The Gamma function

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ABSTRACT. This material is excerpted from §18 and Appendix J of [T].

The Gamma function has been previewed in (15.17)–(15.18), arising in the computation of a natural Laplace transform:

$$f(t) = t^{z-1} \Longrightarrow \mathcal{L}f(s) = \Gamma(z)\, s^{-z}, \tag{18.1}$$

for Re $z > 0$, with

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \quad \text{Re } z > 0. \tag{18.2}$$

Here we develop further properties of this special function, beginning with the following crucial identity:

$$\begin{aligned} \Gamma(z+1) &= \int_0^\infty e^{-t} t^z\, dt \\ &= -\int_0^\infty \frac{d}{dt}(e^{-t})\, t^z\, dt \\ &= z\,\Gamma(z), \end{aligned} \tag{18.3}$$

for Re $z > 0$, where we use integration by parts. The definition (18.2) clearly gives

$$\Gamma(1) = 1, \tag{18.4}$$

so we deduce that for any integer $k \ge 1$,

$$\Gamma(k) = (k-1)\Gamma(k-1) = \cdots = (k-1)!. \tag{18.5}$$

While $\Gamma(z)$ is defined in (18.2) for Re $z > 0$, note that the left side of (18.3) is well defined for Re $z > -1$, so this identity extends $\Gamma(z)$ to be meromorphic on $\{z : \text{Re } z > -1\}$, with a simple pole at $z = 0$. Iterating this argument, we extend $\Gamma(z)$ to be meromorphic on $\mathbb{C}$, with simple poles at $z = 0, -1, -2, \dots$. Having such a meromorphic continuation of $\Gamma(z)$, we establish the following identity.

Proposition 18.1. *For $z \in \mathbb{C} \setminus \mathbb{Z}$ we have*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{18.6}$$

Proof. It suffices to establish this identity for $0 < \mathrm{Re}\, z < 1$. In that case we have

$$\begin{aligned} \Gamma(z)\Gamma(1-z) &= \int_0^\infty \int_0^\infty e^{-(s+t)} s^{-z} t^{z-1}\, ds\, dt \\ &= \int_0^\infty \int_0^\infty e^{-u} v^{z-1} (1+v)^{-1}\, du\, dv \\ &= \int_0^\infty (1+v)^{-1} v^{z-1}\, dv, \end{aligned} \tag{18.7}$$

where we have used the change of variables $u = s + t,\ v = t/s$. With $v = e^x$, the last integral is equal to

$$\int_{-\infty}^\infty (1+e^x)^{-1} e^{xz}\, dx, \tag{18.8}$$

which is holomorphic on $0 < \mathrm{Re}\, z < 1$. We want to show that this is equal to the right side of (18.6) on this strip. It suffices to prove identity on the line $z = 1/2 + i\xi,\ \xi \in \mathbb{R}$. Then (18.8) is equal to the Fourier integral

$$\int_{-\infty}^\infty \Bigl(2 \cosh \frac{x}{2}\Bigr)^{-1} e^{ix\xi}\, dx. \tag{18.9}$$

This was evaluated in §16; by (16.23) it is equal to

$$\frac{\pi}{\cosh \pi\xi}, \tag{18.10}$$

and since

$$\frac{\pi}{\sin \pi(\frac{1}{2} + i\xi)} = \frac{\pi}{\cosh \pi\xi}, \tag{18.11}$$

the demonstration of (18.6) is complete.

Corollary 18.2. *The function $\Gamma(z)$ has no zeros, so $1/\Gamma(z)$ is an entire function.*

For our next result, we begin with the following estimate:

Lemma 18.3. *We have*

$$0 \le e^{-t} - \Big(1 - \frac{t}{n}\Big)^n \le \frac{t^2}{n} e^{-t}, \quad 0 \le t \le n, \tag{18.12}$$

the latter inequality holding provided $n \ge 4$.

Proof. The first inequality in (18.12) is equivalent to the simple estimate $e^{-y} - (1-y) \ge 0$ for $0 \le y \le 1$. To see this, denote the function by $f(y)$ and note that $f(0) = 0$ while $f'(y) = 1 - e^{-y} \ge 0$ for $y \ge 0$.

As for the second inequality in (18.12), write

$$\begin{aligned} \log\Big(1 - \frac{t}{n}\Big)^n &= n \log\Big(1 - \frac{t}{n}\Big) = -t - X, \\ X &= \frac{t^2}{n}\Big(\frac{1}{2} + \frac{1}{3}\frac{t}{n} + \frac{1}{4}\Big(\frac{t}{n}\Big)^2 + \cdots\Big). \end{aligned} \tag{18.13}$$

We have $(1 - t/n)^n = e^{-t-X}$ and hence, for $0 \le t \le n$,

$$e^{-t} - \Big(1 - \frac{t}{n}\Big)^n = (1 - e^{-X})e^{-t} \le X e^{-t},$$

using the estimate $x - (1 - e^{-x}) \ge 0$ for $x \ge 0$ (as above). It is clear from (18.13) that $X \le t^2/n$ if $t \le n/2$. On the other hand, if $t \ge n/2$ and $n \ge 4$ we have $t^2/n \ge 1$ and hence $e^{-t} \le (t^2/n)e^{-t}$.

We use (18.12) to obtain, for $\operatorname{Re} z > 0$,

$$\begin{aligned} \Gamma(z) &= \lim_{n\to\infty} \int_0^n \Big(1 - \frac{t}{n}\Big)^n t^{z-1}\, dt \\ &= \lim_{n\to\infty} n^z \int_0^1 (1-s)^n s^{z-1}\, ds. \end{aligned}$$

Repeatedly integrating by parts gives

$$\Gamma(z) = \lim_{n\to\infty} n^z \frac{n(n-1)\cdots 1}{z(z+1)\cdots(z+n-1)} \int_0^1 s^{z+n-1}\, ds, \tag{18.14}$$

which yields the following result of Euler:

Proposition 18.4. *For* $\operatorname{Re} z > 0$, *we have*

$$\Gamma(z) = \lim_{n\to\infty} n^z \frac{1 \cdot 2 \cdots n}{z(z+1)\cdots(z+n)}, \tag{18.15}$$

Using the identity (18.3), analytically continuing $\Gamma(z)$, we have (18.15) for all $z \in \mathbb{C}$ other than $0, -1, -2, \dots$. In more detail, we have

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \lim_{n\to\infty} n^{z+1} \frac{1}{z} \frac{1 \cdot 2 \cdots n}{(z+1)(z+2)\cdots(z+1+n)},$$

for $\operatorname{Re} z > -1 (z \neq 0)$. We can rewrite the right side as

$$\begin{aligned} &n^z \frac{1 \cdot 2 \cdots n \cdot n}{z(z+1)\cdots(z+n+1)} \\ &= (n+1)^z \frac{1 \cdot 2 \cdots (n+1)}{z(z+1)\cdots(z+n+1)} \cdot \Bigl(\frac{n}{n+1}\Bigr)^{z+1}, \end{aligned}$$

and $(n/(n+1))^{z+1} \to 1$ as $n \to \infty$. This extends (18.15) to $\{z \neq 0 : \operatorname{Re} z > -1\}$, and iteratively we get further extensions.

We can rewrite (18.15) as

$$\Gamma(z) = \lim_{n\to\infty} n^z\, z^{-1}(1+z)^{-1}\Bigl(1+\frac{z}{2}\Bigr)^{-1}\cdots\Bigl(1+\frac{z}{n}\Bigr)^{-1}. \tag{18.16}$$

To work on this formula, we define Euler's constant:

$$\gamma = \lim_{n\to\infty}\Bigl(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\Bigr). \tag{18.17}$$

Then (18.16) is equivalent to

$$\Gamma(z) = \lim_{n\to\infty} e^{-\gamma z}\, e^{z(1+1/2+\cdots+1/n)}\, z^{-1}(1+z)^{-1}\Bigl(1+\frac{z}{2}\Bigr)^{-1}\cdots\Bigl(1+\frac{z}{n}\Bigr)^{-1}, \tag{18.18}$$

which leads to the following Euler product expansion.

Proposition 18.5. *For all $z \in \mathbb{C}$, we have*

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z}\prod_{n=1}^{\infty}\Bigl(1+\frac{z}{n}\Bigr)e^{-z/n}. \tag{18.19}$$

We can combine (18.6) and (18.19) to produce a product expansion for $\sin \pi z$. In fact, it follows from (18.19) that the entire function $1/\Gamma(z)\Gamma(-z)$ has the product expansion

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2\prod_{n=1}^{\infty}\Bigl(1-\frac{z^2}{n^2}\Bigr). \tag{18.20}$$

Since $\Gamma(1-z) = -z\Gamma(-z)$, we have by (18.6) that

$$\sin \pi z = \pi z\prod_{n=1}^{\infty}\Bigl(1-\frac{z^2}{n^2}\Bigr). \tag{18.21}$$

For another proof of this result, see §30, Exercise 2.

Here is another application of (18.6). If we take $z = 1/2$, we get $\Gamma(1/2)^2 = \pi$. Since (18.2) implies $\Gamma(1/2) > 0$, we have

$$\Gamma\Bigl(\frac{1}{2}\Bigr) = \sqrt{\pi}. \tag{18.22}$$

Another way to obtain (18.22) is the following. A change of variable gives

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-t}t^{-1/2}\,dt = \frac{1}{2}\Gamma\Bigl(\frac{1}{2}\Bigr). \tag{18.23}$$

It follows from (10.6) that the left side of (18.23) is equal to $\sqrt{\pi}/2$, so we again obtain (18.22). Note that application of (18.3) then gives, for each integer $k \geq 1$,

$$\Gamma\Bigl(k+\frac{1}{2}\Bigr) = \pi^{1/2}\Bigl(k-\frac{1}{2}\Bigr)\Bigl(k-\frac{3}{2}\Bigr)\cdots\Bigl(\frac{1}{2}\Bigr). \tag{18.24}$$

One can calculate the area A_{n-1} of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ by relating Gaussian integrals to the Gamma function. To see this, note that the argument giving (10.6) yields

$$\int\limits_{\mathbb{R}^n} e^{-|x|^2}\,dx = \Bigl(\int_{-\infty}^\infty e^{-x^2}\,dx\Bigr)^n = \pi^{n/2}. \tag{18.25}$$

On the other hand, using spherical polar coordinates to compute the left side of (18.24) gives

$$\begin{aligned}\int\limits_{\mathbb{R}^n} e^{-|x|^2}\,dx &= A_{n-1}\int_0^\infty e^{-r^2}\,r^{n-1}\,dr \\ &= \frac{1}{2}A_{n-1}\int_0^\infty e^{-t}\,t^{n/2-1}\,dt,\end{aligned} \tag{18.26}$$

where we use $t = r^2$. Recognizing the last integral as $\Gamma(n/2)$, we have

$$A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{18.27}$$

More details on this argument are given at the end of Appendix C.

Exercises

1. Use the product expansion (18.19) to prove that

$$\frac{d}{dz}\,\frac{\Gamma'(z)}{\Gamma(z)} = \sum_{n=0}^\infty \frac{1}{(z+n)^2}. \tag{18.28}$$

Hint. Go from (18.19) to

$$\log \frac{1}{\Gamma(z)} = \log z + \gamma z + \sum_{n=1}^{\infty} \Bigl[\log\Bigl(1 + \frac{z}{n}\Bigr) - \frac{z}{n}\Bigr],$$

and note that

$$\frac{d}{dz} \frac{\Gamma'(z)}{\Gamma(z)} = \frac{d^2}{dz^2} \log \Gamma(z).$$

2. Let

$$\gamma_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log(n+1).$$

Show that $\gamma_n \nearrow$ and that $0 < \gamma_n < 1$. Deduce that $\gamma = \lim_{n\to\infty} \gamma_n$ exists, as asserted in (18.17).

3. Using $(\partial/\partial z)t^{z-1} = t^{z-1} \log t$, show that

$$f_z(t) = t^{z-1} \log t, \quad (\text{Re } z > 0)$$

has Laplace transform

$$\mathcal{L}f_z(s) = \frac{\Gamma'(z) - \Gamma(z) \log s}{s^z}, \quad \text{Re } s > 0.$$

4. Show that (18.19) yields

$$\Gamma(z+1) = z\Gamma(z) = e^{-\gamma z} \prod_{n=1}^{\infty} \Bigl(1 + \frac{z}{n}\Bigr)^{-1} e^{z/n}, \quad |z| < 1. \tag{18.29}$$

Use this to show that

$$\Gamma'(1) = \frac{d}{dz}\bigl(z\Gamma(z)\bigr)\big|_{z=0} = -\gamma. \tag{18.30}$$

5. Using Exercises 3–4, show that

$$f(t) = \log t \Longrightarrow \mathcal{L}f(s) = -\frac{\log s + \gamma}{s},$$

and that

$$\gamma = -\int_0^\infty (\log t) e^{-t}\, dt.$$

6. Show that $\gamma = \gamma_a - \gamma_b$, with

$$\gamma_a = \int_0^1 \frac{1 - e^{-t}}{t}\, dt, \quad \gamma_b = \int_1^\infty \frac{e^{-t}}{t}\, dt.$$

Consider how to obtain accurate numerical evaluations of these quantities.
Hint. Split the integral for γ in Exercise 5 into two pieces. Integrate each piece by parts, using $e^{-t} = -(d/dt)(e^{-t} - 1)$ for one and $e^{-t} = -(d/dt)e^{-t}$ for the other. See Appendix J for more on this.

7. Use the Laplace transform identity (18.1) for $f_z(t) = t^{z-1}$ (on $t \geq 0$, given $\mathrm{Re}\, z > 0$) plus the results of Exercises 5–6 of §15 to show that

$$B(z,\zeta) = \frac{\Gamma(z)\Gamma(\zeta)}{\Gamma(z+\zeta)}, \quad \mathrm{Re}\, z, \mathrm{Re}\, \zeta > 0, \tag{18.31}$$

where the *beta function* $B(z,\zeta)$ is defined by

$$B(z,\zeta) = \int_0^1 s^{z-1}(1-s)^{\zeta-1}\, ds, \quad \mathrm{Re}\, z, \mathrm{Re}\, \zeta > 0. \tag{18.32}$$

The identity (18.31) is known as Euler's formula for the beta function.

8. Show that, for any $z \in \mathbb{C}$, when $n \geq 2|z|$, we have

$$\Bigl(1 + \frac{z}{n}\Bigr) e^{-z/n} = 1 + w_n$$

with $\log(1+w_n) = \log(1+z/n) - z/n$ satisfying

$$\bigl|\log(1+w_n)\bigr| \leq \frac{|z|^2}{n^2}.$$

Show that this estimate implies the convergence of the product on the right side of (18.19), locally uniformly on $\mathbb{C}$.

More infinite products

9. Show that

$$\prod_{n=1}^{\infty} \Bigl(1 - \frac{1}{4n^2}\Bigr) = \frac{2}{\pi}. \tag{18.33}$$

Hint. Take $z = 1/2$ in (18.21).

10. Show that, for all $z \in \mathbb{C}$,

$$\cos \frac{\pi z}{2} = \prod_{\text{odd } n \geq 1} \Bigl(1 - \frac{z^2}{n^2}\Bigr). \tag{18.34}$$

Hint. Use $\cos \pi z/2 = -\sin\big((\pi/2)(z-1)\big)$ and (18.21) to obtain

$$\cos \frac{\pi z}{2} = \frac{\pi}{2}(1-z)\prod_{n=1}^{\infty}\Big(1-\frac{(z-1)^2}{4n^2}\Big). \tag{18.35}$$

Use $(1-u^2)=(1-u)(1+u)$ to write the general factor in this infinite product as

$$\begin{aligned}&\Big(1+\frac{1}{2n}-\frac{z}{2n}\Big)\Big(1-\frac{1}{2n}+\frac{z}{2n}\Big)\\ &\quad=\Big(1-\frac{1}{4n^2}\Big)\Big(1-\frac{z}{2n+1}\Big)\Big(1+\frac{z}{2n-1}\Big),\end{aligned}$$

and obtain from (18.35) that

$$\cos\frac{\pi z}{2}=\frac{\pi}{2}\prod_{n=1}^{\infty}\Big(1-\frac{1}{4n^2}\Big)\cdot\prod_{\text{odd } n\ge 1}\Big(1-\frac{z}{n}\Big)\Big(1+\frac{z}{n}\Big).$$

Deduce (18.34) from this and (18.33).

11. Show that

$$\frac{\sin \pi z}{\pi z}=\cos\frac{\pi z}{2}\cdot\cos\frac{\pi z}{4}\cdot\cos\frac{\pi z}{8}\cdots. \tag{18.36}$$

Hint. Make use of (18.21) and (18.34).

18A. The Legendre duplication formula

The Legendre duplication formula relates $\Gamma(2z)$ and $\Gamma(z)\Gamma(z+1/2)$. Note that each of these functions is meromorphic, with poles precisely at $\{0,-1/2,-1,-3/2,-2,\dots\}$, all simple, and both functions are nowhere vanishing. Hence their quotient is an entire holomorphic function, and it is nowhere vanishing, so

$$\Gamma(2z)=e^{A(z)}\Gamma(z)\Gamma\Big(z+\frac{1}{2}\Big), \tag{18.37}$$

with $A(z)$ holomorphic on $\mathbb{C}$. We seek a formula for $A(z)$. We will be guided by (18.19), which implies that

$$\frac{1}{\Gamma(2z)}=2ze^{2\gamma z}\prod_{n=1}^{\infty}\Big(1+\frac{2z}{n}\Big)e^{-2z/n}, \tag{18.38}$$

and (via results given in §18B)

$$\begin{aligned}\frac{1}{\Gamma(z)\Gamma(z+1/2)}& \\ &= z\Big(z+\frac{1}{2}\Big)e^{\gamma z}e^{\gamma(z+1/2)}\Big\{\prod_{n=1}^{\infty}\Big(1+\frac{z}{n}\Big)e^{-z/n}\Big(1+\frac{z+1/2}{n}\Big)e^{-(z+1/2)/n}\Big\}.\end{aligned} \tag{18.39}$$

Setting

$$1+\frac{z+1/2}{n}=\frac{2z+2n+1}{2n}=\Big(1+\frac{2z}{2n+1}\Big)\Big(1+\frac{1}{2n}\Big), \tag{18.40}$$

and

$$e^{-(z+1/2)/n}=e^{-2z/(2n+1)}e^{-2z[(1/2n)-1/(2n+1)]}e^{-1/2n}, \tag{18.41}$$

we can write the infinite product on the right side of (18.39) as

$$\begin{aligned}&\Big\{\prod_{n=1}^{\infty}\Big(1+\frac{2z}{2n}\Big)e^{-2z/2n}\Big(1+\frac{2z}{2n+1}\Big)e^{-2z/(2n+1)}\Big\} \\ &\qquad\times\Big\{\prod_{n=1}^{\infty}\Big(1+\frac{1}{2n}\Big)e^{-1/2n}\Big\}\times\prod_{n=1}^{\infty}e^{-2z[(1/2n)-1/(2n+1)]}.\end{aligned} \tag{18.42}$$

Hence

$$\begin{aligned}\frac{1}{\Gamma(z)\Gamma(z+1/2)}&=ze^{2\gamma z}e^{\gamma/2}\cdot\frac{e^{2z}}{2}(1+2z)e^{-2z}\times(18.42) \\ &=2ze^{2\gamma z}e^{\gamma/2}\frac{e^{2z}}{4}\Big\{\prod_{k=1}^{\infty}\Big(1+\frac{2z}{k}\Big)e^{-2z/k}\Big\} \\ &\qquad\times\Big\{\prod_{n=1}^{\infty}\Big(1+\frac{1}{2n}\Big)e^{-1/2n}\Big\}\prod_{n=1}^{\infty}e^{-2z[(1/2n)-1/(2n+1)]}.\end{aligned} \tag{18.43}$$

Now, setting $z=1/2$ in (18.19) gives

$$\frac{1}{\Gamma(1/2)}=\frac{1}{2}e^{\gamma/2}\prod_{n=1}^{\infty}\Big(1+\frac{1}{2n}\Big)e^{-1/2n}, \tag{18.44}$$

so taking (18.38) into account yields

$$\begin{aligned}\frac{1}{\Gamma(z)\Gamma(z+1/2)}&=\frac{1}{\Gamma(1/2)\Gamma(2z)}\frac{e^{2z}}{2}\prod_{n=1}^{\infty}e^{-2z[(1/2n)-1/(2n+1)]} \\ &=\frac{1}{\Gamma(1/2)\Gamma(2z)}\frac{e^{2\alpha z}}{2},\end{aligned} \tag{18.45}$$

where

$$\alpha = 1 - \sum_{n=1}^{\infty} \Big(\frac{1}{2n} - \frac{1}{2n+1} \Big) \tag{18.46}$$
$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots$$
$$= \log 2.$$

Hence $e^{2\alpha z} = 2^{2z}$, and we get

$$\Gamma\Big(\frac{1}{2}\Big)\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\Big(z + \frac{1}{2}\Big). \tag{18.47}$$

This is the Legendre duplication formula. Recall that $\Gamma(1/2) = \sqrt{\pi}$.

An equivalent formulation of (18.47) is

$$(2\pi)^{1/2}\Gamma(z) = 2^{z-1/2}\Gamma\Big(\frac{z}{2}\Big)\Gamma\Big(\frac{z+1}{2}\Big). \tag{18.48}$$

This generalizes to the following formula of Gauss,

$$(2\pi)^{(n-1)/2}\Gamma(z) = n^{z-1/2}\Gamma\Big(\frac{z}{n}\Big)\Gamma\Big(\frac{z+1}{n}\Big)\cdots\Gamma\Big(\frac{z+n-1}{n}\Big), \tag{18.49}$$

valid for $n = 3, 4, \dots$.

18B. Convergence of infinite products

Here we record some results regarding the convergence of infinite products, which have arisen in this section. We look at infinite products of the form

$$\prod_{k=1}^{\infty} (1 + a_k). \tag{18.50}$$

Disregarding cases where one or more factors $1 + a_k$ vanish, the convergence of $\prod_{k=1}^{M}(1 + a_k)$ as $M \to \infty$ amounts to the convergence

$$\lim_{M\to\infty} \prod_{k=M}^{N} (1 + a_k) = 1, \quad \text{uniformly in } \ N > M. \tag{18.51}$$

In particular, we require $a_k \to 0$ as $k \to \infty$. To investigate when (18.51) happens, write

$$\prod_{k=M}^{N} (1 + a_k) = (1 + a_M)(1 + a_{M+1}) \cdots (1 + a_N) \tag{18.52}$$
$$= 1 + \sum_j a_j + \sum_{j_1<j_2} a_{j_1}a_{j_2} + \cdots + a_M \cdots a_N,$$

where, e.g., $M \le j_1 < j_2 \le N$. Hence

$$
\begin{aligned}
\Bigl|\prod_{k=M}^{N}(1+a_k)-1\Bigr| &\le \sum_j |a_j| + \sum_{j_1<j_2} |a_{j_1}a_{j_2}| + \cdots + |a_M\cdots a_N| \\
&= \prod_{k=M}^{N}(1+|a_k|)-1 \\
&= b_{MN},
\end{aligned}
\tag{18.53}
$$

the last identity defining b_{MN}. Our task is to investigate when $b_{MN} \to 0$ as $M \to \infty$, uniformly in $N > M$. To do this, we note that

$$
\begin{aligned}
\log(1+b_{MN}) &= \log \prod_{k=M}^{N}(1+|a_k|) \\
&= \sum_{k=M}^{N} \log(1+|a_k|),
\end{aligned}
\tag{18.54}
$$

and use the facts

$$
\begin{aligned}
x \ge 0 &\Longrightarrow \log(1+x) \le x, \\
0 \le x \le 1 &\Longrightarrow \log(1+x) \ge \frac{x}{2}.
\end{aligned}
\tag{18.55}
$$

Assuming $a_k \to 0$ and taking M so large that $k \ge M \Rightarrow |a_k| \le 1/2$, we have

$$
\frac{1}{2}\sum_{k=M}^{N}|a_k| \le \log(1+b_{MN}) \le \sum_{k=M}^{N}|a_k|,
\tag{18.56}
$$

and hence

$$
\lim_{M\to\infty} b_{MN} = 0, \quad \text{uniformly in } N > M \Longleftrightarrow \sum_k |a_k| < \infty.
\tag{18.57}
$$

Consequently,

$$
\begin{aligned}
\sum_k |a_k| < \infty &\Longrightarrow \prod_{k=1}^{\infty}(1+|a_k|) \ \text{ converges} \\
&\Longrightarrow \prod_{k=1}^{\infty}(1+a_k) \ \text{ converges.}
\end{aligned}
\tag{18.58}
$$

Another consequence of (18.57) is the following:

$$
\text{If } 1+a_k \ne 0 \text{ for all } k, \text{ then } \sum |a_k| < \infty \Rightarrow \prod_{k=1}^{\infty}(1+a_k) \ne 0.
\tag{18.59}
$$

We can replace the sequence (a_k) of complex numbers by a sequence (f_k) of holomorphic functions, and deduce from the estimates above the following.

Proposition 18.6. *Let* $f_k : \Omega \to \mathbb{C}$ *be holomorphic. If*

$$\sum_k |f_k(z)| < \infty \quad on \ \ \Omega, \tag{18.60}$$

then we have a convergent infinite product

$$\prod_{k=1}^{\infty} (1 + f_k(z)) = g(z), \tag{18.61}$$

and g *is holomorphic on* Ω*. If* $z_0 \in \Omega$ *and* $1 + f_k(z_0) \neq 0$ *for all* k*, then* $g(z_0) \neq 0$*.*

Another consequence of estimates leading to (18.57) is that if also $g_k : \Omega \to \mathbb{C}$ and $\sum |g_k(z)| < \infty$ on Ω, then

$$\Bigl\{ \prod_{k=1}^{\infty} (1 + f_k(z)) \Bigr\} \prod_{k=1}^{\infty} (1 + g_k(z)) = \prod_{k=1}^{\infty} (1 + f_k(z))(1 + g_k(z)). \tag{18.62}$$

To make contact with the Gamma function, note that the infinite product in (18.19) has the form (18.61) with

$$1 + f_k(z) = \Bigl(1 + \frac{z}{k}\Bigr) e^{-z/k}. \tag{18.63}$$

To see that (18.60) applies, note that

$$e^{-w} = 1 - w + R(w), \quad |w| \le 1 \Rightarrow |R(w)| \le C|w|^2. \tag{18.64}$$

Hence

$$\begin{aligned} \Bigl(1 + \frac{z}{k}\Bigr) e^{-z/k} &= \Bigl(1 + \frac{z}{k}\Bigr)\Bigl(1 - \frac{z}{k} + R\Bigl(\frac{z}{k}\Bigr)\Bigr) \\ &= 1 - \frac{z^2}{k^2} + \Bigl(1 - \frac{z}{k}\Bigr) R\Bigl(\frac{z}{k}\Bigr). \end{aligned} \tag{18.65}$$

Hence (18.63) holds with

$$f_k(z) = -\frac{z^2}{k^2} + \Bigl(1 - \frac{z}{k}\Bigr) R\Bigl(\frac{z}{k}\Bigr), \tag{18.66}$$

so

$$|f_k(z)| \le C \Bigl|\frac{z}{k}\Bigr|^2 \quad \text{for} \ \ k \ge |z|, \tag{18.67}$$

which yields (18.60).

J. Euler's constant

Here we say more about Euler's constant, introduced in (18.17), in the course of producing the Euler product expansion for $1/\Gamma(z)$. The definition

$$\gamma = \lim_{n\to\infty}\Bigl(\sum_{k=1}^{n}\frac{1}{k} - \log(n+1)\Bigr) \tag{J.1}$$

of Euler's constant involves a very slowly convergent sequence. In order to produce a numerical approximation of γ, it is convenient to use other formulas, involving the Gamma function $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt$. Note that

$$\Gamma'(z) = \int_0^\infty (\log t)e^{-t}t^{z-1}\,dt. \tag{J.2}$$

Meanwhile the Euler product formula $1/\Gamma(z) = ze^{\gamma z}\prod_{n=1}^\infty (1+z/n)e^{-z/n}$ implies

$$\Gamma'(1) = -\gamma. \tag{J.3}$$

Thus we have the integral formula

$$\gamma = -\int_0^\infty (\log t)e^{-t}\,dt. \tag{J.4}$$

To evaluate this integral numerically it is convenient to split it into two pieces:

$$\begin{aligned}\gamma &= -\int_0^1 (\log t)e^{-t}\,dt - \int_1^\infty (\log t)e^{-t}\,dt \\ &= \gamma_a - \gamma_b.\end{aligned} \tag{J.5}$$

We can apply integration by parts to both the integrals in (5), using $e^{-t} = -(d/dt)(e^{-t} - 1)$ on the first and $e^{-t} = -(d/dt)e^{-t}$ on the second, to obtain

$$\gamma_a = \int_0^1 \frac{1-e^{-t}}{t}\,dt, \quad \gamma_b = \int_1^\infty \frac{e^{-t}}{t}\,dt. \tag{J.6}$$

Using the power series for e^{-t} and integrating term by term produces a rapidly convergent series for γ_a:

$$\gamma_a = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k\cdot k!}. \tag{J.7}$$

Before producing infinite series representations for γ_b, we note that the change of variable $t = s^m$ gives

$$\gamma_b = m \int_1^\infty \frac{e^{-s^m}}{s}\, ds, \tag{J.8}$$

which is very well approximated by the integral over $s \in [1, 10)$ if $m = 2$, for example.

To produce infinite series for γ_b, we can break up $[1, \infty)$ into intervals $[k, k+1)$ and take $t = s + k$, to write

$$\gamma_b = \sum_{k=1}^\infty \frac{e^{-k}}{k}\, \beta_k, \quad \beta_k = \int_0^1 \frac{e^{-t}}{1 + t/k}\, dt. \tag{J.9}$$

Note that $0 < \beta_k < 1 - 1/e$ for all k. For $k \geq 2$ we can write

$$\beta_k = \sum_{j=0}^\infty \Bigl(-\frac{1}{k}\Bigr)^j \alpha_j, \quad \alpha_j = \int_0^1 t^j e^{-t}\, dt. \tag{J.10}$$

One convenient way to integrate $t^j e^{-t}$ is the following. Write

$$E_j(t) = \sum_{\ell=0}^j \frac{t^\ell}{\ell!}. \tag{J.11}$$

Then

$$E_j(t) = E_{j-1}(t), \tag{J.12}$$

hence

$$\frac{d}{dt}\bigl(E_j(t)e^{-t}\bigr) = \bigl(E_{j-1}(t) - E_j(t)\bigr)e^{-t} = -\frac{t^j}{j!}\, e^{-t}, \tag{J.13}$$

so

$$\int t^j e^{-t}\, dt = -j! E_j(t) e^{-t} + C. \tag{J.14}$$

In particular,

$$\begin{aligned} \alpha_j = \int_0^1 t^j e^{-t}\, dt &= j!\Bigl(1 - \frac{1}{e} \sum_{\ell=0}^j \frac{1}{\ell!}\Bigr) \\ &= \frac{j!}{e} \sum_{\ell=j+1}^\infty \frac{1}{\ell!} \\ &= \frac{1}{e}\Bigl(\frac{1}{j+1} + \frac{1}{(j+1)(j+2)} + \cdots\Bigr). \end{aligned} \tag{J.15}$$

To evaluate β_1 as an infinite series, it is convenient to write

$$
\begin{aligned}
e^{-1}\beta_1 &= \int_1^2 \frac{e^{-t}}{t}\,dt \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_1^2 t^{j-1}\,dt \\
&= \log 2 + \sum_{j=1}^{\infty} \frac{(-1)^j}{j \cdot j!}(2^j - 1).
\end{aligned} \tag{J.16}
$$

To summarize, we have $\gamma = \gamma_a - \gamma_b$, with γ_a given by the convenient series (J.7) and

$$\gamma_b = \sum_{k=2}^{\infty}\sum_{j=0}^{\infty} \frac{e^{-k}}{k}\Big(-\frac{1}{k}\Big)^j \alpha_j + \log 2 + \sum_{j=1}^{\infty} \frac{(-1)^j}{j \cdot j!}(2^j - 1), \tag{J.17}$$

with α_j given by (J.15). We can reverse the order of summation of the double series and write

$$\gamma_b = \sum_{j=0}^{\infty} (-1)^j \zeta_j \alpha_j + \log 2 + \sum_{j=1}^{\infty} \frac{(-1)^j}{j \cdot j!}(2^j - 1). \tag{J.18}$$

with

$$\zeta_j = \sum_{k=2}^{\infty} \frac{e^{-k}}{k^{j+1}}. \tag{J.19}$$

Note that

$$0 < \zeta_j < 2^{-(j+1)} \sum_{k=2}^{\infty} e^{-k} < 2^{-(j+3)}, \tag{J.20}$$

while (J.15) readily yields $0 < \alpha_j < 1/ej$. So one can expect 15 digits of accuracy by summing the first series in (J.18) over $0 \le j \le 50$ and the second series over $0 \le j \le 32$, assuming the ingredients α_j and ζ_j are evaluated sufficiently accurately. It suffices to sum (J.19) over $2 \le k \le 40 - 2j/3$ to evaluate ζ_j to sufficient accuracy.

Note that the quantities α_j do not have to be evaluated independently. Say you are summing the first series in (J.18) over $0 \le j \le 50$. First evaluate α_{50} using 20 terms in (J.15), and then evaluate inductively $\alpha_{49}, \dots, \alpha_0$ using the identity

$$\alpha_{j-1} = \frac{1}{je} + \frac{\alpha_j}{j}, \tag{J.21}$$

equivalent to $\alpha_j = j\alpha_{j-1} - 1/e$, which follows by integration by parts of $\int_0^1 t^j e^{-t}\, dt$.

If we sum the series (J.7) for γ_a over $1 \le k \le 20$ and either sum the series (J.18) as described above or have Mathematica numerically integrate (J.8), with $m = 2$, to high precision, we obtain

$$\gamma \approx 0.577215664901533, \tag{J.22}$$

which is accurate to 15 digits.

We give another series for γ. This one is more slowly convergent than the series in (J.7) and (J.18), but it makes clear why γ exceeds $1/2$ by a small amount, and it has other interesting aspects. We start with

$$\gamma = \sum_{n=1}^{\infty} \gamma_n, \quad \gamma_n = \frac{1}{n} - \int_n^{n+1} \frac{dx}{x} = \frac{1}{n} - \log\Bigl(1 + \frac{1}{n}\Bigr). \tag{J.23}$$

Thus γ_n is the area of the region

$$\Omega_n = \Bigl\{(x, y) : n \le x \le n+1,\ \frac{1}{x} \le y \le \frac{1}{n}\Bigr\}. \tag{J.24}$$

This region contains the triangle T_n with vertices $(n, 1/n), (n+1, 1/n)$, and $(n+1, 1/(n+1))$. The region $\Omega_n \setminus T_n$ is a little sliver. Note that

$$\text{Area } T_n = \delta_n = \frac{1}{2}\Bigl(\frac{1}{n} - \frac{1}{n+1}\Bigr), \tag{J.25}$$

and hence

$$\sum_{n=1}^{\infty} \delta_n = \frac{1}{2}. \tag{J.26}$$

Thus

$$\gamma - \frac{1}{2} = (\gamma_1 - \delta_1) + (\gamma_2 - \delta_2) + (\gamma_3 - \delta_3) + \cdots . \tag{J.27}$$

Now

$$\gamma_1 - \delta_1 = \frac{3}{4} - \log 2, \tag{J.28}$$

while, for $n \ge 2$, we have power series expansions

$$\begin{aligned} \gamma_n &= \frac{1}{2n^2} - \frac{1}{3n^3} + \frac{1}{4n^4} - \cdots \\ \delta_n &= \frac{1}{2n^2} - \frac{1}{2n^3} + \frac{1}{2n^4} - \cdots , \end{aligned} \tag{J.29}$$

the first expansion by $\log(1+z) = z - z^2/2 + z^3/3 - \cdots$, and the second by

$$\delta_n = \frac{1}{2n(n+1)} = \frac{1}{2n^2}\,\frac{1}{1+\frac{1}{n}}, \tag{J.30}$$

and the expansion $(1+z)^{-1} = 1 - z + z^2 - \cdots$. Hence we have

$$\gamma - \frac{1}{2} = (\gamma_1 - \delta_1) + \Bigl(\frac{1}{2} - \frac{1}{3}\Bigr)\sum_{n\ge 2}\frac{1}{n^3} - \Bigl(\frac{1}{2} - \frac{1}{4}\Bigr)\sum_{n\ge 2}\frac{1}{n^4} + \cdots, \tag{J.31}$$

or, with

$$\zeta(k) = \sum_{n\ge 1}\frac{1}{n^k}, \tag{J.32}$$

we have

$$\gamma - \frac{1}{2} = \Bigl(\frac{3}{4} - \log 2\Bigr) + \Bigl(\frac{1}{2} - \frac{1}{3}\Bigr)[\zeta(3) - 1] - \Bigl(\frac{1}{2} - \frac{1}{4}\Bigr)[\zeta(4) - 1] + \cdots, \tag{J.33}$$

an alternating series from the third term on. We note that

$$\begin{aligned} \frac{3}{4} - \log 2 &\approx 0.0568528, \\ \frac{1}{6}[\zeta(3) - 1] &\approx 0.0336762, \\ \frac{1}{4}[\zeta(4) - 1] &\approx 0.0205808, \\ \frac{3}{10}[\zeta(5) - 1] &\approx 0.0110783. \end{aligned} \tag{J.34}$$

The estimate

$$\sum_{n\ge 2}\frac{1}{n^k} < 2^{-k} + \int_2^\infty x^{-k}\,dx \tag{J.35}$$

implies

$$0 < \Bigl(\frac{1}{2} - \frac{1}{k}\Bigr)[\zeta(k) - 1] < 2^{-k}, \tag{J.36}$$

so the series (J.33) is geometrically convergent. If k is even, $\zeta(k)$ is a known rational multiple of π^k. However, for odd k, the values of $\zeta(k)$ are more mysterious. Note that to get $\zeta(3)$ to 16 digits by summing (J.32) one needs to sum over $1 \le n \le 10^8$. On a 1.3 GHz personal computer, a C program does this in 4 seconds. Of course, this is vastly slower than summing (J.7) and (J.18) over the ranges discussed above.

Reference

[T] M. Taylor, Introduction to Complex Analysis. Notes, available on this website.