Singular Integrals and Elliptic Boundary Problems on Rough Domains

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Abstract. These notes discuss results on layer potential methods for elliptic boundary problems, with emphasis on the Dirichlet problem for the Laplace operator. They start by reviewing results for domains with moderately smooth boundary, then for Lipschitz domains, and proceed to discuss results in [HMT], obtained with S. Hofmann and M. Mitrea, for a class of domains we call regular Semmes-Kenig-Toro (SKT) domains, often called chord-arc domains with vanishing constant, and for ε-regular SKT domains, often called chord-arc domains with small constant.

Contents

1. Introduction

We describe results on using layer potentials to solve elliptic boundary problems, with emphasis on the Dirichlet problem

$$Lu = 0 \quad \text{on} \quad \Omega, \qquad u\big|_{\partial\Omega} = f. \tag{1.1}$$

Here we take $L = \Delta - V$. We assume Ω is an open subset of a compact, connected, n-dimensional Riemannian manifold M, whose metric tensor has some moderate regularity. The Laplace-Beltrami operator on M is Δ. We assume $V \in L^\infty(M)$, $V \geq 0$ on M, and $V > 0$ on a set of positive measure on each connected component of $M \setminus \overline{\Omega}$ (which we assume to be nonempty).

The double layer potential attack on (1.1) starts as follows. Under the hypotheses above, $L : H^1(M) \to H^{-1}(M)$ is an isomorphism, whose inverse has an integral kernel:

$$L^{-1}w(x) = \int\limits_M E(x,y)w(y)\, d\mathcal{V}(y), \tag{1.2}$$

$d\mathcal{V}$ denoting the volume element of M. One seeks a function g on $\partial\Omega$ such that (1.1) is solved by

$$u = \mathcal{D}g\big|_{\Omega}, \tag{1.3}$$

where

$$\begin{aligned} \mathcal{D}g(x) &= \int\limits_{\partial\Omega} \partial_{\nu_y} E(x,y) g(y)\, d\sigma(y) \\ &= \int\limits_{\partial\Omega} k(x,y) g(y)\, d\sigma(y), \end{aligned} \tag{1.4}$$

$d\sigma$ denoting the area element of $\partial\Omega$, and ν the outward pointing unit normal to $\partial\Omega$.

In order to survey the structure of the boundary layer approach to (1.1) fairly quickly, we will start with a situation where an analysis of $\mathcal{D}g|_{\partial\Omega}$ is fairly elementary, due to some modest regularity of $\partial\Omega$. Our main goal will then be to discuss the analysis on somewhat rougher domains.

So, for now, we assume $\partial\Omega$ is locally the graph of a C^1 function whose gradient has a modulus of continuity ω; we say $\partial\Omega$ is of class $C^{1,\omega}$. The leading part of the integral kernel in (1.4) behaves like

$$k(x,y) = \langle \nu(y), x-y \rangle\, |x-y|^{-n}, \tag{1.5}$$

in local coordinates, and under this $C^{1,\omega}$ hypothesis, we have

$$|\langle \nu(y), x-y\rangle| \le C\omega(|x-y|)|x-y|, \quad \forall\, x,y \in \partial\Omega, \tag{1.6}$$

hence

$$|k(x,y)| \le C\frac{\omega(d(x,y))}{d(x,y)^{n-1}}, \quad \forall\, x,y \in \partial\Omega. \tag{1.7}$$

Of course, such an estimate does not hold for general $x \in \Omega,\ y \in \partial\Omega$.

Proposition 1.1. *If $k(x,y)$ satisfies (1.7) on $\partial\Omega \times \partial\Omega$, then*

$$Kg(x) = \int\limits_{\partial\Omega} k(x,y) g(y)\, d\sigma(y) \tag{1.8}$$

has the property

$$K : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \ \text{is compact, for} \ \ 1 < p < \infty, \tag{1.9}$$

provided ω satisfies the Dini condition

$$\int_0^1 \frac{\omega(t)}{t}\, dt < \infty. \tag{1.10}$$

Proof. The Schur condition for continuity on $L^p(\partial\Omega)$, for $1 \le p \le \infty$, is

$$\int\limits_{\partial\Omega} |k(x,y)|\, d\sigma(y) \le A, \quad \forall\, x, \tag{1.11}$$

with the same sort of estimate with the roles of x and y reversed. (Then the operator norm is $\le A$.) Now, we dominate the left side of (1.11) by

$$\begin{aligned} C\int\limits_{\partial\Omega} \frac{\omega(d(x,y))}{d(x,y)^{n-1}}\, d\sigma(y) &\approx \sum_{k\ge 0} \int\limits_{\{y: d(x,y)\approx 2^{-k}\}} \frac{\omega(d(x,y))}{d(x,y)^{n-1}}\, d\sigma(y) \\ &\approx \sum_{k\ge 0} \omega(2^{-k}) 2^{k(n-1)} \sigma(\{y : 2^{-k-1} \le d(x,y) \le 2^{-k}\}) \\ &\approx \sum_{k\ge 0} \omega(2^{-k}) 2^{k(n-1)} 2^{-k(n-1)} \\ &= \sum_{k\ge 0} \omega(2^{-k}), \end{aligned} \tag{1.12}$$

and the finiteness of the last sum is equivalent to (1.10). Note that we pass from the second line of (1.12) to the third via

$$\sigma(\{y \in \partial\Omega : d(x,y) \le r\}) \le Cr^{n-1}, \tag{1.13}$$

an estimate that will play an important role in rougher situations.

The argument above gives continuity on $L^p(\partial\Omega)$. To get compactness, one uses the following result.

(1.14) If $|k(x,y)| \le M < \infty$ on $\partial\Omega \times \partial\Omega$, then (1.9) holds.

In fact, this hypothesis implies K is Hilbert-Schmidt, hence compact, on $L^2(\partial\Omega)$, and compactness on L^p for $1 < p < \infty$ follows by interpolation with boundedness on L^1 and L^∞. To get compactness for K satisfying (1.7), one can write

$$k(x,y) = k^{\#}(x,y) + k^b(x,y), \tag{1.15}$$

with $k^{\#}$ as in (1.14) and k^b as in (1.11), with A small, and pass to the limit. This concludes the proof of Proposition 1.1.

It is not hard to show that $\mathcal{D}g|_\Omega$ extends continuously to $\overline{\Omega}$, if g is sufficiently regular. For example,

$$g \in \mathrm{Lip}(\partial\Omega) \Longrightarrow \mathcal{D}g\big|_{\overline{\Omega}_\pm} \in C(\overline{\Omega}_\pm). \tag{1.16}$$

Here and below,

$$\Omega_+ = \Omega, \quad \Omega_- = M \setminus \overline{\Omega}. \tag{1.17}$$

Also, for sufficiently regular g,

$$\mathcal{D}g\big|_{\partial\Omega_\pm} = \Big(\pm\frac{1}{2}I + K\Big)g, \tag{1.18}$$

where K is as in Proposition 1.1, with $k(x,y) = \partial_{\nu_y}E(x,y)$. See Proposition 3.4 for such a result in a rougher setting.

Now we want to use the boundary layer approach to solve (1.1), not for regular f (which can be treated by variational methods, or by maximum principle and barrier arguments), but for rough f, such as $f \in L^p(\partial\Omega)$, a situation for which Proposition 1.1 is relevant. One desires to extend (1.18) to similarly rough g. What one gets is nontangential convergence

$$\lim_{y\to x, y\in\Gamma_\pm(x)} \mathcal{D}g(y) = \Big(\pm\frac{1}{2}I + K\Big)g(x), \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{1.19}$$

Here, for $x \in \partial\Omega$,

$$\Gamma_\pm(x) = \{y \in \Omega_\pm : d(y,x) \le 2\,\mathrm{dist}(y,\partial\Omega)\}. \tag{1.20}$$

The result (1.19) follows from the validity of (1.18) for $g \in \mathrm{Lip}(\partial\Omega)$, which is dense in $L^p(\partial\Omega)$, together with an estimate on the nontangential maximal function

$$\mathcal{N}\mathcal{D}g(x) = \sup_{y \in \Gamma_\pm(x)} |\mathcal{D}g(x)|, \tag{1.21}$$

namely, for $1 < p < \infty$,

$$\|\mathcal{N}\mathcal{D}g\|_{L^p(\partial\Omega)} \le C_p \|g\|_{L^p(\partial\Omega)}. \tag{1.22}$$

Behind (1.22) are estimates of the following sort. Set

$$K_\varepsilon g(x) = \int\limits_{\{y \in \partial\Omega : d(x,y) > \varepsilon\}} k(x,y) g(y)\, d\sigma(y), \tag{1.23}$$

for $\varepsilon > 0$, with $k(x,y)$ as in (1.4), and

$$K_* g(x) = \sup_{\varepsilon > 0} |K_\varepsilon g(x)|. \tag{1.24}$$

Then there exists $C < \infty$ such that

$$\mathcal{N}\mathcal{D}g(x) \le K_* g(x) + C\mathcal{M}g(x), \tag{1.25}$$

where $\mathcal{M}g$ is the Hardy-Littlewood maximal function

$$\mathcal{M}g(x) = \sup_{r>0} \frac{1}{\sigma(B_r(x))} \int\limits_{\partial\Omega} |g(y)|\, d\sigma(y), \quad x \in \partial\Omega. \tag{1.26}$$

It is classical that, for $1 < p < \infty$,

$$\|\mathcal{M}g\|_{L^p(\partial\Omega)} \le C\|g\|_{L^p(\partial\Omega)}, \tag{1.27}$$

and we also have

$$\|K_* g\|_{L^p} \le C\|g\|_{L^p(\partial\Omega)}, \tag{1.28}$$

which, together with (1.27) and (1.25), gives (1.22). In the case of $C^{1,\omega}$ boundary, if (1.10) holds, Proposition 1.1 applies both to K and to

$$\widetilde{K}g(x) = \int\limits_{\partial\Omega} |k(x,y)| g(y)\, d\sigma(y). \tag{1.29}$$

Then, since

$$K_* g(x) \le \widetilde{K}|g|(x), \tag{1.30}$$

the result (1.28) readily follows. In more general situations, which we will discuss below and in subsequent sections, this last part will not apply, but (1.25)–(1.28) will hold, in substantial generality; cf. §3.

In addition to the double layer potential (1.4), the single layer potential, defined by

$$\mathcal{S}g(x) = \int\limits_{\partial\Omega} E(x,y)g(y)\,d\sigma(y), \tag{1.31}$$

plays an important role in the study of (1.1). In such a case, one has, if $\partial\Omega$ is $C^{1,\omega}$, and (1.10) holds,

$$g \in \mathrm{Lip}(\partial\Omega) \Longrightarrow \nabla \mathcal{S}g\big|_{\overline{\Omega}_\pm} \in C(\overline{\Omega}_\pm), \tag{1.32}$$

complementing (1.16). Furthermore, complementing (1.22), for $1 < p < \infty$,

$$\|\mathcal{N}(\nabla \mathcal{S}g)\|_{L^p(\partial\Omega)} \le C\|g\|_{L^p(\partial\Omega)}, \tag{1.33}$$

and, complementing (1.19),

$$\lim_{y\to x, y\in\Gamma_\pm(x)} \partial_{\nu_\pm}\mathcal{S}g(y) = \Big(\mp\frac{1}{2}I + K^*\Big)g(x), \quad \sigma\text{-a.e. } x \in \partial\Omega, \tag{1.34}$$

where K^* is the adjoint of K in (1.19). Hence, for $\partial\Omega$ of class $C^{1,\omega}$, given (1.10), $K^* : L^p(\partial\Omega) \to L^p(\partial\Omega)$ is compact for $1 < p < \infty$.

With this information in hand, we return to the issue of finding a solution to (1.1) of the form (1.3). Clearly any such $u = \mathcal{D}g$ satisfies $Lu = 0$ on $M \setminus \partial\Omega$. By (1.19),

$$u\big|_{\partial\Omega_+} = \Big(\frac{1}{2}I + K\Big)g. \tag{1.35}$$

Thus (1.1) holds provided g solves the equation

$$\Big(\frac{1}{2}I + K\Big)g = f. \tag{1.36}$$

Such solvability is guaranteed by the following.

Proposition 1.2. *If $\partial\Omega$ is $C^{1,\omega}$ and (1.10) holds, then, for all $p \in (1,\infty)$,*

$$\frac{1}{2}I + K : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \ \textit{is an isomorphism.} \tag{1.37}$$

In our current situation, thanks to the compactness of K, we know that $(1/2)I + K$ is Fredholm on $L^p(\partial\Omega)$, of index 0, for each such p. The following is key to proving Proposition 1.2.

Proposition 1.3. *In the setting of Proposition 1.2,*

$$\frac{1}{2}I + K^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \quad \text{is injective,} \tag{1.38}$$

for $p = 2$.

Once this is proven, we also have injectivity in (1.38) for $p \geq 2$. Hence Fredholm theory implies we have an isomorphism in (1.38), for $2 \leq p < \infty$, so we have an isomorphism in (1.37) for $1 < p \leq 2$. Hence the map (1.37) is injective for all $p \in (1, \infty)$, and Fredholm theory implies isomorphism.

The proof of Proposition 1.3 goes as follows. Take $g \in L^2(\partial\Omega)$ and suppose

$$\Bigl(\frac{1}{2}I + K^*\Bigr)g = 0. \tag{1.39}$$

Set

$$u = \mathcal{S}g. \tag{1.40}$$

By (1.34),

$$\partial_\nu \mathcal{S}g\big|_{\partial\Omega_-} = \Bigl(\frac{1}{2}I + K^*\Bigr)g = 0. \tag{1.41}$$

Now we take $v = u\nabla u$, so

$$\operatorname{div} v = |\nabla u|^2 + u\Delta u = |\nabla u|^2 + Vu^2 \quad \text{on} \quad M \setminus \partial\Omega. \tag{1.42}$$

We apply the Gauss-Green theorem

$$\int\limits_{\Omega} \operatorname{div} v \, d\mathcal{V} = \int\limits_{\partial\Omega} \langle \nu, v\rangle \, d\sigma, \tag{1.43}$$

with Ω replaced by Ω_-, to get

$$\int\limits_{\Omega_-} (|\nabla u|^2 + Vu^2) \, d\mathcal{V} = \int\limits_{\partial\Omega_-} u\Bigl(\frac{1}{2}I + K^*\Bigr)g \, d\sigma = 0, \tag{1.44}$$

hence $\nabla u \equiv 0$ on Ω_-, hence $u \equiv 0$ on Ω_-. (The estimate (1.33) helps justify this use of the Gauss-Green theorem.) A result somewhat easier than (1.19) and (1.34) is

$$\mathcal{S}g\big|_{\partial\Omega_\pm}(x) = Sg(x) = \int\limits_{\partial\Omega} E(x,y)g(y) \, d\sigma(y), \quad \sigma\text{-a.e. } x \in \partial\Omega. \tag{1.45}$$

Hence

$$u\big|_{\partial\Omega_+} = 0. \tag{1.46}$$

A second application of the Gauss-Green theorem gives

$$\int\limits_{\Omega_+} (|\nabla u|^2 + V u^2)\, d\mathcal{V} = \int\limits_{\partial\Omega_+} u(\partial_{\nu_+} u)\, d\sigma = 0, \tag{1.47}$$

so $\nabla u \equiv 0$ in Ω, and then (1.46) gives

$$u \equiv 0 \quad \text{on} \quad \Omega, \text{hence on } M. \tag{1.48}$$

Now, by (1.34), if u is given by (1.40), g is equal to the jump of $\partial_\nu u$ across $\partial\Omega$, so we have that (1.39) implies $g = 0$, proving Proposition 1.3.

Therefore, we have Proposition 1.2, hence the following.

Proposition 1.4. *In the setting of Proposition 1.2, given* $1 < p < \infty$ *and* $f \in L^p(\partial\Omega)$*, there exists a solution to (1.1), of the form (1.3), with* $g \in L^p(\partial\Omega)$*, satisfying*

$$\|\mathcal{N}u\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}, \tag{1.49}$$

and

$$\lim_{y\to x, y\in\Gamma_+(x)} u(y) = f(x), \quad \textit{for} \ \ \sigma\textit{-a.e.}\ x \in \partial\Omega. \tag{1.50}$$

There is a corresponding uniqueness result, not quite an immediate consequence of the uniqueness of $g \in L^p(\partial\Omega)$ such that (1.36) holds. We omit details on this. Such uniqueness, in a more general context, is established in various references, including [HMT].

In outline, this is the story for the Dirichlet problem (1.1), on a $C^{1,\omega}$ domain satisfying (1.10), given boundary data $f \in L^p(\partial\Omega)$. There are further results, for boundary data in various L^p-Sobolev spaces and Besov spaces, and with Sobolev space and Besov space estimates on the solution u, which we will not go into here. The discussion above is a prelude to the treatment of domains Ω whose boundaries have less regularity.

We turn now to results on (1.1), for $f \in L^p(\partial\Omega)$, on rougher domains. B. Dahlberg [Dah] demonstrated solvability for $f \in L^2(\partial\Omega)$ of solutions to (1.1), satisfying (1.49)–(1.50), when Ω is a Lipschitz domain, i.e., $\partial\Omega$ is locally the graph of a Lipschitz function. The methods used in [Dah] did not involve layer potentials, but rather maximum principle and Harnack inequality arguments.

A breakthrough by A. P. Calderon [Cal] made it feasible to apply the method of layer potentials to cases rougher than in Proposition 1.1. He demonstrated L^p

bounds on Cauchy integrals on Lipschitz curves in $\mathbb{C}$, with small Lipschitz constant. Via the method of rotations, E. Fabes, M. Jodeit, and N. Riviere [FJR] were able to treat (1.1) on an arbitrary C^1 domain via layer potentials. These authors establish the compactness result on K in this more general setting. In such a case (as well as cases to come), it is useful to write

$$\begin{aligned} Kg(x) &= \mathrm{PV} \int\limits_{\partial\Omega} k(x,y) g(y)\, d\sigma(y) \\ &= \lim_{\varepsilon \searrow 0} K_\varepsilon g(x), \end{aligned} \tag{1.51}$$

with K_ε given by (1.23). The authors establish (1.28) in this setting (this time via Calderon's work). Then (1.25) and (1.22) extend, and so does (1.19). Similarly the results (1.33)–(1.34) on single layers extend, and so do Propositions 1.2–1.4.

The next advance came with the paper [CMM], establishing Calderon's estimates for the Cauchy integral on arbitrary Lipschitz curves, not just those with small Lipschitz constant. Again, via the method of rotations, the results (1.19)–(1.28) and (1.33)–(1.34) work for Lipschitz domains. However, compactness as in (1.9) fails, so a different approach to Proposition 1.2 is required.

A successful attack on Proposition 1.2 appeared in [Ver]. The crucial new ingredient was a "Rellich identity," which had been rediscovered and applied to (1.1) (though not in a way that involved layer potentials) in [JK1]. Such an identity applied to a Lipschitz domain implies that if $u = \mathcal{S}f$, then $\|\nabla_T u\|_{L^2(\partial\Omega)}$ and $\|\partial_\nu u\|_{L^2(\partial\Omega)}$ are comparable, up to controllable terms. In concert with the jump relations (1.34), this gives

$$\|f\|_{L^2(\partial\Omega)} \le C \Big\| \Big(\pm \frac{1}{2} I + K^*\Big) f \Big\|_{L^2(\partial\Omega)} + C \|\mathcal{S}f\|_{H^{1,2}(M)}, \tag{1.52}$$

and using (1.33) one can show that

$$\mathcal{S} : L^2(\partial\Omega) \longrightarrow H^{1,2}(M) \ \text{ is compact.} \tag{1.53}$$

This implies semi-Fredholmness of $\pm(1/2)I + K^*$. This was complemented in [Ver] by a construction yielding dense range. An alternative, used in several subsequent works, extends the Rellich identities to a 1-parameter family of identities, leading to

$$\begin{aligned} \lambda \in \mathbb{R},\ |\lambda| \ge \frac{1}{2} \Rightarrow \lambda I + K^* : L^2(\partial\Omega) \Longrightarrow L^2(\partial\Omega) \\ \text{has closed range and finite dimensional kernel.} \end{aligned} \tag{1.54}$$

Thus each such $\lambda I + K^*$ is semi-Fredholm, with a well defined index. Such an index is continuous in λ, hence constant on $(-\infty, -1/2]$ and on $[1/2, \infty)$. Invertibility is clear for large $|\lambda|$, so the index must be zero. Having this Fredholm property, [Ver] extended Proposition 1.3 to Lipschitz domains, thus showing that (1.37) holds, for $p = 2$. From this, a general result of [Sn] implies (1.37) holds for $|p-2| < \varepsilon(\Omega)$. Hence Proposition 1.4 is established for a Lipschitz domain Ω, for $|p-2| < \varepsilon(\Omega)$. One can interpolate with the endpoint L^∞ result (obtained via the maximum principle) to get Proposition 1.4 for

$$2 - \varepsilon(\Omega) < p \le \infty. \tag{1.55}$$

Actually, [FJR] and [Ver] worked on bounded domains $\Omega \subset \mathbb{R}^n$, with $L = \Delta$, the Euclidean Laplacian $\partial_1^2 + \cdots + \partial_n^2$. In this setting, for Proposition 1.3 to work one needs $\partial\Omega$ connected. (The reader is invited to check the proof, to see why.) To avoid this limitation, one brings in $V \ge 0$, positive on a set of positive measure, at least on each bounded component of $\mathbb{R}^n \setminus \overline{\Omega}$. Of course, once this move is made, it is only natural to let all the coefficients be variable and move to the manifold setting, as we have done here. This approach was taken up in [MT1], and pursued in [MT2]–[MT5], and also in [MMT].

These papers and others have results also on the Neumann boundary problem

$$Lu = 0 \ \text{ on } \ \Omega, \quad \partial_\nu u\big|_{\partial\Omega} = f. \tag{1.56}$$

In such a case, one seeks g on $\partial\Omega$ such that (1.56) is solved by

$$u = \mathcal{S}g. \tag{1.57}$$

This solves (1.56) provided

$$\Bigl(-\frac{1}{2}I + K^*\Bigr)g = f, \tag{1.58}$$

which can be solved for g, given $f \in L^p(\partial\Omega)$, provided

$$-\frac{1}{2}I + K^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \text{ is an isomorphism.} \tag{1.59}$$

Such a result holds for $1 < p < 2 + \varepsilon(\Omega)$, when Ω is a Lipschitz domain, if also $V > 0$ on a set of positive measure in Ω. There is a standard modification of (1.59) in the absence of this extra positivity. We will not dwell on this, or other natural boundary problems, but merely point to the bibliography for such results, and further references.

Our main goal in subsequent sections is to discuss material leading to layer potential attacks on (1.1) on classes of domains that are not Lipschitz. In §§2–4, we deal with successively smaller classes of domains with the following properties:

(1.60) The surface measure σ is finite and the Gauss-Green theorem holds,

(1.61) The double layer potential $\mathcal{D}g$ satisfies (1.19)–(1.28).

with similar results for $\nabla \mathcal{S} g$, and

$$\text{The compactness result (1.9) holds,} \tag{1.62}$$

or variants hold, of use in extensions of Proposition 1.2.

Section 2 deals with domains of finite perimeter, as initially developed by E. De-Giorgi and H. Federer. This is essentially the maximal class of domains for which surface integration over $\partial\Omega$ is meaningful. We mention several versions of the Gauss-Green formula (1.45) valid for such domains. However, these general results require more regularity on the integrand than one has when $u = \mathcal{S}g$ with g merely in $L^2(\partial\Omega)$. We discuss special results that do work in such a case, for those finite perimeter domains that are "Ahlfors regular," a property that incorporates the estimate (1.13).

Section 3 discusses the class of "uniformly rectifiable" (UR) domains, introduced by G. David and S. Semmes, essentially the maximal class of domains for which estimates like (1.28) hold; cf. [D]. For such domains, one also has (1.25), (1.22), and (1.19). (These latter results were demonstrated in [HMT].) In addition, the injectivity result (1.38) holds for this class of domains. (This is also a result of [HMT].) We also discuss some special classes of UR domains, namely Ahlfors regular NTA domains and Ahlfors regular domains that satisfy a 2-sided local John condition.

In §4 we introduce the classes of regular Semmes-Kenig-Toro (SKT) domains and ε-regular SKT domains. These classes arose in work of [Se2]–[Se4] and [KT1]–[KT4], where they were called, respectively, chord-arc domains with vanishing constant, and chord-arc domains with small constant. Classically, a chord-arc domain is a domain $\Omega \subset \mathbb{C}$ such that the arclength of $\partial\Omega$ from $p \in \partial\Omega$ to $q \in \partial\Omega$ is bounded by a constant times $|p - q|$ (the length of the chord). For the higher dimensional variants, the phrase "chord-arc" seems not to capture well the essence of their defining features, so we have proposed this alternative terminology.

In §5 we discuss results of [HMT] on the behavior of a class of operators of the form K and K^* that includes those arising in (1.18) and (1.34), with $\mathcal{D}$ as in (1.4) and $\mathcal{S}$ as in (1.31), when Ω is a regular SKT domain, or more generally an ε-regular SKT domain. These include compactness results when Ω is a regular SKT domain, and operator norm estimates, modulo compacts, when Ω is an ε-regular SKT domain. A key starting point is the proof of compactness in [H] when Ω is a VMO_1 domain with compact boundary. These compactness and almost compactness results, for regular and ε-regular SKT domains, in turn imply that the operators $\pm(1/2)I + K$, and certain more general double layers, are Fredholm of index zero.

In §6 we discuss how the Fredholmness results of §5, together with results discussed in §§2–4, lead to solvability results for (1.1) on these classes of domains. In §6 we also discuss the Neumann boundary condition and briefly mention boundary problems for several systems of equations, treated in [HMT] for regular and ε-regular SKT domains.

2. Finite perimeter domains, Ahlfors regular domains, and the Gauss-Green formula

We will work in the Euclidean space setting here. Translation of this material to the manifold setting is routine. An open set $\Omega \subset \mathbb{R}^n$ is said to have locally finite perimeter provided its characteristic function χ_Ω has the property that

$$\nabla \chi_\Omega = \mu, \tag{2.1}$$

taken a priori as a distribution, is a locally finite $\mathbb{R}^n$-valued measure. In such a case, the Radon-Nikodym theorem implies

$$\mu = -\nu\sigma, \tag{2.2}$$

where σ is a locally finite, positive measure, supported on $\partial\Omega$, and ν is a vector-valued function, defined σ-a.e. on $\partial\Omega$, satisfying $|\nu(x)| = 1$, σ-a.e. The Besicovitch differentiation theorem implies

$$\lim_{r\to 0} \frac{1}{\sigma(B_r(x))} \int\limits_{B_r(x)} \nu \, d\sigma = \nu(x), \tag{2.3}$$

for σ-a.e. x. Via distribution theory, for a vector field $v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, $\langle \operatorname{div} v, \chi_\Omega\rangle = -\langle v, \nabla\chi_\Omega\rangle$, so (2.1)–(2.2) say

$$\int\limits_\Omega \operatorname{div} v \, dx = \int\limits_{\partial\Omega} \langle \nu, v\rangle \, d\sigma, \quad \forall\, v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n). \tag{2.4}$$

We describe some results of Federer and De Giorgi on the structure of σ. Good references for this material include [EG] and [Zie]. First,

$$\sigma = \mathcal{H}^{n-1} \lfloor \partial^*\Omega, \tag{2.5}$$

where $\mathcal{H}^{n-1}$ is $(n-1)$-dimensional Hausdorff measure and $\partial^*\Omega \subset \partial\Omega$ is the reduced boundary of Ω, defined by

$$\partial^*\Omega = \{x : (2.3) \text{ holds, with } |\nu(x)| = 1\}. \tag{2.6}$$

(Remarks supporting (2.3) imply σ is supported on $\partial^*\Omega$.) Second, $\partial^*\Omega$ is countably rectifiable, i.e., it is a countable disjoint union

$$\partial^*\Omega = \bigcup_k M_k \cup N, \tag{2.7}$$

where each M_k is a compact subset of an $(n-1)$-dimensional C^1 surface, to which ν is normal in the usual sense, and $\mathcal{H}^{n-1}(N) = 0$. Given (2.5), then (2.4) yields the Gauss-Green formula

$$\int_\Omega \operatorname{div} v \, dx = \int_{\partial^*\Omega} \langle \nu, v \rangle \, d\mathcal{H}^{n-1}, \tag{2.8}$$

for $v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$. It is useful to know that $\partial^*\Omega \subset \partial_*\Omega \subset \partial\Omega$, where $\partial_*\Omega$ is the measure-theoretic boundary of Ω:

$$\partial_*\Omega = \{x \in \partial\Omega : \limsup_{r\to 0} r^{-n} \mathrm{Vol}(B_r(x) \cap \Omega_\pm) > 0\}, \tag{2.9}$$

where $\Omega_+ = \Omega,\ \Omega_- = \mathbb{R}^n \setminus \overline{\Omega}$. Furthermore, one has

$$\mathcal{H}^{n-1}(\partial_*\Omega \setminus \partial^*\Omega) = 0. \tag{2.10}$$

It is the case that

$$\Omega \text{ has locally finite perimeter} \Leftrightarrow \mathcal{H}^{n-1}(\partial_*\Omega \cap \mathcal{K}) < \infty, \tag{2.11}$$

for all compact $\mathcal{K}$.

Finite perimeter domains exist for which $\partial\Omega \setminus \partial_*\Omega$ is quite large. For example, let $\Omega \subset \mathbb{R}^2$ be a slit disk. Then $\partial^*\Omega = \partial_*\Omega$ is the boundary of the disk, and the slit makes up $\partial\Omega \setminus \partial_*\Omega$. We will generally make the condition

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0 \tag{2.12}$$

an hypothesis for our results. We mention some conditions guaranteeing this below.

Here is a significant class of locally finite perimeter domains. For $A : \mathbb{R}^{n-1} \to \mathbb{R}$, let

$$\Omega = \{x \in \mathbb{R}^n : x_n > A(x')\}, \tag{2.13}$$

where $x = (x', x_n)$. We have:

Proposition 2.1. *Given*

$$A \in C(\mathbb{R}^{n-1}), \quad \nabla A \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}), \tag{2.14}$$

then Ω, defined in (2.13), has locally finite perimeter.

For the proof, take a Friedrichs mollifier $\psi_k \in C_0^\infty(\mathbb{R}^{n-1})$ and set $A_k = \psi_k * A$. Then

$$A_k \longrightarrow A, \quad \text{locally uniformly}, \tag{2.15}$$

so

$$\chi_{\Omega_k} \longrightarrow \chi_\Omega \ \text{ in } \ L^1_{\mathrm{loc}}(\mathbb{R}^n). \tag{2.16}$$

Hence $\nabla \chi_{\Omega_k} \to \nabla \chi_\Omega$ in $\mathcal{D}'(\mathbb{R}^n)$. Also, by the classical Gauss-Green formula for smoothly bounded domains, $\nabla \chi_{\Omega_k} = -\nu_k \sigma_k$, where σ_k is surface area on

$$\Sigma_k = \{x \in \mathbb{R}^n : x_n = A_k(x')\}, \tag{2.17}$$

given in x' coordinates by

$$d\sigma_k(x') = \sqrt{1 + |\nabla A_k(x')|^2}\, dx', \tag{2.18}$$

and ν_k is the downward pointing unit normal to Σ_k. The hypothesis (2.14) implies that $\{\nu_k \sigma_k : k \geq 1\}$ is a bounded set of $\mathbb{R}^n$-valued measures on each set $B_R = \{x \in \mathbb{R}^n : |x| \leq R\}$, so passing to the limit gives $\nabla \chi_\Omega = \mu$, a locally finite $\mathbb{R}^n$-valued measure, hence of the form $\mu = -\nu\sigma$ as in (2.2).

Note that passing to the limit $k \to \infty$ from

$$\int_{\Omega_k} \operatorname{div} v \, dx = \int_{\mathbb{R}^{n-1}} \big\langle (\nabla A_k(x'), -1), v(x', A_k(x')) \big\rangle \, dx', \tag{2.19}$$

valid for $v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, and invoking (2.4), gives

$$\int_{\partial\Omega} \langle \nu, v \rangle \, d\sigma = \int_{\mathbb{R}^{n-1}} \langle \tilde{\nu}(x'), v(x', A(x')) \rangle \, d\sigma(x'), \tag{2.20}$$

where

$$\tilde{\nu}(x') = \frac{(\nabla A(x'), -1)}{\sqrt{1 + |\nabla A(x')|^2}}, \quad d\sigma(x') = \sqrt{1 + |\nabla A(x')|^2}\, dx'. \tag{2.21}$$

The formula (2.20) is valid for all $v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, hence for all $v \in C_0^0(\mathbb{R}^n, \mathbb{R}^n)$.

Regarding the relation between $\partial\Omega$ and $\partial_*\Omega$, it is clear that $\partial\Omega = \partial_*\Omega$ whenever A is locally Lipschitz. We refer to §2.2 of [HMT] for a proof of the following.

Proposition 2.2. *In the setting of Proposition 2.1, the condition (2.12) holds.*

So far, we have discussed the Gauss-Green formula for $v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$. A simple limiting argument extends (2.4), hence (2.8), to $v \in C_0^1(\mathbb{R}^n, \mathbb{R}^n)$. As mentioned in §1, for such purposes as extending Proposition 1.3 to rougher domains, we need such an identity for much rougher vector fields v. Here is a preliminary extension.

Proposition 2.3. *If $\Omega \subset \mathbb{R}^n$ has locally finite perimeter, then (2.4) holds for v in*

$$\mathfrak{D} = \{v \in C_0^0(\mathbb{R}^n, \mathbb{R}^n) : \operatorname{div} v \in L^1(\mathbb{R}^n)\}. \tag{2.22}$$

Proof. Take a Friedrichs mollifier $\varphi_k \in C_0^\infty(\mathbb{R}^n)$ and set $v_k = \varphi_k * v$. Then (2.4) gives

$$\int\limits_\Omega \operatorname{div} v_k \, dx = \int\limits_{\partial\Omega} \langle \nu, v_k \rangle \, d\sigma. \tag{2.23}$$

Since $\operatorname{div} v_k = \varphi_k * \operatorname{div} v$, the left side of (2.23) converges to $\int_\Omega \operatorname{div} v \, dx$ (given $\operatorname{div} v \in L^1$), and since $v_k \to v$ uniformly (given v continuous), the right side of (2.23) converges to $\int_{\partial\Omega} \langle \nu, v \rangle \, d\sigma$, giving the desired result.

It is desirable to have (2.4) for functions defined only on $\overline{\Omega}$. Here is one such result. Let open sets Ω_k satisfy $\overline{\Omega}_k \subset \Omega$, $\Omega_k \nearrow \Omega$. We say $\{\Omega_k : k \geq 1\}$ is a tame approximation to Ω if in addition there exists $C(R) < \infty$ such that, for $R \in [1, \infty)$,

$$\|\nabla \chi_{\Omega_k}\|_{\mathrm{TV}(B_R)} \leq C(R), \quad \forall\, k \geq 1, \tag{2.24}$$

where TV stands for the total variation of a vector measure. The domains treated in Proposition 2.1 have this property. One can take

$$\Omega_k = \{(x', x_n) : x_n > A(x') + k^{-1}\}. \tag{2.25}$$

The following is a partial extension of Proposition 2.3.

Proposition 2.4. *If $\Omega \subset \mathbb{R}^n$ has locally finite perimeter and a tame interior approximation, then (2.4) holds for v in*

$$\widetilde{\mathfrak{D}} = \{v \in C_0^0(\overline{\Omega}, \mathbb{R}^n) : \operatorname{div} v \in L^1(\Omega)\}. \tag{2.26}$$

Proof. In this setting, one can use Proposition 2.3 to show that, for $v \in \widetilde{\mathfrak{D}}$,

$$\int\limits_{\Omega_k} \operatorname{div} v \, dx = -\langle v, \chi_{\Omega_k} \rangle. \tag{2.27}$$

As $k \to \infty$, the left side of (2.27) converges to the left side of (2.4). Using (2.24), one can deduce that the right side of (2.27) converges to $-\langle v, \nabla \chi_\Omega \rangle$, from the fact that distribution theory yields such convergence if $v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

Now vector fields $v = u \nabla u$, with $u = \mathcal{S} g$, $g \in L^2(\partial\Omega)$, arising in the proof of Proposition 1.3, are much rougher than those treated in Proposition 2.4. We introduce a class of domains for which a version of the Gauss-Green formula adequate for an extension of Proposition 1.3 was established in [HMT].

Let $\Omega \subset \mathbb{R}^n$ be an open set, of locally finite perimeter. We say Ω is Ahlfors regular if (2.12) holds and there exist $C_0, C_1 \in (0,\infty)$ such that, for each $r > 0,\ x \in \partial\Omega$,

$$C_0 r^{n-1} \le \sigma(B_r(x)) \le C_1 r^{n-1}. \tag{2.28}$$

If Ω is bounded, we require (2.28) only for $0 < r < \operatorname{diam}\Omega$. To state the next Gauss-Green formula, set

$$\mathfrak{L}^p = \{v \in C^0(\Omega) : \mathcal{N}v \in L^p(\partial\Omega), \text{and } \exists \text{ nontangential limit } v_b,\ \sigma\text{-a.e.}\}. \tag{2.29}$$

Here, as in (1.21), $\mathcal{N}v$ is the nontangential maximal function

$$\mathcal{N}v(x) = \sup_{y\in\Gamma(x)} |v(y)|, \tag{2.30}$$

where, as in (1.20),

$$\Gamma(x) = \{y \in \Omega : d(y,x) \le 2\operatorname{dist}(y,\partial\Omega)\}. \tag{2.31}$$

Nontangential convergence at $x \in \partial\Omega$ means

$$\lim_{y\in\Gamma(x), y\to x} v(y) = v_b(x). \tag{2.32}$$

Here is the result.

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^n$ be bounded and Ahlfors regular. Then*

$$\int_\Omega \operatorname{div} v\, dx = \int_{\partial\Omega} \langle \nu, v_b\rangle\, d\sigma \tag{2.33}$$

for each vector field v satisfying, for some $p > 1$,

$$v \in \mathfrak{L}^p \quad \textit{and} \quad \operatorname{div} v \in L^1(\Omega). \tag{2.34}$$

For this to be meaningful, one needs to know that

$$x \in \overline{\Gamma(x)}, \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{2.35}$$

A domain Ω satisfying (2.35) is said to be weakly accessible. In the course of proving Proposition 2.5 in [HMT], the following is established.

Proposition 2.6. *If $\Omega \subset \mathbb{R}^n$ is Ahlfors regular, then it is a weakly accessible domain.*

See §2.3 of [HMT] for a proof of Proposition 2.5. We mention one ingredient.

Proposition 2.7. *In the setting of Proposition 2.5, there exists $C = C(\Omega)$ such that, for all measurable $u : \Omega \to \mathbb{R}$,*

$$\frac{1}{\delta} \int\limits_{\mathcal{O}_\delta} |u|\, dx \le C \|\mathcal{N} u\|_{L^1(\partial\Omega)}, \quad \forall\, \delta \in (0, diam\, \Omega), \tag{2.36}$$

where

$$\mathcal{O}_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \le \delta\}. \tag{2.37}$$

Note that a special case of (2.36) is that

$$\operatorname{Vol}(\mathcal{O}_\delta) \le C\delta. \tag{2.38}$$

Actually, a shorter direct proof of this is available, involving convolving σ with $\chi_{B_\delta(0)}$. We omit details. Using (2.38), we can establish the following.

Proposition 2.8. *If $\Omega \subset \mathbb{R}^n$ is bounded and Ahlfors regular, then Ω has a tame interior approximation.*

Proof. With $\Omega_s = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge s\}$, set

$$\begin{aligned} \psi_\delta(x) = \delta, \qquad & \qquad x \in \Omega_\delta, \\ \operatorname{dist}(x, \partial\Omega), & \quad x \in \Omega \setminus \Omega_\delta. \end{aligned} \tag{2.39}$$

Then ψ_δ is Lipschitz, $\nabla\psi_\delta$ is supported on $\mathcal{O}_\delta$, and $|\nabla\psi_\delta| = 1$ on $\mathcal{O}_\delta$. A version of the co-area formula (Theorem 5.4.4 of [Zie]) gives

$$\int\limits_{\mathcal{O}_\delta} |\nabla\psi_\delta|\, dx = \int_0^\delta \|\nabla\chi_{\Omega_s}\|_{\mathrm{TV}}\, ds. \tag{2.40}$$

Then, by (2.38), since the left side of (2.40) equals $\operatorname{Vol}(\mathcal{O}_\delta)$, we have

$$\int_0^\delta \|\nabla\chi_{\Omega_s}\|_{\mathrm{TV}}\, ds \le C\delta, \tag{2.41}$$

for all small δ. Thus, for each $k \ge 1$, there exists $s \in (0, 1/k)$ such that $\|\nabla\chi_{\Omega_s}\|_{\mathrm{TV}} \le C$, and Proposition 2.8 follows.

As a consequence, Proposition 2.4 applies to bounded Ahlfors regular domains, but of course Proposition 2.5 is much stronger.

The boundary of an Ahlfors regular domain is a measured metric space of homogeneous type, as studied in [CW]. A lot of tools for harmonic analysis, such as Hardy-Littlewood maximal function estimates, aspects of Calderon-Zygmund theory, atomic Hardy spaces, and theories of $\mathrm{BMO}(\partial\Omega)$ and $\mathrm{VMO}(\partial\Omega)$, are available for such spaces. See §§2.1 and 2.4 of [HMT] for a collection of such results.

In the compact case, the boundary of an Ahlfors regular domain is a natural setting for the result that weakly singular integral operators are compact. We have the following.

Proposition 2.9. *Let $\Omega \subset \mathbb{R}^n$ be Ahlfors regular, and assume $\partial\Omega$ is compact. Let $k : \partial\Omega \times \partial\Omega \to \mathbb{C}$ be measurable and satisfy*

$$|k(x,y)| \le C\frac{\omega(|x-y|)}{|x-y|^{n-1}}, \quad x, y \in \partial\Omega. \tag{2.42}$$

Then

$$Kg(x) = \int\limits_{\partial\Omega} k(x,y)g(y)\,d\sigma(y) \tag{2.43}$$

has the property

$$K : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \ \textit{is compact, for} \ \ 1 < p < \infty, \tag{2.44}$$

provided ω is a modulus of continuity that satisfies the Dini condition

$$\int_0^1 \frac{\omega(t)}{t}\,dt < \infty. \tag{2.45}$$

Proof. Same as that of Proposition 1.1.

An important class of Ahlfors regular domains is the class of BMO_1 domains. The boundary of such a domain is locally the graph of a function $A \in \mathrm{BMO}_1$, that is,

$$A \in C(\mathbb{R}^{n-1}), \quad \nabla A \in \mathrm{BMO}(\mathbb{R}^{n-1}). \tag{2.46}$$

We state the result.

Proposition 2.10. *If $\Omega \subset \mathbb{R}^n$ is a BMO_1 domain, it is Ahlfors regular.*

Since $\mathrm{BMO}(\mathbb{R}^{n-1}) \subset L^p_{\mathrm{loc}}(\mathbb{R}^{n-1})$ for all $p < \infty$, Propositions 2.1–2.2 apply, as do (2.20)–(2.21). Thus, what is to be proved is that, if A satisfies (2.46), then there exist $C_0, C_1 \in (0,\infty)$ such that, for all $x' \in \mathbb{R}^{n-1}$, $r > 0$,

$$C_0 r^{n-1} \le \int\limits_{\{y' : |x'-y'|^2 + |A(x')-A(y')|^2 < r^2\}} \sqrt{1 + |\nabla A(y')|^2}\,dy' \le C_1 r^{n-1}. \tag{2.47}$$

For details, see §2.5 of [HMT].

Remark. For simplicity, we have defined $\mathcal{N}$ by (2.30). More generally, one can take $\alpha \in (0,\infty)$, set

$$\Gamma_\alpha(x) = \{y \in \Omega : d(y,x) \le (1+\alpha)\,\mathrm{dist}(y,\partial\Omega)\}, \tag{2.48}$$

and define

$$\mathcal{N}_\alpha u(x) = \sup_{y \in \Gamma_\alpha(x)} |u(y)|. \tag{2.49}$$

As shown in §2.1 of [HMT], if Ω is Ahlfors regular, $\alpha, \beta \in (0,\infty)$, and $1 < p < \infty$,

$$\|\mathcal{N}_\alpha u\|_{L^p(\partial\Omega)} \approx \|\mathcal{N}_\beta u\|_{L^p(\partial\Omega)}. \tag{2.50}$$

3. Uniformly rectifiable domains and layer potentials

We continue to take $\Omega \subset \mathbb{R}^n$, as the generalization to the manifold setting is routine. We say an open set $\Omega \subset \mathbb{R}^n$ is uniformly rectifiable provided it is Ahlfors regular and the following property holds. There exist $\kappa, M \in (0,\infty)$, called the UR constants of Ω, such that for each $x \in \partial\Omega, R > 0$, there is a Lipschtz map

$$\varphi : B_R^{n-1} \longrightarrow \mathbb{R}^n, \quad B_R^{n-1} = \{y' \in \mathbb{R}^{n-1} : |y'| < R\}, \tag{3.1}$$

such that

$$\|\nabla\varphi\|_{L^\infty} \le M, \quad \mathcal{H}^{n-1}\Big(\partial\Omega \cap B_R(x) \cap \varphi(B_R^{n-1})\Big) \ge \kappa R^{n-1}. \tag{3.2}$$

If $\partial\Omega$ is compact, we require this only for $R \in (0,1]$. For short, we call such Ω a UR domain.

The following result is established in [Dav], Proposition 4 bis.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded UR domain, and take $p \in (1,\infty)$. Then there exists $N \in \mathbb{Z}^+$ and $C \in (0,\infty)$, each depending only on p and the Ahlfors regularity and UR constants of Ω, with the following property. Assume $k \in C^N(\mathbb{R}^n \setminus 0)$ is odd and homogeneous of degree $-(n-1)$. Then, with*

$$\begin{aligned} K_\varepsilon f(x) &= \int\limits_{\partial\Omega\setminus B_\varepsilon(x)} k(x-y)f(y)\,d\sigma(y), \quad x \in \partial\Omega, \\ K_* f(x) &= \sup_{0<\varepsilon\le 1} |K_\varepsilon f(x)|, \end{aligned} \tag{3.3}$$

we have

$$\|K_* f\|_{L^p(\partial\Omega)} \le C_p \big\|k\big|_{S^{n-1}}\big\|_{C^N} \|f\|_{L^p(\partial\Omega)}. \tag{3.4}$$

In the setting of Proposition 3.1, consider the layer potential

$$\mathcal{T} f(x) = \int\limits_{\partial\Omega} k(x-y)f(y)\,d\sigma(y), \quad x \in \Omega. \tag{3.5}$$

In §3.2 of [HMT], the estimate (3.4) is supplemented by the nontangential maximal function estimate

$$\|\mathcal{N}\mathcal{T} f\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty, \tag{3.6}$$

where $\mathcal{N}$ is defined as in (2.30). Indeed, as shown there, one has

$$\mathcal{N}\mathcal{T}f(x) \le K_* f(x) + C\mathcal{M}f(x), \quad x \in \partial\Omega, \tag{3.7}$$

where $\mathcal{M}f$ is the Hardy-Littlewood maximal function. Then (3.6) follows from (3.4) and the standard Hardy-Littlewood maximal function estimate. In more detail, it is shown that

$$\begin{aligned} x \in \partial\Omega,\ y \in \Gamma(x),\ |x-y| = \varepsilon \\ \Longrightarrow |\mathcal{T}f(y) - K_{2\varepsilon}f(x)| \le C\mathcal{M}f(x) \\ \Longrightarrow |\mathcal{T}f(y)| \le K_* f(x) + C\mathcal{M}f(x), \end{aligned} \tag{3.8}$$

which leads to (3.7).

If Ω is a bounded UR domain, then

$$\mathcal{T} : L^p(\partial\Omega) \longrightarrow L^{pn/(n-1)}(\Omega), \quad 1 < p < \infty. \tag{3.9}$$

This is a consequence of (3.6) together with the following, proven in §3.2 of [HMT].

Proposition 3.2. *If $\Omega \subset \mathbb{R}^n$ is a bounded Ahlfors regular domain, then, for $1 < p < \infty,\ u \in C(\Omega)$,*

$$\|u\|_{L^{pn/(n-1)}(\Omega)} \le C\|\mathcal{N}u\|_{L^p(\partial\Omega)}. \tag{3.10}$$

As for operator convergence and pointwise convergence, we have the following.

Proposition 3.3. *In the setting of Proposition 3.1, we have, for each $f \in L^p(\partial\Omega), p \in (1,\infty)$,*

$$\begin{aligned} \lim_{\varepsilon \searrow 0} K_\varepsilon f(x) &= \mathrm{PV} \int_{\partial\Omega} k(x-y) f(y)\, d\sigma(y) \\ &= Kf(x), \end{aligned} \tag{3.11}$$

for σ-a.e. $x \in \partial\Omega$, and

$$K : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \tag{3.12}$$

satisfies the bound (3.4). Furthermore,

$$\lim_{y \to x, y \in \Gamma(x)} \mathcal{T}f(y) = \frac{1}{2i}\hat{k}(\nu(x)) f(x) + Kf(x), \tag{3.13}$$

for σ-a.e. $x \in \partial\Omega$.

The proof, given in §§3.3–3.5 of [HMT], proceeds in stages, starting with such a result for the double layer potential (previewed in §1)

$$\mathcal{D}f(x) = \int_{\partial\Omega} \partial_{\nu_y} E(x-y) f(y)\, d\sigma(y), \tag{3.14}$$

where $E(x-y)$ is the fundamental solution of the Laplace operator Δ. In the Euclidean context, we have

$$\partial_{\nu_y} E(x-y) = \frac{1}{\omega_n} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n}, \tag{3.15}$$

where ω_n is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. The integral kernel $\partial_{\nu_y} E(x-y)$ is a sum $\sum k_j(x-y)\nu_j(y)$, where each k_j satisfies the conditions of Proposition 3.1. With a slight abuse of notation, we set

$$K_\varepsilon f(x) = \int_{\partial\Omega\setminus B_\varepsilon(x)} \partial_{\nu_y} E(x-y) f(y)\, d\sigma(y), \quad K_* f(x) = \sup_{\varepsilon>0} |K_\varepsilon f(x)|. \tag{3.16}$$

As in (3.4) and (3.6), we have, for $1 < p < \infty$,

$$\|K_* f\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}, \quad \|\mathcal{N}\mathcal{D}f\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}. \tag{3.17}$$

We claim that, for each $f \in L^p(\partial\Omega)$,

$$\lim_{\varepsilon\searrow 0} K_\varepsilon f(x) = \mathrm{PV} \int_{\partial\Omega} \partial_{\nu_y} E(x-y) f(y)\, d\sigma(y) = Kf(x), \tag{3.18}$$

and

$$\lim_{y\to x, y\in\Gamma(x)} \mathcal{D}f(y) = \Big(\frac{1}{2}I + K\Big) f(x), \tag{3.19}$$

for σ-a.e. $x \in \partial\Omega$. In view of (3.17), it suffices to prove (3.18)–(3.19) for f in a dense subspace of $L^p(\partial\Omega)$, for example, the space $\mathrm{Lip}(\partial\Omega)$ of Lipschitz continuous functions on the compact set $\partial\Omega$. In fact, we have the following.

Proposition 3.4. *If Ω is a bounded, Ahlfors regular domain, and $f \in \mathrm{Lip}(\partial\Omega)$, then (3.18) holds for each $x \in \partial^*\Omega$, and (3.19) holds for each $x \in \partial^*\Omega$ such that $x \in \overline{\Gamma(x)}$.*

Proof. To establish that

$$\int_{\partial\Omega\setminus B_\varepsilon(x)} \partial_{\nu_y} E(x-y) f(y)\, d\sigma(y) \tag{3.20}$$

converges as $\varepsilon \searrow 0$, add $-f(x)+f(x)$ to $f(y)$. We have

$$\begin{aligned}\lim_{\varepsilon\searrow} \frac{1}{\omega_n} \int\limits_{\partial\Omega\setminus B_\varepsilon(x)} &\frac{\langle \nu(y), y-x\rangle}{|x-y|^n} \big[f(y)-f(x)\big]\, d\sigma(y) \\ &= \frac{1}{\omega_n} \int\limits_{\partial\Omega} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} \big[f(y)-f(x)\big]\, d\sigma(y),\end{aligned} \tag{3.21}$$

by the Lebesgue dominated convergence theorem, given the upper bound in (2.28). To prove (3.18), it remains to establish convergence of (3.20) with $f(y)$ replaced by 1. To get this, pick R such that $\overline{\Omega} \subset B_R(x)$ and set

$$\mathcal{O}_{\varepsilon,R} = B_R(x) \setminus \big(\Omega \cup B_\varepsilon(x)\big). \tag{3.22}$$

Using the Gauss-Green formula (2.4), one can show that, for a.e. $\varepsilon > 0, \mathcal{O}_{\varepsilon,R}$ has finite perimeter and

$$\begin{aligned}\int\limits_{\mathcal{O}_{\varepsilon,R}} \operatorname{div} v(y)\, dy = \int\limits_{\partial\Omega\setminus B_\varepsilon(x)} \langle \nu(y), v(y)\rangle\, d\sigma(y) &+ \int\limits_{\partial B_\varepsilon(x)\cap\Omega^c} \langle \nu(y), v(y)\rangle\, d\sigma(y) \\ &+ \int\limits_{\partial B_R(x)} \langle \nu(y), v(y)\rangle\, d\sigma(y),\end{aligned} \tag{3.23}$$

whenever $v \in C^\infty(\overline{\mathcal{O}_{\varepsilon,R}}, \mathbb{R}^n)$. (Cf. [HMT], §2.2.) This applies to $v(y) = \nabla_y E(x-y)$. Now $\Delta_y E(x-y) = 0$ on $\mathbb{R}^n \setminus \{x\}$, hence on $\mathcal{O}_{\varepsilon,R}$, so (3.23) gives

$$\begin{aligned}&\int\limits_{\partial\Omega\setminus B_\varepsilon(x)} \partial_{\nu_y} E(x-y)\, d\sigma(y) \\ &= \frac{1}{\omega_n} \int\limits_{\partial B_R(x)} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n}\, d\sigma(y) - \frac{1}{\omega_n} \int\limits_{\partial B_\varepsilon(x)\cap\Omega^c} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n}\, d\sigma(y).\end{aligned} \tag{3.24}$$

The first term on the right equals 1. The second term equals

$$-\frac{1}{\omega_n} \varepsilon^{-(n-1)}\, \sigma(\partial B_\varepsilon(x) \cap \Omega^c). \tag{3.25}$$

Now if $x \in \partial^*\Omega$, then the orthogonal complement to $\nu(x)$, translated to pass through x, is an approximate tangent plane, and the limit of (3.25) as $\varepsilon \searrow 0$ (at least on a set of density 1 at 0) equals 1/2. Finally, elementary arguments eliminate the restriction on ε, and we get

$$\lim_{\varepsilon\searrow 0} \int\limits_{\partial\Omega\setminus B_\varepsilon(x)} \partial_{\nu_y} E(x-y)\, d\sigma(y) = \frac{1}{2}, \quad \forall\, x \in \partial^*\Omega. \tag{3.26}$$

This gives (3.18).

We turn to (3.19). Given $x \in \partial^*\Omega \cap \overline{\Gamma(x)}$, we write

$$\lim_{z\to x, z\in\Gamma(x)} \mathcal{D}f(z) = I_1 + I_2 + I_3, \tag{3.27}$$

with

$$\begin{aligned}
I_1 &= \lim_{\varepsilon\searrow 0} \lim_{z\to x, z\in\Gamma(x)} \frac{1}{\omega_n} \int\limits_{\partial\Omega\setminus B_\varepsilon(x)} \frac{\langle \nu(y), y-z\rangle}{|y-z|^n} f(y)\, d\sigma(y), \\
I_2 &= \lim_{\varepsilon\searrow 0} \lim_{z\to x, z\in\Gamma(x)} \frac{1}{\omega_n} \int\limits_{\partial\Omega\cap B_\varepsilon(x)} \frac{\langle \nu(y), y-z\rangle}{|y-z|^n} [f(y)-f(x)]\, d\sigma(y), \\
I_3 &= f(x)\cdot \lim_{\varepsilon\searrow 0} \lim_{z\to x, z\in\Gamma(x)} \frac{1}{\omega_n} \int\limits_{\partial\Omega\cap B_\varepsilon(x)} \frac{\langle \nu(y), y-z\rangle}{|y-z|^n}\, d\sigma(y).
\end{aligned} \tag{3.28}$$

For each $\varepsilon > 0$, the Lebesgue dominated convergence theorem applies to the limit as $\Gamma(x) \ni z \to x$ in I_1. We get

$$\begin{aligned}
I_1 &= \lim_{\varepsilon\searrow 0} \frac{1}{\omega_n} \int\limits_{\partial\Omega\setminus B_\varepsilon(x)} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} f(y)\, d\sigma(y) \\
&= Kf(y),
\end{aligned} \tag{3.29}$$

by (3.18). To handle I_2, note that

$$\begin{aligned}
x, y \in \partial\Omega, z\in\Gamma(x) \Rightarrow |x-y| &\le |z-y| + |z-x| \\
&\le |z-y| + 2\,\mathrm{dist}(z,\partial\Omega) \\
&\le 3|z-y|,
\end{aligned} \tag{3.30}$$

so, for Lipschitz continuous f, the absolute value of the integrand in I_2 is

$$\le \frac{C}{|x-y|^{n-2}}, \tag{3.31}$$

and the Lebesgue dominated convergence theorem applies, to give

$$I_2 = 0. \tag{3.31}$$

Finally, for I_3, use of harmonicity and the Gauss-Green formula, similar to (3.24), gives, for $z \in \Gamma(x) \subset \Omega$,

$$\begin{aligned}
&\int\limits_{\partial\Omega\cap B_\varepsilon(x)} \partial_{\nu_y} E(z-y)\, d\sigma(y) \\
&\quad = -\frac{1}{\omega_n} \int\limits_{B_\varepsilon(x)\cap\Omega^c} \frac{\langle \nu(y), y-z\rangle}{|y-z|^n}\, d\sigma(y),
\end{aligned} \tag{3.33}$$

which tends to

$$-\frac{1}{\omega_n} \int\limits_{B_\varepsilon(x)\cap\Omega^c} \frac{\langle \nu(y), y-x\rangle}{|y-x|^n}\, d\sigma(y), \tag{3.34}$$

as $z \to x$, and parallel to (3.24)–(3.26), this tends to $1/2$ as $\varepsilon \searrow 0$, if $x \in \partial^*\Omega$. This gives (3.19).

The next step in §3.3 of [HMT] is to apply a similar argument to

$$\mathcal{R}_{jk} f(x) = \int\limits_{\partial\Omega} \big[\nu_j(y)\partial_k E(x-y) - \nu_k(y)\partial_j E(x-y)\big] f(y)\, d\sigma(y), \quad x \in \Omega, \tag{3.35}$$

and

(3.36)

$$R_{jk} f(x) = \lim_{\varepsilon\searrow 0} \int\limits_{\partial\Omega\setminus B_\varepsilon(x)} \big[\nu_j(y)\partial_k E(x-y) - \nu_k(y)\partial_j E(x-y)\big] f(y)\, d\sigma(y), \quad x \in \partial\Omega.$$

We get

$$\mathcal{R}_{jk} f\big|_{\partial\Omega} = R_{jk} f, \quad \sigma\text{-a.e. on } \partial\Omega,$$

if Ω is a UR domain. Then results are obtained on $\nabla\mathcal{S}f$, where $\mathcal{S}$ is the single layer potential

$$\mathcal{S}f(x) = \int\limits_{\partial\Omega} E(x-y) f(y)\, d\sigma(y), \tag{3.37}$$

using the identities

$$\partial_j \mathcal{S}f = -\mathcal{D}(\nu_j f) + \mathcal{R}_{jk}(\nu_k f). \tag{3.38}$$

In particular, generalizing (1.34), we have

$$\lim_{y\to x, y\in\Gamma(x)} \partial_\nu \mathcal{S}f(y) = \Big(-\frac{1}{2} I + K^*\Big) f(x), \quad \sigma\text{-a.e. } x \in \partial\Omega. \tag{3.38A}$$

Section 3.4 of [HMT] establishes Proposition 3.3 in case

$$k(x) = \frac{P_\ell(x)}{|x|^{n+\ell-1}}, \tag{3.39}$$

whenever P_ℓ is an odd harmonic polynomial on $\mathbb{R}^n$, homogeneous of (odd) degree $\ell \geq 3$ (the case $\ell = 1$ being covered by the treatment of $\nabla\mathcal{S}f$). We refer the reader to that subsection for details, which involve Clifford analysis, and induction on ℓ.

The proof of Proposition 3.3 is finished in §3.5 of [HMT], using the classical device of expanding (the odd function) $k|_{S^{n-1}}$ in (odd) spherical harmonics. This yields

$$k(x) = \sum_{\ell \geq 1} a_\ell \frac{P_\ell(x)}{|x|^{n+\ell-1}}, \tag{3.40}$$

with P_ℓ as above. If $k|_{S^{n-1}}$ is smooth enough, appropriate norms of $a_\ell P_\ell|_{S^{n-1}}$ decay rather rapidly, giving a convergent operator expansion and finishing the proof of Proposition 3.3.

In fact, §3.5 of [HMT] goes one step further, using such an expansion to treat "variable coefficient" versions of (3.3), (3.5), and (3.11), with $k(x-y)$ replaced by

$$k(x, x-y). \tag{3.41}$$

Here, $k(x,z)$ is continuous on $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$, along with z-derivatives of order $\leq N$, and homogeneous of degree $-(n-1)$ in z, and satisfies $k(x,-z) = -k(x,z)$. In place of (3.40), we have

$$k(x, x-y) = \sum_{\ell \geq 1} a_\ell(x) \frac{P_\ell(x-y)}{|x-y|^{n+\ell-1}}. \tag{3.42}$$

As a consequence, one has

$$Kf(x) = \mathrm{PV} \int_{\partial\Omega} k(x, x-y) f(y)\, d\sigma(y) \tag{3.43}$$

defining a bounded operator on $L^p(\partial\Omega)$ for $1 < p < \infty$, as in (3.12), obtained in (3.11), with

$$K_\varepsilon f(x) = \int_{\partial\Omega \setminus B_\varepsilon(x)} k(x, x-y) f(y)\, d\sigma(y), \tag{3.44}$$

and $K_* f(x) = \sup_{\varepsilon > 0} |K_\varepsilon f(x)|$ satisfies, for $1 < p < \infty$,

$$\|K_* f\|_{L^p(\partial\Omega)} \leq C_p \sup_x \big\| k(x, \cdot)\big|_{S^{n-1}} \big\|_{C^N} \|f\|_{L^p(\partial\Omega)}. \tag{3.45}$$

Also the double layer potential

$$\mathcal{T} f(x) = \int_{\partial\Omega} k(x, x-y) f(y)\, d\sigma(y), \quad y \in \Omega, \tag{3.46}$$

satisfies (3.6), and

$$\lim_{y\to x, y\in\Gamma(x)} \mathcal{T}f(y) = \frac{1}{2i}\hat{k}(x,\nu(x))f(x) + Kf(x), \tag{3.47}$$

for σ-a.e. $x \in \partial\Omega$, where $\hat{k}(x,\xi)$ is the partial Fourier transform of $k(x,z)$ with respect to z. In addition, variants of (3.43)–(3.47) hold with

$$k(y, x-y) \tag{3.47A}$$

in place of $k(x, x-y)$.

Remark. In the setting of Lipschitz domains, passing to such results via the expansion (3.42) formed the starting point for the work of [MT1], on variable coefficient Laplacians on Lipschitz domains, which led to [MT2]–[MT5], [MMT], and other works. As in these papers, it is such a device that allows for a treatment of elliptic boundary problems in the Riemannian manifold setting. See §6 for more on this.

We next record some geometrical conditions on an Ahlfors regular domain Ω that are known to imply that Ω is a UR domain. The following is a result of [DJ].

Proposition 3.5. *Let $\Omega \subset \mathbb{R}^n$ be Ahlfors regular. Assume $\partial\Omega$ satisfies the following "two disks" condition. There exists $C_0 \in (0,\infty)$ such that for each $x \in \partial\Omega$ and $r > 0$, there are two $(n-1)$-dimensional disks, with centers at a distance $\le r$ from x, radius r/C_0, one contained in Ω and the other in $\mathbb{R}^n \setminus \overline{\Omega}$. (If $\partial\Omega$ is compact, one can pick $R_0 \in (0,\infty)$ and restrict attention to $r \in (0, R_0]$.) Then Ω is a UR domain.*

A similar result, but with balls in place of disks, had been established in [Se5].

One important class of UR domains is the class of Ahlfors regular domains that are nontangentially accessible (NTA), a class introduced in [JK2]. An open set $\Omega \subset \mathbb{R}^n$ is an NTA domain provided

$$\Omega \text{ satisfies a 2-sided corkscrew condition,} \tag{3.48}$$

and

$$\Omega \text{ satisfies a Harnack chain condition.} \tag{3.49}$$

The 1-sided corkscrew condition is that there exist $M > 1$ and $R > 0$ such that, if $x \in \partial\Omega$ and $r \in (0,R)$, then there exists $y = y(x,r) \in \Omega$ such that

$$|x-y| < r \quad \text{and} \quad \operatorname{dist}(y,\partial\Omega) > \frac{r}{M}. \tag{3.50}$$

The 2-sided corkscrew condition is that both Ω and $\mathbb{R}^n \setminus \Omega$ have this property. Note that

$$\text{2-sided corkscrew condition} \implies \partial\Omega = \partial_*\Omega. \tag{3.51}$$

A Harnack chain (of type (K, M)) from $x_1 \in \Omega$ to $x_2 \in \Omega$ consists of balls $B_1, \dots, B_K$ such that

$$\begin{aligned} &x_1 \in B_1, \quad x_2 \in B_K, \\ &B_j \cap B_{j+1} \neq \emptyset, \quad B_j \subset \Omega, \\ &B_j \text{ has radius } r_j \text{ and } \frac{r_j}{M} < \operatorname{dist}(B_j, \partial\Omega) < M r_j. \end{aligned} \tag{3.52}$$

The Harnack chain condition on Ω is that there exist $M > 1, R > 0$ such that whenever $r \in (0, R)$, if

$$\begin{gathered} q \in \partial\Omega, \ x_1, x_2 \in B_{r/4}(q) \cap \Omega, \\ \text{and } |x_1 - x_2| < 2^k \cdot \min_j \operatorname{dist}(x_j, \partial\Omega) \end{gathered} \tag{3.53}$$

(with $k \geq 1$) then there is a Harnack chain of type (K, M) from x_1 to x_2, with

$$K \leq Mk, \quad \text{and } \operatorname{diam} B_\ell \geq M^{-1} \min_j \operatorname{dist}(x_j, \partial\Omega). \tag{3.54}$$

It follows from Proposition 3.5 that if $\Omega \subset \mathbb{R}^n$ is Ahlfors regular and satisfies the 2-sided corkscrew condition, then it is a UR domain.

A more general class of domains than NTA is the class satisfying a local John condition. For an open set $\Omega \subset \mathbb{R}^n$, this condition is that there exist $\theta \in (0, 1)$, $R > 0$ ($R = \infty$ if $\partial\Omega$ is not compact) with the following properties. For each $p \in \partial\Omega$, $r \in (0, R)$, there is a point $p_r \in B_r(p) \cap \Omega$ such that

$$B_{\theta r}(p_r) \subset \Omega, \tag{3.55}$$

and, for each $x \in \partial\Omega \cap B_r(p)$, there is a path γ_x from x to p_r, of length $\leq r/\theta$, such that

$$\operatorname{dist}(\gamma_x(t), \partial\Omega) \geq \theta |\gamma_x(t) - x|, \quad \forall\, t. \tag{3.56}$$

If Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ both satisfy a local John condition, we say Ω satisfies a 2-sided local John condition.

The condition on γ_x above is stronger than the corkscrew condition, so if Ω is Ahlfors regular and satisfies a 2-sided John condition, then Ω is a UR domain.

Despite what is stated in the last paragraph, it is shown in §3.1 of [HMT] that

(3.57) Every NTA domain satisfies a local John condition.

We have the following result on BMO_1 domains.

Proposition 3.6. *If* $\Omega \subset \mathbb{R}^n$ *is a* BMO_1 *domain, then it is an NTA domain, hence (since, by Proposition 2.10,* BMO_1 *domains are Ahlfors regular) it is a UR domain.*

Proof. It is shown in [JK2] that if $A \in \Lambda_*(\mathbb{R}^{n-1})$, the Zygmund space, then $\Omega = \{(x', x_n) : x_n > A(x')\}$ is an NTA domain. Proposition 3.6 then follows from the fact that $\mathrm{BMO}_1(\mathbb{R}^{n-1}) \subset \Lambda_*(\mathbb{R}^{n-1})$.

REMARK. By Proposition 3.6 and (3.51) we see that if Ω is a BMO_1 domain, then

$$\partial_* \Omega = \partial \Omega, \tag{3.58}$$

which is stronger than (2.10).

Another important result of [DJ] is that if a bounded domain Ω is Ahlfors regular and NTA, then harmonic measure on $\partial\Omega$ satisfies the A_∞ weight condition. Hence the Poisson integral

$$\mathrm{PI} : C(\partial\Omega) \longrightarrow C(\overline{\Omega}), \tag{3.59}$$

for which $u = \mathrm{PI}\, f$ solves $\Delta u = 0,\ u|_{\partial\Omega} = f$ (in this case, every boundary point is regular) extends continuously to

$$\mathrm{PI} : L^q(\partial\Omega) \longrightarrow \{u \in C^\infty(\Omega) : \Delta u = 0\}, \tag{3.60}$$

for some $q < \infty$. Recent papers, including [KT4] and [MPT], have dubbed such domains "chord-arc domains."

Classically, a chord-arc domain is a planar domain $\Omega \subset \mathbb{R}^2$, for which $\partial\Omega$ is a simple closed curve, which is rectifiable and has the property that there exists $\beta < \infty$ such that, if $p, q \in \partial\Omega$ and $\gamma_{pq} \subset \partial\Omega$ is the shorter of the two arcs joining p to q, then its length $\ell(\gamma_{pq})$ satisfies

$$|p - q| \le \ell(\gamma_{pq}) \le (1 + \beta)|p - q|. \tag{3.61}$$

Clearly (3.61) implies Ahlfors regularity. It is classical that such a planar domain is a quasidisk. It was proven in [Jo] that such Ω is NTA. Lavrientiev proved (3.60) in this setting, in 1936.

Works of [Se2]–[Se4] and [KT1]–[KT4] have dealt with higher dimensional generalizations of "chord-arc domains with small constant," and "chord-arc domains with vanishing constant." We turn to this topic in the next section.

4. Regular SKT domains and variants

At the end of §3, we defined the notion of chord-arc domains $\Omega \subset \mathbb{R}^2$, by (3.61), and noted that domains $\Omega \subset \mathbb{R}^n$ for $n > 2$ are called chord-arc domains in some papers if they are Ahlfors regular and NTA. To proceed further, we recall that a planar domain $\Omega \subset \mathbb{R}^2$ is said to be chord-arc with constant $\delta > 0$ provided there exists $A > 0$ such that, for $p, q \in \partial\Omega$,

$$|p - q| \le A \Longrightarrow \ell(\gamma_{pq}) \le (1 + \delta)|p - q|. \tag{4.1}$$

The domain Ω is said to be vanishing chord-arc if it is chord-arc with constant δ for all $\delta > 0$, so

$$\lim_{p \to q} \frac{\ell(\gamma_{pq})}{|p - q|} = 1, \quad \forall\, q \in \partial\Omega. \tag{4.2}$$

It was shown in [Se2]–[Se3] that if $\Omega \subset \mathbb{R}^2$ is chord-arc with sufficiently small constant $\delta > 0$, then the outward unit normal ν has the property

$$\|\nu\|_{\mathrm{BMO}(\gamma_{pq})} \le C\delta^{1/2}, \tag{4.3}$$

whenever $p, q \in \partial\Omega,\ |p - q| \le A$, with A as in (4.1). Conversely, as shown in these papers, if Ω is bounded by a simple, closed, rectifiable curve and $\nu \in \mathrm{BMO}(\partial\Omega)$, with $\|\nu\|_{\mathrm{BMO}(\gamma_{pq})} \le \eta$ sufficiently small, whenever $|p - q| \le A$, then Ω is chord-arc, with constant $\delta \le C\eta^2$. This work also showed that, for $\Omega \subset \mathbb{R}^2$,

$$\Omega \text{ vanishing chord-arc} \Longrightarrow \nu \in \mathrm{VMO}(\partial\Omega). \tag{4.4}$$

Higher dimensional analogues of these classes of domains were introduced and studied in [Se4]–[Se5] and [KT1]–[KT4]. The definitions of these classes (which we discuss below) involved both the notions of $\nu \in$ BMO, or VMO, and the notion of Reifenberg flatness. However, they did not involve the notion of "chord" or "arc," and there is motivation to give these classes more fitting names than chord-arc domains with small constant (resp., with vanishing constant). On the other hand, a label like "Ahlfors regular NTA domains with BMO bounds (resp., VMO condition) on ν and Reifenberg-flatness condition" won't do. In [HMT] it was proposed to call them ε-regular (resp., regular) Semmes-Kenig-Toro (SKT) domains. We proceed to define these classes of domains.

We start with the notion of Reifenberg flatness. If $\Omega \subset \mathbb{R}^n$ is open, one says $\partial\Omega$ is Reifenberg flat if on all scales, given $q \in \partial\Omega$, $B_r(q) \cap \partial\Omega$ is well approximated, in a Hausdorff distance sense, by an $(n-1)$-plane through q $(\cap B_r(q))$. More precisely, given $q \in \partial\Omega$, set

$$\theta(q, r) = \inf_L \frac{1}{r} D[\partial\Omega \cap B_r(q), L \cap B_r(q)], \tag{4.5}$$

where the inf is over all $(n-1)$-planes through q, and $D[\ ,\ ]$ is Hausdorff distance. Then $\partial\Omega$ is δ-Reifenberg flat if and only if, for each compact $K \subset \partial\Omega$, there exists $R > 0$ such that

$$\sup_{0<r\le R}\ \sup_{q\in K}\ \theta(q,r) \le \delta. \tag{4.6}$$

One requires $\delta < 1/4\sqrt{2}$. One says $\partial\Omega$ is Reifenberg flat with vanishing constant provided it is δ-Reifenberg flat for some $\delta \in (0, 1/4\sqrt{2})$, and

$$\lim_{r\searrow 0}\ \sup_{q\in K}\ \theta(q,r) = 0, \tag{4.7}$$

for each compact $K \subset \partial\Omega$.

These labels are assigned to the domain Ω if, in addition, there is the following separation property: given $K \subset \partial\Omega$ compact, there exists $R > 0$ such that, for each $q \in K,\ r \in (0,R]$, there is an $(n-1)$-plane $L(q,r)$ through q, with normal $n_{q,r}$, such that

$$A = \Big\{x + tn_{q,r} \in B_r(q) : x \in L(q,r),\ t > \frac{r}{4}\Big\} \subset \Omega, \tag{4.8}$$

and

$$B = \Big\{x - tn_{q,r} \in B_r(q) : x \in L(q,r),\ t > \frac{r}{4}\Big\} \subset \mathbb{R}^n \setminus \overline{\Omega}. \tag{4.9}$$

Note. The separation property implies $\partial\Omega = \partial_*\Omega$.

The following was established in [KT1].

Proposition 4.1. *There exists $\delta_n \in (0, 1/4\sqrt{2})$ such that, if $\delta \in (0,\delta_n)$, $\Omega \subset \mathbb{R}^n$ has the separation property, and $\partial\Omega$ is δ-Reifenberg flat, then Ω is an NTA domain.*

In light of this, given $\delta < \delta_n$, one says Ω is a δ-Reifenberg flat domain provided it has the separation property and $\partial\Omega$ is δ-Reifenberg flat. If, in addition, (4.7) holds, one says Ω is a Reifenberg flat domain with vanishing constant.

We now define three classes of SKT domains.

Definition 4.1. Given $\delta \in (0,\delta_n)$, a domain $\Omega \subset \mathbb{R}^n$ is a δ-SKT domain provided

$$\Omega \text{ is Ahlfors regular,} \tag{4.10}$$

$$\Omega \text{ is a } \delta\text{-Reifenberg flat domain,} \tag{4.11}$$

and, given $K \subset \partial\Omega$ compact, there exists $R > 0$ such that

$$\sup_{q\in K} \|\nu\|_{\mathrm{BMO}(B_R(q)\cap\partial\Omega)} < \delta. \tag{4.12}$$

Definition 4.2. In the setting of Definition 4.1, if, in addition,

$$\nu \in \mathrm{VMO}(\partial\Omega), \tag{4.13}$$

we say Ω is a regular SKT domain, while if

$$\mathrm{dist}(\nu, \mathrm{VMO}(\partial\Omega)) < \varepsilon, \tag{4.14}$$

the distance taken in BMO-norm, we say Ω is an ε-regular SKT domain.

Note. If $\delta < \delta_n$ and Ω is a δ-SKT domain, then Ω is Ahlfors regular and NTA; hence it is a UR domain.

Definition 4.1 and the first part of Definition 4.2 are from [KT2], where the domains were called, respectively, δ-chord-arc domains and chord-arc domains with vanishing constant. The following was proved in [KT2].

Proposition 4.2. *Assume $\Omega \subset \mathbb{R}^n$ is Ahlfors regular and δ-Reifenberg flat (with $\delta < \delta_n$). Then the following are equivalent.*

$$\Omega \textit{ is a regular SKT domain.} \tag{4.15}$$

(4.16)
Ω is Reifenberg flat with vanishing constant, and for each compact $K \subset \partial\Omega$,

$$\lim_{r \searrow 0} \sup_{q \in K} \frac{1}{A_{n-1} r^{n-1}} \sigma(B_r(q) \cap \partial\Omega) = 1,$$

where $A_{n-1} =$ area of S^{n-1}.

$$\begin{aligned} &\textit{For each compact } K \subset \partial\Omega, \\ &\lim_{r \searrow 0} \inf_{q \in K} \frac{1}{A_{n-1} r^{n-1}} \sigma(B_r(q) \cap \partial\Omega) \\ &\qquad = \lim_{r \searrow 0} \sup_{q \in K} \frac{1}{A_{n-1} r^{n-1}} \sigma(B_r(q) \cap \partial\Omega) = 1. \end{aligned} \tag{4.17}$$

In counterpoint to these results, we have the following two results, established in §4.2 of [HMT].

Proposition 4.3. *Let $\Omega \subset \mathbb{R}^n$ be Ahlfors regular and satisfy a two-sided local John condition. There exist ε_0 and C_0, depending only on n and the John and Ahlfors regularity constants, with the following significance. If $\varepsilon \in (0, \varepsilon_0)$ and (4.14) holds, then Ω is a δ-SKT domain, with $\delta = C_0\varepsilon$. In particular, Ω is a δ-Reifenberg flat domain and hence a two-sided NTA domain.*

Proposition 4.4. *If $\Omega \subset \mathbb{R}^n$ is Ahlfors regular, the following are equivalent.*

$$\Omega \textit{ satisfies a 2-sided local John condition and } \nu \in \mathrm{VMO}(\partial\Omega), \tag{4.18}$$

$$\Omega \textit{ is 2-sided NTA and } \nu \in \mathrm{VMO}(\partial\Omega), \tag{4.19}$$

$$\Omega \textit{ is a regular SKT domain.} \tag{4.20}$$

In Propositions 4.3–4.4, δ-Reifenberg flatness is part of the conclusion, rather than part of the hypothesis. A key tool in the work of §4.2 of [HMT] is a variant of the Semmes decomposition theorem. Such a decomposition was introduced in [Se3], where it was stated for C^2 surfaces, with estimates independent of this degree of smoothness. A more general result, with hypotheses involving Reifenberg flatness, was given in [KT2]. The decomposition theorem in [HMT] was done in the setting of an Ahlfors regular domain $\Omega \subset \mathbb{R}^n$, satisfying a 2-sided local John condition. Roughly, if the unit normal has small local mean oscillation, then, at an appropriate scale, $\partial\Omega$ agrees with the graph of a function with small Lipschitz constant, except for a small bad set, while staying close to this graph even on the bad set. See [HMT] for further details.

The following identifies a significant class of regular SKT domains.

Proposition 4.5. *If $\Omega \subset \mathbb{R}^n$ is a VMO_1 domain, then it is a regular SKT domain.*

Recall from Propositions 2.10 and 3.6 that if Ω is a BMO_1 domain, then it is Ahlfors regular and NTA. The final ingredient for Proposition 4.5 is that if Ω is VMO_1, then (4.13) holds. More generally, as shown in §2.5 of [HMT], if Ω has the form (2.13), with $A \in \mathrm{BMO}_1$, then

$$\mathrm{dist}(\nu, \mathrm{VMO}(\partial\Omega)) \le C(1 + \|\nabla A\|_{\mathrm{BMO}})\, \mathrm{dist}(\nabla A, \mathrm{VMO}(\mathbb{R}^n)). \tag{4.21}$$

The proof uses the John-Nirenberg inequality.

5. Fredholm properties of layer potentials on regular SKT domains

The following is the major result in §4.5 of [HMT].

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a UR domain, and assume $\partial\Omega$ is compact. Let $k : \mathbb{R}^n \setminus 0 \to \mathbb{R}$ be smooth, even, and homogeneous of degree $-n$. Consider the operator T, defined by*

$$Tf(x) = \mathrm{PV} \int\limits_{\partial\Omega} \langle x-y, \nu(y)\rangle \, k(x-y) f(y) \, d\sigma(y), \quad x \in \partial\Omega \tag{5.1}$$

(which, by results of §3, is bounded on $L^p(\partial\Omega)$ for all $p \in (1,\infty)$). Then

$$\Omega \text{ regular SKT domain} \Rightarrow T : L^p(\partial\Omega) \to L^p(\partial\Omega) \text{ compact}, \ \forall\, p \in (1,\infty). \tag{5.2}$$

More generally, if Ω is Ahlfors regular and satisfies a 2-sided local John condition, and $\partial\Omega$ is compact, then, given $p \in (1,\infty)$, for each $\varepsilon > 0$, there exists $\delta > 0$, depending only on ε, the Ahlfors and John constants of Ω, n, p, and the norm of k in (3.45), such that

$$\mathrm{dist}(\nu, \mathrm{VMO}(\partial\Omega)) < \delta \Longrightarrow \mathrm{dist}(T, \mathcal{K}(L^p(\partial\Omega)) < \varepsilon. \tag{5.3}$$

Here, $\mathcal{K}(L^p(\Omega))$ denotes the set of compact operators on $L^p(\partial\Omega)$, and the dist on the right side of (5.3) is the distance in $\mathcal{L}(L^p(\partial\Omega))$.

Remark. With μ as in (2.1)–(2.2), we can write (5.1) as

$$Tf(x) = -\,\mathrm{PV} \int\limits_{\partial\Omega} k(x-y)(x-y) f(y) \, d\mu(y).$$

An expansion of the form (3.42) allows us to replace $k(x-y)$ in (5.1) by

$$k(x, x-y), \tag{5.4}$$

with $k(x,z)$ continuous on $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$, along with z-dervatives of order $\le N$, homogeneous of degree $-n$ in z, and even in z.

In [H], this result was established in the case that Ω is a VMO_1 domain, with compact boundary. We recall some ingredients of the proof of this, since they also figure in the proof of Proposition 5.1. Suppose Ω has the form (2.13), with $\nabla A \in \mathrm{VMO}_1(\mathbb{R}^{n-1})$. Let $f \in L^p(\partial\Omega)$ have compact support, and consider

$$T_\varepsilon f(x) = \int\limits_{|x-y|>\varepsilon} \langle x-y, \nu(y)\rangle k(x-y) f(y) \, d\sigma(y). \tag{5.5}$$

Then, in $\mathbb{R}^{n-1}$-coordinates, via (2.21), this operator takes the form

(5.6)
$$\widetilde{T}_\varepsilon f(x') = \int\limits_{\vartheta(x',y')>\varepsilon} [A(x')-A(y')-\langle\nabla A(y'),x'-y'\rangle]k(x'-y',A(x')-A(y'))f(y')\,dy',$$

where $\vartheta(x',y')^2 = |x'-y'|^2 + (A(x')-A(y'))^2$. Now the area element on $\partial\Omega$ in these coordinates is $\sqrt{1+|\nabla A(y')|^2}\,dy'$. As shown in [H],

$$A \in \mathrm{BMO}_1(\mathbb{R}^{n-1}) \Rightarrow w = \sqrt{1+|\nabla A|^2} \ \text{ is an } A_p\text{-weight}, \ \forall\, p \in (1,\infty). \tag{5.7}$$

Thus the task is to show that, with

$$T_* f(x') = \sup_{\varepsilon>0} |T_\varepsilon f(x')|, \tag{5.8}$$

we have

$$\|T_* f\|_{L^p(w)} \le C_p(w,A,k)\|f\|_{L^p(w)}, \quad 1<p<\infty, \tag{5.9}$$

given that w is an A_p-weight. Using the homogeneity of k, let us write

(5.10)
$$\widetilde{T}_\varepsilon f(x') = \int\limits_{|x-y|>\varepsilon} \frac{A(x')-A(y')-\langle\nabla A(y'),x'-y'\rangle}{|x'-y'|^n} k\Bigl(\frac{B(x')-B(y')}{|x'-y'|}\Bigr) f(y')\,dy',$$

where

$$B(x') = (x', A(x')), \tag{5.11}$$

so

$$k\Bigl(\frac{B(x')-B(y')}{|x'-y'|}\Bigr) = k\Bigl(\frac{x'-y'}{|x'-y'|}, \frac{A(x')-A(y')}{|x'-y'|}\Bigr). \tag{5.12}$$

Here we changed the region of integration to $|x-y|>\varepsilon$. See p. 502 of [H] for an estimate of the difference. If we set

$$F(x) = \varphi(x)\psi(x')k(x), \tag{5.13}$$

with smooth, radial multipliers satisfying $\varphi(x) = 0$ for $|x| < 1/4$, 1 for $|x| > 1/2$, and $\psi(x') = 1$ for $|x'| \le 2$, 0 for $|x'| \ge 3$, then F is even, F and all its derivatives of order $\le N$ belong to $L^1(\mathbb{R}^n)$, and

$$k\Bigl(\frac{B(x')-B(y')}{|x'-y'|}\Bigr) = F\Bigl(\frac{B(x')-B(y')}{|x'-y'|}\Bigr). \tag{5.14}$$

With the substitution of (5.14) into (5.10), we are in the setting of Theorem 1.10 of [H], which yields

$$\|T_* f\|_{L^p(w)} \le C(n,p,\|w\|_{A_p},F)\|\nabla A\|_{\mathrm{BMO}}(1+\|\nabla B\|_{\mathrm{BMO}})^\nu \|f\|_{L^p(w)}. \tag{5.15}$$

We mention that one ingredient in the proof of (5.15) in [H] is a T1 theorem for rough singular integral operators.

The proof of compactness when Ω is a VMO_1 domain with compact boundary given in [H] is then modeled after the compactness proof for C^1 domains given in [FJR], with (5.15) replacing the Calderon estimate.

We say a little about the proof of Proposition 5.1. The first step is to cover $\partial\Omega$ by open sets $\partial\Omega\cap B_{R_\delta}(x_j)$, on each of which ν has small oscillation, and pass to a subcovering so as to have bounded overlap. Then pick a smooth partition of unity $\{\varphi_j^2\}$ subordinate to this cover, pick $\psi_j \in C_0^\infty(B_{R_\delta}(x_j))$, equal to 1 on supp φ_j, and write

$$T = \sum_{j=1}^N M_{1-\psi_j} T M_{\varphi_j^2} + \sum_{j=1}^N M_{\psi_j} T M_{\varphi_j^2}. \tag{5.16}$$

The first sum in (5.16) is compact on $L^p(\partial\Omega)$, by Proposition 2.9. The bounded overlap property allows one to show that the L^p-operator norm of the second sum is

$$\le C \max_{1\le j\le N} \|M_{\psi_j} T M_{\varphi_j}\|_{\mathcal{L}(L^p(\partial\Omega))}. \tag{5.17}$$

To estimate each of these operators, we use a Semmes decomposition (as described after Proposition 4.4). The estimate (5.15) is applied to the "good" piece, and the "bad" piece is estimated separately, using the fact that this piece has small measure. See §4.5 of [HMT] for details.

In §4.6 of [HMT], the following converse of Proposition 5.1 is established. Let K be the operator that arises in (5.1) when $k(x-y)$ is (a constant multiple of) $|x-y|^{-n}$, so one has the classical double layer, as in (3.15). Also, define R_k by

$$R_k f(x) = \mathrm{PV} \int_{\partial\Omega} \frac{x_k - y_k}{|x-y|^n} f(y)\, d\sigma(y), \tag{5.18}$$

operators bounded on $L^p(\partial\Omega)$ for $1 < p < \infty$, whenever Ω is a UR domain, by results of §3. Here is the result.

Proposition 5.2. *Let $\Omega \subset \mathbb{R}^n$ be Ahlfors regular and satisfy a local two-sided John condition. Assume $\partial\Omega$ is compact. Then Ω is a regular SKT domain if and only if*

$$K \text{ and each } [M_{\nu_j}, R_k] \text{ are compact on } L^2(\partial\Omega). \tag{5.19}$$

We move to the associated layer potential

$$
\begin{aligned}
\mathcal{T}f(x) &= \int\limits_{\partial\Omega} \langle x-y, \nu(y)\rangle k(x-y) f(y)\, d\sigma(y), \quad x \in \mathbb{R}^n \setminus \partial\Omega \\
&= \int\limits_{\partial\Omega} \langle \nu(y), \kappa(x-y)\rangle f(y)\, d\sigma(y).
\end{aligned} \tag{5.20}
$$

It follows from Proposition 3.3 (in the setting of bounded UR domains) that, given $f \in L^p(\partial\Omega)$,

$$
\lim_{y\to x, y\in\Gamma(x)} \mathcal{T}f(y) = \Bigl(\frac{1}{2} a(\nu(x)) I + T\Bigr) f(x), \tag{5.21}
$$

for σ-a.e. $x \in \partial\Omega$, where

$$
a(\xi) = \frac{1}{i}\langle \xi, \hat{\kappa}(\xi)\rangle, \quad \kappa(z) = z k(z). \tag{5.22}
$$

The special class of kernels $k(z)$ of the form

$$
z k(z) = \nabla_z E(z) \tag{5.23}
$$

arises in the classical double layer, when E is radial, and hence, up to a constant factor, the fundamental solution of the Laplace equation. In such a case, $\hat{\kappa}(\xi) = i\xi\hat{E}(\xi)$, so $\langle \nu(x), \hat{\kappa}(\nu(x))\rangle = i\hat{E}(\nu(x))$, and (5.21) boils down to (3.19),

$$
\lim_{y\to x, y\in\Gamma(x)} \mathcal{D}f(y) = \Bigl(\frac{1}{2} I + K\Bigr) f(x). \tag{5.24}
$$

In the more general case, note that $a(\xi)$ is homogeneous of degree 0 in ξ and continuous on $\mathbb{R}^n \setminus 0$. The following is a natural ellipticity condition on (5.21):

$$
a(\xi)^{-1} \in C(\mathbb{R}^n \setminus 0). \tag{5.25}
$$

If (5.25) holds and T is compact, or if its norm modulo compacts (on $L^p(\partial\Omega)$) is sufficiently small, then the operator $(1/2)a(\nu(x))I + T$ is Fredholm of index 0. With this knowledge in hand, we move on to the next section, to discuss the Dirichlet problem and other elliptic boundary problems on regular (and ε-regular) SKT domains.

6. Elliptic boundary problems on regular SKT domains

We started in §1 working on a compact n-dimensional manifold M, equipped with a metric tensor, which in local coordinates takes the form of a positive definite $n\times n$ matrix $(g_{jk}(x))$. In §§2–5 we switched to the setting of n-dimensional Euclidean space $\mathbb{R}^n$. Here, we return to the compact Riemannian manifold setting. We assume M is connected and let $\Omega\subset M$ be a connected open set. We assume $M\setminus\overline{\Omega}$ is not empty; it may have several connected components. We say Ω is finite perimeter, Ahlfors regular, UR, NTA, local John, regular SKT, etc., provided it has such properties in local coordinate systems. Results of §§2–5 extend to this manifold setting in a fairly straightforward way. Local formulas for volumes and surface area change; we have, in local coordinates,

$$d\mathcal{V}(x)=\sqrt{g(x)}\,dx,\quad g(x)=\det(g_{jk}(x)), \tag{6.1}$$

and

$$d\sigma(x)=\rho(x)\,d\sigma_e(x),\quad \rho(x)=\sqrt{g(x)}G(x,n(x))^{1/2}, \tag{6.2}$$

where $d\sigma_e$ is Euclidean surface area, $G(x,\xi)=\sum g^{jk}\xi_j\xi_k$, and n is the unit normal to $\partial\Omega$ relative to the Euclidean metric. Also,

$$\operatorname{div} v=\sum g^{-1/2}\partial_j(g^{1/2}v^j),\quad \Delta u=\sum g^{-1/2}\partial_j(g^{1/2}g^{jk}\partial_k u). \tag{6.3}$$

These changes are all coherent, and the Gauss-Green theorem retains the form (1.43), i.e.,

$$\int\limits_{\Omega}\operatorname{div} v\,d\mathcal{V}=\int\limits_{\partial\Omega}\langle\nu,v\rangle\,d\sigma, \tag{6.4}$$

where ν is the outward unit normal and $\langle\ ,\ \rangle$ the inner product, with respect to the metric tensor (g_{jk}). This identity holds for vector fields of the form (2.22) when Ω has finite perimeter, and it holds for vector fields of the form (2.34) when Ω is Ahlfors regular. In smoother settings, this is classical and well known, and was implicit in the discussion in §1. In these rougher settings, details can be found in §5.3 of [HMT]. Further results in the manifold setting are given in [HMT2].

As in §1, we take $V\in L^\infty(M)$, such that $V\ge 0$ on M and $V>0$ on a set of positive measure on each connected component of $M\setminus\overline{\Omega}$, and set $L=\Delta-V$. In this setting, $L:H^1(M)\to H^{-1}(M)$ is an isomorphism, whose inverse has an integral kernel

$$L^{-1}w(x)=\int\limits_M E(x,y)w(y)\,d\mathcal{V}(y), \tag{6.5}$$

We desire to solve

$$Lu = 0 \quad \text{on} \quad \Omega, \quad u\big|_{\partial\Omega} = f. \tag{6.6}$$

The layer potential attack seeks a function g on $\partial\Omega$ such that (6.6) is solved by

$$u = \mathcal{D}g\big|_{\Omega}, \tag{6.7}$$

where

$$\mathcal{D}g(x) = \int\limits_{\partial\Omega} \partial_{\nu_y} E(x,y) g(y)\, d\sigma(y). \tag{6.8}$$

To begin, we assume Ω is a UR domain.

To make use of results of §3, we need information on $E(x,y)$. A parametrix construction, detailed for progressively rougher metric tensors in [MT1]–[MT5], gives

$$\sqrt{g(x)}\, E(x,y) = e_0(x-y,x) + e_1(y,x), \tag{6.9}$$

where the leading term has the form (for $n \ge 3$)

$$e_0(z,x) = C_n \Bigl(\sum g_{jk}(x) z_j z_k \Bigr)^{-(n-2)/2}, \tag{6.10}$$

and the remainder $e_1(y,x)$ satisfies the following estimates if the metric tensor is Hölder continuous, say $g_{jk} \in C^\alpha$ for some $\alpha \in (0,1)$:

$$|e_1(y,x)| \le C|x-y|^{-(n-2-\alpha)}, \quad |\nabla_y e_1(y,x)| \le C|x-y|^{-(n-1-\alpha)}. \tag{6.11}$$

Cf. Proposition 2.4 of [MT5], which improves (2.70)–(2.71) of [MT2]. Going further, [MT5] and [HMT] treat Dini-type conditions, such as

$$g_{jk} \in C^{0,\omega}, \quad \omega(h) = \Bigl(\log \frac{1}{h}\Bigr)^{-2-b}, \tag{6.12}$$

for $0 < h < 1/2$, where we take $b > 0$. In such a case, one has

$$|e_1(y,x)| \le C \frac{\sigma(|x-y|)}{|x-y|^{n-2}}, \quad |\nabla_y e_1(y,x)| \le C \frac{\beta(|x-y|)}{|x-y|^{n-1}}, \tag{6.13}$$

where, for $0 < h < 1/2$, and with $a \in (0,b)$,

$$\sigma(h) = \Bigl(\log \frac{1}{h}\Bigr)^{-1-b}, \quad \beta(h) = \Bigl(\log \frac{1}{h}\Bigr)^{-1-a}. \tag{6.14}$$

Under these conditions, the contribution of $e_1(y,x)$ to $\mathcal{D}g$ in (6.8) can be handled by elementary means, and the results of §3 apply to the contribution of $e_0(x-y,x)$ to (6.8). We obtain the following results:

$$\|\mathcal{N}\mathcal{D}g\|_{L^p(\partial\Omega)} \le C_p\|g\|_{L^p(\partial\Omega)}, \quad 1<p<\infty, \tag{6.15}$$

$$\mathcal{D}g\big|_{\partial\Omega_\pm}(x) = \Bigl(\pm\frac{1}{2}I+K\Bigr)g(x), \quad \sigma\text{-a.e. } x\in\partial\Omega, \tag{6.16}$$

where

$$Kg(x) = \mathrm{PV}\int\limits_{\partial\Omega} \partial_{\nu_y}E(x,y)g(y)\,d\sigma(y), \tag{6.17}$$

and

$$K : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad 1<p<\infty, \tag{6.18}$$

whenever Ω is a UR domain. Furthermore, as shown in §5.2 of [HMT], we have

$$K = K^\# + K_0, \tag{6.19}$$

where K_0 has weakly singular integral kenel and (by Proposition 2.9) is compact on $L^p(\partial\Omega)$ for $p\in(1,\infty)$, and

$$K^\# g(x) = C_n\ \mathrm{PV}\int\limits_{\partial\Omega} \frac{\langle x-y,n(y)\rangle}{\Gamma(x,x-y)^{n/2}}G(y,n(y))^{-1/2}g(y)\,d\sigma(y), \tag{6.20}$$

where $\Gamma(x,x-y)=\sum g_{jk}(x)(x_j-y_j)(x_k-y_k)$, $\langle\ ,\ \rangle$ denotes the Euclidean inner product, and $n(y)$ is the Euclidean unit normal. It follows from Proposition 5.1 (or rather its variable coefficient extension, as in (5.4)) that $K^\#$ is compact on $L^p(\partial\Omega)$, and hence so is K, when Ω is a regular SKT domain.

As in §1, in addition to the double layer potential $\mathcal{D}$, the single layer potential

$$\mathcal{S}f(x) = \int\limits_{\partial\Omega} E(x,y)f(y)\,d\sigma(y), \quad x\in M\setminus\partial\Omega, \tag{6.21}$$

plays an important role. One wants to estimate $\nabla\mathcal{S}f$ and establish the limiting behavior of $\partial_\nu\mathcal{S}f$. To make this analysis for rough metric tensors, it is convenient to replace (6.9) by

$$\sqrt{g(y)}\,E(x,y) = e_0(x-y,y)+e_1(x,y), \tag{6.22}$$

using the symmetry $E(x,y) = E(y,x)$, and apply results of §3 (dealing with (3.47A)) to $e_0(x-y,y)$, and elementary consequences of (6.11) or (6.13) to the remainder. The result is that, when Ω is a UR domain,

$$\|\mathcal{N}(\nabla \mathcal{S} f)\|_{L^p(\partial\Omega)} \le C_p \|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty, \tag{6.23}$$

and, for $f \in L^p(\partial\Omega),\ p \in (1,\infty)$,

$$(\partial_\nu \mathcal{S} f)_\pm(x) = \Bigl(\mp\frac{1}{2}I + K^*\Bigr) f(x), \quad \sigma\text{-a.e. } x \in \partial\Omega. \tag{6.24}$$

In addition, for $f \in L^p(\partial\Omega)$,

$$(\mathcal{S} f)_+(x) = (\mathcal{S} f)_-(x) = Sf(x), \quad \sigma\text{-a.e. } x \in \partial\Omega, \tag{6.25}$$

where $Sf(x)$ is defined as in (6.21), for $x \in \partial\Omega$.

We have now assembled all the tools needed to prove the following.

Proposition 6.1. *Assume the metric tensor on M satisfies (6.12), with $b > 0$. If Ω is a UR domain, then*

$$\frac{1}{2}I + K^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \ \textit{is injective}, \tag{6.26}$$

for $p = 2$.

Proof. With the tools we have, the proof is formally identical to that of Proposition 1.3. The Gauss-Green theorem of Proposition 2.5 (or rather its variant in the Riemannian manifold setting) replaces the Gauss-Green theorem used in §1. Regarding the applicability of such a result here, we have, for $u = \mathcal{S} g$,

$$\begin{aligned} g \in L^2(\partial\Omega) &\Longrightarrow \mathcal{N}\nabla u \in L^2(\partial\Omega) \\ &\Longrightarrow \mathcal{N} v \in L^p(\partial\Omega), \quad \text{for some} \ \ p > 1, \end{aligned} \tag{6.27}$$

with $v = u\nabla u$, whenever Ω is a UR domain (the first implication via (6.23)). Also

$$\begin{aligned} \mathcal{N}\nabla u \in L^2(\partial\Omega) &\Longrightarrow |\nabla u|^2 \in L^q(\Omega) \\ &\Longrightarrow \operatorname{div} v \in L^q(\Omega), \end{aligned} \tag{6.28}$$

with $q = n/(n-1)$, whenever Ω is Ahlfors regular, by Proposition 3.2 and (1.42). Furthermore, local elliptic regularity results imply u is C^1 on the interior regions Ω and Ω_-, so v is continuous there.

From here, we have the following extension of Proposition 1.2.

Proposition 6.2. *In the setting of Proposition 6.1, if in addition Ω is a regular SKT domain, then, for all $p \in (1, \infty)$,*

$$\frac{1}{2}I + K : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \quad \textit{is an isomorphism.} \tag{6.29}$$

Proof. The hypotheses imply K and K^* are compact on each $L^p(\partial\Omega)$, hence $(1/2)I+K$ and $(1/2)I + K^*$ are Fredholm, of index 0, on each $L^p(\partial\Omega)$. The rest of the proof goes just like the proof of Proposition 1.2. Namely, Proposition 6.1 implies $(1/2)I + K^*$ is injective on $L^p(\partial\Omega)$ for $p \geq 2$, hence we have an isomorphism in (6.26) for $2 \leq p < \infty$, so we have an isomorphism in (6.29) for $1 < p \leq 2$. Hence the map (6.29) is injective for all $p \in (1, \infty)$, and Fredholmness of index 0 implies isomorphism.

From here, to solve (6.6), given $f \in L^p(\partial\Omega)$, we set $g = ((1/2)I + K)^{-1}f$ and then, by (6.16), $u = \mathcal{D}g|_{\partial\Omega}$ is the desired solution. This gives the following extension of Proposition 1.4.

Proposition 6.3. *In the setting of Proposition 6.2, given $1 < p < \infty$ and $f \in L^p(\partial\Omega)$, there exists a solution to (6.6), of the form (6.7) (with $g \in L^p(\partial\Omega)$), satisfying*

$$\|\mathcal{N}u\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}, \tag{6.30}$$

and

$$\lim_{y \to x, y \in \Gamma(x)} u(y) = f(x), \quad \textit{for } \sigma\textit{-a.e. } x \in \partial\Omega. \tag{6.31}$$

Uniqueness also holds. This takes an additional argument, given in §7.1 of [HMT]. It is also shown there that, for $p \in (1, \infty)$,

$$f \in H^{1,p}(\partial\Omega) \Longrightarrow \mathcal{N}(\nabla u) \in L^p(\partial\Omega). \tag{6.32}$$

In such a case, u is constructed in the form

$$u = \mathcal{S}(S^{-1}f). \tag{6.33}$$

The theory of L^p-Sobolev spaces $H^{1,p}(\partial\Omega)$ is less straightforward in this general context than it is for Lipschitz domains. It is developed in the context of Ahlfors regular domains in §3.6 of [HMT], and then further in §4.3, where comparison is made with other works on L^p-Sobolev spaces on metric measure spaces.

We turn to the Neumann problem

$$Lu = 0 \ \text{ on } \ \Omega, \quad \partial_\nu u\big|_{\partial\Omega} = f. \tag{6.34}$$

We look for a solution of the form

$$u = \mathcal{S}g, \tag{6.35}$$

which, by (6.24), leads to seeking a solution to

$$\Bigl(-\frac{1}{2}I + K^*\Bigr)g = f. \tag{6.36}$$

For simplicity, we add to our hypotheses on V the assumption that $V > 0$ on a set of positive measure on Ω. (The well known modification for the case $V \equiv 0$ on Ω works.) In this case, we can interchange the roles of Ω and Ω_-, and the proof of Proposition 6.1 gives

Proposition 6.4. *In the setting of Proposition 6.1, if also $V > 0$ on a set of positive measure on Ω, then*

$$-\frac{1}{2}I + K^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \quad \textit{is injective,} \tag{6.37}$$

for $p = 2$.

This leads to

Proposition 6.5. *In the setting of Proposition 6.4, if in addition Ω is a regular SKT domain, then, for all $p \in (1, \infty)$,*

$$-\frac{1}{2}I + K^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \quad \textit{is an isomorphism.} \tag{6.38}$$

Proof. An argument parallel to that in Proposition 6.2 works, to give that $-(1/2)I + K$ is an isomorphism on $L^p(\partial\Omega)$ for each $p \in (1, \infty)$, which then gives (6.38).

Hence we can solve (6.34) via (6.35)–(6.36), and we have the following.

Proposition 6.6. *In the setting of Proposition 6.5, given $1 < p < \infty$, $f \in L^p(\partial\Omega)$, there exists a solution to (6.34), of the form (6.35), satisfying*

$$\|\mathcal{N}\nabla u\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\partial\Omega)}, \tag{6.39}$$

and

$$\lim_{y \to x, y \in \Gamma(x)} u(y) = f(x), \quad \sigma\text{-a.e. } x \in \partial\Omega. \tag{6.40}$$

To close, we mention that [HMT] has extensions of Propositions 6.3 and 6.6 to the setting of ε-regular SKT domains, for small ε, and for p running over an ε-dependent compact subset of $(1, \infty)$. Also, Sections 6–7 of [HMT] treat boundary problems for the following systems of equations on regular and ε-regular SKT domains:

Stokes system for fluids,

Lamé system for linear elasticity,

Maxwell system for electromagnetic fields.

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