Functions of $\sqrt{-\Delta}$ and the Wave Equation

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If $\varphi \in \mathcal{S}(\mathbb{R})$ is even, the Fourier inversion formula on the operator level gives

$$\varphi(\sqrt{-\Delta}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varphi}(t) \cos t\sqrt{-\Delta}\, dt. \tag{1}$$

In fact, we can lower the requirement that φ be smooth and rapidly decreasing quite a bit; having φ and $\hat{\varphi}$ in $L^1(\mathbb{R})$ is sufficient, and sometimes we can go even beyond that.

We combine (1) with the formula

$$\cos t\sqrt{-\Delta} f(x) = C_n t \Big(\frac{1}{2t}\frac{d}{dt}\Big)^k \big[t^{2k-1} \overline{f}_x(t)\big], \tag{2}$$

for a function f on $\mathbb{R}^n = \mathbb{R}^{2k+1}$, where

$$C_n = \frac{1}{2} \pi^{-(n-1)/2} A_{n-1}. \tag{3}$$

We get

$$\begin{aligned} \varphi(\sqrt{-\Delta}) f(x) &= \frac{C_n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varphi}(t) \Big(\frac{d}{dt}\frac{1}{2t}\Big)^k \big[t^{n-1}\overline{f}_x(t)\big]\, dt \\ &= \frac{C_n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big(-\frac{1}{2t}\frac{d}{dt}\Big)^k \hat{\varphi}(t) \cdot t^{n-1} \overline{f}_x(t)\, dt. \end{aligned} \tag{4}$$

Now

$$\begin{aligned} &\int_0^{\infty} \Phi(r) r^{n-1} \overline{f}_x(r)\, dr \\ &= \frac{1}{A_{n-1}} \int_0^{\infty} \int\limits_{S^{n-1}} f(x - r\omega) \Phi(r) r^{n-1}\, dS(\omega)\, dr \\ &= \frac{1}{A_{n-1}} \int\limits_{\mathbb{R}^n} f(x-y) \Phi(|y|)\, dy. \end{aligned} \tag{5}$$

Hence, using (3), we obtain from (4) that

$$\varphi(\sqrt{-\Delta}) f(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}^n} \Phi_n(|y|) f(x-y)\, dy, \tag{6}$$

for $n = 2k+1$, where

$$\Phi_{2k+1}(r) = \Big(-\frac{1}{2\pi r}\frac{d}{dr}\Big)^k \hat{\varphi}(r). \tag{7}$$

Another way to write (6) is

$$\varphi(\sqrt{-\Delta})\delta(x) = \frac{1}{\sqrt{2\pi}}\Phi_n(|x|), \quad x \in \mathbb{R}^n. \tag{8}$$

Remark. The case $k = 0$ of (7) can be seen directly by the Fourier inversion theorem, without use of the calculations (1)–(5).

We seek an analogous formula for $\varphi(\sqrt{-\Delta})f(x)$ when f is a function on $\mathbb{R}^n$ with $n = 2k$. We get this by the following extension of the method of descent, which gives

$$\cos t\sqrt{-\Delta} f(x) = \cos t\sqrt{-\Delta_{n+1}} F(x,0), \tag{9}$$

with

$$F(x, x_{n+1}) = f(x). \tag{10}$$

From this and (1), we get

$$\begin{aligned}\varphi(\sqrt{-\Delta})f(x) &= \varphi(\sqrt{-\Delta_{n+1}})F(x,0) \\ &= \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}^{2k+1}} \Phi_{2k+1}\big(|(y,y_{n+1})|\big)F(x-y,y_{n+1})\,dy\,dy_{n+1} \\ &= \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}^{2k+1}} \Phi_{2k+1}\big((|y|^2+s^2)^{1/2}\big)f(x-y)\,dy\,ds,\end{aligned} \tag{11}$$

or

$$\varphi(\sqrt{-\Delta})f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}^n} \Phi_{2k}(|y|)f(x-y)\,dy, \tag{12}$$

where

$$\Phi_{2k}(r) = \int_{-\infty}^{\infty} \Phi_{2k+1}(\sqrt{r^2+s^2})\,ds, \tag{13}$$

and Φ_{2k+1} is given by (7). The change of variable $t = \sqrt{r^2+s^2}$ gives

$$\Phi_{2k}(r) = 2\int_r^{\infty} \Phi_{2k+1}(t)\frac{t}{\sqrt{t^2-r^2}}\,dt. \tag{14}$$

We now apply the formula (6) to obtain the Poisson kernel, for $e^{-y\sqrt{-\Delta}}$. We take

$$\varphi(\lambda) = e^{-y|\lambda|}, \tag{15}$$

so

$$\begin{aligned}
\hat{\varphi}(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y|\lambda| - i\lambda t}\, d\lambda \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \Big[e^{-\lambda(y+it)} + e^{-\lambda(y-it)} \Big]\, d\lambda \\
&= \frac{1}{\sqrt{2\pi}} \Big[\frac{1}{y+it} + \frac{1}{y-it} \Big] \\
&= \sqrt{\frac{2}{\pi}} \frac{y}{y^2+t^2}.
\end{aligned} \tag{16}$$

We get

$$e^{-y\sqrt{-\Delta}}\delta(x) = P_n(y,x) = \mathcal{P}_n(y,|x|), \quad x \in \mathbb{R}^n, \tag{17}$$

where, by (6)–(7),

$$\begin{aligned}
\mathcal{P}_{2k+1}(y,r) &= \frac{y}{\pi^{k+1}} \Big(-\frac{1}{2r}\frac{d}{dr} \Big)^k (y^2+r^2)^{-1} \\
&= \frac{k!}{\pi^{k+1}} \frac{y}{(y^2+r^2)^{k+1}}.
\end{aligned} \tag{18}$$

Another way to write this is

$$P_n(y,x) = \widetilde{C}_n \frac{y}{(|x|^2+y^2)^{(n+1)/2}}, \quad \widetilde{C}_n = \pi^{-(n+1)/2}\Gamma\Big(\frac{n+1}{2}\Big), \tag{19}$$

when $n = 2k+1$. To treat $n = 2k$, we can use (13) to write

$$\mathcal{P}_{2k}(y,r) = \frac{k!}{\pi^{k+1}} y \int_{-\infty}^{\infty} \frac{ds}{(y^2+r^2+s^2)^{k+1}}. \tag{20}$$

Setting $A = \sqrt{y^2+r^2}$, we have

$$\int_{-\infty}^{\infty} \frac{ds}{(A^2+s^2)^{k+1}} = A^{-(2k+1)} \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)^{k+1}}, \tag{21}$$

and after a slightly tedious residue calculation of the last integral, it develops that (19) also holds for $n = 2k$.

We next consider the resolvent, $(\mu^2 I - \Delta)^{-1}$, with $\mu > 0$. We have

$$\varphi(\lambda) = (\mu^2 + \lambda^2)^{-1}, \tag{22}$$

so

$$\begin{aligned} \hat{\varphi}(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-i\lambda t}}{\mu^2 + \lambda^2}\, d\lambda \\ &= \sqrt{\frac{\pi}{2}} \frac{1}{\mu} e^{-\mu|t|}, \end{aligned} \tag{23}$$

the latter computation either by residue calculus or from (16) and the Fourier inversion formula. We get

$$(\mu^2 I - \Delta)^{-1} \delta(x) = R_n(\mu, x) = \mathcal{R}_n(\mu, |x|), \quad x \in \mathbb{R}^n, \tag{24}$$

where, by (6)–(7),

$$\mathcal{R}_{2k+1}(\mu, r) = \frac{1}{2\pi^k} \frac{1}{\mu} \Bigl(-\frac{1}{2r}\frac{d}{dr}\Bigr)^k e^{-\mu r}. \tag{25}$$

In particular,

$$\begin{aligned} R_1(\mu, x) &= \frac{1}{2\mu} e^{-\mu|x|}, \\ R_3(\mu, x) &= \frac{1}{4\pi} \frac{e^{-\mu|x|}}{|x|}. \end{aligned} \tag{26}$$

We take up another approach to formulas for $R_n(\mu, x)$, based on the identity

$$(\mu^2 I - \Delta)^{-1} = \int_0^{\infty} e^{-(\mu^2 - \Delta)t}\, dt, \tag{27}$$

which gives

$$\begin{aligned} R_n(\mu, x) &= \int_0^{\infty} e^{-\mu^2 t}\, e^{t\Delta} \delta(x)\, dt \\ &= (4\pi)^{-n/2} \int_0^{\infty} e^{-|x|^2/4t} e^{-\mu^2 t} t^{-n/2}\, dt. \end{aligned} \tag{28}$$

Let us compare the cases $n = 1$ and 3 of (28) with (26). Making the change of variables $y = |x|$ and $|\xi| = \mu$, we have

$$\frac{1}{|\xi|} e^{-y|\xi|} = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-y^2/4t} e^{-t|\xi|^2} t^{-1/2}\, dt, \tag{29}$$

and

$$\frac{1}{y}e^{-y|\xi|} = \frac{1}{\sqrt{4\pi}} \int_0^\infty e^{-y^2/4t} e^{-t|\xi|^2} t^{-3/2}\, dt. \tag{29}$$

Note that applying $\partial/\partial y$ to (29) gives (30). (Also recall the remark below (8), implying the relatively elementary nature of the first formula in (28), which leads to (29).) Regarding (30) as an identity between Fourier multipliers, we have the operator identity

$$e^{-y\sqrt{-\Delta}} = \frac{y}{2\sqrt{\pi}} \int_0^\infty e^{-y^2/4t} e^{t\Delta}\, t^{-3/2}\, dt. \tag{31}$$

This identity, synthesizing the Poisson semigroup $e^{-y\sqrt{-\Delta}}$ from the heat semigroup $e^{t\Delta}$, is called the subordination identity.

If we apply both sides of (31) to $\delta \in \mathcal{E}'(\mathbb{R}^n)$, we get

$$\begin{aligned} P_n(y,x) &= \frac{y}{2\sqrt{\pi}} \int_0^\infty e^{-y^2/4t} e^{-|x|^2/4t} (4\pi t)^{-n/2} t^{-3/2}\, dt \\ &= \frac{y}{(4\pi)^{(n+1)/2}} \int_0^\infty e^{-(|x|^2+y^2)/4t} t^{-(n+3)/2}\, dt \\ &= \frac{y}{(4\pi)^{(n+1)/2}} \int_0^\infty e^{-s(|x|^2+y^2)/4} s^{(n-1)/2}\, ds, \end{aligned} \tag{32}$$

the last identity via the change of variable $s = 1/t$. Noting that

$$\begin{aligned} \int_0^\infty e^{-as} s^{(n-1)/2}\, ds &= a^{-(n+1)/2} \int_0^\infty e^{-u} u^{(n-1)/2}\, du \\ &= a^{-(n+1)/2} \Gamma\Bigl(\frac{n+1}{2}\Bigr), \end{aligned} \tag{33}$$

we have a second demonstration of (19).

We next analyze the operation of partial Fourier inversion of a function on $\mathbb{R}^n$,

$$\begin{aligned} S_R f(x) &= (2\pi)^{-n/2} \int\limits_{|\xi|\le R} \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi \\ &= \chi_R(\sqrt{-\Delta}) f(x), \end{aligned} \tag{34}$$

where

$$\chi_R(\lambda) = \begin{array}{ll} 1, & |\lambda| \le R, \\ 0, & \text{otherwise}. \end{array} \tag{35}$$

Note that

$$\widehat{\chi}_R(t) = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{-it\lambda}\, d\lambda = \sqrt{\frac{2}{\pi}}\, \frac{\sin Rt}{t}, \tag{36}$$

so (1) gives

$$S_R f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin Rt}{t}\, u(t,x)\, dt, \tag{37}$$

with

$$u(t,x) = \cos t\sqrt{-\Delta} f(x). \tag{38}$$

Note that if $n = 1$ and $g(t)$ is even,

$$S_R g(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin Rt}{t} g(t)\, dt, \tag{39}$$

so one presentation of (37) is that, for each $x \in \mathbb{R}^n$,

$$S_R f(x) = S_R u(\cdot, x)\big|_{t=0}. \tag{40}$$

Since $\widehat{\chi}_R$ is not in $L^1(\mathbb{R})$, some care needs to be taken with these formulas, but they are valid for $f \in L^2(\mathbb{R}^n)$ with compact support, and more generally for $f \in \mathcal{E}'(\mathbb{R}^n)$. As in (6)–(7), we have

$$S_R f(x) = D_n^R * f(x), \tag{41}$$

where

$$D_R^{2k+1}(x) = \mathcal{D}_R^{2k+1}(|x|), \tag{42}$$

with

$$\mathcal{D}_R^{2k+1}(r) = \frac{1}{\pi} \Big(-\frac{1}{2\pi r} \frac{d}{dr} \Big)^k \frac{\sin Rr}{r}. \tag{43}$$

Alternatively, directly from (4),

$$S_R f(x) = \frac{C_n}{\pi} \int_{-\infty}^{\infty} \frac{\sin Rt}{t} \Big(\frac{d}{dt} \frac{1}{2t} \Big)^k \big[t^{n-1} \overline{f}_x(t) \big]\, dt, \tag{44}$$

in case $n = 2k + 1$.

Specializing to $n = 3$, we have $C_3 = 2$, and

$$S_R f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin Rt}{t} \frac{d}{dt} \big[t \overline{f}_x(t) \big]\, dt, \quad x \in \mathbb{R}^3. \tag{45}$$

Hence, as in (39),

$$S_R f(x) = S_R g_x(0), \quad g_x(t) = \frac{d}{dt} \big[t \overline{f}_x(t) \big]. \tag{46}$$

Known results for pointwise Fourier inversion of functions on $\mathbb{R}$ give the following.

Proposition 1. *Let $f \in L^2(\mathbb{R}^3)$ have compact support. Fix $x \in \mathbb{R}^3$. Assume $t\overline{f}_x(t)$ is an absolutely continuous function of t, so g_x, defined in (46), belongs to $L^1(\mathbb{R})$. If also g_x is Hölder continuous with positive exponent at $t = 0$, then*

$$\lim_{R\to 0} S_R f(x) = g_x(0). \tag{47}$$

Let's take a close look at the following function, defined for $x \in \mathbb{R}^3$ by

$$\begin{aligned} f(x) = \chi_B(x) = 1&, \quad |x| < 1, \\ 0&, \quad |x| > 1. \end{aligned} \tag{48}$$

It is readily verified that

$$x \neq 0 \Longrightarrow t\overline{f}_x(t) \text{ is Lipschitz continuous (and smooth at } t = 0), \tag{49}$$

and, with $g_x(t)$ given by (46),

$$\begin{aligned} g_x(0) = 1&, \quad |x| < 1, \\ \frac{1}{2}&, \quad |x| = 1, \\ 0&, \quad |x| > 1. \end{aligned} \tag{50}$$

We hence have

$$\begin{aligned} \lim_{R\to\infty} S_R \chi_B(x) = 1&, \quad 0 < |x| < 1, \\ \frac{1}{2}&, \quad |x| = 1, \\ 0&, \quad |x| > 1. \end{aligned} \tag{51}$$

It remains to investigate $S_R \chi_B(0)$. As we will see, convergence fails here. In fact, for f given by (48), we have

$$\begin{aligned} \overline{f}_0(|t|) = 1&, \quad |t| < 1, \\ 0&, \quad |t| > 1, \end{aligned} \tag{52}$$

and hence

$$g_0(t) = \chi_{[-1,1]}(t) + \delta(t+1) - \delta(t-1). \tag{53}$$

Hence

$$\begin{aligned} S_R \chi_B(0) &= \frac{1}{\pi} \int_{-1}^{1} \frac{\sin Rt}{t}\, dt - \frac{2}{\pi} \sin R \\ &= S_R \chi_{[-1,1]}(0) - \frac{2}{\pi} \sin R. \end{aligned} \tag{54}$$

The first term on the last line converges to 1 as $R \to \infty$, but obviously the last term has an oscillatory divergence as $R \to \infty$. This is an illustration of a phenomenon called the Pinsky phenomenon. See [PT] for more details, and references to the literature.

Reference

[PT] M. Pinsky and M. Taylor, Pointwise Fourier inversion: a wave equation approach, J. Fourier Anal. and Appl. 3 (1997), 647–703.