**Remarks on the Gauss-Green Theorem**

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Abstract. These notes cover material related to the Gauss-Green theorem that was developed for work with S. Hofmann and M. Mitrea, which appeared in [HMT].

**Contents**



## 1. Versions of the Gauss-Green formula

Let $\Omega \subset \mathbb{R}^m$ be open. We say $\Omega$ has locally finite perimeter provided

$$\nabla \chi_\Omega = \mu \tag{1.1}$$

is a locally finite $\mathbb{R}^m$-valued measure. It follows from the Radon-Nikodym theorem that $\mu = -\nu\,\sigma$, where $\sigma$ is a locally finite positive measure, supported on $\partial\Omega$, and $\nu \in L^\infty(\partial\Omega, \sigma)$ is an $\mathbb{R}^m$-valued function, satisfying $|\nu(x)| = 1$, $\sigma$-a.e. It then follows from the Besicovitch differentiation theorem that

$$\lim_{r\to 0} \frac{1}{\sigma(B_r(x))} \int\limits_{B_r(x)} \nu\, d\sigma = \nu(x) \tag{1.2}$$

for $\sigma$-a.e. $x$.

Via distribution theory, we can restate (1.1) as follows. Take $v \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^m)$ (a vector field). Then

$$\langle \operatorname{div} v, \chi_\Omega \rangle = -\langle v, \nabla \chi_\Omega \rangle. \tag{1.3}$$

Hence (1.1) is equivalent to

$$\int\limits_\Omega \operatorname{div} v\, dx = \int\limits_{\partial\Omega} \nu \cdot v\, d\sigma. \tag{1.4}$$

Works of Federer and of De Giorgi produced the following results on the structure of $\sigma$, when $\Omega$ has locally finite perimeter. First,

$$\sigma = \mathcal{H}^{m-1} \lfloor \partial^* \Omega, \tag{1.5}$$

where $\mathcal{H}^{m-1}$ is $(m-1)$-dimensional Hausdorff measure and $\partial^*\Omega \subset \partial\Omega$ is the reduced boundary of $\Omega$, defined as the set of points $x$ such that the limit (1.2) holds, with $|\nu(x)| = 1$. (It follows from the remarks leading up to (1.2) that $\sigma$ is supported on $\partial^*\Omega$.) Second, $\partial^*\Omega$ is countably rectifiable; it is a countable disjoint union

$$\partial^*\Omega = \bigcup_k M_k \cup N, \tag{1.6}$$

where each $M_k$ is a compact subset of an $(m-1)$-dimensional $C^1$ surface (to which $\nu$ is normal in the usual sense), and $\mathcal{H}^{m-1}(N) = 0$. Given (1.5), the identity (1.4) yields the Gauss-Green formula

$$\int_\Omega \operatorname{div} v \, dx = \int_{\partial^*\Omega} \nu \cdot v \, d\mathcal{H}^{m-1}, \tag{1.7}$$

for $v \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^m)$.

It is also useful to record some results on sets $\partial_*\Omega \supset \partial_0\Omega \supset \partial^*\Omega$. Good references for this material, as well as the results stated above, are [EG], [Fed3], and [Zie]. First, given a unit vector $\nu_E$ and $x \in \partial\Omega$, set

$$H^{\pm}_{\nu_E}(x) = \{y \in \mathbb{R}^m : \pm\nu_E \cdot (y - x) \geq 0\}. \tag{1.8}$$

Then (cf. [EG], p. 203), for $x \in \partial^*\Omega$, $\Omega^+ = \Omega$, $\Omega^- = \mathbb{R}^m \setminus \Omega$, one has

$$\lim_{r\to 0} r^{-m} \, \mathcal{L}^m\big(B_r(x) \cap \Omega^\pm \cap H^{\pm}_{\nu_E}(x)\big) = 0, \tag{1.9}$$

when $\nu_E = \nu(x)$, as in (1.2). Here $\mathcal{L}^m$ denotes Lebesgue measure on $\mathbb{R}^m$. More generally, a unit vector $\nu_E$ for which (1.9) holds is called the measure-theoretic outer normal to $\Omega$ at $x$. It is easy to show that if such $\nu_E$ exists it is unique. Thus if we define $\partial_0\Omega$ to consist of $x \in \partial\Omega$ for which (1.9) holds, with $\nu_E(x)$ denoting the measure-theoretic outer normal, we have $\partial_0\Omega \supset \partial^*\Omega$ and $\nu_E(x) = \nu(x)$ on $\partial^*\Omega$.

Next, we define $\partial_*\Omega$, the measure-theoretic boundary of $\Omega$, to consist of $x \in \partial\Omega$ such that

$$\limsup_{r\to 0} \, r^{-m} \mathcal{L}^m\big(B_r(x) \cap \Omega^\pm\big) > 0. \tag{1.10}$$

It is clear that $\partial_*\Omega \supset \partial_0\Omega$. Furthermore (cf. [EG], p. 208) one has

$$\mathcal{H}^{m-1}(\partial_*\Omega \setminus \partial^*\Omega) = 0. \tag{1.11}$$

Consequently the Green formula (1.7) can be rewritten

$$\int_{\Omega} \operatorname{div} v \, dx = \int_{\partial_* \Omega} \nu \cdot v \, d\mathcal{H}^{m-1}, \tag{1.12}$$

for $v \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^m)$. The advantage of (1.12) is that the definition of $\partial_*\Omega$ is more straightforward and geometrical than is that of $\partial^*\Omega$.

Remark. Note that $\partial_*\Omega$ is well defined whether or not $\Omega$ has locally finite perimeter. It is known that $\Omega$ has locally finite perimeter if and only if $\mathcal{H}^{m-1}(\partial_*\Omega \cap K) < \infty$ for each compact $K \subset \mathbb{R}^m$. (Cf. [EG], p. 222.)

In general $\partial\Omega \setminus \partial_*\Omega$ can be quite large. It is of interest to know conditions under which $\mathcal{H}^{m-1}(\partial\Omega \setminus \partial_*\Omega) = 0$. One such result will be given in §2.

So far we have discussed the Green formula for $v \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^m)$. A simple limiting argument extends (1.4) and hence (1.7) to $v \in C_0^1(\mathbb{R}^m, \mathbb{R}^m)$; [Fed] emphasizes that (1.7) is true for compactly supported Lipschitz $v$. In fact, we can easily do better. Here is one improvement.

**Proposition 1.1.** *If $\Omega$ has locally finite perimeter, then (1.4) holds for $v$ in*

$$\mathcal{D} = \{v \in C_0^0(\mathbb{R}^m, \mathbb{R}^m) : \operatorname{div} v \in L^1(\mathbb{R}^m)\}. \tag{1.13}$$

*Proof.* Given $v \in \mathcal{D}$, take $\varphi \in C_0^\infty(\mathbb{R}^m)$ such that $\int \varphi \, dx = 1$, set $\varphi_k(x) = k^m \varphi(kx)$, and take $v_k = \varphi_k * v \in C_0^\infty(\mathbb{R}^m)$. Then (1.4) applies to $v_k$, i.e.,

$$\int_{\Omega} \operatorname{div} v_k \, dx = \int_{\partial\Omega} \nu \cdot v_k \, d\sigma. \tag{1.14}$$

Meanwhile, $\operatorname{div} v_k = \varphi_k * (\operatorname{div} v) \to \operatorname{div} v$ in $L^1(\mathbb{R}^m)$ and $\nu \cdot v_k \to \nu \cdot v$ uniformly on $\partial\Omega$, so as $k \to \infty$, the left side of (1.14) converges to the left side of (1.4) while the right side of (1.14) converges to the right side of (1.4).

In many cases one deals with functions defined only on $\overline{\Omega}$, and one would like to avoid assuming they have extensions to $\mathbb{R}^m$ with nice properties. To obtain a result for such functions, we will introduce the following concept. Let open sets $\Omega_k$ satisfy $\overline{\Omega}_k \subset \Omega$, $\Omega_k \subset \Omega_{k+1}$, and $\Omega_k \nearrow \Omega$. We say $\{\Omega_k : k \geq 1\}$ is a tame interior approximation to $\Omega$ if in addition there exists $C(R) < \infty$ such that, for $R \in (0, \infty)$,

$$\|\nabla \chi_{\Omega_k}\|_{\mathrm{TV}(B_R)} \leq C(R), \quad \forall\, k \geq 1. \tag{1.15}$$

To give an example, take $A : \mathbb{R}^{m-1} \to \mathbb{R}$, satisfying

$$A \in C(\mathbb{R}^{m-1}), \quad \nabla A \in L^1_{loc}(\mathbb{R}^{m-1}). \tag{1.16}$$

As will be shown in §2,

$$\Omega = \{(x', x_m) \in \mathbb{R}^m : x_m > A(x')\} \tag{1.17}$$

has locally finite perimeter. The estimates proven there then imply that

$$\Omega_k = \{(x', x_m) \in \mathbb{R}^m : x_m > A(x') + k^{-1}\} \tag{1.18}$$

is a tame interior approximation to $\Omega$. The following is a partial extension of Proposition 1.1.

**Proposition 1.2.** *Assume $\Omega$ has locally finite perimeter and a tame interior approximation. Then (1.4) holds for $v$ in*

$$\widetilde{\mathcal{D}} = \{v \in C_0^0(\overline{\Omega}, \mathbb{R}^m) : \operatorname{div} v \in L^1(\Omega)\}. \tag{1.19}$$

*Proof.* Let $\{\Omega_k\}$ denote a tame interior approximation. Pick $\varphi_k \in C_0^\infty(\Omega)$ to be $\equiv 1$ on a neighborhood of $\overline{\Omega}_k \cap \operatorname{supp} v$, set $v_k = \varphi_k v$, and apply Proposition 1.1 with $\Omega$ replaced by $\Omega_k$ and $v$ by $v_k$, noting that $\operatorname{div} v_k = \varphi_k \operatorname{div} v + \nabla\varphi_k \cdot v$. We have

$$\int\limits_{\Omega_k} \operatorname{div} v \, dx = -\langle v, \nabla\chi_{\Omega_k}\rangle. \tag{1.20}$$

As $k \to \infty$, the left side of (1.20) converges to the left side of (1.4). Meanwhile, we can take $w \in C_0^0(\mathbb{R}^m, \mathbb{R}^m)$, equal to $v$ on $\overline{\Omega}$, and the right side of (1.15) is equal to $-\langle w, \nabla\chi_{\Omega_k}\rangle$. Now $\chi_{\Omega_k} \to \chi_\Omega$ in $L^1_{loc}(\mathbb{R}^m)$, so $\nabla\chi_{\Omega_k} \to \nabla\chi_\Omega$ in $\mathcal{D}'(\mathbb{R}^m)$, and hence

$$\langle w, \nabla\chi_{\Omega_k}\rangle \longrightarrow \langle w, \nabla\chi_\Omega\rangle \tag{1.21}$$

for each $w \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^m)$. The bounds (1.15) then imply that (1.21) holds for each $w \in C_0^0(\mathbb{R}^m, \mathbb{R}^m)$. Hence the right side of (1.20) converges to

$$-\langle w, \nabla\chi_\Omega\rangle = \int\limits_{\partial\Omega} \nu \cdot v \, d\sigma, \tag{1.22}$$

which is the right side of (1.4).

As we will discuss in §3, Proposition 1.2 is not adequate for the Green formulas we need for layer potentials. Such a Green formula will be demonstrated in §5.

Remark. Proposition 1.2 can be compared with the following result, given in

[Fed1], p. 314. Let $\Omega \subset \mathbb{R}^m$ be a bounded open set such that $\mathcal{H}^{m-1}(\partial\Omega) < \infty$. Fix $j \in \{1, \dots, m\}$ and take $f$ such that

$$f \in C(\overline{\Omega}), \quad \partial_j f \in L^1(\Omega). \tag{1.23}$$

Then

$$\int_\Omega \partial_j f \, dx = \int_{\partial_0\Omega} e_j \cdot \nu \, f \, d\mathcal{H}^{m-1}, \tag{1.24}$$

where $e_j$ is the $j$th standard basis vector of $\mathbb{R}^m$. In light of (1.11) one could replace $\partial_0\Omega$ by $\partial^*\Omega$ or by $\partial_*\Omega$ in (1.24). This leads to the identity (1.7) for a vector field $v \in C(\overline{\Omega})$ provided each term $\partial_j v_j$ in $\operatorname{div} v$ belongs to $L^1(\Omega)$. However, the vector fields arising in the applications of Green's formula needed in §§3 and 5 need not have this additional structure, so (1.24) is not applicable.

There is also recent work of [CT] and [CTZ], which we will briefly discuss in Appendix B. We present a result where $\operatorname{div} v$ can be a measure in Appendix C.

We turn our attention to a Green formula for

$$\int_{\Omega\cap B_r} \operatorname{div} v \, dx, \tag{1.25}$$

where $\Omega$ has locally finite perimeter and $B_r = \{x \in \mathbb{R}^m : |x| < r\}$. Assume $v \in C_0^{0,1}(\mathbb{R}^m, \mathbb{R}^m)$. Given $\varepsilon \in (0, r)$, set

$$\begin{aligned} \psi_\varepsilon(x) = {} & 1 && \text{for} \quad |x| \le r - \varepsilon, \\ & 1 - \frac{1}{\varepsilon}(|x| - r + \varepsilon) && \text{for} \quad r - \varepsilon \le |x| \le r, \\ & 0 && \text{for} \quad |x| \ge r. \end{aligned} \tag{1.26}$$

Then, with $\nabla\chi_\Omega = -\nu\,\sigma$, we have

$$\begin{aligned} \int_{\Omega\cap B_r} \operatorname{div} v \, dx &= \lim_{\varepsilon\searrow 0} \int_\Omega \psi_\varepsilon \operatorname{div} v \, dx \\ &= \lim_{\varepsilon\searrow 0} \int_\Omega [\operatorname{div} \psi_\varepsilon v - v \cdot \nabla\psi_\varepsilon] \, dx \\ &= \lim_{\varepsilon\searrow 0} \Bigl( \int \nu \cdot \psi_\varepsilon v \, d\sigma - \int_\Omega v \cdot \nabla\psi_\varepsilon \, dx \Bigr) \\ &= \int_{B_r} \nu \cdot v \, d\sigma + \lim_{\varepsilon\searrow 0} \frac{1}{\varepsilon} \int_{\Omega\cap S_\varepsilon} n \cdot v \, dx, \end{aligned} \tag{1.27}$$

where $n$ is the outward unit normal to $B_r$ and

$$S_\varepsilon = B_r \setminus B_{r-\varepsilon}. \tag{1.28}$$

Consequently,

$$\int\limits_{\Omega\cap B_r} \operatorname{div} v\, dx = \int\limits_{B_r} \nu \cdot v\, d\sigma + D_r^- \Phi(r), \tag{1.29}$$

where

$$\Phi(r) = \int\limits_{\Omega\cap B_r} n \cdot v\, dx. \tag{1.30}$$

Note that

$$\Phi(r) = \int_0^r \int\limits_{\Omega\cap\partial B_s} n \cdot v\, d\mathcal{H}^{m-1}\, ds, \tag{1.31}$$

so

$$D_r^- \Phi(r) = \int\limits_{\Omega\cap\partial B_r} n \cdot v\, d\mathcal{H}^{m-1}, \quad \text{for } \mathcal{L}^1\text{-a.e. } r. \tag{1.32}$$

It is of interest to note that $D_r^- \Phi(r)$ exists for *all* $r \in (0, \infty)$ (under our standing hypothesis on $\Omega$), though the identity (1.32) is valid perhaps not for each $r$, but just for a.e. $r$.

REMARK. Having (1.29), one can bring in (1.5) and write

$$\int\limits_{\Omega\cap B_r} \operatorname{div} v\, dx = \int\limits_{B_r\cap\partial^*\Omega} \nu \cdot v\, d\mathcal{H}^{m-1} + D_r^- \Phi(r). \tag{1.33}$$

It is important to note that (1.5) is not needed to prove (1.29), since (1.29) plays a role in proofs of (1.5).

## 2. Examples of domains with locally finite perimeter

Let $\Omega$ be the region over the graph of a function $A : \mathbb{R}^{m-1} \to \mathbb{R}$:

$$\Omega = \{x \in \mathbb{R}^m : x_m > A(x')\}, \tag{2.1}$$

where $x = (x', x_m)$. We have:

**Proposition 2.1.** *If*

$$A \in C(\mathbb{R}^{m-1}), \quad \nabla A \in L^1_{loc}(\mathbb{R}^{m-1}), \tag{2.2}$$

*then $\Omega$ has locally finite perimeter.*

*Proof.* Pick $\psi \in C_0^\infty(\mathbb{R}^{m-1})$ such that $\int \psi(x')\,dx' = 1$, set $\psi_k(x') = k^{m-1}\psi(kx')$, and set $A_k = \psi_k * A$ and

$$\Omega_k = \{x \in \mathbb{R}^m : x_m > A_k(x')\}. \tag{2.3}$$

Clearly

$$A_k \longrightarrow A, \quad \text{locally uniformly,} \tag{2.4}$$

so

$$\chi_{\Omega_k} \longrightarrow \chi_\Omega \ \text{ in } \ L^1_{loc}(\mathbb{R}^m). \tag{2.5}$$

Hence $\nabla\chi_{\Omega_k} \to \nabla\chi_\Omega$ in $\mathcal{D}'(\mathbb{R}^m)$. Also

$$\nabla\chi_{\Omega_k} = -\nu_k\,\sigma_k, \tag{2.6}$$

where $\sigma_k$ is surface area on

$$\Sigma_k = \{x \in \mathbb{R}^m : x_m = A_k(x')\}, \tag{2.7}$$

given in $x'$-coordinates by

$$d\sigma_k(x') = \sqrt{1 + |\nabla A_k(x')|^2}\,dx', \tag{2.8}$$

and $\nu_k$ is the downward-pointing unit normal to $\Sigma_k$. The hypothesis (2.2) implies that $\{\nu_k\sigma_k : k \geq 1\}$ is a bounded set of $\mathbb{R}^m$-valued measures on each set $B_R = \{x \in \mathbb{R}^m : |x| \leq R\}$, so passing to the limit gives

$$\nabla\chi_\Omega = \mu \tag{2.9}$$

where $\mu$ is a locally finite $\mathbb{R}^m$-valued measure. This proves Proposition 2.1.

Remark. The proof above is an easy extension of the argument in [Zie] that the domain over a Lipschitz graph has locally finite perimeter.

The measure $\mu$ in (2.9) has the form $\mu = -\nu\,\sigma$, as described after (1.1). To obtain a more explicit formula, we invoke (1.4),

$$\int_{\Omega} \operatorname{div} v\, dx = \int \nu \cdot v\, d\sigma, \tag{2.10}$$

together with the elementary identities

$$\int_{\Omega_k} \operatorname{div} v\, dx = \int_{\mathbb{R}^{m-1}} (\nabla A_k(x'), -1) \cdot v(x', A_k(x'))\, dx', \tag{2.11}$$

valid for each $v \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^m)$. (See Appendix A for an elementary proof.) As $k \to \infty$, the left side of (2.11) converges to the left side of (2.10), while the right side of (2.11) converges to

$$\int_{\mathbb{R}^{m-1}} (\nabla A(x'), -1) \cdot v(x', A(x'))\, dx'. \tag{2.12}$$

Hence

$$\int \nu \cdot v\, d\sigma = \int_{\mathbb{R}^{m-1}} \tilde{\nu}(x') \cdot v(x', A(x'))\, d\sigma(x'), \tag{2.13}$$

where

$$\tilde{\nu}(x') = \frac{(\nabla A(x'), -1)}{\sqrt{1 + |\nabla A(x')|^2}}, \quad d\sigma(x') = \sqrt{1 + |\nabla A(x')|^2}\, dx'. \tag{2.14}$$

The formula (2.13) is valid for all $v \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^m)$, hence for all $v \in C_0^0(\mathbb{R}^m, \mathbb{R}^m)$.

Remark. So far in this section we have not used (1.5). At this point we can invoke (1.5), to get

$$\int_{\partial^*\Omega} \nu \cdot v\, d\mathcal{H}^{m-1} = \int_{\mathbb{R}^{m-1}} \tilde{\nu}(x') \cdot v(x', A(x'))\, d\sigma(x'), \tag{2.15}$$

for each $v \in C_0^0(\mathbb{R}^m, \mathbb{R}^m)$.

It is also of interest to see how the decomposition (1.6), asserting countable rectifiability, arises in the context of (2.1)–(2.1). For simplicity, assume $A$ has compact support. Then set

$$f = |A| + |\nabla A|, \quad g = \mathcal{M}f, \tag{2.16}$$

the latter being the Hardy-Littlewood maximal function, and take

$$R^\lambda = \{x \in \mathbb{R}^{m-1} : \mathcal{M}f(x) \le \lambda\}. \tag{2.17}$$

Then $\mathcal{L}^{m-1}(\mathbb{R}^{m-1} \setminus R^\lambda) \le C\lambda^{-1}\|f\|_{L^1}$, and an argument involving the Poincaré inequality yields

$$x, y \in R^\lambda \Longrightarrow |A(x)|,\ |A(x) - A(y)| \le C\lambda|x - y|. \tag{2.18}$$

Using this one writes $\partial\Omega = \cup_k K_k \cup \widetilde{N}$, where each $L_k$ is a Lipschitz graph and $\sigma(\widetilde{N}) = 0$. Passing to (1.6) is then done by decomposing each Lipschitz graph into a countable union of $C^1$ graphs plus a negligible remainder, via Rademacher's theorem and Whitney's extension theorem. See §6.6 of [EG], or Theorem 11.9 and Proposition 12.9 of [T], for details.

Regarding the issue of $\partial\Omega$ versus $\partial_*\Omega$, it is clear that $\partial\Omega = \partial_*\Omega$ whenever $A$ is locally Lipschitz. For more general $A$ satisfying (2.2), we have the following, which is a consequence of the main results of [Tom] and [Fed2].

**Proposition 2.2.** *If $\Omega$ is the region in $\mathbb{R}^m$ over the graph of a function $A$ satisfying (2.2), then*

$$\mathcal{H}^{m-1}(\partial\Omega \setminus \partial_*\Omega) = 0. \tag{2.19}$$

*Proof.* Given a "rectangle" $Q = I_1 \times \cdots \times I_{m-1} \subset \mathbb{R}^{m-1}$, a product of compact intervals, set $K_Q = Q \times \mathbb{R}$. Given the formula (2.14) for $\sigma$, it follows from Theorem 3.17 of [Tom] that

$$\sigma(\partial\Omega \cap K_Q) = \mathcal{I}^{m-1}(\partial\Omega \cap K_Q), \tag{2.20}$$

where $\mathcal{I}^{m-1}$ denotes $(m-1)$-dimensional integral-geometric measure. Furthermore, it is shown in [Fed2] that

$$\mathcal{H}^{m-1}(\partial\Omega \cap K_Q) = \mathcal{I}^{m-1}(\partial\Omega \cap K_Q). \tag{2.21}$$

On the other hand, we have from (1.5) that $\sigma(\partial\Omega \cap K_Q) = \mathcal{H}^{m-1}(\partial^*\Omega \cap K_Q)$, so (2.19) follows.

Remark. The main argument in [Fed2] showed that there exists $C_m < \infty$ such that

$$\mathcal{H}^{m-1}(S) \le C_m \mathcal{I}^{m-1}(S), \tag{2.22}$$

for all Borel sets $S \subset \partial\Omega \cap K_Q$. Federer then invoked his structure theorem to deduce that $\partial\Omega \cap K_Q$ is countably rectifiable and (2.21) holds. We can avoid depending on the structure theorem as follows. Theorem 3.17 of [Tom] also says that if we have a Borel set $S \subset \partial\Omega \cap K_Q$ and set $S^b = \{x \in Q : (x, A(x)) \in S\}$, then, under the hypothesis (2.2),

$$\mathcal{L}^{m-1}(S^b) = 0 \Longrightarrow \mathcal{I}^{m-1}(S) = 0. \tag{2.23}$$

Now as discussed above, $(\partial\Omega \setminus \partial^*\Omega) \cap Q \subset \overline{A}(V) = \{(x, A(x)) : x \in V\}$ with $V$ of Lebesgue measure 0, so (2.22)–(2.23) give (2.19).

## 3. Green formulas for layer potentials

Assume $\Omega \subset \mathbb{R}^m$ is a bounded NTA domain, and $\partial\Omega$ satisfies the Ahlfors regularity condition: there exist $C_j \in (0, \infty)$ such that

$$C_1 r^{m-1} \le \mathcal{H}^{m-1}(\partial\Omega \cap B_r(p)) \le C_2 r^{m-1}, \tag{3.1}$$

for each $p \in \partial\Omega,\ r \in (0, \operatorname{diam}\Omega]$. More generally, $\Omega$ can be a UR domain, as defined in §3 of [HMT]. Take $f \in L^2(\partial\Omega, \sigma)$, with $\sigma = \mathcal{H}^{m-1}\lfloor \partial^*\Omega$, and form

$$u(x) = \mathcal{S}f(x), \tag{3.2}$$

where $\mathcal{S}$ is the single layer potential. Then, for a.e. $x \in \partial\Omega$, $\mathcal{S}f(x) = Sf(x)$ and

$$\lim_{y \to x} \nu(x) \cdot \nabla u(y) = T^* f(x) = \Bigl(-\frac{1}{2} + K^*\Bigr) f(x), \tag{3.3}$$

the limit being from within $\Omega$ via a nontangential approach. We would like to be able to establish that

$$\int_{\Omega} |\nabla u|^2 \, dx = \int_{\partial\Omega} u\, T^* f \, d\sigma. \tag{3.4}$$

Under these hypotheses, $\partial\Omega = \partial^*\Omega$ (modulo a null set), and if we set

$$v(x) = u(x) \nabla u(x), \quad x \in \Omega, \tag{3.5}$$

we have

$$\operatorname{div} v = |\nabla u|^2 + u\Delta u = |\nabla u|^2 \ \text{ on } \ \Omega, \tag{3.6}$$

and, for $x \in \partial\Omega$,

$$\nu(x) \cdot \lim_{y \to x \text{ in } \Gamma(x)} v(y) = u(x) T^* f(x). \tag{3.7}$$

Hence (3.4) would follow from (1.4), if (1.4) were established in this context.

For such $\Omega$, it is a fundamental fact, proven in [HMT], that

$$\|\mathcal{N}(\nabla \mathcal{S} f)\|_{L^p(\partial\Omega)} \le C_p \|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty. \tag{3.8}$$

A more elementary estimate is

$$\mathcal{N}\mathcal{S}f(x) \le C \int_{\partial\Omega} |x - y|^{-(m-2)} |f(y)| \, d\sigma(y), \tag{3.9}$$

if $m \geq 3$ (with a simple modification for $n = 2$), which in turn yields, for all $\delta > 0$,

$$\begin{aligned} \|\mathcal{N}\mathcal{S}f\|_{L^{(m-1)/(m-2)}(\partial\Omega)} &\leq C_\delta \|f\|_{L^{1+\delta}(\partial\Omega)}, \\ \|\mathcal{N}\mathcal{S}f\|_{L^\infty(\partial\Omega)} &\leq C_\delta \|f\|_{L^{m-1+\delta}(\partial\Omega)}, \end{aligned} \tag{3.10}$$

when (3.1) holds. Interpolation gives

$$\|\mathcal{N}\mathcal{S}f\|_{L^{2+\alpha(m)}(\partial\Omega)} \leq C\|f\|_{L^2(\partial\Omega)}, \tag{3.11}$$

with $\alpha(m) > 0$.

In §4 we will establish a result that implies

$$\|\nabla\mathcal{S}f\|_{L^p(\Omega)} \leq C\|f\|_{L^2(\partial\Omega)}, \quad \forall\, p < \frac{2m}{m-1}. \tag{3.12}$$

A similar analysis yields an estimate

$$\|\mathcal{S}f\|_{L^q(\Omega)} \leq C\|f\|_{L^2(\partial\Omega)}, \tag{3.13}$$

for somewhat larger $q$.

Thus, if $v$ is given by (3.5), with $u = \mathcal{S}f$, we have

$$\|\mathcal{N}v\|_{L^p(\partial\Omega)} \leq C\|f\|^2_{L^2(\partial\Omega)}, \quad p = p_m > 1, \tag{3.14}$$

and

$$\|\operatorname{div} v\|_{L^q(\Omega)} \leq C\|f\|^2_{L^2(\partial\Omega)}, \quad q = q_m > 1. \tag{3.15}$$

Thus to establish (3.4) it will suffice to prove that

$$\int_\Omega \operatorname{div} v\, dx = \int_{\partial\Omega} \nu \cdot v\, d\sigma, \tag{3.16}$$

provided that, for some $p > 1$,

$$v \in \mathfrak{L}^p, \quad \text{and} \quad \operatorname{div} v \in L^1(\Omega), \tag{3.17}$$

where

$$\mathfrak{L}^p = \{v \in C(\Omega) : \mathcal{N}v \in L^p(\partial\Omega), \text{and } \exists \text{ nontangential limit } v_b, \sigma\text{-a.e.}\}. \tag{3.18}$$

This will be proven for $\Omega$ satisfying (3.1) in §5.

## 4. Layer potentials on Ahlfors regular sets: elementary estimates

Let $\Omega$ be a bounded open set. Assume $\partial\Omega$ satisfies the Ahlfors regularity condition

$$C_1 r^{m-1} \le \mathcal{H}^{m-1}(\partial\Omega \cap B_r(p)) \le C_2 r^{m-1}, \quad \forall\, p \in \partial\Omega. \tag{4.1}$$

We estimate some integral operators mapping functions on $\partial\Omega$ to functions on $\Omega$. Let

$$K : \overline{\Omega} \times \partial\Omega \longrightarrow \mathbb{R} \tag{4.2}$$

be continuous on

$$\overline{\Omega} \times \partial\Omega \setminus \{(x,x) : x \in \partial\Omega\} \tag{4.3}$$

and satisfy

$$|K(x,y)| \le C|x-y|^{-(m-1)}. \tag{4.4}$$

Define

$$\mathcal{T} f(x) = \int\limits_{\partial\Omega} K(x,y) f(y)\, d\sigma(y), \tag{4.5}$$

where $d\sigma = d\mathcal{H}^{m-1}$. We aim to prove:

**Proposition 4.1.** *Under the hypotheses on $\Omega$ and $K$ given above, we have for $p \in [1,\infty)$,*

$$\mathcal{T} : L^p(\partial\Omega) \longrightarrow L^r(\Omega), \quad \forall\, r < \frac{pm}{m-1}. \tag{4.6}$$

*Proof.* It is elementary that

$$\mathcal{T} : L^1(\partial\Omega) \longrightarrow L^q(\Omega), \quad \forall\, q < \frac{m}{m-1}. \tag{4.7}$$

For this one needs only $\Omega$ bounded and $\mathcal{H}^{m-1}(\partial\Omega) < \infty$. We will show that

$$\mathcal{T} : L^\infty(\partial\Omega) \longrightarrow L^s(\Omega), \quad \forall\, s < \infty. \tag{4.8}$$

Then (4.6) follows by interpolation from (4.7)–(4.8).

To prove (4.8), we will establish an estimate of the form

$$\int\limits_{\partial\Omega} |K(x,y)|\, d\sigma(y) \le \gamma(x), \quad \gamma \in L^s(\Omega), \ \ \forall\, s < \infty. \tag{4.9}$$

Then

$$\Bigl|\int\limits_{\partial\Omega} K(x,y) f(y)\, d\sigma(y)\Bigr| \le \|f\|_{L^\infty}\gamma(x), \tag{4.10}$$

and (4.8) follows. Here is part of (4.9).

**Proposition 4.2.** *In the setting of Proposition 4.1,*

$$\int_{\partial\Omega} |K(x,y)|\, d\sigma(y) \le C \log \frac{2M}{\varphi(x)}, \quad \varphi(x) = dist(x, \partial\Omega), \tag{4.11}$$

*with* $M = diam\,\Omega$.

*Proof.* Given $x \in \Omega,\ \delta = \mathrm{dist}\,(x, \partial\Omega)$, take $p \in \partial\Omega,\ |x-p| = \delta$, and set

$$\begin{aligned} \mathcal{A}_0 &= \{x' \in \partial\Omega : |x'-p| \le 2\delta\}, \\ \mathcal{A}_k &= \{x' \in \partial\Omega : |x'-p| \in (2^k\delta, 2^{k+1}\delta]\}, \quad k \ge 1. \end{aligned} \tag{4.12}$$

By (4.1),

$$\mathcal{H}^{m-1}(\mathcal{A}_k) \le C(2^k\delta)^{m-1}, \tag{4.13}$$

so

$$\int_{\mathcal{A}_k} |K(x,y)|\, d\sigma(y) \le C\frac{(2^k\delta)^{m-1}}{(2^k\delta)^{m-1}} = C. \tag{4.14}$$

Summing (4.14) over $k \ge 0$ such that $2^k\delta \le M$ gives (4.11).

The other half of (4.9), i.e., the fact that $\gamma \in L^s(\Omega)$ for all $s < \infty$, follows from:

**Lemma 4.3.** *Let* $\Omega \subset \mathbb{R}^m$ *be a bounded open set whose boundary is Ahlfors regular. Set*

$$\mathcal{O}_\delta = \{x \in \overline{\Omega} : dist(x, \partial\Omega) \le \delta\}. \tag{4.15}$$

*Then*

$$vol(\mathcal{O}_\delta) \le C\delta. \tag{4.16}$$

*Proof.* Let $B_1$ be the unit ball in $\mathbb{R}^m$, with volume $C_m$, and set

$$\chi_\delta(x) = \frac{1}{C_m\delta^m}\chi_{B_1}\Big(\frac{x}{\delta}\Big), \tag{4.17}$$

so $\int \chi_\delta\, dx = 1$. Set $\mu = \mathcal{H}^{m-1}\lfloor\partial\Omega$ and

$$G_\delta = \mu * \chi_\delta. \tag{4.18}$$

Then

$$\int G_\delta(x)\, dx = \mathcal{H}^{m-1}(\partial\Omega), \quad \forall\, \delta > 0. \tag{4.19}$$

We also have the following. If $p \in \partial\Omega,\ |x-p| = \operatorname{dist}(x,\partial\Omega)$,

$$\operatorname{dist}(x,\partial\Omega) \le \frac{\delta}{2} \Rightarrow G_\delta(x) \ge C\delta^{-m}\mathcal{H}^{m-1}(\partial\Omega \cap B_{\delta/2}(p)) \ge C\delta^{-1}, \tag{4.20}$$

the last inequality by (4.1). Hence

$$\operatorname{vol}(\mathcal{O}_{\delta/2}) \le C\delta \int\limits_{\mathcal{O}_{\delta/2}} G_\delta(x)\,dx \le C\delta\,\mathcal{H}^{m-1}(\partial\Omega). \tag{4.21}$$

This proves (4.16). Thus (4.9) is proven, and hence so is Proposition 4.1.

It is of incidental interest that Lemma 4.3 has the following corollary, related to the setting of Proposition 1.2.

**Corollary 4.4.** *If $\Omega \subset \mathbb{R}^m$ is a bounded open set and $\partial\Omega$ satisfies (4.1), then $\Omega$ has a tame interior approximation.*

*Proof.* Consider $\varphi \in \operatorname{Lip}(\overline{\Omega})$ given by $\varphi(x) = \operatorname{dist}(x,\partial\Omega)$, and set $\Omega_s = \{x \in \Omega : \varphi(x) \ge s\}$. For $\delta > 0$, set

$$\begin{aligned} \psi_\delta(x) = \delta \qquad &\text{on } \ \Omega_\delta \\ \varphi(x) \quad &\text{on } \ \Omega \setminus \Omega_\delta. \end{aligned} \tag{4.22}$$

Thus $\nabla\psi_\delta$ is supported on $\mathcal{O}_\delta$ and $|\nabla\psi_\delta| = 1$ on $\mathcal{O}_\delta$. A version of the co-area formula (Theorem 5.4.4 of [Zie]) gives

$$\int\limits_{\mathcal{O}_\delta} |\nabla\psi_\delta|\,dx = \int_0^\delta \|\nabla\chi_{\Omega_s}\|_{TV}\,ds. \tag{4.23}$$

Thus, by (4.16), since the left side of (4.23) equals $\operatorname{vol}(\mathcal{O}_\delta)$,

$$\int_0^\delta \|\nabla\chi_{\Omega_s}\|_{TV}\,ds \le C\delta. \tag{4.24}$$

Thus, for each $k \ge 1$, there exists $s \in (0,1/k)$ such that $\|\nabla\chi_{\Omega_s}\|_{TV} \le C$. This proves the corollary.

## 5. Green's formula on Ahlfors regular domains

Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with finite perimeter. Assume

$$\mathcal{H}^{m-1}(\partial\Omega \setminus \partial^*\Omega) = 0, \tag{5.1}$$

and denote $\mathcal{H}^{m-1}\lfloor\partial\Omega$ by $\sigma$. For $p \in [1, \infty)$, set

$$\mathfrak{L}^p = \{v \in C(\Omega) : \mathcal{N}v \in L^p(\partial\Omega), \quad \text{and } \exists \ \text{ nontangential limit } \ v_b, \sigma\text{-a.e.}\}. \tag{5.2}$$

Here $L^p(\partial\Omega) = L^p(\partial\Omega, d\sigma)$. We will establish a Green formula for vector fields $v \in \mathfrak{L}^p$ with divergence in $L^1(\Omega)$, when $\Omega$ satisfies the following two conditions. First,

$$\frac{1}{\delta}\int\limits_{\mathcal{O}_\delta} |v|\,dx \le C\|\mathcal{N}v\|_{L^1(\partial\Omega)}, \quad \forall\, v \in \mathfrak{L}^1, \quad 0 < \delta \le \operatorname{diam}\Omega, \tag{5.3}$$

where $\mathcal{O}_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \le \delta\}$. Second,

$$\begin{aligned} &\text{If } \ v \in \mathfrak{L}^p, \ \exists\, w \in \mathfrak{L}^1 \ \text{ such that } \ w_b = v_b \ \text{ and } \ \exists\, w_k \in \operatorname{Lip}(\overline{\Omega}) \\ &\text{such that } \ \|\mathcal{N}(w - w_k)\|_{L^1(\partial\Omega)} \longrightarrow 0. \end{aligned} \tag{5.4}$$

Then we will show that (5.3) and (5.4) hold whenever $\partial\Omega$ is Ahlfors regular.

Before we state our first result, a few comments are useful. First, the condition (5.3) is equivalent to the apparently stronger condition

$$\frac{1}{\delta}\int\limits_{\mathcal{O}_\delta} |v|\,dx \le C\|\mathcal{N}_\delta v\|_{L^1(\partial\Omega)}, \quad \forall\, v \in \mathfrak{L}^1, \quad 0 < \delta \le \operatorname{diam}\Omega, \tag{5.5}$$

where

$$\mathcal{N}_\delta v(x) = \sup\{|v(y)| : y \in \Gamma_x, \ |x - y| \le 2\delta\}, \tag{5.6}$$

as a simple cutoff argument shows. Second, in condition (5.4) it would be equivalent to demand merely that $w_k \in C(\overline{\Omega})$, since elements of $C(\overline{\Omega})$ are easily uniformly approximated by Lipschitz functions.

Here is our first result.

**Proposition 5.1.** *Pick $p \in [1, \infty)$. Assume $\Omega$ is a bounded open set with finite perimeter, satisfying (5.1) and (5.3)–(5.4). Then*

$$\int_{\Omega} \operatorname{div} v \, dx = \int_{\partial\Omega} \nu \cdot v_b \, d\sigma, \tag{5.7}$$

*whenever*

$$v \in \mathfrak{L}^p \quad \textit{and} \quad \operatorname{div} v \in L^1(\Omega). \tag{5.8}$$

*Proof.* Let $\Omega_s = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq s\}$. Let $\varphi_\delta(x) = \operatorname{dist}(x, \partial\Omega_{\delta/2})$, and set

$$\begin{aligned} \chi_\delta(x) = 1 \qquad & \text{on } \ \Omega_\delta, \\ 2\delta^{-1}\,\varphi_\delta(x) \quad & \text{on } \ \Omega \setminus (\Omega_\delta \cup \mathcal{O}_{\delta/2}), \\ 0 \qquad & \text{on } \ \mathcal{O}_{\delta/2}. \end{aligned} \tag{5.9}$$

If $v$ satisfies (5.8), then $\chi_\delta v \in C_0^0(\Omega)$ and $\operatorname{div} \chi_\delta v \in L^1(\Omega)$, so it is elementary that

$$\int_{\Omega} \operatorname{div}(\chi_\delta v) \, dx = 0. \tag{5.10}$$

Hence

$$\begin{aligned} \int_{\Omega} \chi_\delta \ \operatorname{div} v \, dx &= -\int_{\Omega} \nabla\chi_\delta \cdot v \, dx \\ &= \frac{2}{\delta} \int_{\widetilde{\mathcal{O}}_\delta} \nu \cdot v \, dx, \end{aligned} \tag{5.11}$$

where $\nu = -\nabla\varphi_0$ and $\widetilde{\mathcal{O}}_\delta = \mathcal{O}_\delta \setminus \mathcal{O}_{\delta/2}$. The left side of (5.11) converges to the left side of (5.7) as $\delta \to 0$, whenever $\operatorname{div} v \in L^1(\Omega)$. Hence (5.7) is true provided

$$\frac{2}{\delta} \int_{\widetilde{\mathcal{O}}_\delta} \nu \cdot v \, dx \longrightarrow \int_{\partial\Omega} \nu \cdot v_b \, d\sigma, \tag{5.12}$$

as $\delta \to 0$. Of course, by (5.11), the left side of (5.12) does converge as $\delta \to 0$, namely to the left side of (5.7). Hence (5.12) is true whenever (5.7) is true. In particular, (5.12) is true whenever $v \in \operatorname{Lip}(\overline{\Omega})$.

More generally, if $v$ satisfies (5.8), take $w, w_k$ as in (5.4). Given (5.3), we have

$$\Big| \frac{2}{\delta} \int_{\widetilde{\mathcal{O}}_\delta} (\nu \cdot w_k - \nu \cdot w) \, dx \Big| \leq C \|\mathcal{N}(w - w_k)\|_{L^1(\partial\Omega)}, \tag{5.13}$$

and we also have

$$\Big|\int\limits_{\partial\Omega}(\nu\cdot w_k-\nu\cdot w_b)\,d\sigma\Big|\le\|\mathcal{N}(w_k-w)\|_{L^1(\partial\Omega)}. \tag{5.14}$$

Thus, since (5.12) holds for $w_k$, we have

$$\begin{aligned}\frac{2}{\delta}\int\limits_{\widetilde{\mathcal{O}}_\delta}\nu\cdot w\,dx\longrightarrow&\int\limits_{\partial\Omega}\nu\cdot w_b\,d\sigma\\ =&\int\limits_{\partial\Omega}\nu\cdot v_b\,d\sigma,\end{aligned} \tag{5.15}$$

as $\delta\to 0$. Thus to obtain (5.12) for each $v\in\mathfrak{L}^p$, it suffices to show that

$$\frac{2}{\delta}\int\limits_{\widetilde{\mathcal{O}}_\delta}(\nu\cdot v-\nu\cdot w)\,dx\longrightarrow 0. \tag{5.16}$$

Thus it suffices to show that

$$u\in\mathfrak{L}^1,\ u_b=0\Longrightarrow\frac{2}{\delta}\int\limits_{\widetilde{\mathcal{O}}_\delta}|u|\,dx\to 0, \tag{5.17}$$

as $\delta\to 0$. Recalling that (5.3) implies (5.5), we see that it suffices to show that

$$u\in\mathfrak{L}^1,\ u_b=0\Longrightarrow\|\mathcal{N}_\delta u\|_{L^1(\partial\Omega)}\to 0, \tag{5.18}$$

as $\delta\to 0$. Indeed, the hypotheses of (5.18) yield $\mathcal{N}_\delta u(x)\to 0$, $\sigma$-a.e., and furthermore $\mathcal{N}_\delta u\le\mathcal{N}u$ for each $\delta$, so (5.18) follows from the dominated convergence theorem. Proposition 5.1 is proven.

We next show that Ahlfors regularity implies (5.4).

**Proposition 5.2.** *If $\Omega\subset\mathbb{R}^m$ is a bounded open set whose boundary is Ahlfors regular, then (5.4) holds for each $p\in(1,\infty)$.*

*Proof.* Fix $p\in(1,\infty)$. For $f\in L^p(\partial\Omega)$, $x\in\Omega$, set

$$\Psi f(x)=\frac{1}{V(x)}\int\limits_{\partial\Omega}\psi(x,y)f(y)\,d\sigma(y), \tag{5.19}$$

where

$$\psi(x,y)=\Big(1-\frac{|x-y|}{2\varphi(x)}\Big)_+,\quad\varphi(x)=\mathrm{dist}(x,\partial\Omega), \tag{5.20}$$

and

$$V(x) = \int\limits_{\partial\Omega} \psi(x,y)\, d\sigma(y). \tag{5.21}$$

It is readily checked that

$$\Psi : C(\partial\Omega) \longrightarrow C(\overline{\Omega}), \quad \Psi f\big|_{\partial\Omega} = f, \tag{5.22}$$

and that

$$\mathcal{N}\Psi f \leq C\mathcal{M}f, \quad \forall\, f \in L^1(\partial\Omega), \tag{5.23}$$

where $\mathcal{M}f$ is the Hardy-Littlewood maximal function of $f \in L^1(\partial\Omega)$, given that $\partial\Omega$ is Ahlfors regular. Hence

$$\|\mathcal{N}\Psi f\|_{L^p(\partial\Omega)} \leq C_p \|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty. \tag{5.24}$$

We claim that for each $p \in (1,\infty)$,

$$\Psi : L^p(\partial\Omega) \longrightarrow \mathfrak{L}^p, \quad (\Psi f)_b = f \ \ \sigma\text{-a.e.} \tag{5.25}$$

In light of (5.24), only the nontangential convergence of $\Psi f$ to $f$ remains to be justified. This follows by taking $f_k \in C(\partial\Omega)$, $f_k \to f$ in $L^p$-norm, using (5.22) on $f_k$ and (5.24) on $f - f_k$.

Now, to establish (5.4), we argue as follows. Take $v \in \mathfrak{L}^p$, and set $w = \Psi v_b$. By (5.25), $w \in \mathfrak{L}^p$ and $w_b = v_b$. Then take $f_k \in C(\partial\Omega)$ such that $f_k \to v_b$ in $L^p(\partial\Omega)$-norm, and set $\tilde{w}_k = \Psi f_k$. By (5.22), each $\tilde{w}_k \in C(\overline{\Omega})$. By (5.24),

$$\|\mathcal{N}(\tilde{w}_k - w)\|_{L^p(\partial\Omega)} \leq C_p \|f_k - v_b\|_{L^p(\partial\Omega)} \to 0, \tag{5.26}$$

which is stronger than the $L^1$ estimate demanded in (5.4). As mentioned in the paragraph after (5.4), having such continuous functions suffices, since they are easily approximated by Lipschitz functions. Proposition 5.2 is proven.

We now show that Ahlfors regularity implies (5.3). To be definite, for each $x \in \partial\Omega$ set

$$\Gamma_x = \{y \in \Omega : |x-y| \leq 10\,\mathrm{dist}(y,\partial\Omega)\}. \tag{5.27}$$

**Proposition 5.3.** *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with Ahlfors regular boundary. Then (5.3) holds.*

*Proof.* We first note that it suffices to prove that

$$\frac{1}{\delta} \int\limits_{\widetilde{\mathcal{O}}_\delta} |v|\, dx \leq C\|\mathcal{N}v\|_{L^1(\partial\Omega)}, \quad \forall\, v \in \mathfrak{L}^1, \tag{5.28}$$

where $\widetilde{\mathcal{O}}_\delta = \mathcal{O}_\delta \setminus \mathcal{O}_{\delta/2}$, since (5.3) then follows by applying (5.28) with $\delta$ replaced by $2^{-j}\delta$ and summing over $j \in \mathbb{Z}^+$. We now bring in the following lemma.

**Lemma 5.4.** *There exists $K = K_m$ with the following property. For each $\delta \in (0, (diam\,\Omega)/10]$, there exists a covering of $\partial\Omega$ by a collection*

$$\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_K \tag{5.29}$$

*of balls of radius $\delta$, centered in $\partial\Omega$, such that for each $k \in \{1, \dots, K\}$, if $B$ and $B'$ are distinct balls in $\mathcal{C}_k$, their centers are separated by a distance $\geq 10\delta$.*

We postpone the proof of Lemma 5.4, and show how it is used to prove Proposition 5.3.

To begin, take a collection $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_K$ of balls of radius $\delta$ covering $\partial\Omega$, with the properties stated above. Then $\mathcal{C}^\# = \mathcal{C}_1^\# \cup \cdots \cup \mathcal{C}_K^\#$, consisting of balls concentric with those of $\mathcal{C}$ with radius $2\delta$, covers $\mathcal{O}_\delta$. Furthermore, there exists $A = A_m \in (0, \infty)$ such that one can cover each ball $B \in \mathcal{C}_k^\#$ by balls $B_1, \dots, B_A$ of radius $\delta/8$, centered at points in $B$. Now form collections of balls $\mathcal{C}_{k\ell}^\#$, $1 \leq k \leq K$, $1 \leq \ell \leq A$, with each of the balls $B_1, \dots, B_A$ covering $B \in \mathcal{C}_k^\#$, described above, put into a different one of the collections $\mathcal{C}_{k\ell}^\#$. Throw away some balls from $\mathcal{C}_{k\ell}^\#$, thinning them out to a *minimal* collection

$$\widetilde{\mathcal{C}} = \bigcup_{k \leq K, \ell \leq A} \widetilde{\mathcal{C}}_{k\ell} \tag{5.30}$$

covering $\widetilde{\mathcal{O}}_\delta$. For each $(k, \ell)$, any two distinct balls in $\widetilde{\mathcal{C}}_{k\ell}$ have centers separated by a distance $\geq 7\delta$. Each ball $B \in \widetilde{\mathcal{C}}_{k\ell}$ has radius $\delta/8$ and each point $p \in B$ has distance from $\partial\Omega$ lying between $\delta/4$ and $5\delta/4$. For each such $B$, we will compare $\int_B |v|\,dx$ with the integral of $\mathcal{N}v$ over a certain set $\widetilde{\mathfrak{A}}(B) \subset \partial\Omega$, which we proceed to define.

Given $y \in \Omega$, $d(y) = \operatorname{dist}(y, \partial\Omega)$, consider

$$\mathfrak{A}(y) = \{x \in \partial\Omega : y \in \Gamma_x\}. \tag{5.31}$$

There exists $p \in \partial\Omega$ such that $|y - p| = d(y)$, and certainly $p \in \mathfrak{A}(y)$. Also, if (5.27) holds, then

$$\mathfrak{A}(y) \supset B_{9d(y)}(p) \cap \partial\Omega. \tag{5.32}$$

Now, for a ball $B \in \widetilde{\mathcal{C}}_{k\ell}$, set

$$\mathfrak{A}(B) = \{x \in \partial\Omega : B \subset \Gamma_x\}. \tag{5.33}$$

If $B$ is centered at $y$ and $p \in \partial\Omega$ is closest to $y$, then (5.32) holds with $d(y) \geq 3\delta/8$. For each $y' \in B$, $d(y') \geq \delta/4$ and $|y' - p| \leq d(y) + \delta/8$. Now $d(y) \leq (9/8)\delta$, so $|y' - p| \leq (5/4)\delta \leq 5d(y')$, and hence

$$\mathfrak{A}(B) \supset B_{d(y)}(p) \cap \partial\Omega. \tag{5.34}$$

We denote the right side of (5.34) by $\widetilde{\mathfrak{A}}(B)$.

The use of $\widetilde{\mathfrak{A}}(B)$ in establishing (5.28) arises from the estimates

$$\sup_B |v| \le \inf_{\widetilde{\mathfrak{A}}(B)} \mathcal{N}v, \quad \text{and} \quad \mathcal{H}^{m-1}(\widetilde{\mathfrak{A}}(B)) \ge C_1 \delta^{m-1}, \tag{5.35}$$

the latter estimate due to the hypothesis that $\partial\Omega$ is Ahlfors regular. Hence

$$\begin{aligned} \frac{1}{\delta}\int\limits_B |v|\, dx &\le C_m \delta^{m-1} \inf_{\widetilde{\mathfrak{A}}(B)} \mathcal{N}v \\ &\le C \int\limits_{\widetilde{\mathfrak{A}}(B)} \mathcal{N}v\, d\sigma. \end{aligned} \tag{5.36}$$

Furthermore, the separation properties established for balls in each collection $\widetilde{\mathcal{C}}_{k\ell}$ yield

$$B \ne B' \in \widetilde{\mathcal{C}}_{k\ell} \Longrightarrow \widetilde{\mathfrak{A}}(B) \cap \widetilde{\mathfrak{A}}(B') = \emptyset, \tag{5.37}$$

so for each $k \le K,\ \ell \le A$,

$$\begin{aligned} \frac{1}{\delta}\sum_{B\in\widetilde{\mathcal{C}}_{k\ell}} \int\limits_B |v|\, dx &\le C \int\limits_{\cup\widetilde{\mathfrak{A}}(B),B\in\widetilde{\mathcal{C}}_{k\ell}} \mathcal{N}v\, d\sigma \\ &\le C\|\mathcal{N}v\|_{L^1(\partial\Omega)}. \end{aligned} \tag{5.38}$$

Consequently,

$$\begin{aligned} \frac{1}{\delta}\int\limits_{\widetilde{\mathcal{O}}_\delta} |v|\, dx &\le \frac{1}{\delta}\sum_{\ell=1}^{A}\sum_{k=1}^{K}\sum_{B\in\widetilde{\mathcal{C}}_{k\ell}} \int\limits_B |v|\, dx \\ &\le CAK\, \|\mathcal{N}v\|_{L^1(\partial\Omega)}, \end{aligned} \tag{5.39}$$

and Proposition 5.3 is established, modulo the proof of Lemma 5.4, to which we now turn.

*Proof of Lemma 5.4.* Pick a maximal set $\{p_k\} \subset \partial\Omega$ of points that are separated by a distance $\ge \delta$. (Such clearly exists.) Set $\mathcal{C} = \{B_\delta(p_k)\}$. This collection covers $\partial\Omega$, since if $q \in \partial\Omega$ were not covered one could add it to $\{p_k\}$. On the other hand, the balls in $\mathcal{B} = \{B_{\delta/2}(p_k)\}$ are mutually disjoint balls in $\mathbb{R}^m$. Our task is to partition $\mathcal{C}$ as indicated in (5.29).

Tile $\mathbb{R}^m$ with cubes $Q$ of edge $20\,\delta$, edges parallel to the coordinate axes. Each such cube has a natural partition into $20^m$ little cubes (call them cells), of edge $\delta$.

Label the cells in each $Q$ from 1 to $20^m$, in a fashion so that translating one such $Q$ to the position of another preserves the labels of the cells. Note that for each cell $C$ all the balls in $\mathcal{B}$ intersecting $C$ lie in a cube of edge $2\delta$, so since they are mutually disjoint there can be at most $[4^m/V_m]$ of them, where $V_m$ is the volume of the unit ball in $\mathbb{R}^m$.

Now for each $\ell \in \{1, \dots, 20^m\}$, look at each cube $Q$ and pick one ball from $\mathcal{B}$ (say $B_{\delta/2}(p_k)$) that intersects the cell labeled $\ell$ in $Q$. (If no ball in $\mathcal{B}$ intersects the cell, pass it by.) Collect all these balls, as $Q$ varies. Then take the corresponding balls $B_\delta(p_k)$ in $\mathcal{C}$. This forms a collection, which we will denote $\mathcal{C}_{\ell 1}$, of balls from $\mathcal{C}$, whose centers are separated by a distance $\geq 10\delta$. If $\mathcal{C}$ is not exhausted, do this again, to form $\mathcal{C}_{\ell 2}$, for $\ell \in \{1, \dots, 20^m\}$, and repeat, forming $\mathcal{C}_{\ell k},\ k \geq 3$. All the balls in $\mathcal{C}$ will be exhausted by the time $k = [4^m/V_m]$, so the lemma is established, with $K_m = 20^m \cdot [4^m/V_m]$.

Putting together Propositions 5.1–5.3, we have:

**Theorem 5.5.** *Assume $\partial\Omega$ is Ahlfors regular and satisfies (5.1). If, for some $p > 1$,*

$$v \in \mathfrak{L}^p, \quad \textit{and} \quad \operatorname{div} v \in L^1(\Omega), \tag{5.40}$$

*then*

$$\int_\Omega \operatorname{div} v \, dx = \int_{\partial\Omega} \nu \cdot v_b \, d\sigma. \tag{5.41}$$

## 6. Gauss-Green formulas on Riemannian manifolds

Let $\mathbb{R}^m$ carry a continuous metric tensor $(g_{jk})$, in addition to the Euclidean metric tensor $(\delta_{jk})$. A vector field $v = v^j \partial_j$ has divergence $\operatorname{div} v \in \mathcal{D}'(\mathbb{R}^m)$ given by

$$\langle \varphi, \operatorname{div} v\rangle = -\langle \partial_j \varphi, g^{1/2} v^j\rangle, \tag{6.1}$$

where $g = \det(g_{jk})$, and we use the summation convention. If $\operatorname{div} v$ is a locally integrable multiple of Lebesgue measure, or equivalently of $dV = g^{1/2}\, dx$, we identify $\operatorname{div} v$ with $\operatorname{div} v\, dV$. We denote by $\operatorname{div}_0 v$ this quantity associated to $(\delta_{jk})$ rather than $(g_{jk})$, so $\operatorname{div} v = g^{-1/2} \operatorname{div}_0(g^{1/2} v)$, in the locally integrable case.

Let $\Omega \subset \mathbb{R}^m$ be an open set with locally finite perimeter. Assume $v$ belongs to

$$\begin{aligned} \mathcal{D} &= \{v \in C_0^0(\mathbb{R}^m, \mathbb{R}^m) : \operatorname{div} v \in L^1(\mathbb{R}^m)\} \\ &= \{v \in C_0^0(\mathbb{R}^m, \mathbb{R}^m) : \operatorname{div}_0(g^{1/2} v) \in L^1(\mathbb{R}^m)\}. \end{aligned} \tag{6.2}$$

Then

$$\int_\Omega \operatorname{div} v\, dV = \int_\Omega \operatorname{div}_0(g^{1/2} v)\, dx. \tag{6.3}$$

Hence Proposition 1.1 gives

$$\int_\Omega \operatorname{div} v\, dV = \int_{\partial^*\Omega} n \cdot v\, g^{1/2}\, d\sigma, \tag{6.4}$$

where $n$ is the outward-pointing unit normal with respect to the metric $(\delta_{jk})$ and $\sigma$ is $(m-1)$-dimensional Hausdorff measure defined by $(\delta_{jk})$. We claim that

$$\int_{\partial^*\Omega} n \cdot v\, g^{1/2}\, d\sigma = \int_{\partial^*\Omega} \langle \nu, v\rangle_g\, d\sigma_g, \tag{6.5}$$

where $\nu$ is the unit outward-pointing normal determined by $(g_{jk})$, $\langle\ ,\ \rangle_g$ is the inner product determined by $(g_{jk})$, and $\sigma_g$ is $(m-1)$-dimensional Hausdorff measure determined by $(g_{jk})$. The vectors $\nu = \nu^j \partial_j$ and $n = n^j \partial_j$ have asociated covectors

$$\nu^b = \sum \nu_j\, dx_j, \quad n^b = \sum n_j\, dx_j, \tag{6.6}$$

where

$$\nu_j = g_{jk} \nu^k, \quad n_j = \delta_{jk} n^k. \tag{6.7}$$

The covectors $\nu^b$ and $n^b$ are parallel and both have unit length, with respect to their associated metric tensors, so

$$n_j = a\nu_j, \quad a^2 = \langle n^b, n^b \rangle_g = g^{jk} n_j n_k. \tag{6.8}$$

Hence

$$\begin{aligned} n \cdot v \, g^{1/2} &= n_j v^j g^{1/2} \\ &= g^{1/2} \langle n^b, n^b \rangle_g^{1/2} \, \nu_j v^j \\ &= g^{1/2} \langle n^b, n^b \rangle_g^{1/2} \, \langle \nu, v \rangle_g. \end{aligned} \tag{6.9}$$

Thus (6.5) is equivalent to the assertion that

$$\sigma_g = g^{1/2} \langle n^b, n^b \rangle_g^{1/2} \, \sigma \tag{6.10}$$

on measurable subsets of $\partial^*\Omega$. The following result establishes (6.10).

**Proposition 6.1.** *Let $S \subset \mathbb{R}^m$ be a countably rectifiable $(m-1)$-dimensional set in $\mathbb{R}^m$, with measure-theoretic unit normal $n$ determined by the Euclidean structure. If $\sigma$ is $(m-1)$-dimensional Hausdorff measure determined by $(\delta_{jk})$ and $\sigma_g$ is $(m-1)$-dimensional Hausdorff measure determined by $(g_{jk})$, then (6.10) holds on $S$.*

*Proof.* It is clear from the definitions that for each compact $K \subset \mathbb{R}^m$ there exists $C_K \in (1, \infty)$ such that

$$C_K^{-1} \sigma(A) \le \sigma_g(A) \le C_K \sigma(A), \quad A \subset K. \tag{6.11}$$

The hypothesis of countable rectifiability implies there is a disjoint union

$$S = \bigcup_{k \ge 1} M_k \cup N, \tag{6.12}$$

where each $M_k$ is a Borel subset of some $(m-1)$-dimensional $C^1$ submanifold of $\mathbb{R}^m$, while $\sigma(N) = 0$, hence $\sigma_g(N) = 0$. It is elementary that (6.10) holds on each $C^1$ submanifold of $\mathbb{R}^m$, of dimension $m-1$, so by (6.12) it holds on $S$.

Since $\partial^*\Omega$ is countably rectifiable, we have the following variant of Proposition 1.1.

**Proposition 6.2.** *Let $\mathbb{R}^m$ have a continuous metric tensor $(g_{jk})$. Let $\Omega \subset \mathbb{R}^m$ be an open set with locally finite perimeter. Then*

$$\int_\Omega \operatorname{div} v \, dV = \int_{\partial^*\Omega} \langle \nu, v \rangle_g \, d\sigma_g, \tag{6.13}$$

*for all $v \in \mathcal{D}$, defined by (6.2).*

Similarly we can apply Proposition 1.2 to deduce that

$$\int_{\Omega} \operatorname{div}_0(g^{1/2}v)\,dx = \int_{\partial^*\Omega} n\cdot v\,g^{1/2}\,d\sigma, \tag{6.14}$$

whenever $\Omega$ has a tame interior approximation and $v$ belongs to

$$\begin{aligned}\widetilde{\mathcal{D}} &= \{v \in C_0^0(\overline{\Omega},\mathbb{R}^m) : \operatorname{div}_0(g^{1/2}v) \in L^1(\Omega)\} \\ &= \{v \in C_0^0(\overline{\Omega},\mathbb{R}^m) : \operatorname{div} v \in L^1(\Omega)\}.\end{aligned} \tag{6.15}$$

We obtain

**Proposition 6.3.** *If $\mathbb{R}^m$ has a continuous metric tensor $(g_{jk})$ and $\Omega \subset \mathbb{R}^m$ has a tame interior approximation, then (6.13) holds for all $v \in \widetilde{\mathcal{D}}$.*

Also the results of §5 together with Proposition 6.1 yield the following.

**Proposition 6.4.** *If $\mathbb{R}^m$ has a continuous metric tensor $(g_{jk})$ and $\Omega \subset \mathbb{R}^m$ is a bounded domain with Ahlfors regular boundary, then (6.13) holds whenever*

$$\operatorname{div} v \in L^1(\Omega) \quad \textit{and} \quad v \in \mathfrak{L}^p, \tag{6.16}$$

*for some $p > 1$, where*

$$\mathfrak{L}^p = \{v \in C(\Omega) : \mathcal{N}v \in L^p(\partial\Omega) \ \textit{and} \ \exists \ \textit{nontangential limit} \ v_b,\ \sigma\textit{-a.e.}\}. \tag{6.17}$$

Remark. Using partitions of unity, we can extend the scope of these results to $\Omega \subset M$, where $M$ is a smooth manifold with a continuous metric tensor.

We can apply Proposition 6.4 in the following setting. Let $M$ be a compact manifold with a Riemannian metric whose components are continuous with a modulus of continuity $\omega$ satisfying

$$\int_0^1 \frac{\sqrt{\omega(t)}}{t}\,dt < \infty. \tag{6.18}$$

Let $V \in L^\infty(M)$ satisfy $V \geq 0$ on $M$ and $V > 0$ on a set of positive measure. Then let $E(x,y)$ be the integral kernel of $(\Delta - V)^{-1}$ on $L^2(M)$. Let $\Omega \subset M$ be a connected NTA domain with Ahlfors regular boundary, or more generally a connected UR domain, as defined in [HMT]. For $f \in L^p(\partial\Omega)$, set

$$\mathcal{S}f(x) = \int_{\partial\Omega} E(x,y)f(y)\,d\sigma_g(y), \quad x \in \Omega. \tag{6.19}$$

A fundamental result (established in [HMT]) is

$$\|\mathcal{N}(\nabla\mathcal{S}f)\|_{L^p(\partial\Omega)} \le C_p\|f\|_{L^p(\partial\Omega)}, \quad 1<p<\infty, \tag{6.20}$$

and that nontangential limits of $\nabla\mathcal{S}f$ exist $\sigma_g$-a.e. on $\partial\Omega$. In particular,

$$\lim_{y\to x \text{ in } \Gamma_x} \nu(x)\cdot\nabla\mathcal{S}f(y) = \Big(-\frac{1}{2}+K^*\Big)f(x), \quad \sigma_g\text{-a.e.}, \tag{6.21}$$

where $K^* : L^p(\partial\Omega)\to L^p(\partial\Omega)$, for $1<p<\infty$. Also, by Proposition 4.1,

$$\|\nabla\mathcal{S}f\|_{L^p(\Omega)} + \|\mathcal{S}f\|_{L^q(\Omega)} \le C\|f\|_{L^2(\partial\Omega)} \tag{6.22}$$

for some $p,q>2$, and $\|\mathcal{N}\mathcal{S}f\|_{L^r(\partial\Omega)} \le C\|f\|_{L^2(\partial\Omega)}$ for some $r>2$. Now if we take $f\in L^2(\partial\Omega)$ and set

$$u=\mathcal{S}f, \quad v=u\nabla u, \tag{6.23}$$

we have

$$\operatorname{div} v = |\nabla u|^2 + u\Delta u = |\nabla u|^2 + Vu^2, \quad \text{on } \ \Omega. \tag{6.24}$$

Thus Proposition 6.4 applies to give

$$\int_\Omega (|\nabla u|^2 + Vu^2)\,dV = \int_{\partial\Omega} u\Big(-\frac{1}{2}+K^*\Big)f\,d\sigma_g. \tag{6.25}$$

## A. The Gauss-Green formula on smoothly bounded domains

Our goal here is to give a simple, direct proof of the identity

$$\int_{\Omega} \operatorname{div} v \, dx = \int_{\partial\Omega} \nu \cdot v \, d\sigma, \tag{A.1}$$

for $v \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^m)$, when $\Omega \subset \mathbb{R}^m$ has the form

$$\Omega = \{(x', x_m) \in \mathbb{R}^m : x_m > A(x')\}, \tag{A.2}$$

with

$$A \in C^1(\mathbb{R}^m). \tag{A.3}$$

In §2 it was shown how such an identity (actually for $A \in C^\infty(\mathbb{R}^m)$) plus an approximation argument yields such a result for domains of the form (A.2) with much rougher $A$, satisfying merely (2.2). We can deduce (A.1) from the following result.

**Proposition A.1.** *If $\Omega \subset \mathbb{R}^m$ satisfies (A.2)–(A.3) and $f \in C_0^\infty(\mathbb{R}^m)$, and if $e$ is an element of $\mathbb{R}^m$, then*

$$\int_{\Omega} e \cdot \nabla f(x) \, dx = \int_{\partial\Omega} (e \cdot \nu) f \, d\sigma. \tag{A.4}$$

In fact, taking $\{e_j\}$ to be the standard orthonormal basis of $\mathbb{R}^m$, replacing $e$ by $e_j$ and $f$ by $v_j$ (the $j$th component of $v$) in (A.4) and summing, we obtain (A.1).

To begin the proof of (A.4), write

$$\begin{aligned} \int_{\Omega} \frac{\partial f}{\partial x_m} \, dx &= \int_{\mathbb{R}^{m-1}} \Bigl( \int_{x_m > A(x')} \partial_m f(x', x_m) \, dx_m \Bigr) dx' \\ &= -\int_{\mathbb{R}^{m-1}} f(x', A(x')) \, dx' \\ &= \int_{\partial\Omega} (e_m \cdot \nu) f \, d\sigma. \end{aligned} \tag{A.5}$$

The first identity in (A.5) follows from Fubini's theorem, the second identity from the fundamental theorem of calculus, and the third identity from the formula

$$\nu(x) = \bigl(1 + |\nabla A(x')|^2\bigr)^{-1/2} (\nabla A(x'), -1), \tag{A.6}$$

where $x = (x', A(x'))$, which is the standard formula for the downward pointing normal to the $C^1$ surface $x_m = A(x')$, and from the formula

$$d\sigma(x) = \sqrt{1 + |\nabla A(x')|^2}\, dx', \tag{A.7}$$

for the surface area of the graph of a $C^1$ function. This establishes (A.4) for $e = e_m$.

Here are some useful comments regarding the surface area formula (A.7). The $(m-1)$-dimensional surface $\partial\Omega \subset \mathbb{R}^m$ inherits a Riemannian metric tensor. A coordinate system on a Riemannian manifold $M$ such as $\partial\Omega$ produces components $(g_{jk})$ for the metric tensor, and the area element is given by

$$d\sigma(y) = \sqrt{g(y)}\, dy, \tag{A.8}$$

in a coordinate system $y = (y_1, \dots, y_{m-1})$, where $g(y) = \det(g_{jk}(y))$. In case the coordinate system on $\partial\Omega = \{x_m = A(x')\}$ is $x' \mapsto (x', A(x'))$, the formula (A.8) specializes to (A.7). It is also useful to note that the transformation properties for metric tensors and for integrals under $C^1$ diffeomorphisms imply that (A.8) defines area on a Riemannian manifold in a coordinate invariant fashion.

To continue the proof of (A.4), since $v$ has compact support, we can assume $|\nabla A(x')| \le L$ for all $x'$, for some $L < \infty$. Then $\Omega$ has a representation of the form (A.2)–(A.3) in new coordinates obtained by any rotation sufficiently close to the identity. Hence the identity (A.4) holds when $e = e_m$ is replaced by any sufficiently close element of $\mathbb{R}^m$. In particular, it works for $e = e_m + ae_j$, for $1 \le j \le m-1$ and for $|a|$ sufficiently small. Thus we have

$$\int_{\Omega} (e_m + ae_j) \cdot \nabla f(x)\, dx = \int_{\partial\Omega} (e_m + ae_j) \cdot \nu\, f\, d\sigma. \tag{A.9}$$

If we subtract (A.5) from this and divide the result by $a$, we obtain (A.4) for $e = e_j$, for all $j$, and hence (A.4) holds in general. This completes the proof.

Remark 1. Using again (A.6)–(A.7), we can rewrite (A.1) as

$$\int_{\Omega} \operatorname{div} v\, dx = \int_{\mathbb{R}^{m-1}} (\nabla A(x'), -1) \cdot v(x', A(x'))\, dx', \tag{A.10}$$

which is the form used in (2.11).

Remark 2. For a $C^1$ graph, or more generally for a $C^1$ manifold with a continuous metric tensor, one can show that (A.7) and (A.8) coincide with Hausdorff measure, directly from the definitions and the fact that $\mathcal{H}^{m-1}$ coincides with Lebesgue measure on $\mathbb{R}^{m-1}$. Details are given in [T], Propositions 12.6 and 12.7.

## B. Variant of a result of Chen and Torres

Let $\mathcal{O} \subset \mathbb{R}^m$ be open, and set

$$\mathcal{D}^\infty = \{v \in L^\infty(\mathcal{O}, \mathbb{R}^m) : \operatorname{div} v \in L^1(\mathcal{O})\}. \tag{B.1}$$

This is a Banach space with norm

$$\|v\|_{\mathcal{D}^\infty} = \|v\|_{L^\infty(\mathcal{O})} + \|\operatorname{div} v\|_{L^1(\mathcal{O})}. \tag{B.2}$$

Let $\Omega \subset\subset \mathcal{O}$ be an open set with finite perimeter, and take $\sigma$ and $\nu$ as in (1.4)–(1.5). The following result was established in [CT].

**Proposition B.1.** *There is a continuous linear map (called the normal trace)*

$$\tau_\nu : \mathcal{D}^\infty \longrightarrow L^\infty(\partial\Omega, \sigma) \tag{B.3}$$

*such that, for each $\varphi \in \operatorname{Lip}(\mathcal{O})$,*

$$\int_\Omega \varphi \operatorname{div} v \, dx + \int_\Omega \nabla\varphi \cdot v \, dx = \int_{\partial\Omega} \tau_\nu(v)\varphi \, d\sigma. \tag{B.4}$$

*Furthermore, one has*

$$\|\tau_\nu v\|_{L^\infty(\partial\Omega,\sigma)} \le \lim_{\delta\to 0} \|v\|_{L^\infty(U_\delta)}, \tag{B.5}$$

*where $U_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \le \delta\}$, and*

$$v \in \mathcal{D}^\infty \cap C(\mathcal{O}) \Longrightarrow \tau_\nu v = \nu \cdot v\big|_{\partial\Omega}. \tag{B.6}$$

In fact, [CT] has a stronger result, allowing $\operatorname{div} v$ to be a finite measure on $\mathcal{O}$. Note that Proposition B.1 extends Proposition 1.1, but it does not imply Proposition 1.2 or the results of §5.

Proof of Proposition B.1: Pick $\psi \in C_0^\infty(\mathbb{R}^m)$ such that $\int \psi \, dx = 1$, $\psi \ge 0$, and $\psi(x) = 0$ for $|x| \ge 1$, and set $\psi_k(x) = k^m \psi(kx)$. Given $v \in \mathcal{D}^\infty$, set $v_k = \psi_k * v$, which is well defined and smooth on a neighborhood of $\overline{\Omega}$ for large $k$. For each $\varphi \in \operatorname{Lip}(\mathcal{O})$, $\varphi v_k$ is Lipschitz on a neighborhood of $\overline{\Omega}$ and $\operatorname{div}(\varphi v_k) = \varphi \operatorname{div} v_k + \nabla\varphi \cdot v_k$, so (1.4) gives

$$\int_\Omega \varphi \operatorname{div} v_k \, dx + \int_\Omega \nabla\varphi \cdot v_k \, dx = \int_{\partial\Omega} (\nu \cdot v_k)\varphi \, d\sigma. \tag{B.7}$$

Now the left side of (B.7) converges to

$$\int_{\Omega} \varphi \operatorname{div} v\, dx + \int_{\Omega} \nabla\varphi \cdot v\, dx = L_v \varphi, \tag{B.8}$$

as $k \to \infty$, so the right side also converges:

$$\int_{\partial\Omega} (\nu \cdot v_k) \varphi\, d\sigma \longrightarrow L_v \varphi. \tag{B.9}$$

We have $L_v : \mathrm{Lip}(\mathcal{O}) \to \mathbb{C}$ for each $v \in \mathcal{D}^\infty$, and (B.9) gives the estimate

$$\begin{aligned} |L_v \varphi| &\le \limsup_{k\to\infty} \sup_{\partial\Omega} |v_k(x)| \cdot \|\varphi\|_{L^1(\partial\Omega,\sigma)} \\ &\le \lim_{\delta\to 0} \|v\|_{L^\infty(U_\delta)} \cdot \|\varphi\|_{L^1(\partial\Omega,\sigma)}. \end{aligned} \tag{B.10}$$

Since $\varphi \mapsto \varphi|_{\partial\Omega}$ maps $\mathrm{Lip}(\mathcal{O})$ onto a dense linear subspace of $L^1(\partial\Omega,\sigma)$, we see that $L_v$ has a unique extension to a continuous linear functional on $L^1(\partial\Omega,\sigma)$, with norm $\le \lim_{\delta\to 0} \|v\|_{L^\infty(U_\delta)}$. Since $L^1(\partial\Omega,\sigma)' = L^\infty(\partial\Omega,\sigma)$, we have $\tau_\nu(v) \in L^\infty(\partial\Omega,\sigma)$ such that (B.5) holds and

$$L_v \varphi = \int_{\partial\Omega} \tau_\nu(v) \varphi\, d\sigma. \tag{B.11}$$

Comparison with (B.8) gives (B.4). Finally, under the hypothesis in (B.6) we clearly have $v_k \to v$ uniformly on $\partial\Omega$, so (B.6) follows from (B.9).

Remark. Note that $\mathcal{D}^\infty$ in (B.1) is a module over $C_0^\infty(\mathcal{O})$. Hence there is no loss of generality in replacing $\mathcal{O}$ by $\mathbb{R}^m$ in (B.1).

## C. Extension of Proposition 1.1 to $\operatorname{div} v$ a measure

Here we establish the following extension of Proposition 1.1.

**Proposition C.1.** *Assume $\Omega \subset \mathbb{R}^m$ is an open set with locally finite perimeter. Take*

$$v \in C_0^0(\mathbb{R}^m, \mathbb{R}^m), \tag{C.1}$$

*and assume*

$$\operatorname{div} v = \gamma \quad \text{is a finite (signed) measure on} \quad \mathbb{R}^m. \tag{C.2}$$

*Then*

$$\int\limits_{\Omega} d\gamma + \int\limits_{\partial\Omega} \omega \, d\gamma = \int\limits_{\partial\Omega} \nu \cdot v \, d\sigma, \tag{C.3}$$

*with $\omega$ given by (C.10).*

*Proof.* Take $\varphi \in C_0^\infty(\mathbb{R}^m)$ as in the proof of Proposition 1.1, satisfying $\varphi \geq 0$, $\int \varphi \, dx = 1$. Assume for good measure that $\varphi$ is radial. Set $\varphi_k(x) = k^m \varphi(kx)$, and take

$$v_k = \varphi_k * v \in C_0^\infty(\mathbb{R}^m). \tag{C.3A}$$

Then

$$\int\limits_{\Omega} \operatorname{div} v_k \, dx = \int\limits_{\partial\Omega} \nu \cdot v_k \, d\sigma, \tag{C.4}$$

and the right side of (C.4) tends to the right side of (C.3), as $k \to \infty$, since $v_k \to v$ uniformly on $\overline{\Omega}$.

We also have

$$\operatorname{div} v_k = \varphi_k * \gamma, \tag{C.5}$$

hence

$$\begin{aligned} \int\limits_{\Omega} \operatorname{div} v_k \, dx &= \langle \chi_\Omega, \varphi_k * \gamma \rangle \\ &= \langle \varphi_k * \chi_\Omega, \gamma \rangle. \end{aligned} \tag{C.6}$$

By hypothesis, $\gamma$ is a finite (signed) measure. We have

$$0 \le \varphi_k * \chi_\Omega \le 1, \quad \forall\, x \in \mathbb{R}^m. \tag{C.7}$$

We also have, as $k \to \infty$,

$$\begin{aligned} \varphi_k * \chi_\Omega(x) \to 1, &\quad \forall\, x \in \Omega, \\ 0, &\quad \forall\, x \in \mathbb{R}^m \setminus \overline{\Omega}. \end{aligned} \tag{C.8}$$

There remains the examination of the behavior as $k \to \infty$ of

$$\omega_k(x) = \varphi_k * \chi_\Omega(x), \quad \text{for} \;\; x \in \partial\Omega. \tag{C.9}$$

Let $\Gamma \subset \partial\Omega$ denote the set of points in $\partial\Omega$ at which $\omega_k$ converges, and define

$$\omega(x) = \lim_{k \to \infty} \omega_k(x), \quad x \in \Gamma. \tag{C.10}$$

Then $\Gamma$ is a Borel set and $\omega$ is a Borel function. Clearly $0 \le \omega(x) \le 1$ for each $x \in \Gamma$. Shortly we will show that

$$|\gamma|(\partial\Omega \setminus \Gamma) = 0, \tag{C.11}$$

where $|\gamma|$ is the total variation measure associated to $\gamma$, then the dominated convergence theorem gives that the left side of (C.5) converges to the left side of (C.3) as $k \to \infty$, so we have the identity (C.3).

It remains to prove (C.11). A key ingrediant is the following, proved in [CF].

**Lemma C.2.** *Assume $v$ satisfies (C.1)–(C.2), or more generally that*

$$v \in L^\infty(\mathbb{R}^m, \mathbb{R}^m), \quad \textit{with compact support}, \tag{C.12}$$

*and (C.2) holds. Then, for Borel sets $S \subset \mathbb{R}^m$,*

$$\mathcal{H}^{m-1}(S) = 0 \Longrightarrow |\gamma|(S) = 0. \tag{C.13}$$

To proceed, it follows from (1.9) that

$$\partial_0\Omega \subset \Gamma, \;\; \text{and} \;\; \omega = \frac{1}{2} \;\; \text{on} \;\; \partial_0\Omega. \tag{C.14}$$

Going further, we see that

$$x \in \partial\Omega \setminus \partial_*\Omega \Longrightarrow x \in \Gamma \;\; \text{and} \;\; \omega(x) = 0 \;\; \text{or} \;\; 1. \tag{C.15}$$

We can set

$$\Gamma_j = \{x \in \partial\Omega \setminus \partial_*\Omega : \omega(x) = j\}, \quad j \in \{0, 1\}, \tag{C.16}$$

and obtain the following.

**Proposition C.3.** *In the setting of Proposition C.1,*

$$\int_{\Omega} d\gamma + \int_{\Gamma_1} d\gamma + \frac{1}{2} \int_{\partial_0\Omega} d\gamma = \int_{\partial\Omega} \nu \cdot v \, d\sigma. \tag{C.17}$$

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