.6. We sketch the argument. As above, we assume for simplicity that Ω is Ricci flat. We have the Stokes operator A, a self-adjoint operator on W^0, defined by (5.12)–(5.13). As in the proof of Theorem 4.6, we consider the family of projections $J_\varepsilon = \chi(\varepsilon A)$, where $\chi(\lambda)$ is the characteristic function of $[-1,1]$. We approximate the solution u by u_ε, solving

$$\frac{\partial u_\varepsilon}{\partial t} + J_\varepsilon P \operatorname{div}(u_\varepsilon \otimes u_\varepsilon) = -\nu A u_\varepsilon, \quad u_\varepsilon(0) = J_\varepsilon u_0. \tag{5.40}$$

This has a global solution, $u_\varepsilon \in C^\infty\big([0,\infty), \text{Range } J_\varepsilon\big)$. As in (4.32), $\{u_\varepsilon\}$ is bounded in $L^\infty(\mathbb{R}^+, L^2(\Omega))$. Also, as in (4.33),

$$2\nu \int_0^T \|\nabla u_\varepsilon(t)\|^2_{L^2}\, dt = \|J_\varepsilon u_0\|^2_{L^2} - \|u_\varepsilon(T)\|^2_{L^2}, \tag{5.41}$$

for each $T \in \mathbb{R}^+$. Thus, parallel to (4.34), for any bounded interval $I = [0,T]$,

$$\{u_\varepsilon\} \text{ is bounded in } L^2(I, W^1). \tag{5.42}$$

Instead of paralleling (4.36)–(4.39), we prefer to use (5.42) to write

$$\{\nabla_{u_\varepsilon} u_\varepsilon\} \text{ bounded in } L^1\big(I, L^{3/2}(\Omega)\big), \tag{5.43}$$

provided dim $\Omega \le 3$. In such a case, we also have

$$P : W^1 \to H^1(\Omega) \subset L^3(\Omega),$$

and hence

$$P : L^{3/2}(\Omega) \longrightarrow (W^1)^*. \tag{5.44}$$

Also $\{J_\varepsilon\}$ is uniformly bounded on W^1 and its dual $(W^1)^*$, and $A : W^1 \to (W^1)^*$. Thus, in place of (4.37), we have

$$\{\partial_t u_\varepsilon\} \text{ bounded in } L^1(I, (W^1)^*), \tag{5.45}$$

so

$$\{u_\varepsilon\} \text{ is bounded in } H^s(I, (W^1)^*), \quad \forall\, s \in \Big(0, \frac{1}{2}\Big). \tag{5.46}$$

Now we interpolate this with (5.42), to get, for all $\delta > 0$,

$$\{u_\varepsilon\} \text{ bounded in } H^s(I, H^{1-\delta}(\Omega)), \quad s = s(\delta) > 0, \tag{5.47}$$

hence, parallel to (4.40),

$$\{u_\varepsilon\} \text{ is compact in } L^2(I, H^{1-\delta}(\Omega)), \quad \forall\, \delta > 0. \tag{5.48}$$

The rest of the argument follows as in the proof of Theorem 4.6.

We also have results parallel to Propositions 4.9–4.10:

Proposition 5.10. *Let u_1 and u_2 be weak solutions to (5.11), satisfying*

$$u_j \in L^\infty(I, W^0) \cap L^2(I, W^1), \quad u_j \in L^4(I \times \Omega). \tag{5.49}$$

Suppose dim $\Omega = 2$ or 3; if dim $\Omega = 3$, suppose furthermore that

$$u_1 \in L^8\big(I, L^4(\Omega)\big). \tag{5.50}$$

If $u_1(0) = u_2(0)$, then $u_1 = u_2$ on $I \times \Omega$.

The proof of both this result and the following are by the same arguments as used in §4.

Proposition 5.11. *If u is a Leray-Hopf solution and $I = [0, T]$, then*

$$u \in L^4(I \times \Omega) \quad \text{if dim } \Omega = 2 \tag{5.51}$$

and

$$u \in L^{8/3}\big(I, L^4(\Omega)\big) \quad \text{if dim } \Omega = 3. \tag{5.52}$$

Thus we have uniqueness of Leray-Hopf solutions if dim $\Omega = 2$. The following result yields extra smoothness if $u_0 \in W^1$:

Proposition 5.12. *If dim $\Omega = 2$, and u is a Leray-Hopf solution to the Navier-Stokes equations, with $u(0) = u_0 \in W^1$, then, for any $I = [0, T], T < \infty$,*

$$u \in L^\infty(I, W^1) \cap L^2(I, W^2), \tag{5.53}$$

and

$$\frac{\partial u}{\partial t} \in L^2(I, W^0). \tag{5.54}$$

Proof. Let u_j be the approximate solution u_ε defined by (5.40), with $\varepsilon = \varepsilon_j \to 0$. We have

$$\frac{1}{2}\frac{d}{dt}\big\|A^{1/2}u_j(t)\big\|_{L^2}^2 + \nu\|Au_j(t)\|_{L^2}^2 = -\big(\nabla_{u_j}u_j, Au_j\big)_{L^2}, \tag{5.55}$$

upon taking the inner product of (5.40) with Au_ε. Now there is the estimate

$$\big|\big(\nabla_{u_j}u_j, Au_j\big)\big| \le C\|\nabla u_j\|_{L^3}^3. \tag{5.56}$$

To see this, note that since $d \circ (I - P) = 0$, we have, for $u \in W^2$,

$$(\nabla_u u, u)_{V_1} = (d\nabla_u u, du) = \big([d, \nabla_u]u, du\big) + \frac{1}{2}\big([\nabla_u + \nabla_u^*]\, du, du\big),$$

and the absolute value of each of the last two terms is easily bounded by $\int |\nabla u|^3\, dV$.

In order to estimate the right side of (5.56), we use the Sobolev imbedding result

$$H^{1/3}(\Omega) \subset L^3(\Omega), \quad \dim \Omega = 2, \tag{5.57}$$

which implies $\|v\|_{L^3} \le C\|v\|_{L^2}^{2/3}\|v\|_{H^1}^{1/3}$, so

$$\begin{aligned}\|\nabla u_j\|_{L^3}^3 &\le C\|\nabla u_j\|_{L^2}^2 \cdot \|\nabla u_j\|_{H^1} \\ &\le C'(\nu\delta)^{-1}\|\nabla u_j\|_{L^2}^4 + C'\nu\delta\|\nabla u_j\|_{H^1}^2.\end{aligned} \tag{5.58}$$

We have $\|\nabla u_j\|_{H^1}^2 \le C\|Au_j\|_{L^2}^2 + C\|u_j\|_{L^2}^2$, by Proposition 5.2, so if δ is picked small enough, we can absorb the $\|\nabla u_j\|_{H^1}^2$-term into the left side of (5.55). We get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big\|A^{1/2}u_j(t)\big\|_{L^2}^2 + \frac{\nu}{2}\|Au_j(t)\|_{L^2}^2 \\ &\qquad \le C\|A^{1/2}u_j(t)\|_{L^2}^4 + C\Big(\|u_j(t)\|_{L^2}^4 + \|u_j(t)\|_{L^2}^2\Big).\end{aligned} \tag{5.59}$$

We want to apply Gronwall's inequality. It is convenient to set

$$\sigma_j(t) = \big\|A^{1/2}u_j(t)\big\|_{L^2}^2, \quad \Phi(\lambda) = \lambda^4 + \lambda^2. \tag{5.60}$$

The boundedness of u_ε in $L^2_{\text{loc}}(\mathbb{R}^+, W^1)$ (noted in (5.42)) implies that, for any $T < \infty$,

$$\int_0^T \sigma_j(t)\, dt \le K(T) < \infty, \tag{5.61}$$

with $K(T)$ independent of j. If we drop the term $(\nu/2)\|Au_j(t)\|_{L^2}^2$ from (5.59), we obtain

$$\frac{d}{dt}\big\|A^{1/2}u_j(t)\big\|_{L^2}^2 \le C\sigma_j(t)\big\|A^{1/2}u_j(t)\big\|_{L^2}^2 + C\Phi\Big(\|u_j(t)\|_{L^2}\Big), \tag{5.62}$$

and Gronwall's inequality yields

(5.63)

$$\left\|A^{1/2}u_j(t)\right\|_{L^2}^2 \le e^{CK(t)}\left\|A^{1/2}u_0\right\|_{L^2}^2 + Ce^{CK(t)}\int_0^t \Phi\Big(\|u_j(s)\|_{L^2}\Big)\,ds.$$

This implies that u_j is bounded in $L^\infty(I, W^1)$, and then integrating (5.59) implies u_j is bounded in $L^2(I, W^2)$. The conclusions (5.53) and (5.54) follow.

The argument used to prove Proposition 5.12 does not extend to the case in which dim $\Omega = 3$. In fact, if dim $\Omega = 3$, then (5.57) must be replaced by

$$H^{1/2}(\Omega) \subset L^3(\Omega), \quad \dim\,\Omega = 3, \tag{5.64}$$

which implies $\|v\|_{L^3} \le C\|v\|_{L^2}^{1/2}\|v\|_{H^1}^{1/2}$, and hence (5.58) is replaced by

$$\begin{aligned}\|\nabla u_j\|_{L^3}^3 &\le C\|\nabla u_j\|_{L^2}^{3/2}\cdot\|\nabla u_j\|_{H^1}^{3/2}\\ &\le C(\nu\delta)^{-3}\|\nabla u_j\|_{L^2}^6 + C\nu\delta\|\nabla u_j\|_{H^1}^2.\end{aligned}\tag{5.65}$$

Unfortunately, the power 6 of $\|\nabla u_j\|_{L^2}$ on the right side of (5.65) is too large in this case for an analogue of (5.60)–(5.63) to work, so such an approach fails if dim $\Omega = 3$.

On the other hand, when dim $\Omega = 3$, we do have the inequality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|A^{1/2}u_j(t)\|_{L^2}^2 + \frac{\nu}{2}\|Au_j(t)\|_{L^2}^2\\ &\qquad\le C\|A^{1/2}u_j(t)\|_{L^2}^6 + C\big(\|u_j(t)\|_{L^2}^6 + \|u_j(t)\|_{L^2}^2\big).\end{aligned}\tag{5.66}$$

We have an estimate $\|u_j(t)\|_{L^2} \le K$, so we can apply Gronwall's inequality to the differential inequality

$$\frac{1}{2}\frac{d}{dt}Y_j(t) \le CY_j(t)^3 + C(K^6 + K^2)$$

to get a uniform bound on $Y_j(t) = \|A^{1/2}u_j(t)\|_{L^2}^2$, at least on some interval $[0, T_0]$. Thus we have the following result.

Proposition 5.13. *If dim $\Omega = 3$, and u is a Leray-Hopf solution to the Navier-Stokes equations, with $u(0) = u_0 \in W^1$, then there exists $T_0 = T_0(\|u_0\|_{W^1}) > 0$ such that*

$$u \in L^\infty\big([0, T_0], W^1\big) \cap L^2\big([0, T_0], W^2\big), \tag{5.67}$$

and

$$\frac{\partial u}{\partial t} \in L^2\big([0, T_0], W^0\big). \tag{5.68}$$

Note that the properties of the solution u on $[0, T_0] \times \Omega$ in (5.67) are stronger than the properties (5.49)–(5.50) required for uniqueness in Proposition 5.10. Hence we have the following:

Corollary 5.14. *If dim* $\Omega = 3$ *and* u_1 *and* u_2 *are Leray-Hopf solutions to the Navier-Stokes equations, with* $u_0(0) = u_2(0) = u_0 \in W^1$*, then there exists* $T_0 = T_0(\|u_0\|_{W^1}) > 0$ *such that* $u_1(t) = u_2(t)$ *for* $0 \le t \le T_0$*.*

Furthermore, if $u_0 \in W^s$ *with* $s \in (s_0, 2)$ *as in Proposition 5.4, then the strong solution* $u \in C([0,T], W^s)$ *provided by Proposition 5.4 agrees with any Leray-Hopf solution, for* $0 \le t \le \min(T, T_0)$*.*

As we have seen, a number of results presented in §4 for viscous fluid flows on domains without boundary extend to the case of domains with boundary. We now mention some phenomena that differ in the two cases.

The role of the vorticity equation is altered when $\partial\Omega \neq \emptyset$. One still has the PDE for $w = \text{curl}\, u$, for example,

$$\begin{aligned} \frac{\partial w}{\partial t} + \nabla_u w &= \nu \Delta w \qquad (\dim \Omega = 2), \\ \frac{\partial w}{\partial t} + \nabla_u w - \nabla_w u &= \nu \Delta w \qquad (\dim \Omega = 3), \end{aligned} \tag{5.69}$$

but when $\partial\Omega \neq \emptyset$, the initial value $w(0)$ alone does not serve to determine $w(t)$ for $t > 0$ from such a PDE, and a good boundary condition to impose on $w(t,x)$ is not available. This is not a problem in the $\nu = 0$ case, since u itself is tangent to the boundary. For $\nu > 0$, one result is that one can have $w(0) = 0$ but $w(t) \neq 0$ for $t > 0$. In other words, for $\nu > 0$, interaction of the fluid with the boundary can create vorticity.

The most crucial effect a boundary has lies in complicating the behavior of solutions u_ν in the limit $\nu \to 0$. There is no analogue of the ν-independent estimates of Propositions 4.1 and 4.2 when $\partial\Omega \neq \emptyset$. This is connected to the change of boundary condition, from $u_\nu|_{\partial\Omega} = 0$ for ν positive (however small) to $n \cdot u|_{\partial\Omega} = 0$ when $\nu = 0$, n being the normal to $\partial\Omega$. Study of the small-ν limit is important because it arises naturally. In many cases flow of air can be modeled as an incompressible fluid flow with $\nu \approx 10^{-5}$. However, after more than a century of investigation, this remains an extremely mysterious problem. See the next section for further discussion of these matters.

Exercises

1. Show that $\mathcal{D}(A^k) \subset H^{2k}(\Omega, T)$, for $k \in \mathbb{Z}^+$. Hence establish (5.37).
2. Extend the L^2-Sobolev space results of this section to L^p-Sobolev space results.
3. Work out results parallel to those of this section for the Navier-Stokes equations, when the no-slip boundary condition (5.2) is replaced by the "slip" boundary condition:

$$2\nu \,\text{Def}(u)N - pN = 0 \quad \text{on} \quad \partial\Omega, \tag{5.70}$$

where N is a unit normal field to $\partial\Omega$ and $\mathrm{Def}(u)$ is a tensor field of type $(1,1)$, given by (2.60). Relate (5.70) to the identity

$$(\nu\mathcal{L}u - \nabla p, v) = -2\nu(\mathrm{Def}\ u, \mathrm{Def}\ v) \quad \text{whenever} \quad \mathrm{div}\ v = 0.$$

6. Vanishing viscosity limits

In this section we consider some classes of solutions to the Navier-Stokes equations

$$\frac{\partial u^\nu}{\partial t} + \nabla_{u^\nu} u^\nu + \nabla p^\nu = \nu\Delta u^\nu + F^\nu, \quad \mathrm{div}\, u^\nu = 0, \tag{6.1}$$

on a bounded domain, or a compact Riemannian manifold, $\overline{\Omega}$ (with a flat metric), with boundary $\partial\Omega$, satisfying the no-slip boundary condition

$$u^\nu\big|_{\mathbb{R}^+\times\partial\Omega} = 0, \tag{6.2}$$

and initial condition

$$u^\nu(0) = u_0, \tag{6.3}$$

and investigate convergence as $\nu \to 0$ to the solution to the Euler equation

$$\frac{\partial u^0}{\partial t} + \nabla_{u^0} u^0 + \nabla p^0 = F^0, \quad \mathrm{div}\, u^0 = 0, \tag{6.4}$$

with boundary condition

$$u^0 \parallel \partial\Omega, \tag{6.5}$$

and initial condition as in (6.3). We assume

$$\mathrm{div}\, u_0, \quad u_0 \parallel \partial\Omega, \tag{6.6}$$

but do not assume $u_0 = 0$ on $\partial\Omega$.

When $\partial\Omega \neq \emptyset$, the problem of convergence $u^\nu \to u^0$ is very difficult, and there are not many positive results, though there is a large literature. The enduring monograph [Sch] contains a great deal of formal work, much stimulated by ideas of L. Prandtl. More modern mathematical progress includes a result of [Kat], that $u^\nu(t) \to u^0(t)$ in L^2-norm, uniformly in $t \in [0,T]$, provided one has an estimate

$$\nu \int_0^T \int\limits_{\Gamma_{c\nu}} |\nabla u^\nu(t,x)|^2\, dx\, dt \longrightarrow 0, \quad \text{as} \quad \nu \to 0, \tag{6.7}$$

where $\Gamma_\delta = \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) \le \delta\}$. Unfortunately, this condition is not amenable to checking. In [W] there is a variant, namely that such convergence holds provided

$$\nu \int_0^T \int\limits_{\Gamma_{\eta(\nu)}} |\nabla_T u^\nu(t,x)|^2\, dx\, dt \longrightarrow 0, \quad \text{as} \quad \nu \to 0, \tag{6.8}$$

with $\eta(\nu)/\nu \to \infty$ as $\nu \to 0$, where ∇_T denotes the derivative tangent to $\partial\Omega$.

Here we confine attention to two classes of examples. The first is the class of circularly symmetric flows on the disk in 2D. The second is a class of circular pipe flows, in 3D, which will be described in more detail below. Both of these classes are mentioned in [W] as classes to which the results there apply. However, we will seek more detailed information on the nature of the convergence $u^\nu \to u^0$. Our analysis follows techniques developed in [LMNT], [MT1], and [MT2]. See also [Mat], [BW], and [LMN] for other work in the 2D case. Most of these papers also treated moving boundaries, but for simplicity we treat only stationary boundaries here.

We start with circularly symmetric flows on the disk $\Omega = D = \{x \in \mathbb{R}^2 : |x| < 1\}$. Here, we take $F^\nu \equiv 0$. By definition, a vector field u_0 on D is circularly symmetric provided

$$u_0(R_\theta x) = R_\theta u_0(x), \quad \forall\, x \in D, \tag{6.9}$$

for each $\theta \in [0, 2\pi]$, where R_θ is counterclockwise rotation by θ. The general vector field satisfying (6.9) has the form

$$s_0(|x|)x^\perp + s_1(|x|)x, \tag{6.10}$$

with s_j scalar and $x^\perp = Jx$, where $J = R_{\pi/2}$, but the condition $\operatorname{div} u_0 = 0$ together with the condition $u_0 \parallel \partial D$, forces $s_1 = 0$, so the type of initial data we consider is characterized by

$$u_0(x) = s_0(|x|)x^\perp. \tag{6.11}$$

It is easy to see that $\operatorname{div} u_0 = 0$ for each such u_0. Another characterization of vector fields of the form (6.11) is the following. For each unit vector $\omega \in S^1 \subset \mathbb{R}^2$, let $\Phi_\omega : \mathbb{R}^2 \to \mathbb{R}^2$ denote the reflection across the line generated by ω, i.e., $\Phi(a\omega + bJ\omega) = a\omega - bJ\omega$. Then a vector field u_0 on D has the form (6.11) if and only if

$$u_0(\Phi_\omega x) = -\Phi_\omega u_0(x), \quad \forall\, \omega \in S^1,\ x \in D. \tag{6.12}$$

A vector field u_0 of the form (6.11) is a steady solution to the 2D Euler equation (with $F^0 = 0$). In fact, a calculation gives

$$\nabla_{u_0} u_0 = -s_0(|x|)^2 x = -\nabla p_0(x), \tag{6.13}$$

with

$$p_0(x) = \tilde{p}_0(|x|), \quad \tilde{p}_0(r) = -\int_r^1 s_0(\rho)^2 \rho\, d\rho. \tag{6.14}$$

Consequently, in this case the vanishing viscosity problem is to show that the solution u^ν to (6.1)–(6.3) satisfies $u^\nu(t) \to u_0$ as $\nu \to 0$. The following is the key to the analysis of the solution u^ν.

Proposition 6.1. *Given that u_0 has the form (6.11), the solution u^ν to (6.1)–(6.3) (with $F^\nu \equiv 0$) is circularly symmetric for each $t > 0$, of the* *form*

$$u^\nu(t,x) = s^\nu(t,|x|)x^\perp, \tag{6.15}$$

and it coincides with the solution to the linear PDE

$$\frac{\partial u^\nu}{\partial t} = \nu\Delta u^\nu, \tag{6.16}$$

with boundary condition (6.2) and initial condition (6.3).

Proof. Let u^ν solve (6.16), (6.2), and (6.3), with u_0 as in (6.11). We claim (6.15) holds. In fact, for each unit vector $\omega \in \mathbb{R}^2$, $-\Phi_\omega u^\nu(t,\Phi_\omega x)$ also solves (6.16), with the same initial data and boundary conditions as u^ν, so these functions must coincide, and (6.15) follows. Hence $\operatorname{div} u^\nu = 0$ for each $t > 0$. Also we have an analogue of (6.13)–(6.14):

$$\begin{aligned} \nabla_{u^\nu} u^\nu = -\nabla p^\nu, \quad p^\nu(t,x) = \tilde{p}^\nu(t,|x|), \\ \tilde{p}^\nu(t,r) = -\int_r^1 s^\nu(t,\rho)^2 \rho \, d\rho. \end{aligned} \tag{6.17}$$

Hence this u^ν is the solution to (6.1)–(6.3).

To restate matters, for $\Omega = D$, the solution to (6.1)–(6.3) is in this case simply

$$u^\nu(t,x) = e^{\nu t\Delta} u_0(x). \tag{6.18}$$

The following is a simple consequence.

Proposition 6.2. *Assume u_0, of the form (6.11), belongs to a Banach space $\mathfrak{X}$ of $\mathbb{R}^2$-valued functions on D. If $\{e^{t\Delta} : t \ge 0\}$ is a strongly continuous semigroup on $\mathfrak{X}$, then $u^\nu(t,\cdot) \to u_0$ in $\mathfrak{X}$ as $\nu \to 0$, locally uniformly in $t \in [0,\infty)$.*

As seen in Chapter 6, $\{e^{t\Delta} : t \ge 0\}$ is strongly continuous on the following spaces:

$$L^p(D), \quad 1 \le p < \infty, \quad C_o(D) = \{f \in C(\overline{D}) : f = 0 \text{ on } \partial D\}. \tag{6.19}$$

Also, it is strongly continuous on $\mathcal{D}_s = \mathcal{D}((-\Delta)^{s/2})$ for all $s \in \mathbb{R}^+$. We recall from Chapter 5 that

$$\mathcal{D}_2 = H^2(D) \cap H_0^1(D), \quad \mathcal{D}_1 = H_0^1(D), \tag{6.20}$$

and

$$\mathcal{D}_s = [L^2(D), H_0^1(D)]_s, \quad 0 < s < 1. \tag{6.21}$$

In particular, by interpolation results given in Chapter 4,

$$\mathcal{D}_s = H^s(D), \quad 0 < s < \frac{1}{2}. \tag{6.22}$$

We also mention Proposition 7.4 of Chapter 13, which implies this heat semigroup is strongly continuous on

$$C_b^1(\overline{D}) = \{f \in C^1(\overline{D}) : f = 0 \text{ on } \partial D\}. \tag{6.23}$$

On the other hand, if $u_0 \in C(\overline{D})$ but does not vanish on ∂D, then $e^{t\Delta}u_0$ does not converge uniformly to u_0 on $\overline{D}$, as $t \to 0$, though as shown in Corollary 8.2 of Chapter 6, we do have convergence of $e^{t\Delta}u_0$ to u_0 uniformly on compact subsets of D. Thus there is a boundary layer attached to ∂D where uniform convergence fails. We recall Proposition 8.3 of Chapter 6 in the current context.

Proposition 6.3. *Given $u_0 \in C^\infty(\overline{D})$, we have, as $\nu \searrow 0$, locally uniformly in $t \in \mathbb{R}^+$,*

$$\begin{aligned} e^{\nu t\Delta}u_0 \sim u_0(x) &+ \sum_{k\ge 1} \frac{(\nu t)^k}{k!}\Delta^k u_0(x) \\ &- \sum_{j\ge 0} 2b_j(x)(4\nu t)^{j/2} E_j\Big(\frac{\varphi(x)}{\sqrt{4\nu t}}\Big). \end{aligned} \tag{6.24}$$

Here, $b_j \in C^\infty(\overline{D})$, $\varphi(x) = \operatorname{dist}(x, \partial D) = 1 - |x|$, and the special functions $E_j(y)$ are given by

$$E_j(y) = \frac{1}{\sqrt{\pi}} \int_y^\infty e^{-s^2}(s-y)^j\, ds. \tag{6.25}$$

We mention that $b_0 = u_0$ on ∂D and $b_j|_{\partial D} = 0$ for j odd. Also, $E_0(0) = 1/2$. The primary "boundary layer" term is

$$-2u_0(x)E_0\Big(\frac{1-|x|}{\sqrt{4\nu t}}\Big), \tag{6.26}$$

and we see the boundary layer thickness is $\sim \sqrt{4\nu t}$.

We pass from this class of 2D problems to the following class of 3D problems. We look for solutions to (6.1)–(6.3) with $u^\nu = u^\nu(t,x,z), p^\nu = p^\nu(t,x,z)$, $(t,x,z) \in \mathbb{R}^+ \times \Omega$, where $\Omega = D \times \mathbb{R}$, D being the 2D disk as above. Thus Ω is an infinitely long circular pipe. In this case, we consider external force fields of the form

$$F^\nu(t,x,z) = (0, f^\nu(t)), \tag{6.27}$$

so F^ν is parallel to the z-axis, with z-component $f^\nu(t)$. We take initial data of the following form:

$$u^\nu(0,x,z) = u_0(x) = (v_0(x), w_0(x)), \tag{6.28}$$

where v_0 is a vector field on D and w_0 is the z-component of u_0. We require the conditions

$$\operatorname{div} u_0 = 0, \quad u_0 \parallel \partial\Omega, \quad \textit{i.e.}, \quad \operatorname{div} v_0 = 0, \quad v_0 \parallel \partial D, \tag{6.29}$$

and we require the vector field v_0 on D to be circularly symmetric, so, as in (6.11), $v_0(x) = s_0(|x|)x^\perp$, hence

$$u_0(x) = (s_0(|x|)x^\perp, w_0(x)). \tag{6.30}$$

The fact that Ω is infinite is inconvenient. To get the theoretical treatment started, it is convenient to modify the set-up by requiring that solutions be periodic (say of period L) in z, so we replace Ω by $\Omega_L = D \times (\mathbb{R}/L\mathbb{Z})$. In such a case, results of §5 imply that, for each $\nu > 0$, (6.1)–(6.3) has a unique strong, short time solution, given mild regularity hypotheses on $v_0(x)$ and $w_0(x)$ (a solution that, as we will shortly see, persists for all time $t > 0$ under the current hypotheses), and the solution is z-translation invariant, i.e.,

$$u^\nu = (v^\nu(t,x), w^\nu(t,x)), \quad p^\nu = p^\nu(t,x). \tag{6.31}$$

Consequently,

$$\nabla_{u^\nu} u^\nu = (\nabla_{v^\nu} v^\nu, \nabla_{v^\nu} w^\nu), \quad \operatorname{div} u^\nu = \operatorname{div} v^\nu. \tag{6.32}$$

Hence, in the current setting, (6.1) is equivalent to the following system of equations on $\mathbb{R}^+ \times D$:

$$\frac{\partial v^\nu}{\partial t} + \nabla_{v^\nu} v^\nu + \nabla p^\nu = \nu \Delta v^\nu, \quad \operatorname{div} v^\nu = 0, \tag{6.33}$$

$$\frac{\partial w^\nu}{\partial t} + \nabla_{v^\nu} w^\nu = \nu \Delta w^\nu + f^\nu. \tag{6.34}$$

Note that (6.33) is the Navier-Stokes equation for flow on D, which we have just treated. Given initial data satisfying (6.30), we have

$$v^\nu(t,x) = e^{\nu t \Delta} v_0(x), \tag{6.35}$$

where Δ is the Laplace operator on D, with Dirichlet boundary condition. The results of Proposition 6.2, complemented by (6.19)–(6.23) apply, as do those of Proposition 6.3, taking care of (6.33).

It remains to investigate (6.34). For this, we have the initial and boundary conditions

$$w^\nu(0) = w_0, \quad w^\nu\big|_{\mathbb{R}^+ \times \partial D} = 0, \tag{6.36}$$

and we ask whether, as $\nu \searrow 0$, w^ν converges to w^0, solving

$$\frac{\partial w^0}{\partial t} + \nabla_{v^0} w^0 = f^0(t), \quad w^0(0) = w_0. \tag{6.37}$$

We impose no boundary condition on w^0, which is natural since $v^0 = v_0$ is tangent to ∂D.

Before pursuing this convergence question, we pause to observe a class of steady solutions to (6.33)–(6.34) known as *Poiseuille flows.* Namely, given $\alpha \in \mathbb{R} \setminus 0$,

$$u_0(x) = \alpha(0, 1 - |x|^2) \tag{6.38}$$

is such a steady solution, with

$$p^\nu(t,x) = 0, \quad f^\nu(t) = (0, 4\nu\alpha). \tag{6.34}$$

An alternative description is to set

$$p^\nu(t,x,z) = -4\nu\alpha z, \quad f^\nu(t) = 0. \tag{6.40}$$

This latter is a common presentation, and one refers to Poiseuille flow as "pressure driven" However, this presentation does not fit into our set-up, since we passed from the infinite pipe $D \times \mathbb{R}$ to the periodized pipe $D \times (\mathbb{R}/L\mathbb{Z})$, and p^ν in (6.40) is not periodic in z. These Poiseuille flows do fit into our set-up, but we need to represent the force that maintains the flow as an external force.

We return to the convergence problem. For notational convenience, we set

$$X_\nu = \nabla_{v^\nu} = s^\nu(t,|x|)\frac{\partial}{\partial\theta}, \quad X = \nabla_{v^0} = s_0(|x|)\frac{\partial}{\partial\theta}. \tag{6.41}$$

Thus we examine solutions to

$$\begin{aligned} \frac{\partial w^\nu}{\partial t} &= \nu\Delta w^\nu - X_\nu w^\nu + f^\nu(t), \\ w^\nu(0,x) &= w_0(x), \quad w^\nu\big|_{\mathbb{R}^+\times\partial D} = 0, \end{aligned} \tag{6.42}$$

compared to solutions to

$$\frac{\partial w^0}{\partial t} = -Xw^0 + f^0(t), \quad w^0(0,x) = w_0(x). \tag{6.43}$$

We do not assume $w_0|_{\partial D} = 0$. In order to separate the two phenomena that make (6.42) a singular perturbation (6.43), namely the appearance of $\nu\Delta$ on the one hand and the replacement of X by X_ν on the other hand, we rewrite (6.42) as

$$\frac{\partial w^\nu}{\partial t} = (\nu\Delta - X)w^\nu + (X - X_\nu)w^\nu + f^\nu(t), \tag{6.44}$$

and apply Duhamel's formula to get

$$w^\nu(t) = e^{t(\nu\Delta - X)}w_0 + \int_0^t e^{(t-s)(\nu\Delta - X)}\big[(X - X_\nu)w^\nu(s) + f^\nu(s)\big]\,ds. \tag{6.45}$$

By comparison, we can write the solution to (6.43) as

$$w^\nu(t) = e^{-tX}w_0 + \int_0^t f^0(s)\,ds. \tag{6.46}$$

Consequently,

$$w^\nu(t,x) - w^0(t,x) = R_1(\nu,t,x) + R_2(\nu,t,x) + R_3(\nu,t,x), \tag{6.47}$$

where

$$
(6.48)\qquad
\begin{aligned}
R_1(\nu,t,x) &= e^{t(\nu\Delta - X)}w_0 - e^{-tX}w_0,\\
R_2(\nu,t,x) &= \int_0^t \Big[f^\nu(s)e^{(t-s)(\nu\Delta - X)}1 - f^0(s)\Big]\,ds,\\
R_3(\nu,t,x) &= \int_0^t e^{(t-s)(\nu\Delta - X)}(s_0 - s^\nu)\frac{\partial w^\nu}{\partial\theta}\,ds.
\end{aligned}
$$

The term R_2 is the easiest to treat. By radial symmetry,

$$
e^{(t-s)(\nu\Delta - X)}1 = e^{(t-s)\nu\Delta}1,
$$

and we can write

$$
(6.49)\qquad
\begin{aligned}
R_2(\nu,t,x) = &\int_0^t \big[f^\nu(s) - f^0(s)\big]\,ds\\
&+ \int_0^t f^\nu(s)\Big[e^{(t-s)\nu\Delta}1 - 1\Big]\,ds.
\end{aligned}
$$

The uniform asymptotic expansion of the last integrand is a special case of (6.24):

$$
(6.50)\qquad e^{(t-s)\nu\Delta}1 - 1 \sim -\sum_{j\ge 0} 2b_j(x)(\nu(t-s))^{j/2}E_j\Big(\frac{1-|x|}{\sqrt{4\nu(t-s)}}\Big),
$$

with $b_j \in C^\infty(\overline{D})$, $b_0|_{\partial D} = 1$, and E_j as in (6.25). The principal contribution giving the boundary layer effect for the term $R_2(\nu,t,x)$ is

$$
(6.51)\qquad -2\int_0^t f^\nu(s)E_0\Big(\frac{1-|x|}{\sqrt{4\nu(t-s)}}\Big)\,ds.
$$

Methods initiated in [MT1] and carried out for this case in [MT2] produce a uniform asymptotic expansion for $R_1(\nu,t,x)$ almost as explicit as that given above for R_2, but with much greater effort. Here we will be content to present simpler estimates on R_1. Our analysis of

$$
(6.52)\qquad W^\nu(t,x) = e^{t(\nu\Delta - X)}w_0(x)
$$

starts with the following. Recall that X is divergence free and tangent to ∂D.

Lemma 6.4. *Given* $\nu > 0$,

$$
(6.53)\qquad \mathcal{D}((\nu\Delta - X)^j) = \mathcal{D}(\Delta^j), \quad j = 1,2.
$$

Proof. We have, for $\nu > 0$,

$$
(6.54)\qquad \mathcal{D}(\nu\Delta - X) = \{f \in H^2(D) : f\big|_{\partial D} = 0\},
$$

and

$$\mathcal{D}((\nu\Delta - X)^2) = \{f \in H^4(D) : f\big|_{\partial D} = \nu\Delta f - Xf\big|_{\partial D} = 0\}. \tag{6.55}$$

The first space is clearly equal to $\mathcal{D}(\Delta)$. Since X is tangent to ∂D, $f|_{\partial D} = 0 \Rightarrow Xf|_{\partial D} = 0$, so the second space coincides with $\{f \in H^4(D) : f|_{\partial D} = \Delta f|_{\partial D} = 0\}$, which is $\mathcal{D}(\Delta^2)$.

Remark. The analogous identity of domains typically fails for larger j.

To proceed, since W^ν in (6.52) satisfies $\partial_t W^\nu = -XW^\nu + \nu\Delta W^\nu$, we can use Duhamel's formula to write

$$W^\nu(t) = e^{-tX}w_0 + \nu\int_0^t e^{-(t-s)X}\Delta W^\nu(s)\,ds, \tag{6.56}$$

hence

$$\|e^{t(\nu\Delta - X)}w_0 - e^{-tX}w_0\|_{L^p} \le \nu\int_0^t \|\Delta W^\nu(s)\|_{L^p}\,ds. \tag{6.57}$$

The following provides a useful estimate on the right side of (6.57) when $p = 2$.

Lemma 6.5. *Take $w_0 \in \mathcal{D}(\Delta^2) = \mathcal{D}((\nu\Delta - X)^2)$, and construct W^ν as in (6.52). Then there exists $K \in (0,\infty)$, independent of $\nu > 0$, such that*

$$\|\Delta W^\nu(t)\|_{L^2}^2 \le e^{2Kt}\|\Delta w_0\|_{L^2}^2. \tag{6.58}$$

Proof. We have

$$\begin{aligned}\frac{d}{dt}\|\Delta W^\nu(t)\|_{L^2}^2 &= 2\,\mathrm{Re}(\Delta\partial_t W^\nu, \Delta W^\nu)\\ &= 2\,\mathrm{Re}(\nu\Delta^2 W^\nu, \Delta W^\nu) - 2\,\mathrm{Re}(\Delta X W^\nu, \Delta W^\nu)\\ &\le -2\,\mathrm{Re}(\Delta X W^\nu, \Delta W^\nu)\\ &= -2\,\mathrm{Re}(X\Delta W^\nu, \Delta W^\nu) - 2\,\mathrm{Re}([\Delta, X]W^\nu, \Delta W^\nu)\\ &\le 2K\|\Delta W^\nu\|_{L^2}^2,\end{aligned} \tag{6.59}$$

with K independent of ν. The last estimate holds because

$$g \in \mathcal{D}(\Delta) \Rightarrow |(Xg, g)| \le K_1\|g\|_{L^2}^2, \tag{6.60}$$

and

$$\begin{aligned}W^\nu(t) \in \mathcal{D}(\Delta^2) \Rightarrow\ & [\Delta, X]W^\nu(t) \in L^2(D), \quad \text{and}\\ & \|[\Delta, X]W^\nu(t)\|_{L^2} \le \widetilde{K}_2\|W^\nu(t)\|_{H^2} \le K_2\|\Delta W^\nu(t)\|_{L^2}.\end{aligned} \tag{6.61}$$

The estimate (6.58) follows.

We can now prove the following.

Proposition 6.6. *Given* $p \in [1, \infty)$, $w_0 \in L^p(D)$, *we have*

$$e^{t(\nu\Delta - X)} w_0 \longrightarrow e^{-tX} w_0, \quad \text{as} \ \ \nu \searrow 0, \tag{6.62}$$

with convergence in L^p*-norm.*

Proof. We know that $e^{t\nu\Delta}$ is a contraction semigroup on $L^p(D)$ and e^{-tX} is a group of isometries on $L^p(D)$, and we have the Trotter product formula:

$$e^{t(\nu\Delta - X)} w_0 = \lim_{n\to\infty} \Bigl(e^{(t/n)\nu\Delta} e^{-(t/n)X}\Bigr)^n w_0, \tag{6.63}$$

in L^p-norm, hence $e^{t(\nu\Delta - X)}$ is a contraction semigroup on $L^p(D)$. By (6.58) and (6.57), we have L^2 convergence for $w_0 \in \mathcal{D}(\Delta^2)$, which is dense in $L^2(D)$. This gives (6.62) for $p = 2$, by the standard approximation argument, a second use of which gives (6.62) for all $p \in [1, 2]$.

Suppose next that $p \in (2, \infty)$, with dual exponent $p' \in (1, 2)$. The previous results work with X replaced by $-X$, yielding $e^{t(\nu\Delta + x)} g \to e^{tX} g$, as $\nu \searrow 0$, in $L^{p'}$-norm, for all $g \in L^{p'}(D)$. This implies that for $w_0 \in L^p(D)$, convergence in (6.62) holds in the weak* topology of $L^p(D)$. Now, since e^{-tX} is an isometry on $L^p(D)$, we have

$$\|e^{-tX} w_0\|_{L^p} \ge \limsup_{\nu\to 0} \|e^{t(\nu\Delta - X)} w_0\|_{L^p}, \tag{6.69}$$

for each $w_0 \in L^p(D)$. Since $L^p(D)$ is a uniformly convex Banach space for such p, this yields L^p-norm convergence in (6.62).

To produce higher order Sobolev estimates, we have from (6.58) the estimate

$$\|e^{t(\nu\Delta - X)} w_0\|_{\mathcal{D}(\Delta)} \le e^{Kt} \|w_0\|_{\mathcal{D}(\Delta)}, \tag{6.65}$$

first for each $w_0 \in \mathcal{D}(\Delta^2)$, hence for each $w_0 \in \mathcal{D}(\Delta)$. Interpolation with the L^2-estimate then gives

$$\|e^{t(\nu\Delta - X)} w_0\|_{\mathcal{D}((-\Delta)^{s/2})} \le e^{Kt} \|w_0\|_{\mathcal{D}((-\Delta)^{s/2})}, \tag{6.66}$$

for each $s \in [0, 2]$, $w_0 \in \mathcal{D}((-\Delta)^{s/2}) = \mathcal{D}_s$. As noted in (6.22), $\mathcal{D}_s = H^s(D)$ for $0 \le s < 1/2$, so we have

$$\|e^{t(\nu\Delta - X)} w_0\|_{H^s(D)} \le C e^{Kt} \|w_0\|_{H^s(D)}, \quad 0 \le s < \frac{1}{2}, \tag{6.67}$$

with C and K independent of $\nu \in (0, 1]$. We can interpolate the estimate (6.67) with

$$\|e^{t(\nu\Delta - X)} w_0\|_{L^p(D)} \le \|w_0\|_{L^p(D)}, \quad 1 \le p < \infty. \tag{6.68}$$

Using

$$[H^s(D), L^p(D)]_\theta = H^{(1-\theta)s, q(\theta)}(D), \quad \frac{1}{q(\theta)} = \frac{1-\theta}{2} + \frac{\theta}{p}, \tag{6.69}$$

which follows from material in Chapter 13, §6, we have

$$\|e^{t(\nu\Delta - X)} w_0\|_{H^{\sigma,q}(D)} \le C_{\sigma,q} e^{Kt} \|w\|_{H^{\sigma,q}(D)}, \tag{6.70}$$

valid for

$$2 \le q < \infty, \quad \sigma q \in [0, 1). \tag{6.71}$$

Similar arguments give such operator bounds on e^{-tX}. We have the following convergence result.

Propositin 6.7. *Let σ, q satisfy (6.71). Then, for each $t \in (0, \infty)$,*

$$w_0 \in H^{\sigma,q}(D) \Longrightarrow \lim_{\nu \to 0} e^{t(\nu\Delta - X)} w_0 = e^{-tX} w_0, \tag{6.72}$$

in $H^{\sigma,q}$-norm.

Proof. Given $w_0 \in H^{\sigma,q}(D)$, (6.70) implies $\{e^{t(\nu\Delta - X)} w_0 : \nu \in (0, 1]\}$ is bounded in $H^{\sigma,q}(D)$ for each $t \in (0, \infty)$, so there is a weak* limit point. But Proposition 6.6 yields convergence to $e^{-tX} w_0$ in L^q-norm, so $e^{-tX} w_0$ is the only possible weak* limit point. Norm convergence in $H^{\tau,q}(D)$, for each $\tau < \sigma$, then follows from the compactness of the inclusion $H^{\sigma,q}(D) \hookrightarrow H^{\tau,q}(D)$. Taking $\sigma' > \sigma$ such that $\sigma' q < 1$, the argument above yields $e^{t(\nu\Delta - X)} w_0 \to e^{-tX} w_0$ in $H^{\sigma,q}$-norm for each $w_0 \in H^{\sigma',q}(D)$. The conclusion follows by denseness of $H^{\sigma',q}(D)$ in $H^{\sigma,q}(D)$, plus the uniform operator bound (6.70).

This concludes our treatment of $R_1(\nu, t, x)$. As mentioned, more precise results, including boundary layer analyses, are given in [MT1] and [MT2].

We move to an analysis of $R_3(\nu, t, x)$ in (6.48), i.e.,

$$R_3(\nu, t, x) = \int_0^t e^{(t-s)(\nu\Delta - X)} (s_0 - s^\nu) \frac{\partial w^\nu}{\partial \theta} \, ds, \tag{6.73}$$

where w^ν solves (6.34) and (6.36). Note that $\partial/\partial\theta$ commutes with X, X_ν, and Δ, so $z^\nu(t, x) = \partial w^\nu / \partial \theta$ solves

$$\frac{\partial z^\nu}{\partial t} = (\nu\Delta - X_\nu) z^\nu, \quad z^\nu \Big|_{\mathbb{R}^+ \times \partial D} = 0, \quad z^\nu(0, x) = \frac{\partial w_0}{\partial \theta}. \tag{6.74}$$

The maximum principle gives

$$\Big\| \frac{\partial w^\nu}{\partial \theta}(s) \Big\|_{L^\infty(D)} \le \Big\| \frac{\partial w_0}{\partial \theta} \Big\|_{L^\infty(D)}. \tag{6.75}$$

Since the semigroup $e^{t(\nu\Delta - X)}$ is positivity preserving, we have

$$|R_3(\nu, t, x)| \le \|\partial_\theta w_0\|_{L^\infty} \int_0^t e^{(t-s)(\nu\Delta - X)} \big| s_0(|x|) - s^\nu(s, |x|) \big| \, ds. \tag{6.76}$$

Also, by radial symmetry,

$$e^{(t-s)(\nu\Delta - X)} |s_0 - s^\nu| = e^{\nu(t-s)\Delta} |s_0 - s^\nu|, \tag{6.77}$$

so

$$|R_3(\nu,t,x)| \le \|\partial_\theta w_0\|_{L^\infty} \int_0^t e^{\nu(t-s)\Delta} |\tilde{s}_0 - \tilde{s}^\nu| \, ds, \tag{6.78}$$

where

$$\tilde{s}_0(x) = s_0(|x|), \quad \tilde{s}^\nu(t,x) = s^\nu(t,|x|), \tag{6.79}$$

and, we recall from (6.41),

$$v^\nu(t,x) = s^\nu(t,|x|)x^\perp, \quad v_0(x) = s_0(|x|)x^\perp. \tag{6.80}$$

Turning these around, we have

$$s^\nu(t,|x|) = \frac{1}{|x|^2} v^\nu(t,x)\cdot x^\perp, \quad s_0(|x|) = \frac{1}{|x|^2} v_0(x)\cdot x^\perp, \tag{6.81}$$

and also, if $\{e_1, e_2\}$ denotes the standard orthonormal basis of $\mathbb{R}^2$,

$$\begin{aligned} s^\nu(t,r) &= \frac{1}{r} v^\nu(t, re_1)\cdot e_2 \\ &= \frac{1}{r}\int_0^1 \frac{d}{d\sigma} v^\nu(t, r\sigma e_1)\cdot e_2 \, d\sigma \\ &= \int_0^1 e_2 \cdot \nabla_{e_1} v^\nu(t, r\sigma e_1)\, d\sigma, \end{aligned} \tag{6.82}$$

and similarly

$$s_0(r) = \int_0^1 e_2\cdot \nabla_{e_1} v_0(t, r\sigma e_1)\, d\sigma. \tag{6.83}$$

The representation (6.81) is effective away from a neighborhood of $\{x = 0\}$, especially near ∂D, where one reads off the uniform convergence of $s^\nu(t,r)$ to $s_0(r)$ except on the boundary layer discussed above in the analysis of $v^\nu(t,\cdot) \to v_0(\cdot)$, given $v_0 \in C^\infty(\overline{D})$.

The representation (6.82)–(6.83) is effective on a neighborhood of $\{x = 0\}$, for example the disk $\overline{D}_{1/2} = \{x \in \mathbb{R}^2 : |x| \le 1/2\}$, and it shows that $s^\nu(t,r) \to s_0(r)$ uniformly on $r \le 1/2$ provided $v^\nu(t,\cdot) \to v_0(t,\cdot)$ in $C^1(\overline{D}_{1/2})$. Results from Chapter 6, §8 (cf. Propositions 8.1–8.2) imply one has such convergence if $v_0 \in C^1(\overline{D})$, and in particular if $v_0 \in C^\infty(\overline{D})$.

Furthermore, the maximum principle implies

$$e^{\nu t\Delta}|\tilde{s}_0 - \tilde{s}^\nu| \le h_{\nu t} * |\tilde{s}_0 - \tilde{s}^\nu|, \tag{6.84}$$

where $h_{\nu t}$ is the free space heat kernel, given (with $n = 2$) by

$$h_{\nu t}(x) = (4\pi\nu t)^{-n/2} e^{-|x|^2/4\nu t}, \tag{6.85}$$

and $|\tilde{s}_0 - \tilde{s}^\nu|$ is extended by 0 outside $\overline{D}$. We hence have the following boundary layer estimates on R_3.

Proposition 6.8. *Assume $v_0, w_0 \in C^\infty(\overline{D})$. Then, given $T \in (0,\infty)$, we have a uniform bound*

$$|R_3(\nu,t,x)| \le C, \tag{6.86}$$

for $t \in [0,T],\ \nu \in (0,1],\ x \in \overline{D}$. Furthermore, as $\nu \to 0$,

$$R_3(\nu,t,x) \to 0 \ \ \text{uniformly on} \ \ \overline{D} \setminus \Gamma_{\omega(\nu)}, \tag{6.87}$$

as long as

$$\frac{\omega(\nu)}{\sqrt{\nu}} \to \infty. \tag{6.88}$$

We recall $\Gamma_\delta = \{x \in D : \operatorname{dist}(x, \partial D) \le \delta\}$.

Among other results established in [MT1]–[MT2], we mention one here. For $k \in \mathbb{N}$, set

$$\mathcal{V}^k(D) = \{f \in L^2(D) : X_{j_1} \cdots X_{j_\ell} f \in L^2(D),\ \forall\, \ell \le k,\ X_{j_m} \in \mathfrak{X}^1(D)\}, \tag{6.89}$$

where $\mathfrak{X}^1(D)$ denotes the space of smooth vector felds on $\overline{D}$ that are tangent to ∂D. After establishing that

$$f \in \mathcal{V}^k(D) \Longrightarrow \lim_{t \to 0} e^{t\Delta} f = f, \quad \text{in} \ \ \mathcal{V}^k\text{-norm}, \tag{6.90}$$

and

$$f \in \mathcal{V}^k(D) \Longrightarrow \lim_{\nu \searrow 0} e^{t(\nu\Delta - X)} f = e^{-tX} f, \quad \text{in} \ \ \mathcal{V}^k\text{-norm}, \tag{6.91}$$

these works proved the following (cf. [MT2], Proposition 3.10).

Proposition 6.9. *Assume $v_0 \in C^\infty(\overline{D})$ and $w_0 \in C^1(\overline{D})$. Take $k \in \mathbb{N}$ and also assume $w_0 \in \mathcal{V}^k(D)$. Then, for each $t > 0$, as $\nu \searrow 0$,*

$$v^\nu(t,\cdot) \to v_0 \ \ \text{and} \ \ w^\nu(t,\cdot) \to w^0(t,\cdot), \ \ \text{in} \ \ \mathcal{V}^k\text{-norm}. \tag{6.92}$$

Such a result is consistent with Prandtl's principle, that in the boundary layer it is normal derivatives of the velocity field, not tangential derivatives, that blow up as $\nu \to 0$. We mention that the convergence of $R_2(\nu,t,x)$ to 0 in $\mathcal{V}^k$-norm follows from the analysis described in (6.50), and the convergence of $R_1(\nu,t,x)$ to 0 in $\mathcal{V}^k$-norm, given $w_0 \in C^\infty(\overline{D})$, follows from a parallel analysis, carried out in [MT2], but not here. The convergence of $R_3(\nu,t,x)$ to 0 in $\mathcal{V}^k$-norm, given $v_0, w_0 \in C^\infty(\overline{D})$, does not follow from the results on R_3 established here; this requires further arguments.

The two cases analyzed above are much simpler than the general cases, which might involve turbulent boundary layers and boundary layer separation. Another issue is loss of stability of a solution as ν decreases. One can read more about such problems in [Bat], [Sch], [ChM], [OO], and references given there. We also mention [VD], which has numerous interesting illustrations of fluid phenomena, at various viscosities.

Exercises

1. Verify the characterization (6.12) of vector fields of the form (6.11).
2. Verify the calculation (6.13)–(6.14).
3. Produce a proof of (6.90), at least for $k = 1$. Try for larger k.

7. From velocity field convergence to flow convergence

In §6 we have given some results on convergence of the solutions u^ν to the Navier-Stokes equations

$$\begin{aligned}&\frac{\partial u^\nu}{\partial t} + \nabla_{u^\nu} u^\nu + \nabla p^\nu = \nu \Delta u^\nu \quad \text{on} \quad I \times \Omega,\\ &\operatorname{div} u^\nu = 0, \quad u^\nu\big|_{I\times\partial\Omega} = 0, \quad u^\nu(0) = u_0,\end{aligned} \tag{7.1}$$

to the solution u to the Euler equation

$$\begin{aligned}&\frac{\partial u}{\partial t} + \nabla_u u + \nabla p = 0 \quad \text{on} \quad I \times \Omega,\\ &\operatorname{div} u = 0, \quad u \,\|\, \partial\Omega, \quad u(0) = u_0,\end{aligned} \tag{7.2}$$

as $\nu \searrow 0$ (given $\operatorname{div} u_0 = 0,\ u_0 \,\|\, \partial\Omega$).

We now tackle the question of what can be said about convergence of fluid flows generated by the t-dependent velocity fields u^ν to the flow generated by u. Given the convergence results of §6, we are motivated to see what sort of flow convergence can be deduced from fairly weak hypotheses on u, u^ν, and the nature of the convergence $u^\nu \to u$. We obtain some such results here; further results can be found in [DL].

We will make the following hypotheses on the t-dependent vector fields u and u^ν.

(7.3)
$$u \in \operatorname{Lip}([0,T] \times \overline{\Omega}), \quad \operatorname{div} u(t) = 0, \quad u(t) \,\|\, \partial\Omega,$$

(7.4)
$$u^\nu \in \operatorname{Lip}([\varepsilon,T] \times \overline{\Omega}),\ \forall\, \varepsilon > 0, \quad \operatorname{div} u^\nu(t) = 0, \quad u^\nu(t) \,\|\, \partial\Omega,$$

(7.5)
$$u^\nu \in L^\infty([0,T] \times \Omega).$$

Say $\nu \in (0,1]$. Here $\overline{\Omega}$ is a smoothly bounded domain in $\mathbb{R}^n$, or more generally it could be a compact Riemannian manifold with smooth boundary

$\partial\Omega$. We do not assume any uniformity in ν on the estimates associated to (7.4)–(7.5).

The field u defines volume preserving bi-Lipschitz maps

$$\varphi^{t,s} : \overline{\Omega} \longrightarrow \overline{\Omega}, \quad s,t \in [0,T], \tag{7.6}$$

satisfying

$$\frac{\partial}{\partial t}\varphi^{t,s}(x) = u(t,\varphi^{t,s}(x)), \quad \varphi^{s,s}(x) = x. \tag{7.7}$$

Similarly the fields u^ν define volume preserving bi-Lipschitz maps

$$\varphi^{t,s}_\nu : \overline{\Omega} \longrightarrow \overline{\Omega}, \quad s,t \in (0,T], \tag{7.8}$$

satisfying

$$\frac{\partial}{\partial t}\varphi^{t,s}_\nu(x) = u^\nu(t,\varphi^{t,s}_\nu(x)), \quad \varphi^{s,s}_\nu(x) = x. \tag{7.9}$$

Note that

$$\begin{aligned} \varphi^{t,s} \circ \varphi^{s,r} &= \varphi^{t,r}, \quad r,s,t \in [0,T], \\ \varphi^{t,s}_\nu \circ \varphi^{s,r}_\nu &= \varphi^{t,r}_\nu, \quad r,s,t \in (0,T]. \end{aligned} \tag{7.10}$$

Our convergence results will be phrased in terms of strong operator convergence on $L^p(\Omega)$ of operators $\mathcal{S}^{t,0}_\nu$ to $\mathcal{S}^{t,0}$, where

$$\begin{aligned} \mathcal{S}^{t,s} f_0(x) &= f_0(\varphi^{s,t}(x)), \quad s,t \in [0,T], \\ \mathcal{S}^{t,s}_\nu f_0(x) &= f_0(\varphi^{s,t}_\nu(x)), \quad s,t \in (0,T], \end{aligned} \tag{7.11}$$

and $\mathcal{S}^{t,0}_\nu$ (as well as $\varphi^{0,t}_\nu$) will be constructed below. These operators are also characterized as follows. For $f = f(t,x)$ satisfying

$$\frac{\partial f}{\partial t} = -\nabla_{u(t)} f(t), \quad t \in [0,T], \tag{7.12}$$

we set

$$\mathcal{S}^{t,s} f(s) = f(t), \quad s,t \in [0,T]. \tag{7.13}$$

Note that f is advected by the flow generated by $u(t)$. Clearly

$$\mathcal{S}^{t,s} : L^p(\Omega) \longrightarrow L^p(\Omega), \quad \text{isometrically isomorphically}, \ \forall\, s,t \in [0,T]. \tag{7.14}$$

Similarly, for $f^\nu = f^\nu(t,x)$ solving

$$\frac{\partial f^\nu}{\partial t} = -\nabla_{u^\nu(t)} f^\nu(t), \tag{7.15}$$

we set

$$\mathcal{S}^{t,s}_\nu f^\nu(s) = f^\nu(t), \quad s,t \in (0,T], \tag{7.16}$$

and again

$$\mathcal{S}^{t,s}_\nu : L^p(\Omega) \longrightarrow L^p(\Omega), \quad \text{isometrically isomorphically}, \ \forall\, s,t \in (0,T]. \tag{7.17}$$

Note that

$$\begin{aligned} \mathcal{S}^{t,s}\mathcal{S}^{s,r} &= \mathcal{S}^{t,r}, \quad r,s,t \in [0,T], \\ \mathcal{S}_\nu^{t,s}\mathcal{S}_\nu^{s,r} &= \mathcal{S}_\nu^{t,r}, \quad r,s,t \in (0,T]. \end{aligned} \tag{7.18}$$

We will extend the scope of (7.16) to the case $s = 0$. Then we will show that, given (7.3)–(7.5), $p \in [1,\infty),\ t \in [0,T],\ f_0 \in L^p(\Omega)$,

$$\begin{aligned} u^\nu \to u \ \text{ in } \ L^1([0,T], L^p(\Omega)) \\ \Longrightarrow \mathcal{S}_\nu^{t,0} f_0 \to \mathcal{S}^{t,0} f_0 \ \text{ in } \ L^p\text{-norm}. \end{aligned} \tag{7.19}$$

In light of the relationship

$$\mathcal{S}_\nu^{t,0} f_0(x) = f_0(\varphi_\nu^{0,t}(x)), \tag{7.20}$$

which will be established below, this convergence amounts to some sort of convergence

$$\varphi_\nu^{0,t} \longrightarrow \varphi^{0,t} \tag{7.21}$$

for the *backward* flows $\varphi_\nu^{0,t}$.

To construct $\mathcal{S}_\nu^{t,0}$ on $L^p(\Omega)$, we first note that

$$\begin{aligned} &\varepsilon, \delta \in (0,T],\ f_0 \in \mathrm{Lip}(\overline{\Omega}) \\ &\quad \Longrightarrow \|\mathcal{S}_\nu^{\delta,\varepsilon} f_0 - f_0\|_{L^\infty} \le \|u^\nu\|_{L^\infty} \|f_0\|_{\mathrm{Lip}} |\varepsilon - \delta|, \end{aligned} \tag{7.22}$$

which in turn implies

$$\begin{aligned} \|\mathcal{S}_\nu^{t,\delta} f_0 - \mathcal{S}_\nu^{t,\varepsilon} f_0\|_{L^\infty} &= \|\mathcal{S}_\nu^{t,\varepsilon}(\mathcal{S}_\nu^{\varepsilon,\delta} f_0 - f_0)\|_{L^\infty} \\ &= \|\mathcal{S}_\nu^{\varepsilon,\delta} f_0 - f_0\|_{L^\infty} \\ &\le \|u^\nu\|_{L^\infty} \|f_0\|_{\mathrm{Lip}} \cdot |\varepsilon - \delta|. \end{aligned} \tag{7.23}$$

Hence

$$\lim_{\varepsilon \searrow 0} \mathcal{S}_\nu^{t,\varepsilon} f_0 = \mathcal{S}_\nu^{t,0} f_0 \tag{7.24}$$

exists for all $f_0 \in \mathrm{Lip}(\overline{\Omega})$, convergence in (7.24) holding in sup-norm, and a fortiori in L^p-norm. The uniform boundedness from (7.17) then implies that (7.24) holds in L^p-norm for each $f_0 \in L^p(\Omega)$, as long as $p \in [1,\infty)$, so $\mathrm{Lip}(\overline{\Omega})$ is dense in $L^p(\Omega)$. This defines

$$\mathcal{S}_\nu^{t,0} : L^p(\Omega) \longrightarrow L^p(\Omega), \quad 1 \le p < \infty,\ t \in (0,T], \tag{7.25}$$

and we have

$$\|\mathcal{S}_\nu^{t,0} f_0\|_{L^p} = \lim_{\varepsilon \searrow 0} \|\mathcal{S}_\nu^{t,\varepsilon} f_0\|_{L^p} \equiv \|f_0\|_{L^p}, \tag{7.26}$$

so $\mathcal{S}_\nu^{t,0}$ is an isometry on $L^p(\Omega)$ for each $p \in [1,\infty)$.

We note that, parallel to (7.22), for $\varepsilon, \delta \in (0,T],\ x \in \overline{\Omega}$,

$$\mathrm{dist}(\varphi_\nu^{\delta,\varepsilon}(x), x) \le \|u^\nu\|_{L^\infty} \cdot |\varepsilon - \delta|, \tag{7.27}$$

and, parallel to (7.23), if also $t \in (0,T]$,

$$\begin{aligned} \operatorname{dist}(\varphi_\nu^{\varepsilon,t}(x), \varphi_\nu^{\delta,t}(x)) &= \operatorname{dist}(\varphi_\nu^{\varepsilon,\delta}(\varphi_\nu^{\delta,t}(x)), \varphi_\nu^{\delta,t}(x)) \\ &\le \|u^\nu\|_{L^\infty} \cdot |\varepsilon - \delta|. \end{aligned} \tag{7.28}$$

It follows that

$$\varphi_\nu^{0,t}(x) = \lim_{\varepsilon \searrow 0} \varphi_\nu^{\varepsilon,t}(x) \tag{7.29}$$

exists and $\varphi_\nu^{0,t} : \overline{\Omega} \to \overline{\Omega}$ continuously, preserving the volume. Furthermore, we have

$$\mathcal{S}_\nu^{t,0} f_0(x) = f_0(\varphi_\nu^{0,t}(x)), \tag{7.30}$$

first for $f_0 \in \operatorname{Lip}(\overline{\Omega})$, then, by limiting arguments, for all $f_0 \in C(\overline{\Omega})$, and furthermore, for all $f_0 \in L^p(\Omega)$, in which case (7.30) holds, for each $t \in (0,T]$, for a.e. x, i.e., (7.30) is an identity in the Banach space $L^p(\Omega)$.

We derive some more properties of $\mathcal{S}_\nu^{t,0}$. Note from (7.23)–(7.24) that, when $f_0 \in \operatorname{Lip}(\overline{\Omega})$, $\varepsilon \in (0,T]$,

$$\|\mathcal{S}_\nu^{t,0} f_0 - \mathcal{S}_\nu^{t,\varepsilon} f_0\|_{L^\infty} \le \|u^\nu\|_{L^\infty} \|f_0\|_{\mathrm{Lip}} \cdot \varepsilon, \tag{7.31}$$

and hence, by uniform operator boundedness, for each $f_0 \in L^p(\Omega)$, $p \in [1,\infty)$, $s,t \in (0,T]$,

$$\begin{aligned} \mathcal{S}_\nu^{t,0} f_0 &= \lim_{\varepsilon \searrow 0} \mathcal{S}_\nu^{t,\varepsilon} f_0 \quad (\text{in } \ L^p\text{-norm}) \\ &= \lim_{\varepsilon \searrow 0} \mathcal{S}_\nu^{t,s} \mathcal{S}_\nu^{s,\varepsilon} f_0 \quad (\text{by (7.18)}) \\ &= \mathcal{S}_\nu^{t,s} \mathcal{S}_\nu^{s,0} f_0. \end{aligned} \tag{7.32}$$

Hence, $\mathcal{S}_\nu^{s,t} \mathcal{S}_\nu^{t,0} = \mathcal{S}_\nu^{s,0}$ on $L^p(\Omega)$, $\forall\, s,t \in (0,T]$, or equivalently,

$$\mathcal{S}_\nu^{\delta,t} \mathcal{S}^{t,0} = \mathcal{S}_\nu^{\delta,0} \quad \text{on} \quad L^p(\Omega). \tag{7.33}$$

We also have from (7.23)–(7.24) that

$$f_0 \in \operatorname{Lip}(\overline{\Omega}) \Longrightarrow \|\mathcal{S}_\nu^{\delta,0} f_0 - f_0\|_{L^\infty} \le \|u^\nu\|_{L^\infty} \|f_0\|_{\mathrm{Lip}} \cdot \delta, \tag{7.34}$$

and hence, again by uniform operator boundedness and denseness of $\operatorname{Lip}(\overline{\Omega})$,

$$\lim_{\delta \searrow 0} \mathcal{S}_\nu^{\delta,0} f_0 = f_0 \quad \text{in} \quad L^p\text{-norm}, \ \forall\, f_0 \in L^p(\Omega). \tag{7.35}$$

We now want to compare $\mathcal{S}^{t,0} f_0$ with $\mathcal{S}_\nu^{t,0} f_0$. To begin, take

$$f_0 \in \operatorname{Lip}(\overline{\Omega}), \quad f(t) = \mathcal{S}^{t,0} f_0. \tag{7.36}$$

Then $f(t)$ satisfies

$$\frac{\partial f}{\partial t} = -\nabla_{u^\nu(t)} f(t) + \nabla_{u^\nu(t) - u(t)} f(t), \quad f(0) = f_0, \tag{7.37}$$

so Duhamel's formula gives

$$f(t) = \mathcal{S}_\nu^{t,0} f_0 + \int_0^t \mathcal{S}_\nu^{t,s} \nabla_{u^\nu(s)-u(s)} f(s)\, ds. \tag{7.38}$$

Now, by hypothesis (7.3) on u, we see that, for $s \in [0,T]$,

$$\begin{aligned} f_0 \in \mathrm{Lip}(\overline{\Omega}) &\Longrightarrow \|f(s)\|_{\mathrm{Lip}} \le A\|f_0\|_{\mathrm{Lip}} \\ &\Longrightarrow |\nabla_{u^\nu(s)-u(s)} f(s)| \le A\|f_0\|_{\mathrm{Lip}} |u^\nu(s) - u(s)|. \end{aligned} \tag{7.39}$$

Hence

$$\begin{aligned} f_0 \in \mathrm{Lip}(\overline{\Omega}) \Longrightarrow & \|\mathcal{S}^{t,0} f_0 - \mathcal{S}_\nu^{t,0} f_0\|_{L^p} \\ & \le \int_0^t \|\mathcal{S}_\nu^{t,s} \nabla_{u^\nu(s)-u(s)} f(s)\|_{L^p}\, ds \\ & \le A\|f_0\|_{\mathrm{Lip}} \int_0^t \|u^\nu(s) - u(s)\|_{L^p}\, ds. \end{aligned} \tag{7.40}$$

Hence, given $p \in [1,\infty),\ t \in [0,T]$,

$$\begin{aligned} & u^\nu \to u \ \text{ in } \ L^1([0,T], L^p(\Omega)) \\ & \Longrightarrow \mathcal{S}_\nu^{t,0} f_0 \to \mathcal{S}^{t,0} f_0 \ \text{ in } \ L^p\text{-norm}, \end{aligned} \tag{7.41}$$

for all $f_0 \in \mathrm{Lip}(\overline{\Omega})$, and hence, by the uniform operator bounds (7.26) and denseness of $\mathrm{Lip}(\overline{\Omega})$ in $L^p(\Omega)$, we have:

Proposition 7.1. *Under the hypotheses (7.3)–(7.5), given $p \in [1,\infty),\ t \in [0,T]$, convergence in (7.41) holds for all $f_0 \in L^p(\Omega)$.*

In fact, we can improve Proposition 7.1, as follows. (Compare [DL], Theorem II.4.)

Proposition 7.2. *Given $p \in (1,\infty),\ t \in [0,T]$,*

$$\begin{aligned} & f_0 \in L^p(\Omega),\ u^\nu \to u \ \textit{ in } \ L^1([0,T], L^1(\Omega)) \\ & \Longrightarrow \mathcal{S}_\nu^{t,0} f_0 \to \mathcal{S}^{t,0} f_0 \ \textit{ in } \ L^p\textit{-norm.} \end{aligned} \tag{7.42}$$

Proof. By Proposition 7.1, the hypotheses of (7.42) imply

$$\mathcal{S}_\nu^{t,0} f_0 \to \mathcal{S}^{t,0} f_0 \tag{7.43}$$

in L^1-norm. We also know that

$$\|\mathcal{S}_\nu^{t,0} f_0\|_{L^p} = \|\mathcal{S}^{t,0} f_0\|_{L^p} = \|f_0\|_{L^p}, \tag{7.44}$$

for each $\nu \in (0,1],\ t \in [0,T]$. These bounds imply weak* compactness in $L^p(\Omega)$, and we see that convergence in (7.43) holds weak* in $L^p(\Omega)$. Then another use of (7.44), together with the *uniform convexity* of $L^p(\Omega)$ for each $p \in (1,\infty)$ gives convergence in L^p-norm in (7.43).

A. Regularity for the Stokes system on bounded domains

The following result is the basic ingredient in the proof of Proposition 5.2. Assume that $\overline{\Omega}$ is a compact, connected Riemannian manifold, with smooth boundary, that

$$u \in H^1(\Omega, T^*), \quad f \in L^2(\Omega, T^*), \quad p \in L^2(\Omega),$$

and that

$$-\Delta u = f + dp, \quad \delta u = 0, \quad u\big|_{\partial\Omega} = 0. \tag{A.1}$$

We claim that $u \in H^2(\Omega, T^*)$. More generally, we claim that, for $s \geq 0$,

$$f \in H^s(\Omega, T^*) \Longrightarrow u \in H^{s+2}(\Omega, T^*). \tag{A.2}$$

Indeed, given any $\lambda \in [0, \infty)$, it is an equivalent task to establish the implication (A.2) when we replace (A.1) by

$$(\lambda - \Delta)u = f + dp, \quad \delta u = 0, \quad u\big|_{\partial\Omega} = 0. \tag{A.3}$$

In this appendix we prove this result. We also treat the following related problem. Assume $v \in H^1(\Omega, T^*)$, $p \in L^2(\Omega)$, and

$$(\lambda - \Delta)v = dp, \quad \delta v = 0, \quad v\big|_{\partial\Omega} = g. \tag{A.4}$$

Then we claim that, for $s \geq 0$,

$$g \in H^{s+3/2}(\partial\Omega, T^*) \Longrightarrow v \in H^{s+2}(\Omega, T^*). \tag{A.5}$$

Here, for any $x \in \overline{\Omega}$ (including $x \in \partial\Omega$), $T^*_x = T^*_x(\overline{\Omega}) = T^*_x M$, where we take M to be a compact Riemannian manifold without boundary, containing Ω as an open subset (with smooth boundary $\partial\Omega$). In fact, take M to be diffeomorphic to the double of $\overline{\Omega}$.

We will represent solutions to (A.4) in terms of layer potentials, in a fashion parallel to constructions in §11 of Chapter 7. Such an approach is taken in [Sol1]; see also [Lad]. A different sort of proof, appealing to the theory of systems elliptic in the sense of Douglis-Nirenberg, is given in [Tem]. An extension of the boundary-layer approach to Lipschitz domains is given in [FKV]. This work has been applied to the Navier-Stokes equations on Lipschitz domains in [DW]. Here the analysis was restricted to Lipschitz domains with connected boundary. This topological restriction was removed in [MiT]. Subsequently, [Mon] produced strong, short time solutions on 3D domains with arbitrarily rough boundary.

Pick $\lambda \in (0, \infty)$. We now define some operators on $\mathcal{D}'(M)$, so that

$$(\lambda - \Delta)\Phi - dQ = I \text{ on } \mathcal{D}'(M, T^*), \quad \delta\Phi = 0. \tag{A.6}$$

To get these operators, start with the Hodge decomposition on M:

$$d\delta G + \delta dG + P_h = I \quad \text{on } \mathcal{D}'(M, \Lambda^*), \tag{A.7}$$

where P_h is the orthogonal projection onto the space $\mathcal{H}$ of harmonic forms on M, and G is Δ^{-1} on the orthogonal complement of $\mathcal{H}$. Then (A.6) holds if we set

$$\begin{aligned}\Phi &= (\lambda - \Delta)^{-1}(\delta dG + P_h) \in OPS^{-2}(M), \\ Q &= -\delta G \in OPS^{-1}(M).\end{aligned} \tag{A.8}$$

Let $F(x, y)$ and $Q(x, y)$ denote the Schwartz kernels of these operators. Thus

$$(\lambda - \Delta_x)F(x, y) - d_x Q(x, y) = \delta_y(x)I, \quad \delta_x F(x, y) = 0. \tag{A.9}$$

Note that as $\text{dist}(x, y) \to 0$, we have (for dim $\Omega = n \geq 3$)

$$\begin{aligned}F(x, y) &\sim A_0(x, y)\ \text{dist}(x, y)^{2-n} + \cdots, \\ Q(x, y) &\sim B_0(x, y)\ \text{dist}(x, y)^{1-n} + \cdots,\end{aligned} \tag{A.10}$$

where $A_0(\text{Exp}_y v, y)$ and $B_0(\text{Exp}_y v, y)$ are homogeneous of degree zero in $v \in T_y M$.

We now look for solutions to (A.4) in the form of layer potentials:

$$\begin{aligned}v(x) &= \int_{\partial\Omega} F(x, y)\ w(y)\ dS(y) = \mathcal{F}w(x), \\ p(x) &= \int_{\partial\Omega} Q(x, y) \cdot w(y)\ dS(y) = \mathcal{Q}w(x).\end{aligned} \tag{A.11}$$

The first two equations in (A.4) then follow directly from (A.9), and the last equation in (A.4) is equivalent to

$$\Psi w = g, \tag{A.12}$$

where

$$\Psi w(x) = \int_{\partial\Omega} F(x, y) w(y)\ dS(y), \quad x \in \partial\Omega, \tag{A.13}$$

defines

$$\Psi \in OPS^{-1}(\partial\Omega, T^*). \tag{A.14}$$

Note that Ψ is self-adjoint on $L^2(\partial\Omega, T^*)$. The following lemma is incisive:

Lemma A.1. *The operator Ψ is an elliptic operator in $OPS^{-1}(\partial\Omega)$.*

We can analyze the principal symbol of Ψ using the results of §11 in Chapter 7, particularly the identity (11.12) there. This implies that, for $x \in \partial\Omega$, $\xi \in T_x(\partial\Omega)$, ν the outgoing unit normal to $\partial\Omega$ at x,

$$\sigma_\Psi(x, \xi) = C_n \int_{-\infty}^{\infty} \sigma_\Phi(x, \xi + \tau\nu)\ d\tau. \tag{A.15}$$

From (A.8), we have

$$\sigma_\Phi(x,\zeta)\beta = |\zeta|^{-4}\iota_\zeta \wedge_\zeta \beta, \quad \zeta,\beta \in T_x^* M. \tag{A.16}$$

This is equal to $|\zeta|^{-2}P_\zeta^\perp \beta$, where $P_\zeta^\perp$ is the orthogonal projection of T_x^* onto $(\zeta)^\perp$. Thus

$$\sigma_\Phi(x,\zeta)\beta = A(\zeta)\beta - B(\zeta)\beta, \tag{A.17}$$

with

$$A(\zeta)\beta = |\zeta|^{-2}\beta, \quad B(\zeta)\beta = |\zeta|^{-4}(\beta\cdot\zeta)\zeta.$$

Hence

$$\int_{-\infty}^{\infty} A(\xi+\tau\nu)\, d\tau = \int_{-\infty}^{\infty} (|\xi|^2+\tau^2)^{-1}\, d\tau = \gamma_1 |\xi|^{-1}, \tag{A.18}$$

with

$$\gamma_1 = \int_{-\infty}^{\infty} \frac{1}{1+\tau^2}\, d\tau.$$

Also
(A.19)

$$\begin{aligned}\int_{-\infty}^{\infty} B(\xi+\tau\nu)\beta\, d\tau &= \int_{-\infty}^{\infty} \big(|\xi|^2+\tau^2\big)^{-2}\big[(\beta\cdot\xi)\xi + \tau^2(\beta\cdot\nu)\nu\big]\, d\tau \\ &= \gamma_2|\xi|^{-3}(\beta\cdot\xi)\xi + \gamma_3|\xi|^{-1}(\beta\cdot\nu)\nu,\end{aligned}$$

with

$$\gamma_2 = \int_{-\infty}^{\infty} \frac{1}{(1+\tau^2)^2}\, d\tau, \quad \gamma_3 = \int_{-\infty}^{\infty} \frac{\tau^2}{(1+\tau^2)^2}\, d\tau. \tag{A.20}$$

We have

$$\sigma_\Psi(x,\xi) = C_n|\xi|^{-1}\big[\gamma_1 I - \gamma_2 P_\xi - \gamma_3 P_\nu\big], \tag{A.21}$$

where P_ξ is the orthogonal projection of T_x^* onto the span of ξ, and P_ν is similarly defined. Note that $\gamma_2+\gamma_3=\gamma_1,\ 0<\gamma_j$. Hence $0<\gamma_2<\gamma_1$ and $0<\gamma_3<\gamma_1$. In fact, use of residue calculus readily gives

$$\gamma_1 = \pi, \quad \gamma_2 = \frac{\pi}{4}, \quad \gamma_3 = \frac{3\pi}{4}.$$

Thus the symbol (A.21) is invertible, in fact positive-definite. Lemma A.1 is proved.

We also have, for any $\sigma \in \mathbb{R}$,

$$\Psi : H^\sigma(\partial\Omega, T^*) \longrightarrow H^{\sigma+1}(\partial\Omega, T^*), \text{ Fredholm, of index zero.} \tag{A.22}$$

We next characterize Ker Ψ, which we claim is a one-dimensional subspace of $C^\infty(\partial\Omega, T^*)$.

The ellipticity of Ψ implies that Ker Ψ is a finite-dimensional subspace of $C^\infty(\partial\Omega, T^*)$. If $w \in$ Ker Ψ, consider $v = \mathcal{F}w,\ p = \mathcal{Q}w$, defined by

(A.11), on $\Omega \cup \mathcal{O}$ (where $\mathcal{O} = M \setminus \overline{\Omega}$). We have $(\lambda - \Delta)v = dp$ on Ω, $\delta v = 0$ on Ω, and $v\big|_{\partial\Omega} = 0$, so, since solutions to (A.3) are unique for any $\lambda > 0$, we deduce that $v = 0$ on Ω. Similarly, $v = 0$ on $\mathcal{O}$. In other words,

$$\Phi(w\sigma) = 0 \quad \text{on } \Omega \cup \mathcal{O}, \tag{A.23}$$

where σ is the area element of $\partial\Omega$, so $w\sigma$ is an element of $\mathcal{D}'(M, T^*)$, supported on $\partial\Omega$. Since $\Phi \in OPS^{-2}(M)$, $\Phi(w\sigma) \in C(M, T^*)$, so (A.23) implies $\Phi(w\sigma) = 0$ on M. Consequently, by (A.6),

$$w\sigma = dQ(w\sigma) \text{ on } M. \tag{A.24}$$

The right side is equal to $d\delta G(w\sigma) = P_d(w\sigma)$. It follows that $d(w\sigma) = 0$, which uniquely determines w, up to a constant scalar multiple, on each component of $\partial\Omega$, namely as a constant multiple of ν. It follows that

$$w \in \text{ Ker } \Psi \Longleftrightarrow w\sigma = C \, d\chi_\Omega, \tag{A.25}$$

for some constant C, assuming Ω and $\mathcal{O}$ are connected. In our situation, $\mathcal{O}$ is diffeomorphic to Ω, which is assumed to be connected.

Consequently, whenever $g \in H^{s+3/2}(\partial\Omega, T^*)$ satisfies

$$\int_{\partial\Omega} \langle g, \nu \rangle \, dS = 0, \tag{A.26}$$

the unique solution to (A.4) is given by (A.11), with

$$w \in H^{s+1/2}(\partial\Omega, T^*). \tag{A.27}$$

Note that if $\delta v = 0$ on Ω and $v\big|_{\partial\Omega} = g$, then the divergence theorem implies that (A.26) holds. Thus this construction applies to all solutions of (A.4).

Next we reduce the analysis of (A.3) to that of (A.4). Thus, let $f \in H^s(\Omega, T^*)$. Extend f to $\widetilde{f} \in H^s(M, T^*)$. Now let $u_1 \in H^{s+2}(M, T^*)$, $p_1 \in H^{s+1}(M)$ solve

$$(\lambda - \Delta)u_1 = \widetilde{f} + dp_1, \quad \delta u_1 = 0 \quad \text{on } M, \tag{A.28}$$

hence $u_1 = \Phi\widetilde{f}$ and $p_1 = Q\widetilde{f}$. If u solves (A.3), take $v = u - u_1\big|_\Omega$, which solves (A.4), with p replaced by $p - p_1$, and

$$g = -u_1\big|_{\partial\Omega} \in H^{s+3/2}(\partial\Omega, T^*). \tag{A.29}$$

Furthermore, since $\delta u_1 = 0$ on M, we have (A.26), as remarked above.

We are in a position to establish the results stated at the beginning of this appendix, namely:

Proposition A.2. *Assume $u, v \in H^1(\Omega, T^*)$, $f \in L^2(\Omega, T^*)$, $p \in L^2(\Omega)$, and $\lambda > 0$. If*

$$(\lambda - \Delta)u = f + dp, \quad \delta u = 0, \quad u\big|_{\partial\Omega} = 0, \tag{A.30}$$

then, for $s \geq 0$,

$$f \in H^s(\Omega, T^*) \Longrightarrow u \in H^{s+2}(\Omega, T^*), \tag{A.31}$$

and if

$$(\lambda - \Delta)v = dp, \quad \delta v = 0, \quad v\big|_{\partial\Omega} = g, \tag{A.32}$$

then, for $s \geq 0$,

$$g \in H^{s+3/2}(\partial\Omega, T^*) \Longrightarrow v \in H^{s+2}(\Omega, T^*). \tag{A.33}$$

Proof. As seen above, it suffices to deduce (A.33) from (A.32), and we can assume g satisfies (A.26), so

$$v(x) = \int_{\partial\Omega} F(x,y)w(y)\, dS(y), \quad x \in \Omega, \tag{A.34}$$

where $F(x,y)$ is the Schwartz kernel of the operator Φ in (A.6)–(A.8), and

$$g \in H^{s+3/2}(\partial\Omega, T^*) \Longrightarrow w \in H^{s+1/2}(\partial\Omega, T^*). \tag{A.35}$$

Now $V = dv$ satisfies

$$\begin{gathered}(\lambda - \Delta)V = 0, \\ V(x) = \lim_{x' \to x, x' \in \Omega} \int_{\partial\Omega} d_x F(x', y)w(y)\, dS(y) = \mathcal{G}w(x), \quad x \in \partial\Omega,\end{gathered} \tag{A.36}$$

where, parallel to Proposition 11.3 of Chapter 7, we have

$$\mathcal{G} \in OPS^0(\partial\Omega). \tag{A.37}$$

Hence (A.35) implies $\mathcal{G}w \in H^{s+1/2}(\partial\Omega, \Lambda^2 T^*)$. Now standard estimates for the Dirichlet problem (A.36) yield $V \in H^{s+1}(\Omega)$ if $w \in H^{s+1/2}(\partial\Omega)$; hence, if v satisfies (A.32),

$$\Delta v = \delta V \in H^s(\Omega), \quad v\big|_{\partial\Omega} = g, \tag{A.38}$$

and regularity for the Dirichlet problem yields the desired conclusion (A.33). Thus Proposition A.2 is proved.

References

[BaC] H. Bahouri and J. Chemin, Equations de transport relatives a des champs de vecteurs non-lipschitziens et mecanique des fluides, *Arch. Rat. Mech. Anal.* 127(1994), 159–181.

[BM] J. Ball and D. Marcus, Vorticity intensification and transition to turbulence in the three-dimensional Euler equations, *Comm. Math. Phys.* 147(1992), 371–394.

[Bar] C. Bardos, Existence et unicité de la solution de l'équation d'Euler en dimension deux, *J. Math. Anal. Appl.* 40(1972), 769–790.

[Bat] G. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge Univ. Press, Cambridge, 1967.

[BG] J. T. Beale and C. Greengard, Convergence of Euler-Stokes splitting of the Navier-Stokes equations, Preprint, 1992.

[BKM] J. T. Beale, T. Kato, and A. Majda, Remarks on the breakdown of smooth solutions for the 3-d Euler equations, *Comm. Math. Phys.* 94(1984), 61–66.

[BeC] A. Bertozzi and P. Constantin, Global regularity for vortex patches, *Comm. Math. Phys.* 152(1993), 19–28.

[BW] J. Bona and J. Wu, The zero-viscosity limit of the 2D Navier-Stokes equations, *Studies in Applied Math.* 109 (2002), 265–278.

[Bon] V. Bondarevsky, On the global regularity problem for 3-dimensional Navier-Stokes equations, Preprint, 1995.

[BBr] J. Bourguignon and H. Brezis, Remarks on the Euler equations, *J. Func. Anal.* 15(1974), 341–363.

[CKN] L. Caffarelli, R. Kohn, and L. Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations, *Comm. Pure Appl. Math.* 35(1982), 771–831.

[Cha] D. Chae, Weak solutions of the 2-D Euler equations with initial vorticity in L(log L), *J. Diff. Eqs.* 103(1993), 323–337.

[Che1] J. Chemin, Remarques sur l'éxistence globale pour le système de Navier-Stokes incompressible, *SIAM J. Math. Anal.* 23 (1992), 20–28.

[Che2] J. Chemin, Persistence des structures géometriques dans les fluides incompressibles bidimensionnels, *Ann. Ecole Norm Sup. Paris* 26(1993), 517–542.

[Che3] J. Chemin, Fluides Parfaits Incompressibles, *Asterisque* #230, Société Math. de France, 1995.

[ChL] J. Chemin and N. Lerner, Flot de champs de vecteurs non-lipschitziens et équations de Navier-Stokes, *Publ. CNRS* #1062, 1993.

[Cho] A. Chorin, *Vorticity and Turbulence*, Springer-Verlag, New York, 1994.

[ChM] A. Chorin and J. Marsden, *A Mathematical Introduction to Fluid Mechanics*, Springer-Verlag, New York, 1979.

[CFe] P. Constantin and C. Fefferman, Direction of vorticity and the problem of global regularity for the Navier-Stokes equations, *Indiana Math. J.* 42(1993), 775–790.

[CFo] P. Constantin and C. Foias, *Navier-Stokes Equations*, Chicago Lectures in Math., Univ. of Chicago Press, 1988.

[CLM] P. Constantin, P. Lax, and A. Majda, A simple one-dimensional model for the three-dimensional vorticity equation, *CPAM* 38(1985), 715–724.

[Del] J. Delort, Existence de nappes de tourbillon en dimension deux, *J. AMS* 4(1991), 553–586.

[DW] P. Deuring and W. von Wahl, Strong solutions of the Navier-Stokes system in Lipschitz bounded domains, *Math. Nachr.* 171(1995), 111–198.

[DL] R. Diperna and P.-L. Lions, Ordinary differential equations, transport theory, and Sobolev spaces, *Invent. Math.* 98 (1989), 511–547.

[DM] R. DiPerna and A. Majda, Concentration in regularizations for 2-D incompressible flow, *CPAM* 40(1987), 301–345.

[Eb] D. Ebin, A concise presentation of the Euler equations of hydrodynamics, *Comm. PDE* 9(1984), 539–559.

[EbM] D. Ebin and J. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, *Annals of Math.* 92(1970), 102–163.

[EM] L. Evans and S. Müller, Hardy spaces and the two-dimensional Euler equations with nonnegative vorticity, *J. AMS* 7(1994), 199–219.

[FJR] E. Fabes, B. F. Jones, and N. Riviere, The initial boundary value problem for the Navier-Stokes equation with data in L^p, *Arch. Rat. Mech. Anal.* 45(1972), 222–240.

[FKV] E. Fabes, C. Kenig, and G. Verchota, The Dirichlet problem for the Stokes system on Lipschitz domains, *Duke Math. J.* 57(1988), 769–793.

[Fed] P. Federbush, Navier and Stokes meet the wavelet, *Comm. Math. Phys.* 155(1993), 219–248.

[Fer] A. Ferrari, On the blow-up of the 3-D Euler equation in a bounded domain, *Comm. Math. Phys.* 155(1993), 277–294.

[FGT] C. Foias, C. Guillope, and R. Temam, Lagrangian representation of a flow, *J. Diff. Eqs.* 57(1985), 440–449.

[FT] C. Foias and R. Temam, Some analytical and geometric properties of the solutions of the evolution Navier-Stokes equations, *J. Math. Pures et Appl.* 58(1979), 339–368.

[FK] H. Fujita and T. Kato, On the Navier-Stokes initial value problem, *Arch. Rat. Mech. Anal.* 16(1964), 269–315.

[FM] H. Fujita and H. Morimoto, On fractional powers of the Stokes operator, *Proc. Japan Acad.* 16(1970), 1141–1143.

[GM1] Y. Giga and T. Miyakawa, Solutions in L_r of the Navier-Stokes initial value problem, *Arch. Rat. Mech. Anal.* 89(1985), 267–281.

[GS] G. Grubb and V. Solonnikov, Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods, *Math. Scand.* 69(1991), 217–290.

[Hel] H. Helmholtz, On the integrals of the hydrodynamical equations that express vortex motion, *Phil. Mag.* 33(1887), 485–512.

[Hop] E. Hopf, Uber die Anfangwertaufgabe fur die hydrodynamischen Grundgleichungen, *Math. Nachr.* 4(1951), 213–231.

[HM] T. Hughes and J. Marsden, *A Short Course in Fluid Mechanics*, Publish or Perish Press, Boston, 1976.

[Kt1] T. Kato, On classical solutions of two dimensional nonstationary Euler equations, *Arch. Rat. Mech. Anal.* 25(1967), 188–200.

[Kt2] T. Kato, Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$, *J. Func. Anal.* 9(1972), 296–305.

[Kt3] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, *Springer LNM* 448(1974), 25–70.

[Kt4] T. Kato, Strong L^p-solutions to the Navier-Stokes equations in $\mathbb{R}^m$, with applications to weak solutions, *Math. Zeit.* 187(1984), 471–480.

[Kt5] T. Kato, Strong solutions of the Navier-Stokes equation in Morrey spaces, *Bol. Soc. Bras. Mat.* 22(1992), 127–155.

[Kt6] T. Kato, The Navier-Stokes equation for an incompressible fluid in $\mathbb{R}^2$ with a measure as the initial vorticity, *Diff. Integral Equations* 7 (1994), 949–966.

[Kt7] T. Kato, Remarks on the zero viscosity limit for nonstationary Navier-Stokes flows with boundary, *Seminar on PDE* (S. S. Chern, ed.), Springer-Verlag, New York, 1984.

[KL] T. Kato and C. Lai, Nonlinear evolution equations and the Euler flow, *J. Func. Anal.* 56(1984), 15–28.

[KP] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, *CPAM* 41(1988), 891–907.

[Kel] J. Kelliher, Vanishing viscosity and the accumulation of vorticity on the boundary, *Commun. Math. Sci.* 6 (2008), 869–880.

[Lad] O. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969.

[Lam] H. Lamb, *Hydrodynamics*, Dover, New York, 1932.

[Ler] J. Leray, Etude de diverses équations integrales non linéaires et de quelques problèmes que pose d'hydrodynamique, *J. Math. Pures et Appl.* 12(1933), 1–82.

[Li] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Lineaires*, Dunod, Paris, 1969.

[LMN] M. Lopes Filho, A. Mazzucato, and H. Nessenzveig Lopes, Vanishing viscosity limits for incompressible flow inside rotating circles, *Physica D. Nonlin. Phenom.* 237 (2008), 1324–1333.

[LMNT] M. Lopes Filho, A. Mazzucato, H. Nussenzveig Lopes, and M. Taylor, Vanishing viscosity limits and boundary layers for circularly symmetric 2D flows, *Bull. Braz. Math. Soc.* 39 (2008), 471–513.

[Mj] A. Majda, Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables, *Appl. Math. Sci.* #53, Springer-Verlag, 1984.

[Mj2] A. Majda, Vorticity and the mathematical theory of incompressible fluid flow, *CPAM* 38(1986), 187–220.

[Mj3] A. Majda, Mathematical fluid dynamics: The interaction of nonlinear analysis and modern applied mathematics, *Proc. AMS Centennial Symp.* (1988), 351–394.

[Mj4] A. Majda, Vorticity, turbulence, and acoustics in fluid flow, *SIAM Review* 33(1991), 349–388.

[Mj5] A. Majda, Remarks on weak solutions for vortex sheets with a distinguished sign, *Indiana Math. J.* 42(1993), 921–939.

[MP] C. Marchioro and M. Pulvirenti, *Mathematical Theory of Incompressible Nonviscous Fluids*, Springer-Verlag, New York, 1994.

[Mat] S. Matsui, Example of zero viscosity limit for two-dimensional nonstationary Navier-Stokes flow with boundary, *Japan J. Indust. Appl. Math.* 11 (1994), 155–170.

[MT1] A. Mazzucato and M. Taylor, Vanishing viscosity plane parallel channel flow and related singular perturbation problems, *Analysis and PDE* 1 (2008), 35–93.

[MT2] A. Mazzucato and M. Taylor, Vanishing viscosity limits for a class of circular pipe flows, Preprint, 2010.

[MF] R. von Mises and K. O. Friedrichs, *Fluid Dynamics*, Appl. Math. Sci. 5, Springer-Verlag, New York, 1971.

[MiT] M. Mitrea and M. Taylor, Navier-Stokes equations on Lipschitz domains in Riemannian manifolds, *Math. Annalen* 321 (2001), 955–987.

[Mon] S. Monniaux, Navier-Stokes equations in arbitrary domains: the Fujita-Kato scheme, *Math. Res. Letters* 13 (2006), 455–461.

[OO] H. Ockendon and J. Ockendon, *Viscous Flow*, Cambridge Univ. Press, Cambridge, 1995.

[OT] H. Ockendon and A. Tayler, *Inviscid Fluid Flow*, Appl. Math. Sci. #43, Springer-Verlag, New York, 1983.

[PT] L. Prandtl and O. Tietjens, *Applied Hydro- and Aerodynamics*, Dover, New York, 1934.

[Saf] P. Saffman, *Vortex Dynamics*, Cambridge Univ. Press, Cambridge, 1992.

[Sch] H. Schlichting, *Boundary Layer Theory*, 8th ed., Springer-Verlag, New York, 2000.

[Se1] J. Serrin, Mathematical principles of classical fluid dynamics, *Encycl. of Physics*, Vol. 8, pt. 1, pp. 125–263, Springer-Verlag, New York, 1959.

[Se2] J. Serrin, The initial value problem for the Navier-Stokes equations, in *Non-linear Problems*, R.E.Langer, ed., Univ. of Wisc. Press, Madison, Wisc., 1963, pp. 69–98.

[Sol1] V. Solonnikov, On estimates of the tensor Green's function for some boundary-value problems, *Dokl. Akad. Nauk SSSR* 130(1960), 988–991.

[Sol2] V. Solonnikov, Estimates for solutions of nonstationary Navier-Stokes equations, *J. Soviet Math.* 8(1977), 467–529.

[T1] M. Taylor, *Pseudodifferential Operators and Nonlinear PDE*, Birkhäuser, Boston, 1991.

[T2] M. Taylor, Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations, *Comm. PDE* 17(1992), 1407–1456.

[Tem] R. Temam, *Navier-Stokes Equations*, North-Holland, New York, 1977.

[Tem2] R. Temam, On the Euler equations of incompressible perfect fluids, *J. Func. Anal.* 20(1975), 32–43.

[Tem3] R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, SIAM, Philadelphia, Penn., 1983.

[VD] M. Van Dyke, *An Album of Fluid Motion*, Parabolic Press, Stanford, Calif., 1982.

[vWa] W. von Wahl, *The Equations of Navier-Stokes and Abstract Parabolic Equations*, Vieweg & Sohn, Braunschweig, 1985.

[W] X. Wang, A Kato type theorem on zero viscosity limit of Navier-Stokes flows, *Indiana Univ. Math. J.* 50 (2001), 223–241.

[Wol] W. Wolibner, Un théorème d'existence du mouvement plan d'un fluide parfait, homogene, incompressible, pendant un temps infiniment long, *Math. Zeit.* 37(1933), 698–726.

[Yud] V. Yudovich, Non-stationary flow of an ideal incompressible fluid, *J. Math. and Math. Phys.* 3(1963), 1032–1066.