Boundary Problems for Wave Equations With Grazing and Gliding Rays

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Chapter 1: Introduction

Models for linear wave propagation in a domain Ω with boundary $\partial\Omega$ are given by linear hyperbolic equations

$$Pu = 0 \quad \text{on} \quad \Omega, \tag{1.0.1}$$

with initial conditions, and also with boundary conditions

$$B_j u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.0.2}$$

where $B_j u$ may involve u and various derivatives on $\partial\Omega$. The most basic example of this is the second order equation often called 'the wave equation':

$$\Big(\frac{\partial^2}{\partial t^2} - \Delta\Big) u = 0 \quad \text{on} \quad \Omega, \tag{1.0.3}$$

where Δ is the Laplace operator:

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}. \tag{1.0.4}$$

We impose in this case initial conditions on $u(t,x)$ at $t = 0$:

$$u(0,x) = f(x), \quad \partial_t u(0,x) = g(x). \tag{1.0.5}$$

For example, u could describe the vibrations of a membrane, stretched across a drum of shape $\mathcal{O}$, with $\Omega = \mathbb{R} \times \mathcal{O}$. If the membrane is firmly attached to the boundary, then the appropriate boundary condition for (1.0.3) is the Dirichlet boundary condition:

$$u\big|_{\partial\Omega} = 0. \tag{1.0.6}$$

Equation (1.0.3) also models the linearized theory of sound propagation, with u representing air pressure. In this case, if $\mathcal{O}$ is a region and $\partial\mathcal{O}$ a hard wall, then the pressure u satisfies at the boundary the Neumann condition

$$\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0. \tag{1.0.7}$$

To take another example, the electromagnetic field (E, B) satisfies the wave equation (1.0.3) in a vacuum:

$$(\partial_t^2 - \Delta)E = 0, \quad (\partial_t^2 - \Delta)B = 0. \tag{1.0.8}$$

If $\mathcal{O}$ is a vacuum region and $\partial\mathcal{O}$ is a perfect conductor, then a set of boundary conditions making (1.0.8) a well posed problem for the electromagnetic field is

$$\begin{aligned} &\nu \times E = 0, \quad \operatorname{div} E = 0, \\ &\nu \cdot B = 0, \quad \nu \times \operatorname{curl} B = 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.0.9}$$

where ν denotes the unit normal to $\partial\mathcal{O}$. We will encounter more examples in this monograph.

Waves tend to move in a given direction, at a given speed, and exhibit a rather geometrical behavior. This is particularly true of waves at high frequency, or equivalently of waves with singularities. The movement of singular wave fronts is governed by what are called the laws of geometrical optics.

It is the purpose of this book to present analytical tools developed to study the propagation of such wave fronts, for certain classes of domains with smooth boundary, with particular emphasis on how the propagation is influenced by the boundary.

In §1.1 we will describe in more detail what sort of wave propagation phenomena we are going to study. Then we introduce some basic tools from linear PDE in §1.2 and §1.3, which suffice to treat the simplest of these phenomena, including transversal reflection of waves at a boundary. The bulk of our study concerns two types of situations where wave fronts are tangent to the boundary, i.e., situations with grazing and gliding rays. We give in §1.4 some model examples of this, due to F.G. Friedlander, which can be analyzed via separation of variables, and which provide a clue to solutions in more general cases. The basic form of the solution to grazing and gliding ray problems is introduced in §1.5. Constructing the phase functions and amplitudes that appear in the parametrices described in §1.5 is closely related to some problems in symplectic geometry which we discuss in §1.6. Carrying out the details of this construction and drawing some conclusions will take up the rest of the book. In §1.7 we give an overview of the various geometrical and analytical problems that will be treated in the course of this analysis.

§1.1: The propagation phenomenon

One of the most striking features of solutions to wave equations is the geometrical character of propagation, particularly for ‘impulsive’ initial data. The simplest example of this is the propagation of a plane wave solution, such as

$$u(t, x) = \delta(t - x \cdot \omega) \tag{1.1.1}$$

to the wave equation (1.0.3) on $\mathbb{R} \times \mathbb{R}^n$, where $\omega \in S^{n-1}$. Another example is given by the ‘fundamental solution’ to (1.0.3), with initial data

$$u(0, x) = 0, \quad \partial_t u(0, x) = \delta_p(x). \tag{1.1.2}$$

When $\mathcal{O} = \mathbb{R}^3$, the fundamental solution is

$$u(t, x) = (4\pi t)^{-1} \delta(|t| - |x|), \tag{1.1.3}$$

while if $\mathcal{O} = \mathbb{R}^2$ it is

$$
\begin{aligned}
u(t,x) &= c(\operatorname{sgn} t)(t^2 - |x|^2)^{-1/2} && \text{if } \ |x| < |t| \\
&\qquad\qquad 0 && \text{if } \ |x| > |t|.
\end{aligned} \tag{1.1.4}
$$

These fundamental solutions illustrate several important phenomena, such as finite propagation speed. The phenomenon we are emphasizing here is the location of the singularities of u, which is the set $|x| = |t|$ in (1.1.3) and (1.1.4). The normals to this surface form the wave front set of the solution, at fixed t, and one sees that these normals propagate at unit speed, in straight lines, as the 'light rays,' in this case.

Such a phenomenon also holds for initial data with simple singularities along a more general surface S, as was first pointed out by Huygens, who argued as follows. The solution will be a superposition of fundamental solutions and its singularity will be located on the *envelope* of the family of spheres of radius $|t|$, centered at the various points of S. For given t, this envelope is the set S_t of points of distance $|t|$ from S. One can obtain S_t by following normals to S to a distance $|t|$. Making this intuitive argument precise is a pleasant exercise in distribution theory. Carrying it out rests on the explicit hold we have in (1.1.3)–(1.1.4) on the fundamental solution. For more general wave equations, e.g., on curved space, one may not have a closed form expression for the fundamental solution. Obtaining a good approximation to the fundamental solution is then essentially equivalent to analyzing the propagation of singularities phenomenon. We go over some analytical tools which are effective for this problem in the next two sections.

We have illustrated the propagation phenomenon in a region without boundary, namely $\mathbb{R}^n$. The method of images provides an illustration of the simplest sort of reflection of waves. For example, consider the wave equation (1.0.3) on $\mathbb{R} \times \mathcal{O}$ where $\mathcal{O} = \{x \in \mathbb{R}^3 : x_3 \leq 0\}$, with boundary condition $u|_{x_3=0} = 0$. If we take initial data of the form (1.1.2), with $p = (a, b, c),\ c < 0$, we get the solution, for $t \geq 0$,

$$
u(t,x) = (4\pi t)^{-1} \Big[\delta(t - |x - p|) - \delta(t - |x - p'|) \Big] \Big|_{x_3 \leq 0}, \tag{1.1.5}
$$

where $p' = (a, b, -c)$. Note that the second term in (1.1.5) represents the reflected wave and is nonzero only for $|t| \geq |c| = \operatorname{dist}(p, \partial\mathcal{O})$. If we match every ray from p with its mirror image ray from p', we see that, at the boundary, the reflection law is given in terms of rays by: angle of incidence equals angle of reflection, again a basic law of geometrical optics. This law continues to hold for curved boundaries. The analysis required to prove this will be sketched in §1.3, for the case when the ray hits the boundary transversally.

More subtle analytical problems arise in describing the situation when rays are tangent to the boundary; such rays are said to be glancing. Here there are no truly simple models with closed form solutions to guide one's intuition. The purpose of this book is to present the analysis of wave propagation for two types of rays tangent to the boundary, known respectively as grazing and gliding rays.

For the wave equation (1.0.3) on $\mathbb{R} \times \mathcal{O}$ with $\mathcal{O} \subset \mathbb{R}^n$, a grazing ray is a ray (i.e., a straight line) that intersects $\partial\mathcal{O}$ tangentially, having exactly second order contact, and locally lying in $\overline{\mathcal{O}}$. There is a natural generalization for other hyperbolic equations, involving the notion of bicharacteristic, which we will discuss in the next section. A gliding ray arises when there is such a line, tangent to $\partial\mathcal{O}$ with exactly second order contact, lying locally in $\overline{\mathbb{R}^n \setminus \mathcal{O}}$. In such a case, there are nearby rays which are reflected arbitrarily often at the boundary, leading in the limit to a curve which glides along the boundary, carrying singularities. These two cases are illustrated in the figure below.

§1.2: Pseudodifferential operators and singularities

Pseudodifferential operators, first used to study elliptic PDE, provide a fundamental tool for studying singularities of distributions. We recall some definitions and basic properties here.

A pseudodifferential operator has the form

$$p(x, D)u = \int p(x, \xi) e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi. \tag{1.2.1}$$

The amplitude, or symbol, $p(x, \xi)$, is said to belong to the symbol space $S^m_{\rho,\delta}$ provided

$$|D^\beta_x D^\alpha_\xi p(x, \xi)| \le C_{\alpha\beta} \langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}, \tag{1.2.2}$$

where $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$. We then say $p(x, D) \in OPS^m_{\rho,\delta}$. We will consider only situations in which

$$0 \le \delta < \rho \le 1; \tag{1.2.3}$$

the case $\rho = 1, \delta = 0$ is most common. We will denote by S^m_{cl}, or just S^m, the subspace of $S^m_{1,0}$ consisting of symbols with an asymptotic expansion

$$p(x, \xi) \sim \sum_{j\ge 0} p_j(x, \xi), \tag{1.2.4}$$

where $p_j(x, \xi)$ is homogeneous of degree $m - j$ in ξ, for $|\xi| \ge 1$, and the meaning of (1.2.4) is that the difference between $p(x, \xi)$ and the sum of the first N terms on the right belongs to $S^{m-N}_{1,0}$. Given (1.2.4), $p_0(x, \xi)$ is called the principal symbol of $p(x, D)$.

We have the following central algebraic property. If $p_j(x, \xi) \in S^{m_j}_{\rho,\delta}$, then

$$p_1(x, D)p_2(x, D) = q(x, D) \in OPS^{m_1+m_2}_{\rho,\delta} \tag{1.2.5}$$

with

$$q(x,\xi) \sim p_1(x,\xi)p_2(x,\xi) + \sum_{\alpha>0} \frac{(-i)^{|\alpha|}}{\alpha!}\, \partial_\xi^\alpha p_1(x,\xi)\cdot \partial_x^\alpha p_2(x,\xi). \tag{1.2.6}$$

In particular,

$$q(x,\xi) - p_1(x,\xi)p_2(x,\xi) \in S^{m_1+m_2-(\rho-\delta)}_{\rho,\delta}. \tag{1.2.7}$$

In case $\rho = 1, \delta = 0$, we then have for the commutator

$$[p_1(x,D), p_2(x,D)] = C(x,D) \in OPS^{m_1+m_2-1}_{1,0} \tag{1.2.8}$$

and

$$C(x,\xi) = -i\{p_1,p_2\}(x,\xi) \text{ mod } S^{m_1+m_2-2}_{1,0} \tag{1.2.9}$$

where $\{p_1, p_2\}$ denotes the Poisson bracket.

The definition (1.2.1) easily gives

$$p(x,D) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{1.2.10}$$

An integration by parts argument yields

$$p(x,D) : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n). \tag{1.2.11}$$

There is L^2-boundedness:

$$p(x,\xi) \in S^0_{\rho,\delta} \Longrightarrow p(x,D) : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n), \tag{1.2.12}$$

valid for $\delta \le \rho$, as long as $\delta < 1$. More generally we have Sobolev space mapping properties:

$$p(x,\xi) \in S^m_{\rho,\delta} \Longrightarrow p(x,D) : H^s(\mathbb{R}^n) \longrightarrow H^{s-m}(\mathbb{R}^n). \tag{1.2.13}$$

The property (1.2.6) shows that, if $\phi_1, \phi_2 \in C_0^\infty(\mathbb{R}^n)$ have disjoint support, then $\phi_1 p(x,D)\phi_2 \in OPS^{-\infty}$; hence it maps $H^{-\infty}(\mathbb{R}^n)$ to $H^\infty(\mathbb{R}^n)$. From this it follows that $p(x,D)$ has the 'pseudolocal' property:

$$\text{sing supp } p(x,D)u \subset \text{ sing supp } u. \tag{1.2.14}$$

The algebraic property (1.2.7) allows one to construct a parametrix for an elliptic operator. That is, if $p(x,\xi) \in S^m_{\rho,\delta}$ and $|p(x,\xi)| \ge C|\xi|^m$ for $|\xi|$ large, then we have $a(x,\xi) \in S^{-m}_{\rho,\delta}$, equal to $p(x,\xi)^{-1}$ for $|\xi|$ large, and

$$a(x,D)p(x,D) = I + r_1(x,D), \quad p(x,D)a(x,D) = I + r_2(x,D) \tag{1.2.15}$$

with $r_j(x,\xi) \in S^{-(\rho-\delta)}_{\rho,\delta}$. Now one can produce $s_j(x,\xi) \in S^{-(\rho-\delta)}_{\rho,\delta}$ such that

$$I + s_j(x,D) \sim I - r_j(x,D) + r_j(x,D)^2 - \cdots \tag{1.2.16}$$

and then

$$\begin{aligned}&(I + s_1(s,D))a(x,D)p(x,D)\\ &= p(x,D)a(x,D)(I + s_2(x,D)) = I \text{ mod } OPS^{-\infty}.\end{aligned} \tag{1.2.17}$$

In view of (1.2.14), we have local elliptic regularity:

$$p(x,D) \text{ elliptic } \implies \text{ sing supp } p(x,D)u = \text{ sing supp } u. \tag{1.2.18}$$

In our mention of wave fronts in §1.1, we indicated that these objects have a direction as well as a location. We now define the wave front set $WF(u)$ of a distribution u on $\mathbb{R}^n$ to be a closed conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ characterized as follows; $(x_0,\xi_0) \notin WF(u)$ provided there exists $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\varphi(x_0) \neq 0$, and $p(x,\xi) \in S^m$, such that $|p(x_0,\xi)| \geq C|\xi|^m$ for ξ in some conic neighborhood of ξ_0, and such that

$$p(x,D)(\varphi u) \in \mathcal{C}^\infty. \tag{1.2.19}$$

It follows from elliptic regularity that the image of $WF(u)$ under the natural projection $\pi(x,\xi) = x$ is precisely supp u.

When one develops pseudodifferential operators on a manifold M, one finds that the principal symbol of an operator in OPS^m is well defined on the cotangent bundle, $T^*M \setminus 0$. Thus, for $u \in \mathcal{D}'(M)$, $WF(u)$ is naturally defined as a closed conic subset of $T^*M \setminus 0$. 'Microlocal analysis' consists of emphasizing properties of distributions and operators in terms of the properties of their wave front sets, symbols, etc., on conic subsets of $T^*M \setminus 0$.

For example, given $P = p(x,D) \in OPS^m$, with homogeneous principal symbol $p_0(x,\xi)$, the characteristic set of P is the closed conic set $\{(x,\xi) : p_0(x,\xi) = 0\}$, denoted Char P. One says P is microlocally elliptic on the complement of Char P. The microlocal version of elliptic regularity is that

$$WF(u) \subset WF(Pu) \cap \text{ Char } P. \tag{1.2.20}$$

Refining the pseudo-local property (1.2.14), we have the microlocal property

$$WF(Pu) \subset WF(u). \tag{1.2.21}$$

The property (1.2.21) is valid whenever $P \in OPS^m_{\rho,\delta}$, granted (1.2.3), or even more generally, granted $\rho > 0$ and $\delta < 1$.

When $P \in OPS^m$ has real principal symbol $p(x,\xi)$, the integral curves of the Hamilton vector field Ξ_p contained in Char P are called null bicharacteristic curves, or simply rays. Hörmander's propagation of singularities theorem for such P states that, if $\gamma : [a,b] \to T^*\mathbb{R}^n \setminus 0$ is a null bicharacteristic of P, disjoint from $WF(Pu)$, and if $\gamma(a) \in WF(u)$, then $\gamma(b) \in WF(u)$. In some cases this can be demonstrated by constructing a parametrix, using Fourier integral operators, as we discuss in the next section. The extension of Hörmander's theorem to solutions to boundary problems is the subject of most of this book. The natural question is what to do when a ray passes over the boundary. Generally speaking, one's goal is to show that singularities propagate along reflected rays. We say two rays γ_j are related by reflection if $\gamma_j(s_j) = (x_j, \xi_j) \in T^*_{x_j}\Omega \setminus 0$ are such that $x_1 = x_2 \in \partial\Omega$ and $\xi_1 - \xi_2$ annihilates vectors tangent to $\partial\Omega$. If η denotes the restriction of ξ_j to $T^*_{x_j}\partial\Omega$, and if $\eta \neq 0$, we say the ray γ_j passes over $(x_j, \eta) \in T^*\partial\Omega \setminus 0$.

§1.3: Basic geometrical optics

One way to construct parametrices for hyperbolic equations away from a boundary is with the use of Fourier integral operators. The most basic sort of Fourier integral operator is one of the form

$$Au(x) = \int a(x,\xi) e^{i\phi(x,\xi)} \hat{u}(\xi)\, d\xi, \tag{1.3.1}$$

where the amplitude $a(x,\xi)$ belongs to a symbol class such as S^m or $S^m_{\rho,\delta}$, and the phase function $\phi(x,\xi)$ is real valued and homogeneous of degree one in ξ, and satisfies $d_x\phi(x,\xi) \neq 0$ on a conic neighborhood of the support of $a(x,\xi)$. Such Fourier integral operators can be used to transform the operator P (at least 'microlocally') to the operator $i\partial/\partial x_1$, for which propagation of singularities is transparent. Alternatively, particularly for hyperbolic equations, the solution to an initial value problem, at least near the initial surface, can be written as a linear combination of such Fourier integral operators. In either case, one is left with the problem of describing what a Fourier integral operator does to the wave front set of a distribution to which it is applied. This problem is dealt with in Appendix C. There it is shown that $WF(Au) \subset C(WF(u))$, where C is the canonical transformation defined by

$$C(d_\xi\phi(x,\xi), \xi) = (x, d_x\phi(x,\xi)).$$

As an example of this representation of the solution operator, note that the solution to the wave equation (1.0.3) on $\mathbb{R} \times \mathbb{R}^n$, with initial data (1.0.5), is given by

$$u(t,x) = \int e^{ix\cdot\xi} \cos t|\xi|\ \hat{f}(\xi)\, d\xi + \int e^{ix\cdot\xi}\ |\xi|^{-1} \sin t|\xi|\ \hat{g}(\xi)\, d\xi. \tag{1.3.2}$$

Thus for each t we have a sum of two Fourier integral operators with phase functions

$$\phi_\pm(t,x,\xi) = x\cdot\xi \pm t|\xi|. \tag{1.3.3}$$

As a warm-up for a calculation we shall do in §1.5, let us work out a local parametrix for the initial value problem for a general second order hyperbolic equation

$$p(x,D)u = 0, \tag{1.3.4}$$

with initial data on $x_n = 0$:

$$u(x',0) = f, \quad \frac{\partial}{\partial x_n} u(x',0) = g; \quad f, g \in \mathcal{E}'(\mathbb{R}^{n-1}). \tag{1.3.5}$$

We denote by $p_2(x,\xi)$ the principal symbol of $p(x,D)$; it is a second order homogeneous polynomial in ξ. We can think of $p_2(x,\xi)$ as a family of quadratic forms in ξ, parametrized by x. Let $\langle\ ,\ \rangle$ denote the associated family of symmetric bilinear forms in ξ (with x dependence implicit but suppressed), so

$$\langle \xi, \xi \rangle = p_2(x,\xi). \tag{1.3.6}$$

If $p(x,D)$ is hyperbolic, then the form $\langle\ ,\ \rangle$ is Lorentzian, and the initial value problem is well posed if the hypersurface $\{x_n = 0\}$ is spacelike with respect to $\langle\ ,\ \rangle$. We construct a solution mod $\mathcal{C}^\infty$ to (1.3.4)–(1.3.5) of the form

$$u(x) = \sum_{j=1}^{2} \int\limits_{\mathbb{R}^{n-1}} a_j(x,\xi) e^{i\phi_j(x,\xi)} \hat{F}_j(\xi)\, d\xi \tag{1.3.7}$$

where F_j will be related to f, g below. We want $a_j(x,\xi) \in S^0$, say

$$a_j(x,\xi) \sim \sum_{\nu \ge 0} a_{j\nu}(x,\xi) \tag{1.3.8}$$

with $a_{j\nu}(x,\xi)$ homogeneous of degree $-\nu$ in ξ. To begin, there is the computation of $p(x,D)(a\, e^{i\phi})$, if

$$P = p(x,D) = \sum a_{jk}(x)\partial_j\partial_k + \sum b_j(x)\partial_j + c(x), \tag{1.3.9}$$

which comes from

$$\begin{aligned} \partial_j(ae^{i\phi}) &= i(\partial_j\phi)ae^{i\phi} + (\partial_j a)e^{i\phi}, \\ \partial_j\partial_k(ae^{i\phi}) &= -(\partial_j\phi)(\partial_k\phi)ae^{i\phi} + i[(\partial_j\phi)(\partial_k a) + (\partial_k\phi)(\partial_j a)]e^{i\phi} \\ &\qquad + i(\partial_j\partial_k\phi)ae^{i\phi} + (\partial_j\partial_k a)e^{i\phi}. \end{aligned} \tag{1.3.10}$$

This gives

$$\begin{aligned} p(x,D)(ae^{i\phi}) = &-\Big[\sum_{j,k} a_{jk}(x)(\partial_j\phi)(\partial_k\phi)\Big]ae^{i\phi} + i\sum_{j,k} a_{jk}(x)(\partial_j\partial_k\phi)ae^{i\phi} \\ &+ 2i\sum_{j,k} a_{jk}(x)(\partial_j\phi)(\partial_k a)e^{i\phi} + \sum_{j,k} a_{jk}(x)(\partial_j\partial_k a)e^{i\phi} \\ &+ i\sum b_j(x)(\partial_j\phi)ae^{i\phi} + \sum b_j(x)(\partial_j a)e^{i\phi} + c(x)ae^{i\phi}, \end{aligned} \tag{1.3.11}$$

or, in a more invariant form,
(1.3.12)

$$p(x,D)(ae^{i\phi}) = \big\{-p_2(x,d_x\phi)a + i[2\langle d_x\phi, d_xa\rangle + (P^b\phi)a] + (Pa)\big\}e^{i\phi} = be^{i\phi}.$$

We have set

$$P^b = P - c(x). \tag{1.3.13}$$

The condition that the term of highest order of homogeneity in b vanish is

$$p_2(x, d_x\phi) = 0, \tag{1.3.14}$$

which is the eikonal equation. In Chapter 4 we present a solution to the eikonal equation via Hamilton-Jacobi theory. We specify

$$\phi_j(x', 0, \xi) = x' \cdot \xi, \quad x', \xi \in \mathbb{R}^{n-1}, \tag{1.3.15}$$

with the idea in mind that, for $x = (x', 0)$, (1.3.7) is given by a pseudodifferential operator. With $\phi(x,\xi) = x' \cdot \xi + x_n\psi(x,\xi)$, we see that (1.3.14) demands

$$p_2(x', 0; \xi, \psi) = 0 \ \text{ at } \ x_n = 0. \tag{1.3.15A}$$

If P is hyperbolic and $\{x_n = 0\}$ is spacelike, then for each $\xi \neq 0$, (1.3.15A) has two distinct real solutions $\psi_j(x', 0, \xi)$. The eikonal equation (1.3.14) consequently has two local solutions ϕ_j, with $\partial\phi_j/\partial x_n = \psi_j(x,\xi)$ at $x_n = 0$.

Granted (1.3.14), the term in (1.3.12) homogeneous of degree $1-k,\ k \geq 0$, is

$$2i\langle d_x\phi_j, d_xa_{jk}\rangle + i(P^b\phi_j)a_{jk} + Pa_{j,k-1}, \tag{1.3.16}$$

where we adopt the convention that $a_{j,-1} = 0$. Thus, we have the following transport equation for a_{jk} in terms of $a_{j,k-1}$.

$$2\langle d_x\phi_j, d_xa_{jk}\rangle + (P^b\phi_j)a_{jk} = iPa_{j,k-1}. \tag{1.3.17}$$

This is solved by integrating along the trajectories of the vector field Z defined by

$$Zf = 2\langle d_x\phi_j, d_xf\rangle. \tag{1.3.18}$$

Convenient initial conditions to pick are

$$a_{j0}(x', 0, \xi) = 1, \quad a_{jk}(x', 0, \xi) = 0 \ (k \geq 1). \tag{1.3.19}$$

Now one can show that, if $a_j(x,\xi) \in S^0$ is picked satisfying (1.3.8) and u is given by (1.3.7), then $Pu \in \mathcal{C}^\infty$. One can correct by a $\mathcal{C}^\infty$ term to get u solving (1.3.4).

We next want to arrange (1.3.5). Indeed, granted (1.3.15) and (1.3.19), by (1.3.7) we have

$$u(x',0) = \int_{\mathbb{R}^{n-1}} [\hat{F}_1(\xi) + \hat{F}_2(\xi)] e^{ix'\cdot\xi}\, d\xi = F_1 + F_2. \tag{1.3.20}$$

and

$$\begin{aligned} \frac{\partial}{\partial x_n} u(x',0) &= \int (i\frac{\partial \phi_1}{\partial x_n} + \frac{\partial a_1}{\partial x_n}) e^{ix'\cdot\xi} \hat{F}_1\, d\xi + \int (\frac{\partial \phi_2}{\partial x_n} + \frac{\partial a_2}{\partial x_n}) e^{ix'\cdot\xi} \hat{F}_2\, d\xi \\ &= T_1 F_1 + T_2 F_2. \end{aligned} \tag{1.3.21}$$

Thus $T_j \in OPS^1$ is a pseudodifferential operator with principal symbol equal to

$$\tau_{j1}(x',\xi) = i\frac{\partial \phi_j}{\partial x_n}\Big|_{x_n=0}. \tag{1.3.22}$$

Our initial value problem is hence equivalent to

$$\begin{aligned} F_1 + F_2 &= f, \\ T_1 F_1 + T_2 F_2 &= g. \end{aligned} \tag{1.3.23}$$

We can use the first equation to eliminate F_2, obtaining

$$(T_1 - T_2) F_1 = g - T_2 f. \tag{1.3.24}$$

The operator $T_1 - T_2 \in OPS^1$ is elliptic, granted

$$\Big|\frac{\partial \phi_1}{\partial x_n} - \frac{\partial \phi_2}{\partial x_n}\Big| \geq C_1|\xi|, \text{ on } x_n = 0, \tag{1.3.25}$$

which follows for the solutions ϕ_j of (1.3.14)–(1.3.15), provided $\{x_n = 0\}$ is spacelike. Consequently, we can certainly solve (1.3.24), and hence (1.3.23), at least mod $\mathcal{C}^\infty$, for F_1 and F_2. Again, by adding a $\mathcal{C}^\infty$ term we can correct for this last error and get the solution to (1.3.4)–(1.3.5).

We now consider situations where the surface $S = \{x_n = 0\}$ is not spacelike; for example, it could be the lateral surface $\partial\Omega$ for the wave equation (1.0.3) on $\Omega = \mathbb{R} \times \mathcal{O}$. Consider in this case the eikonal equation (1.3.14)–(1.3.15). In this case there are several possibilities for the equation (1.3.15A). Given $x_n = 0,\ \xi \neq 0$, we have either two distinct real roots $\psi_j(x,\xi)$, two complex roots, or one double real root. In these three respective cases, we have the following three geometrical possibilities. Either two bicharacteristic rays for P pass over $(x',\xi) \in T^*S \setminus 0$, or no rays, or one ray.

In the first case, one says (x',ξ) belongs to the hyperbolic set $\mathcal{G}_h$ in $T^*S \setminus 0$. In such a case, we can let u consist of a single term in (1.3.7), say the term $j = 1$, with $F = f$, and produce u satisfying $u|_S = f$, with singularities along one family of rays. In this case, the rays pass over S transversally, and the geometrical optics construction treats transversal reflection of singularities.

In the second case, one says (x', ξ) belongs to the elliptic set $\mathcal{G}_e$ in $T^*S \setminus 0$. In such a case, we do not necessarily have an exact solution to the eikonal equation, but a formal power series construction yields a pair of complex valued functions $\phi_j(x, \xi)$, with $\phi_j(x', 0, \xi) = x' \cdot \xi$ and $\partial\phi_j/\partial x_n = \psi_j(x', 0, \xi)$ at $x_n = 0$, satisfying the eikonal equation (1.3.14) to infinite order at $x_n = 0$. Say Im $\psi_1 > 0$ and Im $\psi_2 < 0$. Then $e^{i\phi_2(x,\xi)}$ blows up as $|\xi| \to \infty$, for $x_n > 0$, and so this is not useful for constructing a parametrix, if $x_n > 0$ defines Ω. Again we take the single term $j = 1$ in (1.3.7). The vector field Z given by (1.3.18) is not real, so one also solves the transport equations only to infinite order at $x_n = 0$. We can rewrite the resulting product $a_1(x, \xi)e^{i\phi_1(x,\xi)}$ as

$$b(x, \xi)e^{ix'\cdot\xi} \quad \text{with} \quad b(x, \xi) = a_1(x, \xi)e^{ix_n\psi_1(x,\xi)}. \tag{1.3.26}$$

Since Im $\psi_1(x, \xi) \geq C|\xi|$, we can establish that, for $x_n \geq 0$,

$$x_n^\ell |D_{x_n}^k D_{x'}^\beta D_\xi^\alpha b(x, \xi)| \leq C_{\alpha\beta k\ell} \langle\xi\rangle^{|\alpha|+k-\ell}. \tag{1.3.27}$$

Thus $\int b(x, \xi)\hat{f}(\xi)e^{ix'\cdot\xi}\, d\xi$ acts like a Poisson integral, and indeed this construction is microlocally the same as the construction of the Poisson integral, solving the Dirichlet problem for an elliptic PDE.

In the third case, one says (x', ξ) belongs to the glancing set $\mathcal{G}_g$ in $T^*S \setminus 0$. This is the case to which the rest of the book is devoted.

§1.4: Glancing rays; Friedlander's example

The simplest case of the wave equation that arises naturally in which tangential rays are involved is the case $\Omega = \mathbb{R} \times \mathcal{O}$ where $\mathcal{O} = \mathbb{R}^n \setminus B$ with $B = \{x \in \mathbb{R}^n : |x| \leq 1\}$, or in the complementary case, $\mathcal{O} = B$. The first case is the case of scattering by a sphere in Euclidean space. It is amenable to separation of variables, and is reducible to a problem in harmonic analysis on the cylinder $\mathbb{R} \times S^{n-1}$. This analysis goes back at least to Watson [Wa]; the analysis in Nussensweig [Nu] is more complete. We shall present a treatment of this special case, from a more contemporary perspective, in Appendix B.

In the case just described, of scattering by a sphere, the bicharacteristics that intersect the boundary $\partial\Omega = \mathbb{R} \times S^{n-1}$ tangentially have exactly second order contact with the boundary, and remain in $\overline{\Omega}$. These are grazing rays.

F.G. Friedlander [Fr] analyzed the following example of a boundary problem with grazing rays. On the region

$$\Omega_+ = \{x \in \mathbb{R}^{n+1} : x_{n+1} > 0\}, \tag{1.4.1}$$

consider the equation

$$Pu = (\partial_{n+1}^2 + x_{n+1}\partial_1^2 + \partial_1\partial_n)u = 0 \tag{1.4.2}$$

with boundary condition

$$u\big|_{x_{n+1}=0} = f \in \mathcal{E}'(\mathbb{R}^n). \tag{1.4.3}$$

There is a unique solution u satisfying

$$u \in \mathcal{C}^\infty \text{ for } x_n << 0. \tag{1.4.4}$$

In this example, as well as in the preceding example of scattering by a sphere, the analysis is simplified by the presence of a large group of symmetries. However, Friedlander's example was designed for maximum simplicity, partly because we have a large *commutative* group of symmetries, the translations in $(x_1, \dots, x_n)$, whereas the noncommutativity of the rotation group complicates the harmonic analysis that is effective in the analysis of scattering by a sphere. Taking the partial Fourier transform with respect to $x' = (x_1, \dots, x_n)$, we obtain from (1.4.2) the ODE (with $y = x_{n+1}$)

$$\frac{d^2}{dy^2}\,\hat{u}(y,\xi) = (y\xi_1^2 + \xi_1\xi_n)\hat{u}. \tag{1.4.5}$$

Another advantage of Friedlander's example is that this ODE has a simpler structure than the Bessel equations obtained by applying separation of variables to the problem of scattering by a sphere. The change of variable $s = (y\xi_1^2 + \xi_1\xi_n)\xi_1^{-4/3}$ puts (1.4.5) in the form of the Airy equation

$$A''(s) + sA(s) = 0, \tag{1.4.6}$$

One solution to Airy's equation is given by

$$Ai(s) = \int_{-\infty}^{\infty} e^{i(st+t^3/3)}\,dt. \tag{1.4.7}$$

Any solution to (1.4.6) is entire; the functions

$$A_\pm(s) = Ai(e^{\pm 2\pi i/3}s) \tag{1.4.8}$$

are also seen to solve (1.4.6). These Airy functions are discussed in detail in Appendix A. One important feature that distinguishes among them is their various asymptotic behavior as $s \to +\infty$. One sees that, if $\hat{f}(\xi)$ is supported in a cone $|\xi'| < \pm K\xi_1$, where $\xi' = (\xi_2, \dots, \xi_n)$, then the solution to (1.4.2)–(1.4.4) is given by

$$u(x) = \int_{\mathbb{R}^n} A_\pm(\zeta)A_\pm(\zeta_0)^{-1}e^{ix'\cdot\xi}\hat{f}(\xi)\,d\xi, \tag{1.4.9}$$

with

$$\zeta = \zeta(x, \xi) = |\xi_1|^{-1/3}\xi_n + x_{n+1}|\xi_1|^{2/3} \tag{1.4.10}$$

and

$$\zeta_0 = \zeta\big|_{x_{n+1}=0} = |\xi_1|^{-1/3}\xi_n. \tag{1.4.11}$$

We can also consider the boundary problem (1.4.2)–(1.4.4) in the region

$$\Omega_- = \{x \in \mathbb{R}^{n+1} : x_{n+1} < 0\}. \tag{1.4.12}$$

In this case, rays tangent to $\partial\Omega_-$ stay in the *complement* of Ω_-, still having second order contact with $\partial\Omega_-$. We have multiply reflected rays, which tend in the limit to curves traveling along the boundary, known as gliding rays. In this case, the solution to (1.4.2)-(1.4.4) can be given by an integral analogous to (1.4.9), using the Airy function Ai rather than $A_\pm$. Since $Ai(s)$ has real zeros, it is convenient to translate into the complex domain to avoid them. This can be accomplished by setting $v = e^{-Tx_n}u$, so the equation for v is obtained by replacing ∂_n by $\partial_n - T$ in (1.4.2). Again, the partial Fourier transform yields Airy's equation, and (with f replaced by $e^{-Tx_n}f$) we get

$$v(x) = \int_{\mathbb{R}^n} Ai(\zeta)Ai(\zeta_0)^{-1}e^{ix'\cdot\xi}\hat{f}(\xi)\,d\xi, \tag{1.4.13}$$

with

$$\zeta = \zeta(x, \xi) = \xi_1^{-1/3}(\xi_n + iT) + x_{n+1}\xi_1^{2/3} \tag{1.4.14}$$

and

$$\zeta_0 = \xi_1^{-1/3}(\xi_n + iT), \tag{1.4.15}$$

assuming $\hat{f}(\xi)$ is supported on a cone $|\xi'| < K\xi_1$.

A qualitative analysis of a class of Fourier-Airy integral operators generalizing (1.4.9) and (1.4.13) occupies Chapters 5 and 6. These more general operators will be described in the next section, when we write down more general grazing and gliding ray parametrices.

The wave equation (1.0.3) on the region $\mathbb{R} \times B$, where B is the unit ball in Euclidean space, furnishes another model example of a boundary problem with gliding rays. This example is also amenable to separation of variables, and there is a literature on this example. However, we have not seen a treatment in sufficient analytical detail to make manifest the propagation of singularities in a neighborhood of gliding rays.

§1.5: Outline of the glancing ray parametrices

For general grazing ray problems, we will construct parametrices of the form

$$u(x) = \int \big[gA_\pm(\zeta) + ihA'_\pm(\zeta)\big] A_\pm(\zeta_0)^{-1} e^{i\theta} \hat{F}(\xi)\, d\xi, \tag{1.5.1}$$

where the phase functions $\theta(x,\xi)$, $\zeta(x,\xi)$ are real valued and homogeneous in ξ, of degree 1 and 2/3, respectively, and

$$\zeta_0(x,\xi) = \zeta(x,\xi)|_{\partial\Omega} = \xi_1^{-1/3}\xi_n. \tag{1.5.2}$$

The amplitudes $g(x,\xi)$ and $h(x,\xi)$ belong to the symbol classes S^0 and $S^{-1/3}$, respectively. The phase functions θ, ζ solve an eikonal equation, and the amplitudes g, h satisfy a sequence of transport equations. We will derive here such equations for the case of $Pu = 0$ where P is a second order differential operator,

$$P = p(x,D) = \sum a_{jk}(x)\partial_j\partial_k + \sum b_j(x)\partial_j + c(x). \tag{1.5.3}$$

Parallel to the computation producing (1.3.11), we have (with $f_j = \partial_j f$),

$$\partial_j\big(gA(\zeta)e^{i\theta}\big) = i\theta_j gA(\zeta)e^{i\theta} + \zeta_j gA'(\zeta)e^{i\theta} + g_jA(\zeta)e^{i\theta}, \tag{1.5.4}$$

and

$$\begin{aligned}\partial_j\partial_k\big(gA(\zeta)e^{i\theta}\big) =& \big[-(\theta_j\theta_k + \zeta\zeta_j\zeta_k)g + i(\theta_j g_k + \theta_k g_j) + i\theta_{jk}g + g_{jk}\big]A(\zeta)e^{i\theta} \\ &+ \big[i(\theta_j\zeta_k + \theta_k\zeta_j)g + (\zeta_j g_k + \zeta_k g_j) + \zeta_{jk}\big]A'(\zeta)e^{i\theta},\end{aligned} \tag{1.5.5}$$

where the Airy equation (1.4.6) has been used to replace $A''(\zeta)$ by $-\zeta A(\zeta)$. Consequently, for P given by (1.5.3), we have

$$\begin{aligned}P\big(gA(\zeta)e^{i\theta}\big) =& \Big[\big(\langle d\theta, d\theta\rangle + \zeta\langle d\zeta, d\zeta\rangle\big)g + 2i\langle d\theta, dg\rangle + i(P^b\theta)g + (Pg)\Big]A(\zeta)e^{i\theta} \\ &+ \big[2i\langle d\theta, d\zeta\rangle g + 2\langle d\zeta, dg\rangle + (P^b\zeta)g\big]A'(\zeta)e^{i\theta}.\end{aligned} \tag{1.5.6}$$

Here, as before, $\langle \xi, \eta\rangle$ is the form polarizing $p_2(x,\xi)$, so $\langle \xi, \xi\rangle = p_2(x,\xi)$ and

$$\langle \xi, \eta\rangle = \sum a_{jk}(x)\xi_j\eta_k. \tag{1.5.7}$$

Also, as before, $P^b = P - c(x)$. Similarly, we compute

$$\begin{aligned}&P\big(hA'(\zeta)e^{i\theta}\big) = \\ &\big[-2i\zeta\langle d\theta, d\zeta\rangle h - 2\zeta\langle d\zeta, dh\rangle - \langle d\zeta, d\zeta\rangle h - \zeta(P^b\zeta)h\big]A(\zeta)e^{i\theta} \\ &+ \Big[-\big(\langle d\theta, d\theta\rangle + \zeta\langle d\zeta, d\zeta\rangle\big)h + 2i\langle d\theta, dh\rangle + i(P^b\theta)h + Ph\Big]A'(\zeta)e^{i\theta}.\end{aligned} \tag{1.5.8}$$

Thus we see that

$$P\big(gA(\zeta)e^{i\theta}+ihA'(\zeta)e^{i\theta}\big)=aA(\zeta)e^{i\theta}+bA'(\zeta)e^{i\theta} \tag{1.5.9}$$

where

$$\begin{aligned} a =&\big(\langle d\theta,d\theta\rangle+\zeta\langle d\zeta,d\zeta\rangle\big)g+2\zeta\langle d\theta,d\zeta\rangle h\\ &+2i\langle d\theta,dg\rangle-2i\zeta\langle d\zeta,dh\rangle-i\langle d\zeta,d\zeta\rangle h+i(P^b\theta)g-i\zeta(P^b\zeta)h+Pg, \end{aligned} \tag{1.5.10}$$

and

$$\begin{aligned} b=&\, 2i\langle d\theta,d\zeta\rangle g-i\big(\langle d\theta,d\theta\rangle+\zeta\langle d\zeta,d\zeta\rangle\big)h\\ &+2\langle d\zeta,dg\rangle-2\langle d\theta,dh\rangle+(P^b\zeta)g-(P^b\theta)h+iPh. \end{aligned} \tag{1.5.11}$$

We should like the amplitudes a and b to have asymptotic expansions that formally vanish. Requiring the top order term to vanish yields the eikonal equation for θ,ζ :

$$\begin{aligned} \langle d\theta,d\theta\rangle+\zeta\langle d\zeta,d\zeta\rangle&=0,\\ \langle d\theta,d\zeta\rangle&=0. \end{aligned} \tag{1.5.12}$$

If we set

$$g\sim\sum_{\nu\ge o}g_\nu,\quad h\sim\sum_{\nu\ge 0}h_\nu,$$

with g_ν homogeneous of degree $-\nu$ in ξ and h_ν homogeneous of degree $-1/3-\nu$, the transport equations become

$$\begin{aligned} &2\langle d\theta,dg_\nu\rangle-2\zeta\langle d\zeta,dh_\nu\rangle-\langle d\zeta,d\zeta\rangle h_\nu+(P^b\theta)g_\nu-\zeta(P^b\zeta)h_\nu=iPg_{\nu-1},\\ &2\langle d\zeta,dg_\nu\rangle-2\langle d\theta,dh_\nu\rangle\qquad\qquad\qquad\;\;+(P^b\zeta)g_\nu-(P^b\theta)h_\nu=-iPh_{\nu-1}. \end{aligned} \tag{1.5.13}$$

A thorough discussion of these eikonal and transport equations will be given in Chapter 4.

In the study of grazing ray problems for other sorts of differential equations one is led to similar eikonal and transport equations. Examples include various first order systems, treated in Chapter 12.

For general gliding ray problems we will construct parametrices of the form

$$u(x)=\int[gAi(\zeta)+ihAi'(\zeta)]Ai(\zeta_0)^{-1}e^{i\theta}\hat F(\xi)\,d\xi. \tag{1.5.14}$$

One way to deal with the fact that $Ai(s)$ has real zeros is to evaluate $\theta,\zeta,g,$ and h at $(\xi_1,\dots,\xi_{n-1},\xi_n+iT)$. In particular, $\zeta_0=\xi_1^{-1/3}(\xi_n+iT)$. These functions satisfy the same eikonal and transport equations as above, at real ξ, and an almost analytic continuation in ξ_n is taken.

In addition to solving the eikonal equations so that ζ satisfies (1.5.2) on $\partial\Omega$, we will also produce a solution to the transport equations such that

$$h(x,\xi) = 0 \text{ for } x \in \partial\Omega, \tag{1.5.15}$$

while $g(x,\xi)$ is elliptic. Thus, in (1.5.1) and in (1.5.14), u and F are related by

$$u|_{\partial\Omega} = \int g e^{i\theta} \hat{F}(\xi)\, d\xi = J(F). \tag{1.5.16}$$

The operator J is an elliptic Fourier integral operator. Thus the parametrix for a Dirichlet boundary problem, $u|_{\partial\Omega} = f$, is given by (1.5.1) or (1.5.14), with $F = J^{-1}f$, where J^{-1} denotes a microlocal parametrix for J.

Constructing a parametrix for the Neumann problem (1.0.7) involves examining the Neumann operator, defined as follows. If $f \in \mathcal{E}'(\partial\Omega)$ and u is the solution to our boundary problem, given by either (1.5.1) or (1.5.14), then $Nu = (\partial u/\partial\nu)|_{\partial\Omega}$. Thus to solve the Neumann problem $\partial u/\partial\nu|_{\partial\Omega} = g$, one is reduced to solving $Nf = g$ for f.

Applying ∂_ν to (1.5.1), we obtain an expression for the Neumann operator in the grazing case:

$$N = J(A\Phi_\pm + B)J^{-1}. \tag{1.5.17}$$

Here J is as in (1.5.16), $A \in OPS^{2/3}$, is elliptic, with principal symbol ζ_ν (which is shown to be > 0), and $B \in OPS^0$. The operators $\Phi_\pm$ are Fourier multipliers:

$$(\Phi_\pm f)\hat{\ }(\xi) = \Phi_\pm(\zeta_0)\hat{f}(\xi), \quad \Phi_\pm(s) = \frac{A'_\pm(s)}{A_\pm(s)}, \tag{1.5.18}$$

which can be seen to belong to $OPS^{1/3}_{1/3,0}$ microlocally on $|\xi'| < K\xi_1$. In the gliding case one gets

$$N = J(A\Phi i_T + B)J^{-1}, \tag{1.5.19}$$

where

$$\Phi i_T(\xi) = \Phi i(\xi_1^{-1/3}(\xi_n + iT)), \quad \Phi i(s) = \frac{Ai'(s)}{Ai(s)}. \tag{1.5.20}$$

§1.6: Symplectic geometry behind the parametrix

There is a deep geometrical reason underlying the similarity of the general grazing and gliding ray parametrices described in §1.5 and those for Friedlander's examples given in §1.4, which will facilitate solution of the eikonal and transport equations. Under the grazing and gliding hypotheses, the hypersurfaces $Q_1 = T^*_{\partial\Omega}\overline{\Omega}$

and $P_1 = \text{Char } P = \{(x,\xi) : p_2(x,\xi) = 0\}$ have glancing intersection. By definition, hypersurfaces defined by $p = 0$ and $q = 0$ have glancing intersection at m provided $\{p,q\} = 0$ at m while $\{p,\{p,q\}\} \neq 0$ and $\{q,\{q,p\}\} \neq 0$ at m. The main theorem established in Chapter 3 is that (in any given dimension) any two pairs of hypersurfaces with glancing intersection are locally symplectically equivalent, via a homogeneous canonical transformation (in the homogeneous case). Consequently, there is a microlocally defined homogeneous canonical transformation

$$\chi : T^*(\mathbb{R}^{n+1}) \longrightarrow T^*\Omega \tag{1.6.1}$$

taking $Q_0 = \{x_{n+1} = 0\}$ to Q_1 and taking P_0, the characteristic set of Friedlander's operator (1.4.2), to P_1.

Now, on Q_1 (and similarly on Q_0), the symplectic form gives a Hamilton foliation. Let this determine an equivalence relation $\sim$. Then $Q_1 \cap P_1/\sim$ has the structure of a symplectic manifold with boundary, and is naturally isomorphic to the closure of the 'hyperbolic set' in $T^*(\partial\Omega)$, the region over which real rays pass, and similarly $Q_0 \cap P_0/\sim$ is naturally isomorphic to the closure of the hyperbolic region in $T^*(\partial\mathbb{R}^{n+1}_+)$. Thus we get a canonical transformation

$$\chi_J : T^*(\partial\mathbb{R}^{n+1}_+) \longrightarrow T^*(\partial\Omega), \tag{1.6.2}$$

defined in the hyperbolic regions, smooth up to the boundary, which consists of the grazing directions.

The map χ_J has the important property that it intertwines the 'billiard ball maps' $\delta_0^\pm$ and $\delta^\pm$. Here, the billiard ball maps $\delta^\pm : T^*(\partial\Omega) \to T^*(\partial\Omega)$, defined on the hyperbolic region, continuous up to the boundary, smooth in the interior, are defined at a point (x_0,ξ_0) by taking the two rays that lie over this point, in the variety $P_1 = \text{ Char } P$, and following the null bicharacteristics through these points until you pass over $\partial\Omega$ again, projecting such a point onto $T^*(\partial\Omega)$. For Friedlander's example, $\delta_0^\pm$ has the specific formula

$$\delta_0^\pm(x,\xi) = \Big(x_1 \pm \frac{2}{3}\Big(\frac{\xi_n}{\xi_1}\Big)^{3/2}, x_2, \dots, x_{n-1}, x_2 \mp 2\Big(\frac{\xi_n}{\xi_1}\Big)^{1/2}, \xi\Big). \tag{1.6.3}$$

The way in which χ helps one solve the eikonal equations is discussed in detail in Chapter 4. We mention here that one can arrange that $\theta|_{\partial\Omega}$ generate the canonical transformation χ_J, which is hence the canonical transformation associated with the Fourier integral operator J in (1.5.16).

In our first papers on the grazing ray problem ([M2],[T3]), we did not take this approach, and our solutions to the eikonal equation were not shown to have the property (1.5.2). Taylor [T3], taking a cue from Ludwig [Lud2], made use of the weaker result that one could arrange

$$\zeta(x,\xi)\big|_{\partial\Omega} = \xi_1^{-1/3}\xi_n + r(x,\xi) \tag{1.6.4}$$

with

$$\xi_1^{-2/3} r(x,\xi) = O(|\xi_n/\xi_1|^N), \tag{1.6.5}$$

for all N, with a similar weakening of (1.5.15). This result leads to an equation slightly different from (1.5.16) for F, in the solution to the Dirichlet problem, and one can go quite far with it, though some finer information is less accessible via (1.6.4) than by (1.5.2). This approach to the grazing ray problem is discussed in Chapter X of the book [T1]. The refined result (1.5.2) is particularly incisive in our treatment of the gliding ray problem. Eskin [Es1] has proposed a construction of parametrices in the gliding ray case which makes use of the weaker result (1.6.4). In this case the boundary equations which arise have a more complicated nature than (1.5.16), and are more difficult to analyze.

§1.7: Plan of the book

The first order of business will be to establish the geometrical results, especially equivalence of glancing hypersurfaces, lying behind the construction of phase functions and amplitudes in the parametrices. This task is divided into two parts. In Chapter 2 we establish results purely in differential analysis, and in Chapter 3 bring in the symplectic form. Equivalence of glancing hypersurfaces will be established as a consequence of putting the billiard ball map $\delta^{\pm}$ into normal form. The maps $\delta^{\pm}$ are a special case of folding canonical relations; we also establish results on putting more general folding canonical relations into normal form.

In Chapter 4 we solve the eikonal and transport equations (1.5.12)–(1.5.13). As mentioned, it is important to produce solutions to the eikonal equation such that ζ satisfies (1.5.2), and we also produce solutions to the transport equations such that h satisfies (1.5.15). For other uses, we also produce other solutions to the transport equations with various relations between g and h specified on $\partial\Omega$.

In Chapters 5 and 6 we examine analytical properties of the parametrices (1.5.1) and (1.5.14), beginning with a study of the Fourier multipliers $\Phi_{\pm}$ and Φi_T. As mentioned, $\Phi_{\pm}$ are pseudodifferential operators. On the other hand, Φi_T will be seen to be a very singular sort of Fourier integral operator, with an infinite number of canonical relations accumulating at a 'gliding' canonical relation. This reflects the geometrical difference between grazing and gliding problems.

Using the assembled tools, we carry out the parametrix construction, for Dirichlet boundary conditions, in Chapter 7. In that chapter we also produce $\mathcal{A}_+/\mathcal{A}_-$ as a microlocal model for the initial value problem with homogeneous Dirichlet boundary conditions, in the grazing case.

In Chapter 8 we tackle the Neumann boundary problem. We analyze the Neumann operator, establishing formulas (1.5.17) and (1.5.19). The operator $A\Phi_{\pm}+B$ is a hypoelliptic operator in $OPS^1_{1/3,0}$; $A\Phi i_T + B$ is more complicated, but as we show, its parametrix has a form not essentially more complicated then that of Φi_T itself.

We end Chapter 8 by showing that different choices of solutions to the transport equations yield different expressions for the Neumann operator. This leads to operator identities of a rather subtle nature, which we exploit in Chapters 9 and 10. Chapter 9 studies a calculus of operators containing $A\Phi_{\pm}+B$, using Airy operator identities. Some of these results can be obtained using $S^m_{1/3,0}$ symbol calculus, but combinatorial complications in symbol expressions seem to hide some of the features of the Airy operator calculus. For the operator calculus containing $A\Phi i_T+B$, studied in Chapter 10, these Airy operator identities play an even more substantial role, as the ordinary pseudifferential and Fourier integral calculi are not effective.

Using the results on Airy operator calculus from Chapters 9 and 10, we tackle a number of other boundary problems in Chapters 11 and 12, including transmission problems, the boundary problem (1.0.9) arising from Maxwell's equations, and various types of coercive and non-coercive boundary problems for first order systems.

At the end are several appendices. In Appendix A we derive needed properties of the Airy functions $Ai(z)$ and $A_{\pm}(z)$, and associated Airy quotients. In Appendix B we work out the problem of scattering by a sphere, via separation of variables. We show how the results so obtained can be put in the form (1.5.1) using the uniform asymptotic expansion of Bessel functions for large order and argument. Appendix C covers background material on wave front sets on bounded domains, and Appendix D discusses some results on Fourier integral operators with singular phases, of use in Chapters 5 and 6.

Chapter 2: Folds, involutions and folding relations

As mentioned in Chapter 1, the basic geometric structure that arises in the study of boundary problems, with bicharacteristics simply tangent to the boundary, is a folding Lagrangian relation. The reduction of such relations to normal form is carried out in the next chapter. This is a conjugation problem in symplectic geometry. The underlying analytic problem, of the reduction to normal form of a folding relation, is solved in this chapter, without the symplectic structure. This somewhat simplifies the initial discussion. It should be noted however that the stability, in the sense of singularity theory, of the symplectic problem is lost in the case of a general folding relation.

The fundamental result of this chapter is Proposition 2.3.10 which gives a normal form (under change of coordinates) for a pair of smooth involutions (both orientation reversing) which fix pointwise a certain hypersurface, and which have different linearizations at a base point on the hypersurface.

In the first section we discuss Whitney folds and their relation to involutions. The way folds arise from the tangency of a vector field to a hypersurface is examined in §2.2 and then in §2.3 the basic notion of a folding relation is introduced; the fundamental result is then stated in terms of folding relations and equivalently in terms of involutions. The reduction to normal form for two involutions is carried out, in the sense of Taylor series, in §2.4. To remove the 'flat' error terms a convergence argument is needed (the Taylor series does not converge in general, even for real analytic problems). This is developed as a 'scattering problem' in §2.5. The proof of the normal form for pairs of involutions is given in §2.6; this is briefly extended in §2.7 to give the normal form for folding relations.

The last two sections contain some extensions of the normal form theorem. In §2.8 problems concerning the existence of functions with specified relation between their even and odd parts under the two involutions are solved; these will later be applied to the solution of transport equations, in Chapter 4. In §2.9 the whole discussion is (easily) extended to the homogeneous case.

§2.1: Folds

Let Y and Z be $\mathcal{C}^\infty$ manifolds of the same dimension, k. Although all considerations here will be local it is convenient to maintain invariant notation as much as possible, to suggest always the freedom to make coordinate changes. Consider a $\mathcal{C}^\infty$ map from Y to Z:

$$F : Y \longrightarrow Z. \tag{2.1.1}$$

The graph of F is a $\mathcal{C}^\infty$ submanifold of the product $Z \times Y$:

$$G = \operatorname{gr}(F) = \{(z, y) \in Z \times Y; F(y) = z\}. \tag{2.1.2}$$

Indeed, a $\mathcal{C}^\infty$ submanifold $G \subset Z \times Y$ is, near $m \in G$, the graph of a $\mathcal{C}^\infty$ map from a neighborhood of $\bar{y} \in Y$ to Z if and only if the projection

$$\pi_Y : G \longrightarrow Y \text{ is a diffeomorphism near } m \text{ with } \pi_Y(m) = \bar{y}. \tag{2.1.3}$$

Now, if (2.1.3) holds then the image of $\bar{y}$ is $\bar{z} = \pi_Z(m)$. Of course the map defined by G is a local diffeomorphism from a neighborhood of $\bar{y}$ to a neighborhood of $\bar{z}$ if and only if the analogue of (2.1.3) holds for the other projection:

$$\pi_Z : G \longrightarrow Z \text{ is a diffeomorphism near } m. \tag{2.1.4}$$

The simplest case in which (2.1.3) holds but (2.1.4) is violated is that of a map F with Whitney fold at y. The map F as in (2.1.1) is said to have a Whitney fold at $\bar{y}$ provided the differential:

$$F_* : T_{\bar{y}}Y \longrightarrow T_z Z \text{ has rank } k-1, \tag{2.1.5}$$

and provided furthermore that, if ν_Y and ν_Z are $\mathcal{C}^\infty$ k-forms non-zero at $\bar{y}$ and $\bar{z}$ respectively, then

$$F^*\nu_Z = f\nu_Y, \quad f(\bar{y}) = 0, \quad df(\bar{y}) \neq 0. \tag{2.1.6}$$

A classical result of Whitney shows that such a map may be brought to normal form by diffeomorphisms in Y and Z. In fact, there are local coordinates $y_1, \dots, y_k$ in Y, based at $\bar{y}$, and $z_1, \dots, z_k$ in Z, based at $\bar{z}$ with respect to which:

$$F(y_1, \dots, y_k) = (y_1, \dots, y_{k-1}, y_k^2) = (z_1, \dots, z_k). \tag{2.1.7}$$

The proof of (2.1.7) is elementary. First note that the function f in (2.1.6) is well-defined up to a non-vanishing $\mathcal{C}^\infty$ multiple so the singular surface of F is well-defined by

$$S = \{f(y) = 0\}. \tag{2.1.8}$$

From (2.1.6), S is a $\mathcal{C}^\infty$ hypersurface in Y near $\bar{y}$ and from (2.1.5) F is a diffeomorphism of a neighborhood of $\bar{y}$ in S to a hypersurface $S_Z = F(S)$ in Z through $\bar{z}$. Now N^*S_Z is the annihilator of the range of F_* on T_yY, $y \in S$, so if g is a defining function for S_Z, F^*g must vanish with its differential at S. Thus $F^*g = g'f$, g' being $\mathcal{C}^\infty$. In fact:

$$F^*g = h \cdot f^2, \; h \text{ being } C^\infty \text{ near } \bar{y} \text{ with } h(\bar{y}) \neq 0, \tag{2.1.9}$$

since from (2.1.6) g' must vanish simply on S. In particular the image under F of a neighborhood of $\bar{y}$ lies on one side of S_Z so the normal to S_Z is oriented at $\bar{z}$. Now (2.1.7) can be stated in the strengthened form that if $z_1, \dots, z_{k-1}, z_k$ are any

FIGURE 2.1, A FOLD

coordinates in Z near $\bar{z}$ in which S_Z is locally defined with the correct orientation by $z_k = 0$, (so if $g = z_k$ in (2.1.9) then $h > 0$) there are local coordinates $y_1, \dots, y_k$ in Y with respect to which (2.1.7) holds. In fact this construction gives two different coordinate systems of this type in Y, $y_j = F^* z_j, j < k$, and $y_k = \pm h^{1/2} f$, if $g = z_k$ in (2.1.8).

This last ambiguity in the coordinates corresponds to the fact that a fold map F defines a $\mathcal{C}^\infty$ involution on Y, near $\bar{y}$, and on the graph G:

$$\mathcal{I} = \mathcal{I}_0 \text{ where } \mathcal{I}_0(y_1, \dots, y_k) = (y_1, \dots, y_{k-1}, -y_k) = (z_1, ..., z_k). \tag{2.1.10}$$

Indeed, $\mathcal{I}$ is invariantly defined on Y near $\bar{y}$ by

$$\mathcal{I}(y) = y' \text{ if } f(y) = f(y') \text{ and } y \neq y' \text{ unless } y = y' \in S. \tag{2.1.11}$$

As an involution on the graph G of F, $\mathcal{I}$ is defined by using the diffeomorphism (2.1.4) intrinsically by:

$$\mathcal{I}(m') = m'' \text{ if } \pi_Z(m') = \pi_Z(m''), \quad m' \neq m'' \text{ unless } m' = m'' \in S. \tag{2.1.12}$$

Here we abuse notation somewhat by regarding S as a submanifold of G. As has just been shown the reduction of the fold map to the normal form (2.1.7) reduces the corresponding involution to the normal form (2.1.10). If $\mathcal{I}$ is any $\mathcal{C}^\infty$ involution defined near some point $m \in G$, with the appropriate formal properties:

$$\mathcal{I} \cdot \mathcal{I} = \text{Id} \text{ near } m, \ \mathcal{I} = \text{Id} \text{ on a hypersurface } S \text{ and } \mathcal{I} \text{ exchanges the two sides of } S,$$

then $\mathcal{I}$ can be reduced to the normal form (2.1.10) in some coordinates $y_1, \dots, y_k$ in G. Indeed if $y'_j, j = 1, \dots, k-1$ are $\mathcal{C}^\infty$ functions with independent differentials on

S at m the even parts $y_j = (\mathcal{I}^* y_j' + y_j')/2$ are invariant under $\mathcal{I}$ and are independent on S and together with the $\mathcal{I}$-odd function $y_k = (\mathcal{I}^* y_k' - y_k')/2$, where S is defined by $y_k' = 0$, give such coordinates.

More significantly the reduction of the fold map to normal form (2.1.7) can be deduced directly from the reduction of the corresponding involution $\mathcal{I}$ to the normal form (2.1.10). This an easy consequence of the following result which is left as an exercise.

Exercise 2.1.13: Any $\mathcal{C}^\infty$ function, f, defined near $0 \in \mathbb{R}^k$ which is even (resp. odd) under $\mathcal{I}_0$ can be written in the form

$$f(y_1, \dots, y_{k-1}, y_k) = g(y_1, \dots, y_{k-1}, y_k^2) \quad (\text{resp. } y_k g(y_1, \dots, y_{k-1}, y_k^2)) \tag{2.1.14}$$

for some $\mathcal{C}^\infty$ function g defined near $0 \in \mathbb{R}^k$. Is g uniquely defined by (2.1.14)?

§2.2: Tangency of vector fields

The manner in which maps with fold singularity arise in the study of boundary problems is through the examination of the properties of vector fields tangent to some hypersurface. Consider a manifold M, of dimension k, with base point $m \in M$. Suppose that, near m, M has a foliation with one dimensional leaves. That is, M is equipped with a $\mathcal{C}^\infty$ vector bundle

$$\mathcal{W} \subset TM, \ \dim_{\text{fibre}} \mathcal{W} = 1. \tag{2.2.1}$$

The leaves of the foliation near m are just the integral curves of any $\mathcal{C}^\infty$ section, V, of $\mathcal{W}$ not vanishing at m. Of course any other $\mathcal{C}^\infty$ section is a multiple of V near m, so the (unparametrized) curves are independent of this choice. Now suppose in addition that a hypersurface in M, passing through m, is given:

$$m \in K \hookrightarrow M, \ \operatorname{codim}(K) = 1. \tag{2.2.2}$$

The simplest, and generic, state of affairs is when $\mathcal{W}$ is transversal to K.

$$T_m K + \mathcal{W}_m = T_m M. \tag{2.2.3}$$

In this case, always locally near m, each leaf of M passes through a unique point of K, so restricting M to be a small neighborhood of m there is a map

$$\nu : M \longrightarrow K, \tag{2.2.4}$$

sending each point of M to the point on K on the same curve of M. Then ν has surjective differential at m, so every $\mathcal{C}^\infty$ function on K is the restriction to K of a $\mathcal{C}^\infty$ function on M which is constant on the leaves of M. The transversality condition (2.2.3) can be restated as:

$$V\kappa(m) \neq 0, \ \text{if } \kappa = 0 \text{ on } K, \ d\kappa(m) \neq 0, \ V \text{ a section of } \mathcal{W}, \ V(m) \neq 0. \tag{2.2.5}$$

Thus if K is any hypersurface through m transversal to M the map (2.2.4) can be considered as a local isomorphism

$$M/\mathcal{W} \longleftrightarrow K,$$

from the space of leaves of M. This gives $M/\mathcal{W}$ a $\mathcal{C}^\infty$ structure which is clearly independent of the choice of transversal K.

Another way of stating these elementary results is that if K is transversal to $\mathcal{W}$ at m and V is a section of M with $V(m) \neq 0$ then there are local coordinates, $x_1, \dots, x_k$, in M in which m is the origin and in terms of which

$$K = \{x_1 = 0\}, \quad V = \frac{\partial}{\partial x_1} \text{ spans } \mathcal{W}. \tag{2.2.6}$$

Suppose more generally that $K \hookrightarrow M$ is any submanifold passing through m. Then sending each point of K to the leaf through it gives a map:

$$\rho_K : K \longrightarrow M/\mathcal{W}. \tag{2.2.7}$$

This map is always $\mathcal{C}^\infty$ since it is just the inclusion in M followed by projection onto $M/\mathcal{W}$. In view of (2.2.5), the simplest case of the failure of (2.2.3) is described by:

$$V\kappa(m) = 0, \quad V[V\kappa] \neq 0 \text{ at } m, \quad V, \kappa \text{ as in (2.2.5)}, \tag{2.2.8}$$

which is the case of most interest here. This corresponds to the integral curves of $\mathcal{W}$ being simply tangent to K. The foliation $\mathcal{W}$ near a point of simple tangency as in (2.2.8) can always be reduced by a coordinate transformation to the model case in $M = \mathbb{R}^k$, with coordinates $x_1, \dots, x_k$:

$$K = \{x_1 = 0\}, \quad \mathcal{W} = \operatorname{Span}\left\{\frac{\partial}{\partial x_k} - 2x_k \frac{\partial}{\partial x_1}\right\}. \tag{2.2.9}$$

Figure 2.2

For this example the integral curves are the parabolas $x_1+x_k^2=$const, $x_j=$const, $1<j<k$. The reduction to the normal form (2.2.9) is essentially equivalent to the following fact:

$$\rho_K \text{ has a Whitney fold at } m\in K \iff (2.2.8) \text{ holds.} \tag{2.2.10}$$

If κ is a defining function for K then set $\theta=V\kappa$, where V is some non-vanishing section of M. By hypothesis (2.2.8) the surface

$$T=\{\theta=V\kappa=0\}$$

is transversal to $\mathcal{W}$. Taking coordinates as in (2.2.6) (with different numbering) gives

$$V=\frac{\partial}{\partial x_k},\ \theta=\frac{\partial\kappa}{\partial x_k}=x_k\Longrightarrow\kappa=\frac{1}{2}x_k^2+g(x_1,x_2,\dots,x_{k-1}).$$

Since $d\kappa(m)\neq 0$, by hypothesis, $2g$ can be introduced as a new variable in place of one of the $x_p,\ p<k$. Relabelling it as x_1, ensures that $K=\{x_1+x_k^2=0\}$. A further coordinate change in which x_1 is replaced by $x_1+x_k^2$ reduces K and $\mathcal{W}$ to the form (2.2.9). Not only does this prove the reducibility to (2.2.9) but shows that, given (2.2.8), ρ_K has a Whitney fold as claimed in (2.2.10).

Conversely, suppose that K is a $\mathcal{C}^\infty$ hypersurface through m such that the map ρ_K in (2.2.7) has a Whitney fold. Thus K has an involution, $\mathcal{J}$, exchanging the points identified by ρ_K. Let $x_1,x_2,\dots,x_{k-1}$, be coordinates in $M/\mathcal{W}$ such that $z_k,\ \rho_K^*x_p,\ 1<p<k$, form a coordinate system on K where $z_k^2=\rho_K^*x_1$ and z_k is odd under $\mathcal{J}$. Such coordinates exist because ρ_K has a fold. Now, extend $x_1,x_2,\dots,x_{k-1}$ to functions $x_1',x_2,\dots,x_{k-1}$ on M constant on the leaves of $\mathcal{W}$. Let x_k be a $\mathcal{C}^\infty$ extension off K of z_k, and set $x_1=x_1'-x_k^2$. Since $x_1'=x_k^2$ on K, $x_1=0$ defines K. Moreover, $x_1+x_k^2,x_2,\dots,x_{k-1}$ are constant on the leaves of $\mathcal{W}$, so (2.2.9) holds in these coordinates, and hence (2.2.8) holds too. Summarizing this simple construction gives:

Lemma 2.2.11. *Let $K\subset M$ be a hypersurface simply tangent to the one-dimensional foliation $\mathcal{W}$ along the submanifold S, in the sense of* (2.2.8) *or* (2.2.10)*. Assume $(x_1,\dots,x_{k-1})$ are coordinates in $M/\mathcal{W}$ near m with the image of the fold set in K given by $S=\{z_k=0\}$ where $(z_2,\dots,z_k)$ are coordinates in K such that $\rho_K^*x_j=z_j, 1<j<k, \rho_K^*x_1=z_k^2$. Then there are coordinates $(x_1,\dots,x_k)$ in M with respect to which* (2.2.9) *holds, K can be identified with its original coordinates as $\{x_1=0\}$ and on K, $z_j=x_j,\ 1<j<k, x_k=z_k^2$.*

Exercise 2.2.12: Suppose $\mathcal{W}$ is a 1-dimensional foliation and that K is a hypersurface, to which W is simply tangent in the sense of (2.2.8), at a point m. Show that any $\mathcal{C}^\infty$ function on K invariant under the involution, associated to ρ_K by (2.2.10), can be extended to a $\mathcal{C}^\infty$ function, near m, constant on the leaves of $\mathcal{W}$. Is this extension locally unique?

§2.3: Folding relations

We have been considering here the properties of maps with simple Whitney folds. We must however consider the more complicated problem of a $\mathcal{C}^\infty$ relation G from Y to Z, manifolds of the same dimension k, with folds in both directions. That is

$$G \subset Z \times Y, \quad \pi_Y,\ \pi_Z \text{ have folds at } m \in G. \tag{2.3.1}$$

The generic case of such a structure would correspond to the two fold hypersurfaces $S_Y,\ S_Z \subset G$ being transversal at m. This is not the case which we consider here because, as will be discussed in the next chapter, the presence of a symplectic structure imposes additional constraints. Instead we shall examine the case of a folding relation defined as follows.

Definition 2.3.2. *A $\mathcal{C}^\infty$ submanifold $G \subset Z \times Y$, where $\dim(Z) = \dim(Y) = \dim(G)$, is a folding relation at $m \in G$ if near m the two projections π_Y and π_Z from G are both folds and have the same singular hypersurface $S \subset G$.*

The main result of this chapter is:

Theorem 2.3.3. *If $G \subset Z \times Y$ is a folding relation at m then there are local coordinates $y_1, \ldots, y_k$ in Y near $\pi_Y(m)$ and local coordinates $z_1, \ldots, z_k$ in Z near $\pi_Z(m)$ with respect to which $G = G_0$ where:*

$$G_0 = \{(z_1, \ldots, z_k, y_1, \ldots, y_k) \in \mathbb{R}^k \times \mathbb{R}^k;\ z_k = (z_1 - y_1)^2,\ z_j = y_j,\ j \geq 2\}. \tag{2.3.4}$$

Figure 2.3

The principal step in the reduction of a folding relation to the normal form (2.3.4) is the reduction of the two involutions on G to simultaneous normal forms. On G_0 we can take as coordinates

$$y_1, z_1, z_2, \dots, z_{k-1} \tag{2.3.5}$$

and in terms of these coordinates the two involutions are

$$\begin{cases} (y_1, z_1, z_2, \dots, z_{k-1}) \longmapsto (y_1, 2y_1 - z_1, z_2, \dots, z_{k-1}) & \text{from } \pi_Y \\ (y_1, z_1, z_2, \dots, z_{k-1}) \longmapsto (2z_1 - y_1, z_1, z_2, \dots, z_{k-1}) & \text{from } \pi_Z. \end{cases} \tag{2.3.6}$$

In terms of the less symmetric coordinates

$$t_1 = \frac{1}{2} y_1, \quad t'' = (y_2, \dots, y_{k-2}), \quad t_k = y_1 - z_1. \tag{2.3.7}$$

these two involutions become

$$\mathcal{I}_0(t) = (t_1, t'', -t_k) \tag{2.3.8}$$

and

$$\mathcal{J}_0(t) = (t_1 + t_k, t'', -t_k). \tag{2.3.9}$$

Observe, from (2.1.5), (2.2.8) that the null space of the differential of a map at a fold point is precisely the -1-eigenspace of the corresponding involution. Since they agree on the common singular hypersurface S, the two involutions $\mathcal{I}$ and $\mathcal{J}$ defined on a folding relation G by the two projections could have differentials the same at m if and only if the differentials of the projections have the same null space. However, if π_Y and π_Z have a common null space then the inclusion:

$$\pi_Z \times \pi_Y : G \hookrightarrow Z \times Y$$

cannot have injective differential, which violates the assumption that G is a $\mathcal{C}^\infty$ submanifold. Thus the two involutions satisfy the hypothesis in the following proposition.

Proposition 2.3.10. *Let G be a $\mathcal{C}^\infty$ manifold of dimension k, $m \in G$ a point and $\mathcal{I}$ and $\mathcal{J}$ two $\mathcal{C}^\infty$ involutions defined on a neighborhood of m both point wise fixing a $\mathcal{C}^\infty$ hypersurface S through m and exchanging the two sides of S. Then if $\mathcal{I}$ and $\mathcal{J}$ have different differentials at m there exist local coordinates t_1, t'', t_k, $t'' = (t_2, \dots, t_{k-1})$ in G near m with respect to which $\mathcal{I} = \mathcal{I}_0$, $\mathcal{J} = \mathcal{J}_0$, the normal forms being given by* (2.3.8) *and* (2.3.9)*.*

The result in Proposition 2.3.10 is a conjugation problem. That is, we seek a local diffeomorphism F fixing the origin, thought of as a coordinate transformation, such that

$$\mathcal{I} = F^{-1} \cdot \mathcal{I}_0 \cdot F \text{ and } \mathcal{J} = F^{-1} \cdot \mathcal{J}_0 \cdot F. \tag{2.3.11}$$

Obtaining the normal form (2.3.8) alone is straightforward as noted above. In the next section the preliminary material required to solve the problem in the sense of formal power series is developed. That is, we initially try only to find F so that (2.3.11) is true only in the sense of Taylor series at the common fixed hypersurface of the two involutions. In §2.5 a convergence argument is given which allows the flat remainder terms in (2.3.11) to be removed. This is actually carried out in §2.6.

EXERCISE 2.3.12: Let $\mathcal{U}$ be a two dimensional foliation (so integrable) near some point m and suppose that it is simply tangent to a hypersurface K along a submanifold $S \subset K$ of codimension 1, in the sense that each non-vanishing section of $\mathcal{U}$ is simply tangent to K along S. Select a commuting basis of $\mathcal{U}$ and show that the two involutions defined on K by (2.2.10) satisfy the hypotheses of Proposition 2.3.10. Use this result to show that local coordinates can be introduced in terms of which

$$
\begin{gathered}
K=\{x_1=0\}, \quad S=\{x_k=x_1=0\}, \\
\mathcal{U}_1=\operatorname{Span}\Big\{\frac{\partial}{\partial x_k}-2x_k\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_k}+\frac{\partial}{\partial x_{k-1}}-2x_k\frac{\partial}{\partial x_1}\Big\}.
\end{gathered} \tag{2.3.13}
$$

§2.4: FORMAL POWER SERIES

Consider first the linear version of Proposition 2.3.10.

Lemma 2.4.1. *If $\mathcal{I}$ and $\mathcal{J}$ are two different linear involutions on a real vector space, E, both point wise fixing a hyperspace S and neither being the identity, then there are linear coordinates in which* (2.3.8)*,* (2.3.9) *hold.*

Proof. The reduction discussed above gives linear coordinates in which $\mathcal{I}=\mathcal{I}_0$ is given by (2.3.8). Thus S is given by $\{t_k=0\}$ and $\mathcal{J}$ is the identity on S. Consider $e_k=(0,...,0,1)$ spanning the -1 eigenspace of $\mathcal{I}_0$:

$$\mathcal{J}e_k=-e_k+e',$$

where $S\ni e'\neq 0$, since $\mathcal{J}\cdot\mathcal{J}=\mathrm{Id}$ and $\mathcal{J}\cdot\mathcal{I}_0\neq\mathrm{Id}$. Changing basis in S so $e'=(1,0,...,0)$ gives (2.3.9) without disturbing (2.3.8) and proves the Lemma.

Now consider the model case, with involutions (2.3.8), (2.3.9). Let $\mathcal{D}_p\subset\mathcal{C}^\infty(\mathbb{R}^k)$ be the space of $\mathcal{C}^\infty$ functions vanishing to order p at S, i.e.,

$$g\in\mathcal{D}_p \iff g=t_k^p g',\ g'\in\mathcal{C}^\infty(\mathbb{R}^k) \text{ and } \mathcal{D}_\infty=\bigcap_p\mathcal{D}_p. \tag{2.4.2}$$

For $\mathcal{K}=\mathcal{I}_0$ and $\mathcal{J}_0$ consider the subspaces

$$\mathcal{D}_pE(\mathcal{K}),\ \mathcal{D}_pO(\mathcal{K})\subset\mathcal{D}_p$$

of functions respectively even and odd under the involution $\mathcal{K}$.

Lemma 2.4.3. *For any $q \geq p = 0, 1, 2, \ldots, \infty$,*

$$\mathcal{D}_{2p} = \mathcal{D}_{2p}E(\mathcal{I}) + \mathcal{D}_{2p}E(\mathcal{J}) + \mathcal{D}_{2q+2}, \tag{2.4.4}$$

$$\mathcal{D}_{2p+1} = \mathcal{D}_{2p+1}O(\mathcal{I}) + \mathcal{D}_{2p+1}O(\mathcal{J}) + \mathcal{D}_{2q+1}. \tag{2.4.5}$$

Proof. Consider (2.4.4) for $q = p$. From the definition (2.4.2) $g \in \mathcal{D}_{2p}$, modulo terms in $\mathcal{D}_{2p+2}$, is just:

$$g \equiv g(t')t_k^{2p} + h(t')t_k^{2p+1}. \tag{2.4.6}$$

The first term is even under $\mathcal{I}_0$ so can be absorbed in the first term on the right in (2.4.4). Moreover if $f(t')$ is any $\mathcal{C}^\infty$ function then from Taylor's formula

$$-f(t_1, t'')t_k^{2p} + f(t_1 + \frac{1}{2}t_k, t'')t_k^{2p} \equiv \frac{1}{2}\frac{\partial f(t')}{\partial t_1} \cdot t_k^{2p+1} \text{ modulo } \mathcal{D}_{2p+2}. \tag{2.4.7}$$

Since the first and second terms on the left in (2.4.7) are in $\mathcal{D}_{2p}E(\mathcal{I})$ and $\mathcal{D}_{2p}E(\mathcal{J})$ respectively, it is only necessary to solve the ordinary differential equation

$$\frac{\partial f(t')}{\partial t_1} = 2h(t') \tag{2.4.8}$$

to see that the second term in (2.4.6) is also in the sum on the right in (2.4.4). The converse is obvious, so this proves (2.4.4) for $q = p$.

Iterating this special case gives (2.4.4) for every finite $q \geq p$. Moreover, the construction is such that for a fixed $g \in \mathcal{D}_{2p}$, the decomposition (2.4.4) can be obtained for each $q \geq p$:

$$g \equiv f_{\mathcal{I},q} + f_{\mathcal{J},q} \mod \mathcal{D}_{2q+2},$$

with the sequences $\{f_{\mathcal{I},q}\}$, $\{f_{\mathcal{J},q}\}$ converging in the sense of formal power series:

$$f_{\mathcal{I},q} \longrightarrow \sum_{p \geq r \geq q} g_r(t')t_k^{2r} \tag{2.4.9}$$

and similarly for $f_{\mathcal{J},q}$. Using Borel's Lemma these can be summed to give $\mathcal{C}^\infty$ functions invariant under the two involutions, so proving (2.4.4) for $q = \infty$.

The result, (2.4.5), dealing with functions odd under the two involutions follows in the same way. The only difference between the two cases is that (2.4.4) shows that, modulo $\mathcal{D}_\infty$, any function g can be written as the sum of functions even under the two involutions, whereas in the odd case (2.4.5) it is, of course, necessary that the function g must vanish on S.

Next we consider the analogous result for vector fields. Let $\mathcal{V}_p$ be the space of $\mathcal{C}^\infty$ vector fields on $\mathbb{R}^k$ which are tangent to $S = \{t_k = 0\}$ to order p in the sense that:

$$V \in \mathcal{V}_p \iff V : \mathcal{D}_q \longrightarrow \mathcal{D}_{q+p}, \quad \forall\, q \geq 0.$$

Thus $V \in \mathcal{V}_p$ if it is of the form

$$V = \sum_{j=1}^{k} a_j \frac{\partial}{\partial t_j}, \tag{2.4.10}$$

with $a_j \in \mathcal{D}_p,\ j < k,\ a_k \in \mathcal{D}_{p+1}$. As above we shall use the notation:

$$V \in \mathcal{V}_p E(\mathcal{K}) \iff V \in \mathcal{V}_p \ \&\ \mathcal{K}_* V = V,\ \mathcal{K} = \mathcal{I}_0 \text{ or } \mathcal{J}_0,$$

for the spaces of vector fields invariant under the two model involutions.

Lemma 2.4.11. *For any $q \geq p \geq 0$,*

$$\mathcal{V}_{2p} = \mathcal{V}_{2p} E(\mathcal{I}) + \mathcal{V}_{2p} E(\mathcal{J}) + \mathcal{V}_{2q}.$$

Proof. Observe that

$$\begin{aligned} &(\mathcal{I}_0)_* \partial_j = \partial_j,\ 1 \leq j < k, \quad (\mathcal{I}_0)_* \partial_k = -\partial_k, \\ &(\mathcal{J}_0)_* \partial_j = \partial_j,\ 1 \leq j < k, \quad (\mathcal{J}_0)_* (\partial_k - \frac{1}{2}\partial_1) = -(\partial_k - \frac{1}{2}\partial_1). \end{aligned} \tag{2.4.12}$$

Thus with the decomposition (2.4.10) $V \in \mathcal{V}_{2p} E(\mathcal{I})$ if

$$a_j \in \mathcal{D}_{2p} E(\mathcal{I}),\ 1 \leq j < k, \quad a_k \in \mathcal{D}_{2p+1} O(\mathcal{I}),$$

and similarly, $V \in \mathcal{V}_{2p} E(\mathcal{J})$ if

$$a_1 + \frac{1}{2} a_k,\ a_j \in \mathcal{D}_{2p} E(\mathcal{J}),\ 1 < j < n, \quad a_k \in \mathcal{D}_{2p+1} O(\mathcal{J}).$$

Thus to decompose a general vector field V, in (2.4.10), as desired first use (2.4.4) to decompose the coefficients $a_j,\ 1 < j < k$. Then use (2.4.5) to decompose a_k:

$$a_k = a_{\mathcal{I},k} + a_{\mathcal{J},k},$$

with the $a_{\mathcal{K},k}$ odd under $\mathcal{K}_0$. Using (2.4.4) again it can be arranged that

$$a_1 + \frac{1}{2} a_{\mathcal{J},k} = a_{\mathcal{I},1} + a_{\mathcal{J},2},$$

with the $a_{\mathcal{K},1}$ even under $\mathcal{K}$. This ensures that

$$V = V_{\mathcal{I}} + V_{\mathcal{J}} \mod \mathcal{V}_\infty,$$

with the $V_{\mathcal{K}}$ invariant under $\mathcal{K}$, proving the Lemma.

Exercise 2.4.13: Let f be a $\mathcal{C}^\infty$ function which is invariant under both the model involutions $\mathcal{I}_0$ and $\mathcal{J}_0$. Show that the Taylor series of f at $t_k = 0$ is of the form

$$\sum_{k=0}^{\infty} f_j(t_2, \ldots, t_{k-1}) t_k^{2k}. \tag{2.4.14}$$

Show that, in a given neighbourhood of 0, there are infinitely many such $\mathcal{C}^\infty$ invariant functions with the same Taylor series (2.4.14).

§2.5: A scattering problem

As a second preliminary step to the proof of Proposition 2.3.10 we shall prove a result analogous to the existence of wave operators in scattering theory. The connection between such existence results and conjugation problems is well-known – see in particular the work of Sternberg [St1].

On $\mathbb{R}^k$ let T be a map which is equal, to infinite order, to a shift in a particular hyperplane and at infinity:-

$$\text{(2.5.1)} \quad \begin{cases} T(y_1, \dots, y_k) = (y_1 + 1 + Z_1(y), y_2 + Z_2(y), \dots, y_k + Z_k(y)), \\ \text{where } \ \forall\, p \in \mathbb{N}, \alpha \in \mathbb{N}^k \ \exists\, C_{\alpha,p} \text{ s.t. } |D_y^\alpha Z_j(y)| \le C_{\alpha,p} |y_k|^p (1 + |y|)^{-2p}. \end{cases}$$

In particular this implies that T is invertible in a suitably small strip $|y_k| \le \delta$ for some $\delta > 0$ with the inverse having similar estimates to (2.5.1) in that region, for the deviation from the inverse of the shift. Setting

$$T_0(y) = (y_1 + 1, y_2, \dots, y_k),$$

we define 'intertwining operators' as limits, when they exist,

$$\text{(2.5.2)} \qquad W_\pm(y) = \lim_{n \to \pm\infty} T_0^{-n} T^n.$$

These are analogous to 'wave operators' in the theory of scattering of waves.

Proposition 2.5.3. *Given $c > 0$ there exists $\epsilon > 0$ such that in the region $R_\pm(c) = \{|t_k| < \epsilon, \mp t_1 < c\}$ the limit (2.5.2) exists in the Schwartz topology, i.e. the maps $W_\pm : R_\pm(c) \longrightarrow \mathbb{R}^k$ are $\mathcal{C}^\infty$ and such that*

$$\text{(2.5.4)} \qquad T_0^{-n} \cdot T^n(y) = W_\pm(y) + E_n(y), \quad E_n(y) = (E_{n,1}(y), \dots, E_{n,k}(y)),$$

where for each $p \in \mathbb{N}$, $\alpha \in \mathbb{N}^k$ there exists $C_{\alpha,p}$ such that

$$\text{(2.5.5)} \qquad \begin{cases} |D_y^\alpha E_{n,\ell}(y)| \le C_{\alpha,p} n^{-p} (1 + |y|)^{-2p} |y_k|^p, \quad \forall\ \pm n \ge 0, \\ |D^\alpha (W_{\pm,\ell}(y) - y_\ell)| \le C_{\alpha,p} (1 + |y|)^{-2p} |y_k|^p, \end{cases}$$

on $R_\pm(c)$.

One of the simplest ways that a sequence of maps with the property in (2.5.4) and (2.5.5) might arise is as the sequence of values at integer times of the solution of a system of differential equations. Since we shall in fact reduce the proof of Proposition 2.5.3 to this case we first record a suitable result of this type.

Lemma 2.5.6. *Let $W(r, y)$ be a $\mathcal{C}^\infty$ vector field on $\mathbb{R}^k$ depending smoothly on a parameter $r \in \mathbb{R}$ and satisfying estimates*

(2.5.7)

$$|D_r^\ell D_y^\alpha W_j(r, y)| \le C_{\ell,\alpha,p} (1 + |r| + |y|)^{-2p} |y_k|^p, \quad \forall\ p, \ell \in \mathbb{N},\ \alpha \in \mathbb{N}^k,\ j = 1, \dots, k.$$

Then the solution to

$$\frac{d}{dr}Y_j(r,y) = W_j(r,Y), \quad Y_j(0,y) = y_j, \quad j = 1,\dots,k \tag{2.5.8}$$

defines a family of $\mathcal{C}^\infty$ *maps* $Q(r) : \mathbb{R}^k \ni y \longmapsto Y(r,y) \in \mathbb{R}^k$ *which are invertible, such that the* Y_j *satisfy the estimates* (2.5.7) *for all* $\ell > 0$ *and furthermore*

$$\lim_{t\to\infty} Q(r) = Q(\infty) \tag{2.5.9}$$

exists in the sense of (2.5.4) *and* (2.5.5).

Proof. The main point is that the solution of (2.5.8) must exist for all r. Certainly it exists locally so it suffice to prove that it remains bounded, uniformly in r and locally uniformly in y. From (2.5.7) there is an absolute bound on the coefficients of W :

$$|W_j(r,\cdot)| \le C(1+|r|)^{-2}, \tag{2.5.10}$$

which shows, by integration of (2.5.8), that the solution is so bounded.

The estimates (2.5.7) for $\ell > 0$ and with W_j replaced by Y_j now follow by differentiation of (2.5.8). Integrating (2.5.8) then gives:

$$Y_j(r,y) = y_j + \int_0^\infty W_j(r,Y(r,y))\,dr - \int_r^\infty W_j(r,Y(r,y))\,dr. \tag{2.5.11}$$

The first two terms on the right give the limiting map $Q(\infty)$. Clearly the error, the last term in (2.5.11), decreases rapidly with all its derivatives as $|r| \to \infty$ or $|y| \to \infty$ and also as $|y_k| \to 0$, giving the estimate (2.5.5) and proving the lemma.

Proof of Proposition 2.5.3. The first step is to obtain T by integration of a (parameter dependent) vector field. Set $S = T_0^{-1}T$. By assumption $S(y) = y + F(y)$ where

$$|D_y^\alpha F_j(y)| \le C_{\alpha,p}(1+|y|)^{-2p}|y_k|^p, \quad \forall\ p. \tag{2.5.12}$$

Thus $T(y) = (S_1(y)+1, S_2(y),\dots,S_k(y))$. We shall interpolate between $T(y)$ and the identity by setting

$$T^{(r)}(y) = (y_1 + \phi(r)F_1(y) + r, y_2 + \phi(r)F_2(y),\dots,y_k + \phi(r)F_k(y)), \tag{2.5.13}$$

where $\phi \in \mathcal{C}^\infty(\mathbb{R})$ has $0 \le \phi(r) \le 1,\ \phi(r) = 0$ for $r < 1/4$ and $\phi(r) = 1$ for $r > 3/4$. Certainly, from (2.5.13), $T^{(r)}$ is a diffeomorphism from a region $|y_k| \le \delta$ onto its range, for all $r \in [0,1)$.

Differentiating the components of $T^{(r)}$ gives

$$\frac{d}{dr}T_j^{(r)}(y) = \begin{cases} 1 + \phi'(r)F_1(y), & j = 1,\\ \phi'(r)F_j(y), & j > 1.\end{cases} \tag{2.5.14}$$

Since $T^{(r)}$ is a diffeomorphism this defines a vector field, in some strip, by

$$W_j(r, T^{(r)}(y)) = \phi'(r)F_j(y) + \delta_{j1}. \tag{2.5.15}$$

Setting $W_j(r, y) = \delta_{j1} + V_j(r, y)$ it is clear that

$$|D_r^\ell D_y^\alpha V_j(r, y)| \le C_{\ell,\alpha,p}(1 + |y|)^{-2p}|y_k|^p, \quad \forall\, p, \alpha, \ell. \tag{2.5.16}$$

As we shall see below the integral curves of W stay near $y_k = 0$ so for simplicity we shall assume that the V_j have been extended globally to vanish in $|y_k| > 1$ and still satisfy (2.5.15).

It is important to note that W is constant, equal to ∂_1 near both $r = 0$ and $r = 1$. It can therefore be extended to be periodic of period one in r and the estimates (2.5.15) then hold uniformly for $r \in (-\infty, \infty)$. Moreover from (2.5.14) it follows immediately that

$$T^{(n)}(y) = T^n(y), \quad \forall\, n \in \mathbb{N}. \tag{2.5.17}$$

This is the interpolating curve of diffeomorphisms which will be used below.

Consider next the 1-parameter family of diffeomorphisms

$$S^{(r)} = T_0^{-r}T^{(r)},\ T_0^r(y) = (y_1 + r, y_2, \ldots, y_k), \quad \forall\, r \in \mathbb{R}. \tag{2.5.18}$$

We can immediately find the vector field which generates $S^{(r)}$ by differentiation:

$$\begin{aligned} \frac{d}{dr}S_j^{(r)}(y) &= \frac{d}{dr}T_j^{(r)}(y) - \delta_{j1} \\ &= W_j(r, T_j^{(r)}(y)) - \delta_{j1} \\ &= V_j(r, S_1^{(r)}(y) + r, S_2^{(r)}(y), \ldots, S_k^{(r)}(y)). \end{aligned} \tag{2.5.19}$$

Thus the vector field generating $S^{(r)}$ is simply $V_j(r, y_1 + r, y_2, \ldots, y_k)$. For any constant $c > 0$ we can truncate this near $y_1 = -c$ and in $r < 0$ setting

$$\tilde{V}_j = \phi(r + 1)\phi(y_1 + c + 1)V_j(r, y_1 + r, y_2, \ldots, y_k). \tag{2.5.20}$$

Now, as $|y|$ or $r \to \infty$ this vector field, $\tilde{V}$ vanishes rapidly with all derivatives, so Lemma 2.5.6 applies to it. The global diffeomorphisms $\tilde{S}^{(r)}$ generated by $\tilde{V}$ therefore satisfy (2.5.9). In particular these diffeomorphisms approach the identity rapidly and uniformly as $|y_k| \to \infty$. Thus the extension of V, after (2.5.16), is immaterial and

$$\tilde{S}^{(r)}(y) = S^{(r)}(y) \text{ in } y_1 > -c,\ r > 0,\ |y_k| < \varepsilon(c), \varepsilon(c) > 0. \tag{2.5.21}$$

Thus Lemma 2.5.6 applies directly to prove the half of Proposition 2.5.3 dealing with $R_+(c)$. Since the other case is clearly similar this completes the proof.

The limiting maps $W_\pm$ defined by Proposition 2.5.3 are invertible in $R_+(c)$ for any c. Moreover

$$T_0 \cdot (T_0^{-n-1} \cdot T^{n+1}) = (T_0^{-n} \cdot T^n) \cdot T. \tag{2.5.22}$$

Passing to the limit on both sides gives the desired intertwining property

$$T_0 \cdot W_\pm = W_\pm T. \tag{2.5.23}$$

Exercise 2.5.24: Assuming that T satisfies the hypotheses of Proposition 2.5.3 show that W_+ is uniquely fixed in some region $R_+(c)$ by the estimates (2.5.5) and the conjugation condition (2.5.23) (which may be required to hold in a larger strip).

§2.6: Proof of Proposition 2.3.10

As remarked after the statement of the Proposition it can be assumed that $\mathcal{I} = \mathcal{I}_0$ is already in the form (2.3.8). Let $\mathcal{J}_0$ be the involution (2.3.9), to which $\mathcal{J}$ is to be reduced, whilst preserving the form of $\mathcal{I}_0$. Since $\mathcal{J}$ fixes $S = \{t_k = 0\}$ pointwise,
(2.6.1)

$$\mathcal{J}^* g(t') = g(t') + Q(t') \cdot (\frac{\partial g(t')}{\partial t'}) \cdot t_k + O(t_k^2),\ \mathcal{J}^* t_k = -T(t') \cdot t_k + O(t_k^2), \quad \forall\, g \in \mathcal{C}^\infty,$$

with $T > 0$, $Q = (Q_1, .., Q_{k-1}) \neq 0$, at 0, since the involutions are different. Moreover from $\mathcal{J} \cdot \mathcal{J} = \mathrm{Id}$, it follows that $T = 1$. There are $k-2$ independent solutions of

$$Q \cdot \frac{\partial}{\partial t'} Z_j = 0, \quad j = 2, ..., k-1, \tag{2.6.2}$$

and a further independent solution of

$$Q \cdot \frac{\partial}{\partial t'} Z_1 = 1.$$

Together the change of variables from $t = (t_1, t'', t_k)$ to $(Z_1(t'), Z''(t'), t_k)$ reduces $\mathcal{J}$ to the form (2.6.1) with $T = 1$, $Q = (1, 0, ..., 0)$:

$$\mathcal{J}^* t_j = t_j + \delta_{j1} \cdot t_k + O(t_k^2),\ 1 \le j < k,\ \mathcal{J}^* t_k = -t_k + T'(t') \cdot t_k^2 + O(t_k^3). \tag{2.6.3}$$

Next we use the freedom to make a change of variables preserving the form of $\mathcal{I}_0$ to ensure that $T'(t') = 1$. Clearly, as long as $g(t') > 0$ the transformation
(2.6.4)

$$(t_1, t'', t_k) \longmapsto (h(t_1, t''), t'', g(t') t_k), \quad t'' = (t_2, \ldots, t_{k-1}),\ \frac{\partial h(t_1, t'')}{\partial t_1} = g(t_1, t'')$$

does leave $\mathcal{I}_0$ unchanged. Moreover the relationship between h and g means that
(2.6.5)

$$\mathcal{J}^* h(t_1, t'') = h(t_1, t'') + \frac{\partial h(t_1, t'')}{\partial t_1} \cdot t_k + O(t_k^2) = h(t_1, t'') + (g(t_1, t'') \cdot t_k) + O(t_k^2),$$

so the pull-back under $\mathcal{J}$ of the each of the first $k-1$ of the new coordinates still satisfies (2.6.3). On the other hand for the last coordinate we obtain

$$\mathcal{J}^*(g(t')t_k) = -(g(t')t_k) + \Big[T(t')g(t') - \frac{\partial g(t_1,t')}{\partial t_1}\Big] \cdot t_k^2 + O(t_k^3).$$

Thus if we choose g to satisfy the initial value problem

$$\frac{\partial g(t_1,t')}{\partial t_1} = T(t')g(t'), \quad g\big|_{\{t=0\}} = 1, \tag{2.6.6}$$

then in the new coordinates (2.6.4) we get (2.6.3) with $T' = 0$.

In fact we shall proceed to show that this achieves the desired reduction up to the next higher order. Applying $\mathcal{J}$ again and using (2.6.3) shows that

$$\mathcal{J}^*(t_j) = t_j + \delta_{j1} + F_j(t') \cdot t_k^2 + O(t_k^3) \Longrightarrow$$
$$(\mathcal{J}^2)^*(t_j + \delta_{j1}\frac{1}{2}t_k) = (t_j + \delta_{j1}\frac{1}{2}t_k) + 2F_j(t') \cdot t_k^2 + O(t_k^3), \quad 1 \le j < k.$$

It therefore follows from the fact that $\mathcal{J}$ is an involution that

$$\mathcal{J}^*(t_k) + t_k,\ \mathcal{J}^*(t_j) - t_j,\ \mathcal{J}^*(t_1) - t_1 - t_k = O(t_k^3), \quad 1 < j < k. \tag{2.6.7}$$

Now, (2.6.7) can be restated as saying that

$$\mathcal{J} = F^{-1}\mathcal{J}_0 F, \ \text{ near } 0, \ \text{with } (\mathrm{Id} - F^*)\mathcal{D}_p \subset \mathcal{D}_{p+2}, \quad \forall\ p. \tag{2.6.8}$$

Thus the local diffeomorphism F is close to the identity. In particular there exists a global 1-parameter family of diffeomorphisms G_t, $t \in [0,1]$ with

$$\begin{gathered} G_1 = F \text{ near } 0,\ G_0 = \mathrm{Id},\ G_t = \mathrm{Id} \text{ outside a compact set,} \\ (\mathrm{Id} - G_t^*)\mathcal{D}_p \subset \mathcal{D}_{p+2}, \quad \forall\ p. \end{gathered} \tag{2.6.9}$$

In place of the original $\mathcal{J}$ we now consider

$$\mathcal{J}_t = G_t^{-1} \cdot \mathcal{J}_0 \cdot G_t, \tag{2.6.10}$$

since $\mathcal{J}_1 = \mathcal{J}$ near 0.

The last condition in (2.6.9) means that G_t is generated by a 1-parameter family of vector fields $V_t \in \mathcal{W}_2$, to which Lemma 2.4.11 applies. Let $\mathcal{W}_{\mathcal{I}}(t)$ and $\mathcal{W}_{\mathcal{J}}(t)$ be the $\mathcal{I}$-even and $\mathcal{J}$-even parts of $\mathcal{W}(t)$, modulo $\mathcal{W}_4$. Let $H_{\mathcal{I},t}$ and $H_{\mathcal{J},t}$ be the 1-parameter families of diffeomorphisms they generate. Define

$$G'_t = H_{\mathcal{J},t}^{-1} \cdot G_t \cdot H_{\mathcal{I},t}^{-1} \Longleftrightarrow G_t = H_{\mathcal{J},t} \cdot G'_t \cdot H_{\mathcal{I},t}. \tag{2.6.11}$$

This 1-parameter family of diffeomorphisms is generated by the family of vector fields:

$$V'(t) = W_1(t) + W_2(t) + W_3(t),$$

where $W_1(t)$ is the push-forward under $G_t \cdot H_{\mathcal{I},t}^{-1}$ of the generator of $H_{\mathcal{J},t}^{-1}$, $W_2(t)$ is the push-forward under $H_{\mathcal{I},t}^{-1}$ of $V(t)$, the generator of G_t, and $W_3(t)$ is the generator of $H_{\mathcal{I},t}^{-1}$. Each of these maps has the last property of G_t in (2.6.9), so

$$W_1(t) \equiv V_{\mathcal{J}},\ W_2(t) \equiv V(t),\ W_3(t) \equiv -V_{\mathcal{I}}(t),\quad \text{modulo } \mathcal{W}_4.$$

Thus $[\operatorname{Id} - G_t']^* \mathcal{D}_p \subset \mathcal{D}_{p+4},\ \forall\ p$.

Using Lemma 2.4.11 repeatedly in this way to further factorize G_t gives a sequence of factorizations improving (2.6.11),

$$G_t = H_{\mathcal{J},t,m} \cdot G_{t,m} \cdot H_{\mathcal{I},t,m}, \tag{2.6.12}$$

where $(\operatorname{Id} - G_{t,m}^*)\mathcal{D}_p \subset \mathcal{D}_{p+m},\ \forall\ p$, and $H_{\mathcal{K},t,m}$ are generated by vector fields in $\mathcal{V}_2 E(\mathcal{K})$. Moreover, as $m \to \infty$ these vector fields converge in the sense of formal power series at S. That is, successive terms vanish to higher and higher order. Thus, using Borel's Lemma to sum, (2.6.12) can be improved to a factorization (2.6.11) in which the $H_{\mathcal{K},t}$ commute with $\mathcal{K}$ and G_t' is a 1-parameter family of global diffeomorphisms, equal to the identity outside a compact set and to infinite order at S. Inserting this into (2.6.10), the factor $H_{\mathcal{J},t}$ commutes with $\mathcal{J}_0$ and so cancels and the factor $H_{\mathcal{I},t}$ can be carried out as a change of coordinates under which $\mathcal{I}_0$ is preserved. This reduces $\mathcal{J}$ to the form (2.6.8) where now in (2.6.10) $G_t = \operatorname{Id}$ outside a compact set and to infinite order at S.

To remove this last factor we shall use Proposition 2.5.3. To do so consider the singular transformation

$$R(t_1, t'', t_k) = (s, t'', t_k),\ s = \frac{t_1}{t_k}. \tag{2.6.13}$$

This gives a $\mathcal{C}^\infty$ diffeomorphism

$$\begin{aligned} R : & D_c = \{(t_1, t'', t_k) \in \mathbb{R}^k; t_1 > ct_k,\ t_k > 0\} \\ & \longleftrightarrow E_c = \{(s, t'', t_k) \in \mathbb{R}^k; s > c, t_k > 0\}, \end{aligned} \tag{2.6.14}$$

for any constant c.

Lemma 2.6.15. *Conjugation by R transforms the set of all $\mathcal{C}^\infty$ maps $\mathcal{E} : \overline{D_c} \longrightarrow \mathbb{R}^k$ which are diffeomorphisms onto their ranges and are equal to the identity outside a compact set and to all orders at $t_k = 0$ into the set of all $\mathcal{C}^\infty$ maps $\mathcal{E}' : \overline{E_c} \longrightarrow \mathbb{R}^k$ which are diffeomorphisms onto their ranges and are equal to the identity to infinite order at $t_k = 0$, in $t_k > C$ and $t_1 > C/t_k$ for some $C > 0$ and rapidly as $t_1 \to \infty$ in the sense that if $f = (\mathcal{E}')^* t_j - t_j,\ 1 < j \in k$ or $f = (\mathcal{E}')^* s - s$ then*

$$|D_s^\ell D_{t''}^\beta D_{t_k}^p (f)| \le C_{\ell,\beta,p,N} (1 + |s|)^{-N}, \quad \forall\ N. \tag{2.6.16}$$

Proof. The regions where the maps are the identity correspond under R so it is enough to check the regularity properties of

$$\mathcal{E}' = R \cdot \mathcal{E} \cdot R^{-1} \tag{2.6.17}$$

in terms of those of $\mathcal{E}$ and conversely. Write

$$\mathcal{E}(t_1, t'', t_k) = \big(t_1 + t_k T_1(t), t'' + T''(t), t_k(1 + T_k(t))\big) \tag{2.6.18}$$

where the $T_j(t)$ are $\mathcal{C}^\infty$ with compact supports and vanish to all orders at $t_k = 0$. Then

$$\mathcal{E}'(s, t'', t_k) = \left(s\frac{1 + T_1(st_k, t'', t_k)}{1 + T_k(st_k, t'', t_k)}, t'' + T''(st_k, t'', t_k), t_k + t_k T_k(st_k, t'', t_k)\right). \tag{2.6.19}$$

This shows that $\mathcal{E}'$ is $\mathcal{C}^\infty$ up to $t_k = 0$ and the rapid decrease in (2.6.16) is also easy to see, since as $s \to \infty$, in the support, $t_k < C/s \to 0$.

In the other direction we can suppose that $\mathcal{E}'$ is of the form

$$\mathcal{E}'(s, t'', t_k) = \big(s + G_1(s, t'', t_k), t'' + G''(s, t'', t_k), t_k(1 + G_k(s, t'', t_k))\big),$$

where the G_j vanish rapidly as $s \to \infty$ or $t_k \to 0$. Then

$$\begin{aligned}\mathcal{E}(t_1, t'', t_k) &= \\ \Big(\big(t_1 + t_k G_1(\frac{t_1}{t_k}, t'', t_k)\big)&\big(1 + G_k(\frac{t_1}{t_k}, t'', t_k)\big), \\ t'' + G''(\frac{t_1}{t_k}, t'', t_k)&, t_k\big(1 + G_k(\frac{t_1}{t_k}, t'', t_k)\big)\Big).\end{aligned} \tag{2.6.20}$$

Thus it is enough to show that if f is a $\mathcal{C}^\infty$ function satisfying the estimates in (2.6.16) and vanishing to all orders at $t_k = 0$ then $F(t_1/t_k, t'', t_k)$ is $\mathcal{C}^\infty$. Direct differentiation shows that it suffices to show the boundedness of $t_k^{-p} F(t)$ for all p but this follows immediately from (2.6.16). This proves the lemma.

Consider the composite map

$$\mathcal{T} = \mathcal{J} \cdot \mathcal{I}_0. \tag{2.6.21}$$

By the reduction to this point this map is equal to the corresponding model

$$\mathcal{T}_0 = \mathcal{J}_0 \cdot \mathcal{I}_0, \quad \mathcal{T}_0(t_1, t'', t_k) = (t_1 + t_k, t'', t_k), \tag{2.6.22}$$

to infinite order at $t_k = 0$ and outside a compact set. The singular change of variable (2.6.13) transforms $\mathcal{T}_0$ to

$$T_0 = R \cdot \mathcal{T}_0 \cdot R^{-1}, \quad T(s, t'', t_k) = (s + 1, t'', t_k),$$

just the shift considered in (2.5.2). Applying Lemma 2.6.15 the conjugate of the map $\mathcal{T}$,

$$T = R \cdot \mathcal{T} \cdot R^{-1}, \tag{2.6.23}$$

therefore satisfies the hypotheses of Proposition 2.5.3. By (2.5.23) the 'intertwining operator' W_+ conjugates T to the model operator T_0 and the estimates (2.5.5) show that Lemma 2.6.15 applies to W_+.

Thus, the original map $\mathcal{T}$ is conjugate to $\mathcal{T}_0$ under a map $Q_+ = R^{-1} \cdot W_+ \cdot R$, defined in $t_1 > 0$ and equal to the identity to all orders at $t_k = 0$:

$$\mathcal{T} = Q_+ \cdot \mathcal{T}_0 \cdot Q_+^{-1}. \tag{2.6.24}$$

By selecting the origin so that the base point is in $\{t_1 > 0\}$ this gives a local conjugation to the model in $t_k \geq 0$. Now define the full conjugation map by

$$Q(t) = \begin{cases} Q_+(t), & t_k \geq 0, \\ \mathcal{I}_0 \cdot Q_+ \cdot \mathcal{I}_0(t), & t_k \leq 0. \end{cases} \tag{2.6.25}$$

Since Q_+ is the identity to all orders at $t_k = 0$ this is a $\mathcal{C}^\infty$ local diffeomorphism. Moreover, directly from (2.6.25)

$$Q \cdot \mathcal{I}_0 = \mathcal{I}_0 \cdot Q. \tag{2.6.26}$$

Rewriting (2.6.24) in the form

$$\mathcal{J} \cdot \mathcal{I}_0 \cdot Q_+ = Q_+ \cdot \mathcal{J}_0 \cdot \mathcal{I}_0,$$

and using (2.6.25) shows that

$$\mathcal{J} \cdot Q = Q \cdot \mathcal{J}_0 \tag{2.6.27}$$

in $t_k > 0$. The same identity holds in $t_k < 0$ since

$$\mathcal{J} \cdot (\mathcal{J} \cdot Q_+) = (Q_+ \cdot \mathcal{J}_0) \cdot \mathcal{J}_0 = \mathcal{J} \cdot (Q \cdot \mathcal{J}_0).$$

Thus Q conjugates the involution $\mathcal{J}$ to normal form, whilst simultaneously preserving the form of $\mathcal{I}_0$. This completes the proof of the proposition.

Exercise 2.6.28: Assume the hypotheses of Proposition 2.3.10 and in addition let L be a $\mathcal{C}^\infty$ submanifold of G through m which is invariant under $\mathcal{I}$ but has tangent space $T_m L$ not invariant under the differential of $\mathcal{J}$ at m. Show that there are local coordinates in terms of which the involutions are reduced to normal form and

$$L = \{t_2 = \cdots = t_p = 0\} \text{ for some } p \leq k. \tag{2.6.29}$$

§2.7: Proof of Theorem 2.3.3

Theorem 2.3.3 is a straightforward consequence of Proposition 2.3.10.

Let $t_1, \dots, t_k$ be coordinates on G as described in Proposition 2.3.10, in which the two involutions take the reduced form (2.3.8), (2.3.9). Since $x_j = t_j,\ 1 \le j < k$ and $x_k = t_k^2$ are invariant under $\mathcal{I}_0$ they project to Y as $\mathcal{C}^\infty$ functions on the image of πY. These functions can be extended across S to give a coordinate system on Y. Similarly $y_j = t_j + \frac{1}{2}t_k,\ y_j = t_j,\ 1 < j < k$ and $y_k = t_k^2$ project to Z to $\mathcal{C}^\infty$ functions which may be extended across S to give a coordinate system in Z. Thus, the defining functions of G_0 certainly vanish on G and as they are independent, $G = G_0$ locally. This completes the proof of Theorem 2.3.3.

Exercise 2.7.1: Let G be a folding relation between two manifolds of the same dimension, show that in suitable local coordinates G is the union of the graphs of two maps:

$$\delta_\pm(y_1, y_2, \dots, y_k) = (y_1 \pm y_k^{1/2}, y_2, \dots, y_k). \tag{2.7.2}$$

§2.8: Parity equations

We consider the following functional equation for the two standard involutions (2.3.8), (2.3.9). Given $\mathcal{C}^\infty$ functions $c(t)$ and $g(t)$, both even under $\mathcal{I}_0$, does there exist a $\mathcal{C}^\infty$ function a(t) satisfying:

$$\frac{1}{2t_k}[\mathcal{I}_0^* a - a] = \frac{1}{2}c(t)[\mathcal{I}_0^* a + a] + g(t) \text{ and } \mathcal{J}_0^* a = a? \tag{2.8.1}$$

Proposition 2.8.2. *If $c(t)$ and $g(t)$ are $\mathcal{C}^\infty$ function defined near* 0 *and invariant under $\mathcal{I}_0$ there is a $\mathcal{C}^\infty$, $\mathcal{J}_0$-invariant, function $a(t)$ satisfying* (2.8.1) *and the normalization condition*

$$a(0) = 1. \tag{2.8.3}$$

The solution to this problem amounts to an extension of the formal power series and convergence arguments used above to prove the equivalence theorem.

Proof. First consider this at the formal level, that is in terms of formal power series at $t_k = 0$. If $g(t) = t_k^{2p} g_p(t')$, a term in the Taylor series of an $\mathcal{I}_0$-invariant function, we look for:

$$a(t) = a_p(t_1 + \frac{1}{2}t_k, t'')t_k^{2p}$$

Ignoring, in the first instance, higher order terms, equation (2.8.1) becomes:

$$\frac{\partial a_p(t')}{\partial t_1} = c(t', 0)a_p(t') + g_p(t'). \tag{2.8.4}$$

If this is satisfied then the error in (2.8.1) is an additional, $\mathcal{I}_0$-even, term $g(t)$ vanishing to order $2p$ at least at $t_k = 0$.

Proceeding inductively, this shows that there is a formal power series:

$$a(t) = \sum_0^\infty a_p(t_1 + \frac{1}{2}t_k, t'')t_k^{2p}, \tag{2.8.5}$$

satisfying (2.8.1) in the formal power series sense. From (2.8.4) each $a_p(t')$ can be freely prescribed at $t_1 = 0$, so certainly (2.8.3) can also be satisfied. Thus indeed the problem can always be solved in the sense of formal power series.

Using Borel's Lemma, as before, the series (2.8.5) can be summed to give a $\mathcal{J}_0$-even solution of (2.8.1) modulo terms vanishing to all orders at $t_k = 0$. By the linearity of the problem we can therefore assume that g vanishes to all orders at $S = \{t_k = 0\}$. We can also assume it vanishes in $t_1 < -1$, since the problem is local. Similarly we can suppose that $c(t)$ has support near 0. Suppose that the solution also vanishes identically in $t_1 < -1$.

Using the requirement that $\mathcal{J}_0^* a = a$ the equation (2.8.1) can be rewritten:

$$a = rB^*a + r'B^*g,\ B(t) = (t_1 - t_k, t'', t_k) = \mathcal{T}_0^{-1} \quad \text{in } t_k > 0, \tag{2.8.6}$$

$$r(t) = \frac{1 - t_k B^*c(t)}{1 + t_k B^*c(t)}, \quad r'(t) = \frac{-2t_k}{1 + t_k B^*c(t)}. \tag{2.8.7}$$

Iterating this identity gives

$$a = \sum_0^{M+1} R_m \cdot (B^{m+1})^* g + R_{(M)}(B^{M+1})^* a, \tag{2.8.8}$$

where the coefficients are

$$R_m(t) = \left[\prod_{j=0}^{m-1} (B^j)^* r\right] \cdot (B^m)^* r', R_{(M)} = \prod_{j=0}^{M} (S^j)^* r. \tag{2.8.9}$$

Near any point in $t_k \neq 0$ unless M satisfies

$$M \leq \frac{2}{t_k}, \tag{2.8.10}$$

the last term in (2.8.8) is zero by hypothesis on the support of a. Nor does increasing M change the sum, since $g = 0$ in $t_1 < -1$ also by hypothesis.

That is, the solution we seek, in $t_k > 0$ is simply

$$a = \sum_0^{M+1} R_m \cdot (B^{m+1})^* g. \tag{2.8.11}$$

This is a finite sum of at most $2/t_k$ non-zero terms at any point near 0. Naturally we need to show that this series converges absolutely with all derivatives.

Directly from (2.8.7)

$$|1-r(t)| \le c|t_k|. \tag{2.8.12}$$

Thus the coefficients R_m in (2.8.9) satisfy bounds:

$$|R_m(t)| < (1+c|t_k|)^M < C \ \text{ if (2.8.10) holds.} \tag{2.8.13}$$

Since this last estimate does indeed hold for all non-zero terms, the coefficients in (2.8.8) are uniformly bounded. Moreover g vanishes to all orders at $t_k = 0$, and vanishes rapidly as $t_1 \to -\infty$ (has indeed bounded support) so for any p there exist C_p such that:

$$|(B^j)^* g(t)| \le C_p |t_k^{2p}| (1 + |t_1 - 1 - jt_k|)^{-p} \le C_p |t_k|^p (1+j)^{-p}, \quad j < \frac{2}{t_k}.$$

Thus, uniformly in $t_k > 0$ near 0 the sum (2.8.8) converges rapidly, so defining a continuous function $a(t)$ which vanishes to all orders as $t_k \downarrow 0$.

Since each of the coefficients R_m is given as a product of up to M terms, differentiation of the series (2.8.8) a fixed number, N, times gives a sum of NM series, each of the same type as in (2.8.8) itself, except that a fixed number of the factors in (2.8.9), or g, is replaced by some derivative. These series clearly all satisfy the same type of estimates, uniformly for fixed N. Thus it follows that all derivatives of a, given by (2.8.8) are continuous and vanish rapidly at $t_k = 0$. Now, the invariance condition $\mathcal{J}_0^* a = a$ can be used to define a in $\{t_k < 0\}$, near 0 so that a vanishes in $\{t_1 < \mp 1, \pm t_k > 0\}$.

Clearly the function so constructed is $\mathcal{C}^\infty$ and satisfies (2.8.1), (2.7.2) under these additional assumptions on g. This completes the proof the proposition.

Exercise 2.8.14: Suppose that $b(t)$ and $c(t)$ are $\mathcal{I}_0$-invariant $\mathcal{C}^\infty$ functions near 0 with $c(0) \ne 0$. Show that Proposition 2.8.2 remains true if (2.8.1) is replaced by

$$\frac{1}{2} t_k b(t)[\mathcal{I}_0^* a - a] = \frac{1}{2} c(t)[\mathcal{I}_0^* a + a] + g(t). \tag{2.8.15}$$

§2.9: Homogeneous relations

In the next chapter symplectic versions of the results above are considered. First let us briefly note the homogeneous versions. Thus suppose that Y and Z are both conic manifolds, in the sense that they are $\mathbb{R}^+$-bundles over reduced manifolds Y', Z'. A relation, G, from Y to Z is said to be homogeneous if it intertwines the $\mathbb{R}^+$-actions on the two manifolds. This is the same as demanding that G be a conic submanifold of the product $Z \times Y$, where the $\mathbb{R}^+$-action is the diagonal one:

$$Z \times Y \ni (z, y) \longrightarrow (az, ay) \in Z \times Y, a \in \mathbb{R}^+.$$

Definition 2.9.1. *A homogeneous relation $G \subset Z \times Y$ between two conic manifolds Y and Z is said to be a homogeneous folding relation at $m \in Z \times Y$ if it is a folding relation at m and the generator of the $\mathbb{R}^+$-action, at m, is not in the range of the map:*

$$(2.9.2) \qquad (\mathrm{Id} - \mathcal{I}_* \cdot \mathcal{J}_*) \quad on \quad T_m G.$$

In fact from the normal form Theorem 2.3.3 above it is clear that for any point in the singular surface S, the range of the map (2.9.2) is 1-dimensional. Clearly it defines a $\mathcal{C}^\infty$ line bundle along S, which we shall denote:

$$(2.9.3) \qquad L \subset TS.$$

Referring back to the proof of Lemma 2.4.11 the importance of L can be seen, since the construction of the splitting of a function into $\mathcal{I}$- and $\mathcal{J}$-even parts uses integration along the integral curves of L. Thus the independence condition in Definition 2.9.1:

(2.9.4) the radial actions on S and L are linearly independent.

allows the splitting of homogeneous functions into homogeneous $\mathcal{I}$- and $\mathcal{J}$-even parts. In fact we can proceed more directly to prove:

Proposition 2.9.5. *If $G \subset Z \times Y$ is a homogeneous folding relation at $m = (\bar{z}, \bar{y})$ then there are homogeneous diffeomorphisms defined in open cones Γ_Y, Γ_Z around $\bar{z}$ and $\bar{y}$:*

$$\tau_Y : \Gamma_Y \longrightarrow \mathbb{R}^k \setminus 0, \quad \tau_Z : \Gamma_Z \longrightarrow \mathbb{R}^k \setminus 0,$$

taking $\bar{y}$ and $\bar{z}$ to $(0, 1, 0, ..., 0)$ and reducing G to the coordinate form:
(2.9.6)
$$G_h = (z_1, \dots, z_k, y_1, \dots, y_k) \in \mathbb{R}^{2k} \setminus 0; \ z_j = y_j, \ 1 < j \le k, y_k = y_2^{-1}(y_1 - z_1)^2.$$

Similarly there is a homogeneous version of Proposition 2.3.10 from which Proposition 2.9.5 follows easily:

Proposition 2.9.7. *Let G be a conic manifold on which there are two $\mathcal{C}^\infty$ homogeneous involutions $\mathcal{I}$ and $\mathcal{J}$ both fixing, point wise, a hypersurface S through m, neither the identity and with different differentials at m, then provided (2.9.4) holds there is a homogeneous diffeomorphism from a cone, M, around m:*

$$\tau : \Gamma \longrightarrow \mathbb{R}^k \setminus 0, \quad k = \dim(G),$$

taking m to $\bar{m} = (0, 1, 0, \dots, 0)$ and conjugating $\mathcal{I}$ and $\mathcal{J}$ to the normal forms $\mathcal{I}_0$, $\mathcal{J}_0$ in (2.3.8), (2.3.9).

Proof. Using Proposition 2.3.10 this normal form can be achieved by some, possibly non-homogeneous, diffeomorphism taking m to 0. Translation in t_2 commutes with (2.8.8), (2.8.9) so one can take the base point to be $\bar{m}$. Then L is spanned by $\partial/\partial t_1$. The independence condition (2.8.8) means that there is some $\mathcal{C}^\infty$ function $f(t_2, t_3, \ldots, t_{k-1})$ which satisfies

$$(rf)(\bar{m}) \neq 0, \quad f(\bar{m}) = 0.$$

Since f is invariant under $\mathcal{I}_0$ and $\mathcal{J}_0$, the surface $F = \{f = 0\}$ is also invariant. The functions $t'_j = t_j, k \neq 2$ on F, and $t'_2 = 1$, then extend uniquely to be homogeneous of degree one in some cone M around $\bar{m}$, i.e. as solutions of

$$rt'_j = t'_j, \quad j = 1, \ldots, k.$$

The homogeneity of $\mathcal{I}_0$ and $\mathcal{J}_0$ now shows that they commute with the change of coordinates from t to t', since they do so when restricted to F. This completes the proof of the Proposition.

The reader is left to make the obvious modifications to the proof above of Theorem 2.3.3 in order to give a proof of Proposition 2.9.5.

Exercise 2.9.8: Show that under the assumption of Proposition 2.8.2, or Exercise 2.8.14, if g is also homogeneous, of any real degree s and c is homogeneous of degree 0, then a homogeneous solution can be found to the parity equation.

Chapter 3: Folding Lagrangian relations, billiard ball maps and glancing hypersurfaces

In this chapter symplectic versions of the conjugation result of Chapter 2 are proved and applied, in particular to show the equivalence of glancing hypersurfaces in a symplectic manifold. The basic method used is the 'homotopy method' introduced by Moser ([Mo]) in a proof of Darboux' theorem. These results are in fact generalizations of Darboux' theorem since the general principle is to first make the reduction to normal form of the geometric invariants, using Chapter 2, and then subsequently to reduce the symplectic form by using diffeomorphisms which preserve the geometrical configurations.

In the first section we briefly recall some of the basic notions of symplectic geometry, including a proof of Darboux' theorem. Next we introduce the fundamental notion of a folding Lagrangian relation, in §3.2 and show in §3.3 that the conjugation of such a folding relation to normal form, as given by Theorem 2.3.3 can be carried out with a symplectic transformation. In §3.4 the conjugation theorem is reformulated in terms of billiard ball maps. The case of primary interest below concerns homogeneous problems and homogeneous symplectic equivalence is shown in §3.5. In §3.6 and §3.7 the conjugation theorem is applied to the reduction to normal form of pairs of glancing hypersurfaces.

§3.1: Darboux' theorem

A symplectic form, ω, on a $\mathcal{C}^\infty$ manifold, M, is a closed 2–form which is non–degenerate in the sense that at each $p \in M$,

$$v \in T_pM \text{ and } \omega_p(v,w) = 0 \ \forall\ w \in T_pM \Longrightarrow v = 0. \tag{3.1.1}$$

In particular this implies that M is of even dimension. A symplectic manifold is just a manifold with a specified symplectic form. The basic form of Darboux' theorem, which we extend below, is:

Theorem 3.1.2. *If ω is a closed non-degenerate 2-form near $p \in M$ then there are local coordinates $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ based at p in which*

$$\omega = \omega_D = \sum_{j=1}^{n} d\xi_j \wedge dx_j. \tag{3.1.3}$$

Proof. Linear algebra allows us to introduce a basis for T_pM, $v_1, \dots, v_n, w_1, \dots, w_n$ such that

$$\omega_p(v_i, v_j) = 0,\ \omega_p(\omega_i, v_j) = \delta_{ij},\ \omega_p(w_i, w_j) = 0, \quad \forall\ i,j = 1, \dots, n. \tag{3.1.4}$$

Such a basis can always be realized as the basis induced by local coordinates, x_i', ξ_i', i.e. $v_j = \partial/\partial x_j'$ and $w_j = \partial/\partial \xi_j'$. In these coordinates

$$\omega_p = \sum_{j=1}^{n} d\xi_j' \wedge dx_j', \tag{3.1.5}$$

since this is precisely the meaning of (3.1.4). Thus the difference

$$\mu = \omega - \omega_D, \tag{3.1.6}$$

where ω_D is the form in (3.1.3), is a closed 2–form vanishing at p.

Poincaré's lemma shows that

$$\mu = d\nu \text{ near } p, \tag{3.1.7}$$

where ν is a $\mathcal{C}^\infty$ 1–form which also vanishes at p. Indeed, ν can be obtained by radial integration (see the proof of Lemma 3.2.11 below). Set

$$\omega_t = (1-t)\omega + t\omega_D,$$

so $\omega_0 = \omega$, $\omega_1 = \omega_D$ and $d\omega_t/dt = \mu$. Since $\omega = \omega_D$ at p the form ω_t is non–degenerate (and hence symplectic) in a fixed neighbourhood of M_p of p for all $t \in [0,1]$. Thus we can invert the coefficient matrix of ω_t and solve

$$\omega_t(V(t), \cdot) = \nu, \tag{3.1.8}$$

with $V(t)$ a $\mathcal{C}^\infty$ vector field on M, near p, with coefficients depending smoothly on t; moreover $V(t)$ vanishes at p.

Integration of $V(t)$, as a parameter–dependent vector field, gives a local 1–parameter family of diffeomorphisms, χ_t, fixing p. The defining equation can be written in terms of the pull–back operation on functions as

$$\frac{d}{dt}\chi_t^* f = \chi_t^*[V(t)f], \quad \forall\, f \in \mathcal{C}^\infty(M_p), \quad \chi_0 = \operatorname{Id}. \tag{3.1.9}$$

From the formula for the Lie derivative,

$$\begin{aligned} \frac{d}{dt}\chi_t^*\omega_t &= \chi_t^*\mathcal{L}_{V(t)}\omega_t + \chi_t^*\frac{d\omega_t}{dt} \\ &= \chi_t^*\left\{(d\omega_t)(V(t),\cdot) + d(\omega_t(V(t),\cdot) - d\nu\right\} \\ &= d\chi_t^*[\omega_t(V(t),\cdot) - \nu] = 0. \end{aligned} \tag{3.1.10}$$

Since $\chi_0 = \mathrm{Id}$,

$$\chi_1^*\omega_1 = \chi_1^*\omega_D = \omega_0 = \omega. \tag{3.1.11}$$

This proves the theorem.

EXERCISE 3.1.12: Generalize this proof to show that if H is a smooth hypersurface through the point p then coordinates can be found in which (3.1.3) holds and in addition

$$H = \{x_n = 0\}. \tag{3.1.13}$$

As noted above we shall generalize Darboux' theorem further in this direction, i.e., find additional geometric structures which can be brought to normal form simultaneously with the symplectic form. First we recall a little more of the structure implicit in a symplectic manifold.

If f is a (real–valued) $\mathcal{C}^\infty$ function on a symplectic manifold the Hamilton vector field of f, written H_f, is defined by

$$\omega(H_f, \cdot) = -df. \tag{3.1.14}$$

The antisymmetry of ω means that $H_f f = 0$, i.e., H_f is always tangent to the surfaces $\{f = \text{const}\}$. If $df(p) \neq 0$ and $f(p) = 0$ then $H = \{f = 0\}$ is a smooth hypersurface through p. Any $\mathcal{C}^\infty$ function vanishing on H is of the form af with a $\mathcal{C}^\infty$. From (3.1.14) it follows that

$$H_{af} = aH_f + fH_a \text{ and hence } H_{af} = aH_f \text{ on } H. \tag{3.1.15}$$

Thus the integral curves of H_f (but not their parametrizations) are well defined on H; they are called Hamilton curves, or bicharacteristics, or leaves of the Hamilton foliation.

Let H and G be two $\mathcal{C}^\infty$ hypersurfaces in a symplectic manifold, M, and suppose $p \in H \cap G$. The two hypersurfaces are said to meet transversally at p if

$$T_pH + T_pG = T_pM. \tag{3.1.16}$$

This just means that they are not tangent, so there is a vector tangent to H at p which is not tangent to G. Using the symplectic structure we can impose a stronger condition, namely

(3.1.17) The Hamilton foliation of H is not tangent to G at p.

If this is the case we say that H and G meet symplectically transversally at p. Notice that the condition (3.1.17) is symmetric between H and G since if h and g are defining functions for the two hypersurfaces

$$\{h, g\} = H_h g = -H_g h = \omega(H_g, H_h). \tag{3.1.18}$$

This expression is the Poisson bracket of h and g.

EXERCISE 3.1.19: Let H and G be two $\mathcal{C}^\infty$ hypersurfaces in a symplectic manifold M and suppose $p \in H \cap G$ is a point at which they meet symplectically transversally. Show that there are local coordinates based at p in which (3.1.3) holds and

$$H = \{x_n = 0\}, \quad G = \{\xi_n = 0\}. \tag{3.1.20}$$

Notice that in any local coordinates in which (3.1.3) holds (Darboux coordinates) the Hamilton vector field of a function is given by

$$H_f = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j} \right) \tag{3.1.21}$$

and the Poisson bracket by

$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} \right). \tag{3.1.22}$$

EXERCISE 3.1.23: Prove the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0, \tag{3.1.24}$$

for all $\mathcal{C}^\infty$ functions f, g, h and deduce that

$$H_{\{f,g\}} = [H_f, H_g]. \tag{3.1.25}$$

We also briefly recall the classification of submanifolds of a symplectic manifold. A submanifold $Y \subset M$ is said to be *involutive* (in some quarters 'coisotropic') if for every function $f \in \mathcal{C}^\infty(M)$ which vanishes on Y the Hamilton vector field H_f is tangent to Y. We have already noted that every hypersurface is involutive. A submanifold is said to be *isotropic* if the symplectic form vanishes identically when pulled back to it. The dimension of an involutive submanifold is at least half the dimension of M and that of an isotropic submanifold is at most half the dimension of M. For manifolds of half the dimension of M the two concepts coincide and these, specially important, submanifolds are *Lagrangian*. In each of these three cases there is a local form of Darboux' theorem:

EXERCISE 3.1.26: Show that if Y is an involutive (resp. isotropic) submanifold through the point p in a symplectic manifold of dimension $2n$, then there are Darboux' coordinates based at p in terms of which

$$\begin{aligned} &Y = \{\xi_1 = \cdots = \xi_k = 0\}, \quad k \le n, \\ &(\text{resp. } Y = \{x_1 = \cdots = x_k = 0, \xi_1 = \cdots = \xi_n = 0\}). \end{aligned} \tag{3.1.27}$$

Exercise 3.1.28: Show that a submanifold is involutive or isotropic (or both i.e. Lagrangian) if and only if the symplectic form, when pulled back to it, has at every point minimal rank for submanifolds of that dimension.

Exercise 3.1.29: Consider a pair of smooth hypersurfaces, H and G, intersecting transversally at a point p in the symplectic manifold M, i.e., satisfying (3.1.16). Consider the opposite extreme to (3.1.17), where the Hamilton foliation of H is tangent to G at each point of the intersection. Show that this condition is symmetric between H and G and equivalent to demanding that the intersection $H \cap G$ be an involutive submanifold. Show that an analogue of Exercise 3.1.19 continues to hold, namely that there are Darboux coordinates in which

$$H = \{x_n = 0\}, \quad G = \{x_{n-1} = 0\}. \tag{3.1.30}$$

§3.2: Folding Lagrangian relations

Let Y and Z be two symplectic manifolds of the same dimension, $2n$, with symplectic forms ω_Y and ω_Z. A Lagrangian submanifold with respect to the difference symplectic structure, $\omega_Y - \omega_Z$, on the product:

$$\iota_\Lambda : \Lambda \hookrightarrow Z \times X, \quad \iota_\Lambda^* \sigma_Z = \iota_\Lambda^* \sigma_Y, \qquad \dim(\Lambda) = 2n, \tag{3.2.1}$$

is called a Lagrangian relation from Y to Z. Let $\lambda \in \Lambda$ be a point at which either projection:

$$\pi_X : \Lambda \longrightarrow X,\ X = Y \text{ or } Z. \tag{3.2.2}$$

has injective differential. Then Λ locally defines a map from $X = Y$ or Z to the other space $X' = Z$ or Y. This map must always be symplectic, because of (3.2.1), and hence locally invertible. Thus, if one of the projections has injective (or surjective) differential then both are local diffeomorphisms. That is if Λ is Lagrangian then

$$\text{the singular sets of the two projections are the same, } S \subset \Lambda. \tag{3.2.3}$$

If Λ is a Lagrangian relation then it is folding at $\lambda \in \Lambda$ if:

$$\text{the two projections (3.2.2) have Whitney folds at } \lambda. \tag{3.2.4}$$

If one projection is singular then, by (3.2.3) so is the other. However it is possible to find a Lagrangian relation with one projection having a fold and with the other more singular, having for example a Whitney cusp (see [Me5]).

A folding Lagrangian relation is certainly a folding relation in the sense of Definition 2.3.2, since the folds have the same singular set S. Following the discussion of maps with fold singularities in Chapter 2, if Λ is a folding Lagrangian relation then there are two involutions $\mathcal{I}$ and $\mathcal{J}$ defined on Λ, near λ.

Lemma 3.2.5. *The two involutions defined on Λ near λ are symplectic in the sense that if*

$$\sigma' = \pi_Y^* \sigma_Y = \pi_Z^* \sigma_Z, \tag{3.2.6}$$

then

$$\mathcal{I}^* \sigma' = \mathcal{J}^* \sigma' = \sigma'. \tag{3.2.7}$$

Moreover, the differentials at λ of $\mathcal{I}$ and $\mathcal{J}$ are different.

Proof. The identity (3.2.6), implicit in the definition of the form σ', follows immediately from (3.2.1). The symplectic condition (3.2.7) follows trivially away from S where $\mathcal{I}$ (or $\mathcal{J}$) is locally the projection followed by the appropriate local inverse. By continuity it therefore holds everywhere. The independence of the differentials follows from the remarks after the statement of Theorem 2.3.3.

We now pass to the symplectic version of Theorem 2.3.3:

Theorem 3.2.8. *Let Y and Z be symplectic manifolds of the same dimension, $2n$, with symplectic forms σ_Y, σ_Z and let $\Lambda \subset Z \times Y$ be a Lagrangian relation folding at $\lambda \in \Lambda$. Then there are local Darboux coordinates (x, ξ) in Y and (y, η) in Z, i.e. coordinates in which the symplectic forms are reduced to the Darboux form:*

$$\sigma_0 = \sum_{j=1}^{n} d\xi_j \wedge dx_j, \tag{3.2.9}$$

in which Λ is given by:

$$\Lambda_0 = \{x_j = y_j,\ 1 \le j < n,\ \xi = \eta, (x_n - y_n)^2 = \xi_n\}. \tag{3.2.10}$$

The main technical step in the proof of this result is a suitable invariant Poincaré Lemma.

Lemma 3.2.11. *Let μ be a $\mathcal{C}^\infty$ p-form in a neighborhood of $0 \in \mathbb{R}^k$ invariant under the two maps $\mathcal{I}_0$, $\mathcal{J}_0$ in (2.3.8), (2.3.9). Then if μ is closed there exists a $\mathcal{C}^\infty$ (p-1)-form, τ, invariant under $\mathcal{I}_0$ and $\mathcal{J}_0$ with $d\tau = \mu$; if μ vanishes when pulled-back (respectively restricted) to $S = \{t_k = 0\}$ then τ can be chosen to vanish when pulled-back (respectively restricted) to S too.*

Proof. The standard proof of the Poincaré Lemma suffices to give this result. Thus in the coordinates z, contract μ with the radial vector field, giving

$$\nu = \mu(z \cdot \frac{\partial}{\partial z}, \cdot) = \sum_I \nu_I \, dz^I,$$

and then integrate by parts radially:

$$\tau = \sum_I \int_0^1 \nu_I(rz) \frac{dr}{r} \, dz^I.$$

The radial vector field is invariant under both linear maps so τ is invariant and satisfies $d\tau = \mu$ as may be checked by direct computation. Since $\{t_k = 0\}$ is radial, if the coefficients of μ vanish there so do those of F, proving the Lemma.

Exercise 3.2.12: Consider the set

$$X = \{(\omega, w) \in \mathbb{S}^{n-1} \times \mathbb{R}^n; \omega \cdot w = 0\}. \tag{3.2.13}$$

The set of oriented rays in $\mathbb{R}^n$ can be identified with X, by the formula

$$x = w + t\omega, \quad t \in \mathbb{R}. \tag{3.2.14}$$

The cotangent space of the sphere $\iota : \mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n$ can also be identified with X by using the Euclidean structure to identify a 1–form by

$$\alpha = \iota^*(w \cdot dy), \quad \text{at } \omega \in \mathbb{S}^{n-1}. \tag{3.2.15}$$

Then the 'ray relation' (for the sphere) is

(3.2.16)

$$G \subset X \times X, \; G = \{(\omega, w, \theta, u) \in X \times X; u = \omega - (\omega \cdot \theta)\theta, \; \exists\, t \in \mathbb{R}, \text{ s.t. } \theta = w + t\omega\}.$$

Show that this is a folding Lagrangian relation with respect to the symplectic structure on X from the identification with $T^*\mathbb{S}^{n-1}$.

Figure 3.1

§3.3: Proof of Theorem 3.2.8

Notice that if Λ takes the form (3.2.10) then in terms of the coordinates in Λ,

$$z_j = x_j,\ 1 \le j \le n,\ \zeta_p = \xi_p,\ 1 \le p < n,\ \zeta_n = x_n - y_n, \tag{3.3.1}$$

$$\mathcal{I}_s(z,\zeta) = (z, \zeta', -\zeta_n), \tag{3.3.2}$$

$$\mathcal{J}_s(z,\zeta) = (z', z_n - 2\zeta_n, \zeta', -\zeta_n). \tag{3.3.3}$$

where $\zeta' = (\zeta_1, \dots, \zeta_{n-1})$, etc. Thus the involutions take the form (2.3.8), (2.3.9) apart from different notation for the coordinates. In any case since Λ_0 in (3.2.10) is the same as G_0 in (2.3.4), apart from such a trivial renumbering of the coordinates, Λ can be reduced to Λ_0 using Theorem 2.3.3. The remaining task is to reduce simultaneously the symplectic forms on Y and Z to the Darboux form (3.2.9). Consider first the symplectic form σ_Y on Y. Restricted to the fold, $\xi_n = 0$, this is a closed 2-form of maximal rank, $2n-2$. Let $L \subset TS$ be the line bundle defined as the range of the map (2.9.2).

Lemma 3.3.4. *If Λ_0 in* (3.2.10) *is Lagrangian with respect to $\sigma_Z - \sigma_Y$, then L, transferred to $S \subset X$ ($X = Y$ or Z), is the kernel of σ_X restricted to S, i.e., the Hamilton foliation.*

Proof. By definition if $\mathcal{F} = \mathcal{I} \cdot \mathcal{J}$ then the fiber at $p \in S \subset Y$ is

$$L_p = (\pi_Y)_* \cdot (Id - \mathcal{F}_*)[T_z],\ m \in \Lambda,\ \pi_Y(m) = p,$$

Thus it is enough to show that on Λ, the line bundle is just

$$L \text{ is the kernel of } \sigma' \text{ pulled back to } S. \tag{3.3.5}$$

Let L' be the kernel of σ' on S. This is clearly one dimensional. Using Lemma 3.2.5 and the fact that $\mathcal{F}$ is the identity on S, if $v = w - \mathcal{F}_* w$ is in the range of $Id - \mathcal{F}_*$ then,

$$\sigma'(v, w') = \sigma'(w, w') - \sigma'(\mathcal{F}_* w, \mathcal{F}_* w') = 0,\ \forall\ w' \in T_z S.$$

Thus $L \subset L'$ so these bundles are equal and the Lemma is proved.

Now computation directly using (3.3.2) and (3.3.3) shows that for $\Lambda = \Lambda_0$, L on $S \subset Y$ is spanned by $\partial/\partial x_n$, independently of the symplectic form. Then Lemma 3.3.4 shows that the symplectic form σ_Y projects to a symplectic form on the local quotient manifold $S/\operatorname{Span}\{\partial/\partial x_n\}$. Using the standard form of Darboux' theorem, the restriction of σ_Y to $S = \{\xi_n = 0\}$ can be reduced by a change of variables amongst the $(x_1, \dots, x_{n-1}, \xi_1, \dots, \xi_{n-1})$ to the form coming from (3.2.9). This gives:

$$\sigma_Y = \alpha \wedge d\xi_n + \sum_{j=1}^{n-1} dx_j \wedge d\xi_j + \beta,$$

where β is a 2-form vanishing at S and

$$\alpha = \sum_{j=1}^{n} (a_j\, dx_j + b_j\, d\xi_j) \tag{3.3.6}$$

is a $\mathcal{C}^\infty$ 1-form with $a_n(x, \xi') \neq 0$, by the assumed non-degeneracy of σ_Y. Since the change of variables can be chosen to be of the form

$$(x', x_n, \xi', \xi_n) \mapsto (X'(x', \xi'), x_n, \Xi'(x', \xi'), \xi_n),$$

if carried out in both factors $Y, Z \subset \mathbb{R}^{2n}$ this leaves the form of Λ_0 (and hence the two involutions) the same. Denoting the new variables again by (x, ξ), (y, η) and changing the sign of x_n and ξ_n if necessary it can be assumed that $a_n > 0$. The further change of variables

$$(x, \xi) \mapsto (x', x_n a_n(x', 0, \xi'), \xi', \xi_n a_n^{1/2}(x', 0, \xi')),$$

again in both manifolds Y, Z leaves Λ_0 unchanged and ensures that

$$\sigma_Y = \sigma_0 \text{ on } S \text{ and at } 0. \tag{3.3.7}$$

In view of (3.3.7) the 2-form $\mu = \sigma_Y - \sigma_0$, is closed and vanishes on S and at 0. Thus the lift to Λ_0:$\mu' = \pi_Y^*(\mu)$ is a closed 2-form vanishing on S and invariant under $\mathcal{J}_s$ and $\mathcal{I}_s$. Using Lemma 3.2.11, there is an $\mathcal{I}_s$-, $\mathcal{J}_s$-invariant 1-form, τ', on Λ_0 vanishing at the base point and such that

$$d\tau' = \mu'. \tag{3.3.8}$$

As an $\mathcal{I}_s$–invariant form τ' descends to Y as a $\mathcal{C}^\infty$ 1-form defined on the half space $\xi_n \geq 0$. Extending this to a $\mathcal{C}^\infty$ 1-form τ_1 gives

$$\beta = d\tau_1 - \mu = 0 \text{ in } \xi_n > 0.$$

Thus β is a closed 2-form vanishing in $\xi_n > 0$. Using the standard proof of the Poincaré Lemma it follows that $\beta = d\tau_2$ for a $\mathcal{C}^\infty$ 1-form τ_2 also vanishing in $\{\xi_n > 0\}$. Thus, $\tau = \tau_1 - \tau_2$ satisfies

$$d\tau = \mu, \pi_Y^* \tau = \tau'. \tag{3.3.9}$$

Following the homotopy proof of Darboux' theorem define a $\mathcal{C}^\infty$ vector field $V(t)$ by

$$\sigma_{Y,t}(V(t),\cdot) = -\tau = -d_t\sigma_{Y,t}, \tag{3.3.10}$$

where $\sigma_{Y,t} = (1-t)\sigma_0 + t\sigma_Y,\ t \in [0,1]$. From (3.3.7), $\sigma_{Y,t}$ consists of symplectic forms near 0, so $V(t)$ is well-defined by (3.3.10). Now, F is normal to $S = \{\xi_n = 0\}$, from which it follows that $V(t)$ is tangent to S, and is therefore the push-forward of a $\mathcal{C}^\infty$ vector field, $V'(t)$, on Λ_0; moreover $V(t)$ vanishes at 0. Lifting (3.3.10) to Λ_0 and recalling the $\mathcal{J}_s$–invariance of τ' it follows that $V'(t)$ is invariant under both involutions. Thus $V'(t)$ pushes forward to a $\mathcal{C}^\infty$ 1-parameter family of vector fields, $W(t)$, on Z, in $\eta_n \geq 0$. Clearly $W(t)$ satisfies the same conditions as $V(t)$:

$$\sigma_{Z,t}(W(t),\cdot) = -\tau_Z,\ \pi_Z^*\tau_Z = \tau',\ (\pi_Z)_*V'(t) = W(t), \tag{3.3.11}$$

and extends across $S = \{\eta_n = 0\}$ uniquely with these properties. Let $P_{X,t}, X = Y, Z$, be the 1-parameter families of local diffeomorphisms generated by $V(t)$ and $W(t)$; since $V(t)$ and $W(t)$ vanish at 0 they preserve the base point. From (3.3.10), (3.3.11) it follows that

$$P_{X,t}^*\sigma_{X,t} = \sigma_0 \text{ for } t \in [0,1],$$

since

$$\frac{d}{dt}P_{Y,t}^*\sigma_{Y,t} = P_{Y,t}^*\mathcal{L}_{V(t)}\sigma_{Y,t} + P_{Y,t}^*\frac{d}{dt}\sigma_{Y,t}.$$

The Lie derivative is

$$\mathcal{L}_V\alpha = d[\alpha(V)] + d\alpha(V,\cdot),$$

so from (3.3.10), (3.3.9), $d(P_{Y,t}^*\sigma_{Y,t})/dt = 0$. Thus $P_{X,t}$, as a family of local diffeomorphisms on the corresponding $X = Y, Z$, reduces the symplectic form to (3.2.9), as desired. The product family $P_{Z,t} \times P_{Y,t}$ of diffeomorphisms is generated by $W(t) \oplus V(t)$, as a vector field on $Z \times Y$. Since this projects to $V(t)$ and $W(t)$ under π_Y and π_Z respectively it must be equal to $V'(t)$ at Λ_0; hence tangent to Λ_0 and invariant under both involutions. Thus it follows that Λ_0 is invariant under $P_Z \times P_Y$. This completes the proof of Theorem 3.2.8.

§3.4: Billiard ball maps

As a relation Λ_0 in (3.2.10) defines two maps from Y to Z, in the half-space $\{\xi_n \geq 0\}$:

$$\delta_0^{\pm}(x,\xi) = (x', x_n \pm 2\xi_n^{1/2}, \xi) = (y,\eta). \tag{3.4.1}$$

preserving the Darboux form (3.2.9). Of course correspondingly a general folding Lagrangian relation between Y and Z defines locally near any point of the fold two maps:

$$\delta^{\pm} = \pi_Z \cdot \pi_Y^{\pm} : \pi_Y(\Lambda) \longrightarrow \pi_Z(\Lambda), \tag{3.4.2}$$

where $\pi_Y^{\pm} : \pi_Y(\Lambda) \longrightarrow \Lambda$ are the two local continuous inverses to π_Y on Λ. The distinction between the two inverses corresponds to a choice of normal direction to S in Λ. Thus, Theorem 3.2.8 shows that local coordinates can be introduced in Y and Z with respect to which the symplectic forms are reduced to (3.2.9) and the maps to (3.4.1). In fact it is clear that this is an assertion equivalent to Theorem 3.2.8.

Of particular interest is the case where $Z = Y$ so that Λ is a relation from Y to itself.

Definition 3.4.3. *A Lagrangian relation, Λ, from the symplectic manifold Y to itself is said to be a billiard ball map near $y \in Y$ if Λ is folding near $\lambda = (\bar{y}, \bar{y})$ and in addition, near λ, Λ is its own inverse (or transpose).*

The singular symplectic maps (3.4.2) will also be called billiard ball maps if they arise from a Lagrangian relation as in Definition 3.25. We have the following consequence of Theorem 3.2.8.

Corollary 3.4.4. *If Λ is a billiard ball map from the symplectic manifold Y to itself, near $y \in Y$, then coordinates (x, ξ) can be introduced in Y, the same in the two factors, with respect to which $\bar{y} = 0$, $\delta^{\pm}$ takes the form* (3.4.1) *and the symplectic form becomes* (3.2.9).

Proof. Let $\mathcal{K} : Y \times Y \longrightarrow Y \times Y$ be the involution $\mathcal{K}(z, z') = (z', z)$. By assumption $\mathcal{K}(\lambda) = \lambda$ and the additional hypothesis on Λ is just:

$$\mathcal{K} : \Lambda \longrightarrow \Lambda. \tag{3.4.5}$$

Now, if σ' is the pull back to Λ of the symplectic form on Y by either of the projections then $\mathcal{K}^*\sigma' = \sigma'$, and hence as the manifold of degeneracy of σ', S must be mapped to itself by $\mathcal{K}$. Thus, $\mathcal{K}$ defines an involution on S, preserving the restriction of the symplectic form. Projecting to the symplectic manifold, S (identified with its image in Y) modulo its local foliation, $\mathcal{K}$ must give a symplectic involution with a fixed point. Such a map is necessarily the identity nearby. It follows that on S $\mathcal{K}$ must be a translation, in Darboux coordinates in which S is given by $\xi_n = 0, \mathcal{K}(x, \xi') = (x' + h(x', \xi'), \xi')$. Again, as an involution fixing a point,

$$\mathcal{K} \text{ is the identity on } S \text{ near } y. \tag{3.4.6}$$

Reviewing the proof of Theorem 3.2.8 it is apparent that as a consequence of (3.4.6) one can take $W(t) = V(t)$ in (3.3.11). Thus the coordinate transformation to obtain the reduced form (3.2.9), (3.2.10) and (3.4.1) can be the same in the two copies of Y, proving the Corollary.

Since the definition of a billiard ball map involves considerable inherent symmetry, it is worthwhile noting that there are indeed many examples:

Corollary 3.4.7. *If* $\Lambda \subset Z \times Y$ *is a folding Lagrangian relation at* λ *then the composite relation:*

$$\Lambda^t \cdot \Lambda = \Sigma \cup \mathrm{Id} \subset Y \times Y, \tag{3.4.8}$$

where Id *is the identity relation and* Σ *is a billiard ball map.*

Proof. This is obvious from the normal form theorems, and is also easily proved directly.

Remark 3.4.9: Suppose $\Lambda \subset Z \times Y$ is a folding Lagrangian relation. Suppose Σ is defined by (3.4.8) and (y, η) are Darboux coordinates in Y with respect to which the billiard ball map is given by (3.4.1). Then there are Darboux coordinates (x, ξ) in Z with respect to which Λ becomes Λ_0 as in (3.2.10). Indeed if $\gamma^\pm$ are the two continuous inverses to π_Y and $\mathcal{J}$ is the involution associated to π_Z then

$$\delta^\pm = \pi_Y \cdot \mathcal{J} \cdot \gamma^\pm.$$

Now the (y, η) lift to Λ as $\mathcal{C}^\infty$ functions $Y = \pi_Y^* y,\ H_k = \pi_Y^* \eta_k,\ k < n,\ H_n^2 = \pi_Y^* \eta_n$, with (Y, H) forming a coordinate system. Then (3.4.1) shows that with respect to these coordinates, $\mathcal{J}$ takes the reduced form (3.3.3). Following the last part of the proof of Theorem 3.2.8 now gives suitable coordinates in Z.

Exercise 3.4.10: Show how Exercise 3.2.12 can be generalized for an arbitrary closed compact embedded hypersurface in $\mathbb{R}^n$ with positive principal curvatures. Show further that if G is defined in analogy to (3.2.16) then $G \cdot G^t$ is the 'scattering relation' for the obstacle, i.e., relates two oriented rays if and only if they meet at a point on the surface, lie in a plane containing the normal to the surface and make equal (but opposite) angles with the normal in that plane.

Figure 3.2

§3.5: Homogeneity

The billiard ball maps which are considered below have an additional property which should be preserved under the reduction to normal form, namely homogeneity. Recall that the cotangent bundle T^*X is a symplectic manifold where the symplectic form is

$$\omega = d\alpha.$$

Here, α is the canonical 1-form. A point in T^*X is by definition a 1-form γ_x, at some point $x \in X$. Then by definition

$$\alpha = \pi^*\gamma_x \text{ at } (x, \gamma_x) \in T^*X,$$

with $\pi : T^*X \longrightarrow X$ the natural projection. If $x = (x_1, \dots, x_n)$ are local coordinates in X then setting

$$\gamma_x = \sum_{j=1}^{n} \xi_j \, dx_j$$

gives local 'canonical coordinates' (x, ξ) in T^*X. With respect to these local coordinates, $\alpha = \xi \cdot dx$ and ω is given by (3.2.9). The manifold $T^*X \setminus 0$, where 0 represents the zero section of T^*X, is therefore a symplectic manifold with a locally free $\mathbb{R}^+$–action, from multiplication in the fibers. The infinitesimal generator $r = \xi \cdot \partial/\partial\xi$ of this action satisfies

$$\omega(r, \cdot) = \alpha, \text{ i.e.,} \mathcal{L}_r\omega = r\rfloor d\omega + d(r\rfloor\omega) = d\alpha = \omega. \tag{3.5.1}$$

Any symplectic manifold with a locally free $\mathbb{R}^+$-action satisfying the second of these conditions is called a homogeneous symplectic manifold. Since there is a homogeneous version of Darboux' theorem, showing that a homogeneous symplectic form is locally diffeomorphic to any $T^*X \setminus 0$, of the same dimension, by a symplectic map intertwining the $\mathbb{R}^+$-actions, there is little point in considering, for local questions, the general case of a homogeneous symplectic manifold. A symplectic map, defined in some open cone ($\mathbb{R}^+$-invariant set) M,

$$\chi : \Gamma \longrightarrow T^*X' \setminus 0, \quad \Gamma \subset T^*X \setminus 0,$$

is called a canonical transformation if it intertwines the $\mathbb{R}^+$-actions. A homogeneous Lagrangian relation, for the difference symplectic structure is therefore called a canonical relation.

Theorem 3.5.2. *Suppose that $\Lambda \subset (T^*X \setminus 0) \times (T^*X \setminus 0)$ is a billiard ball map near $(\bar{m}, \bar{m})$ which is a canonical relation, i.e., is invariant under the diagonal $\mathbb{R}^+$-action on $(T^*X \setminus 0) \times (T^*X \setminus 0)$. Then, provided the $\mathbb{R}^+$-action is not tangent to the line bundle L on the fold S, Λ can be reduced to the normal form*

$$\begin{aligned}\Lambda_h = \{(y, \eta, x, \xi) \in T^*\mathbb{R}^n \setminus 0 \times T^*\mathbb{R}^n \setminus 0; x'' = (x_2, \dots, x_{n-1}) = y'', \xi = \eta,\\ (x_1 - y_1) = -\frac{1}{3}(x_n - y_n)^3, \xi_1(x_n - y_n)^2 = \xi_n\},\end{aligned} \tag{3.5.3}$$

by a canonical transformation from $T^*X \setminus 0$ to $T^*\mathbb{R}^n \setminus 0$, the same in each factor and taking m to $(0,\xi_0)$, $\xi_0 = (1,0,\dots,0)$.

Proof. This result is essentially just Corollary 3.4.4 with the additional requirement that homogeneity be preserved. Notice however that (3.5.3) gives a different normal form, in place of (3.4.1) for the billiard ball maps $\delta^{\pm}$, namely:

$$\delta_h^{\pm}(x,\xi) = \Big(x_1 \pm \frac{2}{3}\Big(\frac{\xi_n}{\xi_1}\Big)^{3/2}, x'', x \mp 2\Big(\frac{\xi_n}{\xi_1}\Big)^{1/2}, \xi\Big). \tag{3.5.4}$$

This is necessarily more complicated than (3.4.1), which is not homogeneous.

By using the symplectic reduction result, Corollary 3.4.4, the homogeneous billiard ball map can be reduced to the symplectic normal form (3.2.9), (3.2.10) by symplectic transformation from $T^*X \setminus 0$ to $T^*\mathbb{R}^n \setminus 0$, taking $\bar{m}$ to 0. Let r' be the image under this transformation of the radial vector field. By hypothesis r', whilst tangent to the fold, is not tangent to the Hamilton foliation of S. Thus at 0,

$$r'(0) = a\frac{\partial}{\partial x_n} + \rho,$$

where ρ is in the span of the $\partial/\partial x'$, $\partial/\partial \xi'$ and is not zero. By a symplectic transformation amongst the (x',ξ'), leaving x_n and ξ_n unchanged, ρ can be reduced to $\partial/\partial\xi_n$, for example. Thus it can be arranged that

$$r'(0) = a\frac{\partial}{\partial x_n} + \frac{\partial}{\partial \xi_n}, \tag{3.5.5}$$

by carrying out a symplectic transformation in both factors which does not affect the normal form for Λ. Let χ_1 be the symplectic transformation reducing Λ to Λ_0, and giving (3.5.5) for the new local $\mathbb{R}^+$-action. Let χ_2 be a transformation of this type which reduces the homogeneous model Λ_h to the same normal form Λ_0 and takes the $\mathbb{R}^+$-action to r'' where (3.5.5) also holds for $r''(0)$. Then the difference $V = r' - r''$ is a $\mathcal{C}^\infty$ vector field vanishing at 0. In fact it is Hamiltonian, since from the homogeneity conditions,

$$\omega_1(r',\cdot) = \alpha', \quad \omega_1(r'',\cdot) = \alpha'',$$

both satisfy $d\alpha = \omega_1$. Thus,

$$\omega_1(r'-r'',\cdot) = d\phi, \quad \phi(0) = 0,$$

fixes a unique $\mathcal{C}^\infty$ function ϕ near 0. Consider the 1-parameter family of vector fields $r_t = (1-t)r' + tr''$, $t \in [0,1]$. Because of (3.5.5) these are all transversal to $\{\xi_1 = 0\}$ near 0. Thus the family of differential equations:

$$r_t\psi_t - \psi_t = -\phi, \quad \psi = 0 \text{ on } \{\xi_1 = 0\} \tag{3.5.6}$$

has a unique family of solutions, $\mathcal{C}^\infty$ near 0 and in the parameter t.

By assumption r' and r'' are both tangent to S, hence so is r_t. It follows that ϕ is constant along the leaves of the foliation of S. Moreover, if V $(= \partial/\partial x_n)$ is the Hamilton vector field of a defining function for S,

$$\omega_1(V, \cdot) = d\xi_n,$$

then applying the Lie derivative of r_t to this defining equation shows that:

$$\omega_1([V, r_t], \cdot) + \omega_1(V, \cdot) = \gamma\, d\xi_n, \text{ at } S,$$

i.e., $[V, r_t] = \gamma' V$ on S is tangent to the Hamilton foliation. Applying V to the equation (3.5.6) it follows that ψ is constant on the leaves of the Hamilton foliation of S, since the initial surface $\{\xi_1 = 0\}$ on which ψ vanishes is fibred by these leaves. This means that the Hamilton vector field of ψ_t, $W(t)$,

$$\omega_1(W(t), \cdot) = d\psi_t,$$

is tangent to S. Lifting (3.5.6) to Λ_0 it is clear that ϕ must be invariant under both involutions, as is r_t. Thus ψ must also be invariant under these involutions as its initial data is invariant. Lifting the defining equation for $W(t)$ to Λ_0, which is possible since it is tangent to S, it follows that it too is invariant under both involutions. Thus the 1-parameter family of symplectic diffeomorphisms, P_t, generated by $W(t)$, if carried out in both factors leaves Λ_0 invariant. Now the original choice (3.5.6) of ψ_t ensures that:

$$d_t P_t^* \alpha_t = 0,$$

if $\alpha_t = (1 - t)\alpha' + t\alpha''$, since $d\alpha_t = \omega_1$ and

$$\omega_1(W(t), \cdot) + d(\omega_1(W(t), \cdot) + (\alpha' - \alpha'') = d[\psi_t - r_t\psi_t + \phi] = 0.$$

Thus if $\chi = P_1$ then $\chi^*\alpha'' = \alpha'$ and therefore $\chi_* r' = r''$.

Combining these three transformations, first from the general homogeneous billiard ball map to the symplectic model then changing r' to r'' and then mapping to the homogeneous model gives a local reduction to the homogeneous model which intertwines the local $\mathbb{R}^+$-actions. This composite symplectic transformation then extends to give the desired canonical transformation in a conic neighborhood of the base points. This completes the proof of Theorem 3.5.2.

Although we have chosen to prove a conjugation result for a homogeneous billiard ball map essentially the same proof applies to the general case of a homogeneous folding canonical relation. For later use we record this homogeneous version of Theorem 3.2.8.

Theorem 3.5.7. *Let C be a folding canonical relation from T^*X to T^*Y near λ_0, with X and Y manifolds of dimension n. That is, C is a canonical relation which is folding and for which the $\mathbb{R}^+$–action is not tangent to the fold. Then there are local canonical transformations ξ_X and χ_Y from $T^*\mathbb{R}^n$ to T^*X and T^*Y respectively, taking $\pi_X(\lambda_0)$ and $\pi_Y(\lambda_0)$ to $(0,\xi_0)$, $\xi_0=(1,0,\dots,0)$ and such that near λ_0*

$$C=\chi_Y\cdot\Lambda_h\cdot\chi_X^{-1}, \tag{3.5.8}$$

with the model given by (3.5.3).

Exercise 3.5.9: Assume the same geometric setup as in Exercise 3.4.10 but consider in addition a time variable, t and its dual variable τ. Then

$$Y=(T^*\mathbb{R}_t\times X)\setminus 0\cong(T^*\mathbb{R}\times T^*S^{n-1})\setminus 0 \tag{3.5.10}$$

is a homogeneous symplectic manifold. On Y the 'time-augmented' ray relation is fixed by

$$\begin{gathered}G=\{(t,\tau,\omega,w,t',\tau',\theta,u)\in Y\times Y;w=\tau\hat{w},\ u=\tau'\hat{u},\\ t'=t,\ \tau=\tau',\ \theta=t\omega+w,\ u=\omega-(\omega\cdot\theta)\theta\}.\end{gathered} \tag{3.5.11}$$

Show that this is a homogeneous folding canonical relation. Check that $G\cdot G^t$ is the union of the graphs of two continuous maps defined in the region $|w|\le 1$, one being the identity and the other, γ, the 'time-augmented scattering relation' in the sense that $\gamma(t,\tau,\omega,\hat{w})=(t',\tau,\theta,\hat{u})$ means that (ω,w,θ,u) is in the scattering relation discussed in Exercise 3.4.10 and $t'-t$ is the 'time delay' (see [Gu1])

$$\begin{gathered}t'-t=\text{signed dist}(P,\{\tau\cdot y=0\})-\text{signed dist}(P,\{\omega\cdot x=0\}),\\ P\text{ the point of reflection.}\end{gathered} \tag{3.5.12}$$

§3.6: Glancing hypersurfaces

The main application of these conjugation theorems is to reduce to normal form the geometry of boundary problems with simply glancing rays. Thus, let Y be a symplectic manifold with symplectic form σ. In Y consider two hypersurfaces passing through a point m:

$$P\hookrightarrow Y,\quad Q\hookrightarrow Y,\quad m\in P\pitchfork Q=K.$$

The transversality condition on the intersection, stated symbolically as $P\pitchfork Q$, is just:

$$T_mP+T_mQ=T_mY. \tag{3.6.1}$$

Let $p, q \in C^\infty(Y)$ be defining functions, near m, for P and Q. Recall that the Hamilton vector field of p, defined by:

$$\sigma(H_p, \cdot) = dp,$$

spans the bicharacteristic foliation of P, the kernel in TP of $\iota_P^*\sigma$. Thus the condition

$$H_p q = -H_q p = \{p, q\} \neq 0, \text{ at } m \tag{3.6.2}$$

is precisely the condition (2.2.8) for K as a hypersurface in P with respect to its bicharacteristic foliation or equivalently for K as a hypersurface in Q with respect to its bicharacteristic foliation.

Definition 3.6.3. *Two hypersurfaces P, Q in a symplectic manifold (Y, σ) are said to be glancing at m if they meet transversally at m and in terms of (any) local defining functions:*

$$\{p, q\} = 0 \text{ at } m, \tag{3.6.4}$$

$$\{p, \{p, q\}\} \neq 0, \ \{q, \{q, p\}\} \neq 0 \text{ at } m. \tag{3.6.5}$$

If $Y = T^*X$ then P, Q are homogeneous glancing hypersurfaces at $m \in T^*X \setminus 0$ if they are conic, glancing and the transversality condition (3.6.1) *is strengthened to:*

$$N_m P, N_m Q \text{ and the fundamental 1-form } \alpha \text{ are linearly independent at } m. \tag{3.6.6}$$

Since the two parts of (3.6.5) give precisely the condition (2.2.10) for K as a submanifold of P, with respect to its bicharacteristic foliation, and of Q with respect to its bicharacteristic foliation, Lemma 2.2.11 can be applied. Now we wish to show that any pair of glancing hypersurfaces can be brought to normal form by a symplectic transformation, or a canonical transformation in the homogeneous case. In the homogeneous case a simple model (Friedlander's example [Fr1]) is given by:

$$Q_0 = \{q_0(x, \xi) = x_{n+1} = 0\}, \ P_0 = \{p_0(x, \xi) = \xi_{n+1}^2 - x_{n+1}\xi_1^2 - \xi_1\xi_n = 0\} \tag{3.6.7}$$

which is a pair of hypersurfaces glancing at the point

$$\bar{m} = (0, \xi_0) \in T^*\mathbb{R}^{n+1} \setminus 0, \ \xi_0 = (1, 0, \dots, 0). \tag{3.6.8}$$

In the purely symplectic case the simpler model:

$$Q_s = \{q_s(x, \xi) = x_n = 0\}, \ P_s = \{p_s(x, \xi) = \xi_n^2 - x_n - \xi_{n-1} = 0\}, \tag{3.6.9}$$

where $\bar{m} = 0 \in T^*\mathbb{R}^n$, is available.

Theorem 3.6.10. *If P and Q are hypersurfaces glancing at m in a symplectic manifold, Y, of dimension $2n$ then there is a symplectic transformation from a neighborhood N of m in Y,*

$$\tau : N \longrightarrow T^*\mathbb{R}^n, \tag{3.6.10A}$$

mapping m to 0, and P, Q to the model (3.6.7). *If P and Q are homogeneous glancing hypersurfaces in $T^*X \setminus 0$, X of dimension $n+1$, then there is a canonical transformation from a conic neighborhood M of m:*

$$\chi : \Gamma \longrightarrow T^*\mathbb{R}^{n+1} \setminus 0,$$

taking m to $\bar{m}$ and P and Q to the model (3.6.9).

In the proof of this theorem the following simple symplectic version of Lemma 2.2.11 will be useful.

Lemma 3.6.11. *Suppose $M \subset Y$ is a hypersurface in a symplectic manifold of dimension $2n$ and $K \subset M$ is a hypersurface such that at $m \in K$ the Hamilton foliation $\mathcal{M}$ of M is simply tangent to K. Let (x', ξ') be Darboux coordinates in $Y_M = M/\mathcal{M}$ and (z', τ') coordinates in K such that $\nu^*(x', \xi') = (z', \tau'', \tau_{n-1}^2)$, in terms of the natural projection $\nu : K \longrightarrow Y_M$. Then there exist coordinates (t, x', ξ'), based at m, on M, with respect to which:*

$$K = \{\xi_{n-1} = t^2\},\ z' = x', \tau'' = \xi'', \tau_{n-1} = t \text{ on } K, \tag{3.6.12}$$

and

$$\iota_M^* \sigma = \sum_{j=1}^{n-1} d\xi_j \wedge dx_j, \qquad \text{where } \iota_M : M \hookrightarrow Y. \tag{3.6.13}$$

Proof. Applying Lemma 2.2.11 directly gives local coordinates (t, X', Ξ') in P with respect to which K is the surface $\Xi_{n-1} = 0$, and on it $z' = X', \tau'' = \Xi'', \tau_{n-1} = t$, M is spanned by $\partial/\partial t - 2t\partial/\partial \Xi_{n-1}$ and if Y_M is identified with $t = 0$ then $(x', \xi') = (X', \Xi')$ on $t = 0$. Thus defining the coordinates

$$x' = X', \quad \xi'' = \Xi'', \quad \xi_{n-1} = \Xi_{n-1} + t^2$$

gives (3.6.12), M is then spanned by $V = \partial/\partial t$ and identifying Y_M with $\{t = 0\}$ (3.6.13) holds after pulling back to $\{t = 0\}$. Since σ, on M, is a 2-form with kernel spanned by V (3.6.13) must hold at $\{t = 0\}$. As σ is closed, it must satisfy the differential equation

$$\mathcal{L}_V \sigma = 0 \text{ on } M,$$

which has a unique solution, (3.6.13), with the given initial condition at $\{t = 0\}$, completing the proof of the Lemma.

Exercise 3.6.14: Consider the two hypersurfaces

$$Q_s = \{x_n = 0\},\ P = \{p(x, \xi) = \xi_n^2 - x_n x_{n-1} - \xi_{n-1}\}. \tag{3.6.15}$$

Show that these surfaces intersect transversally and that in $x_{n-1} \neq 0$ this intersection is glancing with respect to the standard symplectic structure on $\mathbb{R}^{2n}$ but that this is not the case at 0. Sketch the bicharacteristic curves on P.

§3.7: Proof of Theorem 3.6.10

Let $Y_Q = Q/H_q$ be the local quotient of Q by its bicharacteristic foliation, a symplectic manifold. The presence of the second, glancing, hypersurface P defines a billiard ball map on Y_Q by setting:

(3.7.1)
$$\Sigma = \{(y_2, y_1) \in Y_Q \times Y_Q; \exists \text{ a bicharacteristic on } P \text{ meeting both } y_1 \text{ and } y_2\}.$$

To prove that Σ has the stated properties consider instead the pure intersection relation:

$$\Lambda = \{(z, y) \in Y_P \times Y_Q; y \text{ and } z \text{ meet}\}, \tag{3.7.2}$$

where y and z are considered as curves on P and Q. Of course bicharacteristics on Q and P can only meet at K and through each point of K there is precisely one bicharacteristic on P and one on Q. Thus, if

$$\iota_K : K \longrightarrow P \times Q \text{ is the diagonal embedding,}$$

Λ is the image of K :

$$\Lambda = (\nu_P \times \nu_Q) \cdot \iota_K . \tag{3.7.3}$$

where ν_P and ν_Q are the projections from P and Q to Y_P and Y_Q. From the equivalence of (3.2.7) and (3.2.9) discussed above, both the projections on K are folds at m, with differentials having kernel the corresponding Hamilton field, H_p or H_q at m, both of which are tangent to K by hypothesis. Since these two vector fields are independent (dp and dq are independent by assumption) ι_Λ has injective differential, hence

$$\iota_\Lambda : K \longleftrightarrow \Lambda \text{ near } m \tag{3.7.4}$$

is a local embedding of K with image Λ. Thus Λ is a Lagrangian relation from Y_Q to Y_P, since it is a smooth manifold of the appropriate dimension, on which the difference of the 2-forms vanishes, by (3.7.3), and from which the two projections are folds. Thus the results of Chapter 2 and their symplectic extensions discussed above can be applied. In fact from the two definitions (3.7.1) and (3.7.2) it is immediate that

$$\Lambda^t \cdot \Lambda = \mathrm{Id} \cup \Sigma, \tag{3.7.5}$$

so Theorem 3.4.7 provides a proof that Σ is indeed a billiard ball map. Using Theorem 3.2.8 let

$$(y_1, \ldots, y_{n-1}, \eta_1, \ldots, \eta_{n-1}) \text{ and } (x_1, \ldots, x_{n-1}, \xi_1, \ldots, \xi_{n-1})$$

be Darboux coordinates in Y_P and in Y_Q respectively, with the property:

$$K \cong \Lambda = \Lambda_h, \tag{3.7.6}$$

where the identification is (3.7.4). Thus the coordinates (z', ζ') in K:

$$\begin{gathered} z_j = \pi_Q^* x_j,\ 1 \le j \le n-1, \quad \zeta_p = \pi_Q^* \xi_p,\ 1 \le p \le n-2, \\ \zeta_{n-1} = \pi_Q^* y_{n-1} - \pi_P^* x_{n-1}, \end{gathered} \tag{3.7.7}$$

reduce the two involutions $\mathcal{I}$, corresponding to Y_Q and $\mathcal{J}$, corresponding to Y_P, to the normal forms (3.3.2), (3.3.3). First apply Lemma 3.6.11 to

$$K \hookrightarrow Q \hookrightarrow Y,$$

around the base point m, with respect to the Darboux coordinates (x', ξ') in Y_Q and (z', ζ') in K given by (3.7.7). Replace the additional variable by $\xi_n = t$, to give coordinates (x', ξ) on Q, with respect to which the symplectic form is (3.6.13) and

$$K = \{\xi_{n-1} = \xi_n^2 \text{ in } Q\}, \quad z' = x',\ \zeta'' = \xi'',\ \xi_n = \zeta_{n-1}^2 \text{ on } K. \tag{3.7.8}$$

Next consider

$$K \hookrightarrow P \hookrightarrow Y,$$

around the same point m. On K consider the coordinates (Z', ζ'), where in terms of (3.7.7),

$$Z'' = z'', \quad Z_{n-1} = z_{n-1} + \zeta_{n-1} = \pi_P^* y_{n-1} \ \text{ on } \ K. \tag{3.7.9}$$

Thus, (Z', ζ') satisfy the hypotheses of Lemma 3.6.11 with respect to the Darboux coordinates (y', η') on Y_P. Let (t, y', η') be the coordinates on P given by the application of Lemma 3.6.11. Consider the functions:

$$x'' = y'',\ x_{n-1} = y_{n-1} - t,\ x_n = \eta_{n-1} - t^2,\ \xi'' = \eta'',\ \xi_n = t \text{ on } P. \tag{3.7.10}$$

Clearly these form a coordinate system, (x, ξ'', ξ_n) on P, based at m. Moreover in these coordinates, K is $\{x_n = 0\}$, and the symplectic form is

$$\sigma = \sum_{j=1}^{n-2} d\xi_j \wedge dx_j + d(\xi_n + x_{n-1}) \wedge d(x_n - \xi_n^2), \tag{3.7.11}$$

as follows from (3.7.10) and the fact that (3.6.13) holds in terms of the coordinates (t, y', η'). Combining (3.6.12), (3.7.8), (3.7.9) and (3.7.10) it also follows that the functions x'', x_{n-1}, ξ'', ξ_n defined on both Q and P are consistent at the transversal intersection K. Since x_n vanishes at K in P and $\xi_{n-1} - \xi_n^2$ vanishes at K in Q there are local coordinates (x, ξ) in Y near m with respect to which P and Q are reduced to the special forms (3.6.10) and the symplectic form is (3.6.13) on Q, with respect to (x, ξ') and (3.7.11) on P with respect to (x, ξ'', ξ_n). The final stage in the proof of the Theorem is to make a further coordinate change in Y, near $m = 0$, reducing the symplectic form σ to the Darboux form. Since (3.6.12) holds on $Q = Q_s$ and (3.7.11) holds on $P = P_s$, it follows that $\sigma = \sigma_0$, given by (3.2.9) at 0, modulo at most a multiple of $dx_n \wedge d\xi_{n-1}$. This does not affect the non-degeneracy, so the homotopy:

$$\sigma_t = (1-t)\sigma_0 + t\sigma \quad \text{for} \quad t \in [0,1] \tag{3.7.12}$$

consists of symplectic forms near 0. Moreover $\sigma_t - \sigma_0$ vanishes when restricted to either P_s or Q_s. Using the radial integration proof of the Poincaré Lemma to construct μ such that

$$d\mu = \sigma_0 - \sigma, \tag{3.7.13}$$

in coordinates in which P_s and Q_s are coordinate hypersurfaces, μ also vanishes when restricted to P or Q and at 0. Thus, the $\mathcal{C}^\infty$ 1-parameter family of vector fields $V(t)$ defined by:

$$\sigma_t(V(t), \cdot) = \mu \tag{3.7.14}$$

is tangent to both P_s and Q_s and vanishes at 0. It follows that the 1-parameter family of local diffeomorphisms, P_t, obtained by integration of $V(t)$ from the identity, leaves P and Q fixed and by (3.7.12), (3.7.13) and (3.7.14) satisfies

$$d_t P_t^* \sigma_t = 0 \Longrightarrow P_1^* \sigma = \sigma_0.$$

This completes the proof of the purely symplectic part of Theorem 3.6.10.

The homogeneous part of the Theorem can be proved in very much the same way as Theorem 3.5.2 was deduced from the symplectic case. Thus suppose P and Q are homogeneous glancing hypersurfaces in $T^*X \setminus 0$ through m. Applying the symplectic conjugation result just proved there is a symplectic transformation reducing P and Q to the model (3.6.9) and the symplectic form to the Darboux form (3.2.9). Let r' be the infinitesimal generator of the (local) $\mathbb{R}^+$-action on the image space. Since P and Q are by assumption conic, r' must be tangent to P_s and Q_s and hence to K_s, their intersection. The obvious homogeneity of the involutions on K implies that r' is invariant under the two involutions $\mathcal{I}_s$, $\mathcal{J}_s$ on K_s. Let

$$S_s = \{\xi_n = \xi_{n-1} = x_n = 0\} \subset K_s$$

be the fold set of the two involutions on K_s. Clearly r' must also be tangent to S_s. Now $\partial/\partial x_{n-1}$ spans the part of the sum of the Hamilton foliations tangent to K_s. From the independence condition on the $\mathbb{R}^+$-action.

$$r'(0) = a\,\frac{\partial}{\partial x_{n-1}} + \rho,$$

where ρ is in the span of $\partial/\partial dx''$ and $\partial/\partial\xi''$ and is non-zero. A symplectic transformation just amongst the (x'',ξ'') coordinates, leaving $x_j,\ \xi_j,\ j \geq n-1$ fixed can be found to reduce ρ to $\partial/\partial\xi_1$, and obviously leaves P_s and Q_s invariant. Thus it can be arranged that

$$r'(0) = a\,\frac{\partial}{\partial x_{n-1}} + \frac{\partial}{\partial \xi_1}. \tag{3.7.15}$$

In particular there is such a transformation $(\chi')^{-1}$ from the homogeneous model (3.6.7) to the symplectic model (3.6.9), with (3.7.15) valid. Let r'' be the image of the generator of the $\mathbb{R}^+$–action on (3.6.9) under $(\chi)^{-1}$. Consider the two 1-forms:

$$\alpha' = \sigma_0(r,\cdot)', \quad \alpha'' = \sigma_0(r'',\cdot).$$

Following the corresponding argument in the proof of Theorem 3.5.2 verbatim gives a symplectic transformation fixing the base point and reducing α' to α'', and hence r' to r''. The same commutation argument on (3.5.6) discussed there to show that this transformation preserves S now shows that this transformation preserves both P_s and Q_s. Composing the three symplectic transformations then gives the desired canonical transformation which reduces the general pair of homogeneous glancing hypersurfaces to the homogeneous model (3.6.9). This completes the proof of Theorem 3.6.10.

Exercise 3.7.16: Consider the billiard ball map (3.7.1) in the model case of Friedlander's example (3.6.9). In terms of the coordinates

$$y = (x',\xi') = (x_1,\dots,x_{n-1},\xi_1,\dots,\xi_{n-1}),$$

show that the two maps defined by Σ are

$$\delta_\pm(x',\xi') = \exp\Big(\pm\frac{4}{3}H_{\zeta^{3/2}}\Big), \tag{3.7.17}$$

$$\zeta = \xi_{n-1}. \tag{3.7.18}$$

(Cf. (3.4.1)). Conclude that for a general billiard ball map there is always a $\mathcal{C}^\infty$ function ζ for which (3.7.17) holds. Such a function is called an interpolating Hamiltonian for the billiard ball map; is it unique? Show, by examining the model (3.6.7), that in the homogeneous case the interpolating Hamiltonian can be taken to be homogeneous of degree 2/3.

Chapter 4: Eikonal and transport equations

As the discussion in Chapter 1 indicates, the first step in the construction of a microlocal parametrix for a boundary problem with glancing rays is the solution of the eikonal equation. The second step is the solution of the closely related transport equations, which are essentially linearizations of the eikonal equations. If $p_2(x, \xi)$ is the principal symbol of the operator we need to find solutions θ, ζ to:

$$\langle d_x\theta, d_x\theta\rangle + \zeta\langle d_x\zeta, d_x\zeta\rangle = 0, \tag{4.0.1}$$

$$\langle d_x\theta, d_x\zeta\rangle = 0. \tag{4.0.2}$$

Here $\langle\cdot,\cdot\rangle$ is the symmetric bilinear form obtained by polarization of the second order homogeneous polynomial $p_2(x,\cdot)$. This form is Lorentzian if $p(x, D)$ is hyperbolic. The system (4.0.1), (4.0.2) can be reduced to the standard eikonal equation:

$$p_2(x, d_x\phi^{\pm}) = 0, \tag{4.0.3}$$

by introducing the singular phase functions:

$$\phi^{\pm} = \theta \pm \frac{2}{3}(-\zeta)^{3/2}. \tag{4.0.4}$$

Thus (4.0.1) and (4.0.2) are obtained, respectively, just by taking the sum and the difference of the equations (4.0.3).

In a certain sense (4.0.3) is the more fundamental equation. In particular to generalize the problem to higher order operators it is only necessary to generalize (4.0.3) directly to:

$$p(x, d_x\phi^{\pm}) = 0, \tag{4.0.5}$$

where $p : T^*X \setminus 0 \to \mathbb{R}$ is homogeneous of some degree, m, and has simple zeros. Of course it is easy, by Hamilton–Jacobi theory, to find many smooth solutions to the eikonal equation (4.0.5). Solutions with the singularity (4.0.4) arise from solving the initial–value problem for (4.0.5) off an initial surface which does not have the usual transversality condition, corresponding to the fact that there are bicharacteristics tangent to the boundary.

For the model problem, Friedlander's example, with p as defining P_0 in (3.6.7), (4.0.5) has the solution:

$$\phi_0^{\pm} = \theta_0 \pm \frac{2}{3}(-\zeta_0)^{3/2}, \tag{4.0.6}$$

where:

$$\theta_0 = \sum_{j=1}^{n} x_j\xi_j, \quad \zeta_0 = (\xi_n + x_{n+1}\xi_1)\xi_1^{-1/3}, \tag{4.0.7}$$

as can be seen by direct computation. This solution serves very much as a guide to the general construction.

Remark. Starting with this chapter, we reverse a sign convention used in Chapter 3. The result is that the model billiard ball maps $\delta_h^{\pm}(x,\xi)$ are defined for $\xi_n \le 0$ rather than for $\xi_n \ge 0$ (we still take $\xi_1 > 0$), by

$$\delta_h^{\pm}(x,\xi) = \Big(x_1 \pm \frac{2}{3}\Big(-\frac{\xi_n}{\xi_1}\Big)^{3/2}, x'', x_n \pm 2\Big(-\frac{\xi_n}{\xi_1}\Big)^{1/2}, \xi\Big). \tag{4.0.8}$$

We trust the reader will adjust to this change without pain.

§4.1: Geometric reduction

As in Chapter 3 it is useful to take a symmetric position, considering the two hypersurfaces,

$$P = \{(x,\xi); p(x,\xi) = 0\}, \tag{4.1.1}$$

which is the characteristic surface for p, and the bounding hypersurface,

$$Q = \{(x,\xi); x \in B, \xi \ne 0\}, \tag{4.1.2}$$

on the same footing. Here we are working in some canonical coordinates, and $B \subset \mathbb{R}^n$ is a fixed hypersurface forming the boundary. We shall suppose that P and Q are glancing hypersurfaces at some point $m \in P \cap Q$. Thus, if Q is defined by $q(x,\xi) = q(x) = 0$ then

$$dp, dq \text{ and } \alpha = \sum_{j=1}^{n+1} \xi_j \, dx_j \text{ are linearly independent at } m, \tag{4.1.3}$$

and

$$\{p,q\}(m) = 0, \tag{4.1.4}$$

while

$$\{p,\{p,q\}\}(m) \ne 0 \text{ and } \{q,\{q,p\}\}(m) \ne 0. \tag{4.1.5}$$

Note that (4.1.4) is just the condition that the null bicharacteristic of P, i.e., Hamilton curve of p, through m be tangent to Q (or equivalently to B after projection to $\mathbb{R}^n$). The first condition in (4.1.5) requires this tangency to be simple. The second condition in (4.1.5) is also easily interpreted in the case that $p = p_2$ is a second order polynomial in ξ. Indeed it is then the condition that the operator $p_2(x,D)$ be non-characteristic with respect to the hypersurface B. To see this just take local coordinates $x_1, \dots, x_n$ in which B is given by $\{x_1 = 0\}$, with $x = 0$ at m. Since (4.1.5) is coordinate-invariant the second condition becomes:

$$\frac{\partial^2 p}{\partial \xi_1^2} \neq 0 \quad \text{at } 0. \tag{4.1.6}$$

That is the coefficient of ξ_1^2 should be non-zero, i.e., $p_2(x, D)$ should be non-characteristic with respect to B.

The results of Chapter 3, especially Theorem 3.6.10, can be applied directly to P and Q defined by (4.1.1), (4.1.2) subject to (4.1.3), (4.1.4) and (4.1.5). Thus, there is a canonical transformation,

$$\chi : \Gamma \longrightarrow T^*X \setminus 0, \quad \Gamma \subset T^*\mathbb{R}^{n+1} \setminus 0, \tag{4.1.7}$$

defined in a conic neighborhood, Γ, of $\bar{m}$ taking the standard pair (P_h, Q_h) defined in (3.6.9), to (P, Q) and taking $\bar{m}$ to m. As was already noted in the construction of χ, the fact that χ, which is symplectic, maps Q_h onto Q means that it defines a local canonical transformation (i.e., a homogeneous symplectic transformation) from the quotient space of Q_h, modulo its Hamilton fibration, to the corresponding quotient space of Q. Notice that when a hypersurface $Q \subset T^*X \setminus 0$ is just the lift of a hypersurface in the base, $B \subset X$, then this quotient is naturally identified as the cotangent space of the hypersurface:

$$Q/\mathbb{R}H_q \cong T^*B. \tag{4.1.8}$$

Thus, χ induces a canonical transformation

$$\begin{aligned} &\chi_\partial : \gamma \longrightarrow T^*B \setminus 0, \quad \gamma \subset T^*\mathbb{R}^n. \\ &\gamma = \{(x_1, \dots, x_n, \xi_1, \dots, \xi_n) \in T^*\mathbb{R}^n; \\ &\qquad (x_1, \dots, x_n, 0, \xi_1, \dots, \xi_n, \xi_{n+1}) \in \Gamma \text{ for some } \xi_{n+1}\}. \end{aligned} \tag{4.1.9}$$

This boundary transformation must intertwine the billiard ball maps $\delta_h^\pm$, defined by P_h and Q_h and $\delta^\pm$, defined by P and Q. The billiard ball maps for the standard pair, P_h, Q_h are:

$$\delta_h^\pm(x, \xi) = \Big(x_1 \pm \frac{2}{3}\Big(-\frac{\xi_n}{\xi_1}\Big)^{3/2}, x_2, \dots, x_{n-1}, x_n \pm 2\Big(-\frac{\xi_n}{\xi_1}\Big)^{1/2}, \xi\Big) \tag{4.1.10}$$

where $(x, \xi) = (x_1, \dots, x_n, \xi_1, \dots, \xi_n)$. We will use this normal form to produce solutions of the eikonal equations of the desired form.

Before doing so we shall strengthen the transversality condition (4.1.3). As we shall see later this does not limit the parametrix construction since the general case can be treated by first making a transformation in the boundary variables. Observe first that (4.1.3) is just the independence of H_p, H_q and the radial direction of

$T^*X \setminus 0$ at m. Since H_p is tangent to Q, by hypothesis, and H_q spans the kernel of the projection to T^*B at m, (4.1.3) just states that the image of H_p in $T^*_x B$, where $\pi(m) = x$, is not in the radial direction. The strengthened form of this condition we need is:

$$H_p \text{ is not tangent to the fibre } T_{\bar{x}}X \text{ at } \bar{x} = \pi(m). \tag{4.1.11}$$

This condition is useful since it allows the canonical transformation (4.1.7) to be so chosen that:

$$\chi_\partial^*(d\xi_j), \quad j = 1, \ldots, n, \quad \text{are linearly independent on } T^*_{\bar{x}}B \text{ at } m.$$

To see this just follow the reasoning above to conclude from (4.1.11) that the projection of H_p into T^*B is not tangent to the fibre $T^*_{\bar{x}}B$. Since this image is the direction of the Hamilton vector field on the fold set it follows that the corresponding direction, $\partial/\partial x_n$ is not tangent to

$$\chi_\partial^{-1}(T^*_{\bar{x}}B) = \mathcal{H}. \tag{4.1.12}$$

Now $\mathcal{H}$ is Lagrangian, and conic, so $d\xi_n \neq 0$ on $\mathcal{H}$ at $\bar{m}$. From the fact that $\mathcal{H}$ is conic it also follows that $\partial/\partial\xi_1$ is tangent to $\mathcal{H}$ at $\bar{m}$. Thus, by a canonical transformation fixing x_1, ξ_1, x_n and ξ_n it is possible to arrange that the $d\xi_j$ be independent on the image of $\mathcal{H}$. Since such a transformation can be extended to leave the normal form P_h, Q_h completely unchanged this shows that (4.1.11) allows us to assume (4.1.12).

Now, consider the mapping:

$$P \longrightarrow \mathbb{R}^n, \quad P \ni p \longmapsto \big(\xi_1(\chi^{-1}(p)), \ldots, \xi_n(\chi^{-1}(p))\big) \in \mathbb{R}^n,$$

defined by sending p to the values of the ξ_j, $j = 1, \ldots, n$, at the image point under (4.1.7). Combining this map with the projection from P to the base gives a map:

$$Y : P \longrightarrow X \times \mathbb{R}^n. \tag{4.1.13}$$

Lemma 4.1.14. *The map* (4.1.13)*, defined from a reduction of the glancing pair* P*,* Q *to normal form* P_h*,* Q_h *by a canonical transformation satisfying* (4.1.12)*, is a fold at* m*. The fold set meets the boundary* Q *transversally in* $\{\xi_n = 0\}$*.*

Proof. If $q \in C^\infty(X)$ is a defining function for B then $dq \neq 0$ on P, near m, since this is part of the transversality condition (4.1.3). Thus it is enough to show that the restriction of (4.1.13) to the intersection of P and Q is a fold. This is just the map:

$$Y' : P \cap Q \longrightarrow B \times \mathbb{R}^n, \quad Y' = Y\big|_{P\cap Q}.$$

Now, the map from $P \cap Q$ to T^*B is indeed a fold and since Y' is this projection followed by the replacement of the fibre variables by the ξ_j, $j = 1, \ldots, n$, the transversality condition (4.1.12) shows that Y', and hence Y, is a fold, with the fold set having the properties stated.

§4.2: Hamilton–Jacobi theory

We briefly recall how Hamilton-Jacobi theory can be used to construct a parametrizing phase function in case H_p is transversal to Q. Then $P \cap Q$ projects locally diffeomorphically to T^*B under (4.1.8). The fibration of T^*B by the (coordinate) Lagrangian submanifolds $\eta =$ const can therefore be lifted to a fibration (always locally) of $P \cap Q$. The H_p-flow-out of these leaves gives a fibration of P, by Lagrangian submanifolds near m; the leaves Λ_η are still parametrized by η. Now the canonical 1-form α restricts to be closed in Λ_η (since $\omega = 0$.) Moreover the Λ_η project diffeomorphically to the base, so the base variables can be used as coordinates on Λ_η. Thus there is a function $\Phi(x, \eta)$ fixed by

$$\begin{aligned} d\Phi &= \alpha \quad \text{on each } \Lambda_\eta \\ \Phi &= 0 \quad \text{on } T^*_{\pi(m)}B. \end{aligned} \tag{4.2.1}$$

The second condition normalizes the constant on each leaf, since the fibre through m is certainly transversal to each Λ_η.

By construction $\Phi(x, \eta)$ is $\mathcal{C}^\infty$ and $d_x\Phi(x, \eta) = \xi(x, \eta)$ where $\Lambda_\eta = \{(x, \xi(x, \eta).\}$ This can be restated as saying that

$$\begin{gathered} p\big(x, d_x\Phi(x, \eta)\big) \equiv 0, \\ (x, \eta) \longleftrightarrow \big(x, d_x\Phi(x, \eta)\big) \in P. \end{gathered} \tag{4.2.2}$$

In this sense Φ is a phase function parametrizing P. It can be used to construct parametrices for boundary problem for the operator P. In the case of interest here H_p is not transversal to the boundary so we have to work harder to construct a suitable Lagrangian fibration of P and then the resulting phase function is not smooth; the main effort is to keep the singularity as simple as possible.

Consider in P the submanifolds defined by the constancy of the $\xi_j,\ j = 1, \ldots, n$:

$$\Lambda_\xi = \{p \in P;\ Y(p) = (\cdot, \xi)\}.$$

These are Lagrangian submanifolds of $T^*X \setminus 0$ foliating P near m; the fact that they are Lagrangian follows by making the transformation χ which shows that the ξ_j and P_h Poisson commute. The canonical 1-form:

$$\alpha = \sum_{j=1}^{n+1} \xi_j \, dx_j$$

restricted to P is therefore closed on each of the submanifolds Λ_ξ near m. Thus there is a $\mathcal{C}^\infty$ function Φ on P such that:

$$d(\Phi|_{\Lambda_\xi}) = \alpha|_{\Lambda_\xi}, \ \text{for } \xi \text{ near } \xi_0. \tag{4.2.3}$$

In fact Φ is locally unique up to a normalization constant on each fibre. Let $T \subset P$ be a conic submanifold of dimension $\mathbb{R}^n$ transversal to the fibration by Λ_ξ, i.e., meeting each one transversally in a point. Clearly one can choose T so that:

$$(4.2.4) \qquad T \text{ is contained in the fold of } Y.$$

The transversality means that Φ satisfying (4.2.3) is uniquely fixed by requiring:

$$(4.2.5) \qquad \Phi = 0 \text{ on } T.$$

Consider again the projection Y. Note that the image $Y(P)$ is a half-space (near $\bar{x}$) of the form:

$$(4.2.6) \qquad Y(P) = \{\xi_n \leq x_{n+1} f(x, \xi)\}$$

for some smooth function f, $f(m) \neq 0$. Since Y is a fold, Φ can be written uniquely in the form:

$$(4.2.7) \qquad \Phi = Y^* \Big(\theta \pm \frac{2}{3}(-\zeta)^{3/2}\Big), \text{ where } \theta, \zeta : Y(P) \longrightarrow \mathbb{R} \text{ are } \mathcal{C}^\infty.$$

The odd part of Φ vanishes to second order (and hence to third order) at the fold because Φ is the integral of a smooth 1-form on each leaf Λ_ξ.

Proposition 4.2.8. *Let P and $Q \subset T^*X \setminus 0$ be hypersurfaces as in* (4.1.1), (4.1.2) *satisfying* (4.1.3), (4.1.4) *and* (4.1.11) *at some point $m \in P \cap Q$. Then there exists a canonical transformation reducing P, Q to the normal form P_h, Q_h and satisfying* (4.1.12). *If Y in* (4.1.13) *is defined from this transformation and T satisfies* (4.2.4) *then the function Φ defined by* (4.2.3), (4.2.5) *in an open cone around $(\bar{x}, \xi_0)$ is of the form:*

$$\Phi = Y^* \Big(\theta \pm \frac{2}{3}(-\zeta)^{3/2}\Big),$$

where $\theta, \zeta : Y(P) \longrightarrow \mathbb{R}$, are smooth, homogeneous of degrees one and two-thirds, $Y^*\zeta$ is a defining function for the fold and:

$$(4.2.9) \qquad \zeta = \xi_1^{-1/3} \xi_n \quad on \quad Y(P) \cap (B \times \mathbb{R}^n).$$

Proof. In view of the analysis above it only remains to show that when Φ is written in the form (4.2.7) the term ζ is a defining function for the fold and that (4.2.9) holds. Notice that (4.2.9) is a formula for ζ, at the boundary, which is independent of the choice of either χ, the reduction to normal form, or T, the submanifold used in the normalization of Φ. Keeping χ fixed for the moment, let Φ_1 and Φ_2 be two smooth solutions of (4.1.13) corresponding to different surfaces T_1 and T_2 of normalization, both satisfying (4.2.4). Thus the difference:

$$w = \Phi_1 - \Phi_2$$

is constant on each leaf of Λ_ξ, in particular it is a function only of ξ. The involution defined by Y preserves the fibration by the Λ_ξ, essentially by definition, so the Y-odd part of w is also a function of ξ only. From the form of the fold of Y it follows that the odd part of w vanishes identically. That is, the two solutions Φ_1 and Φ_2 have the same odd part. As noted above, over the boundary B, the involution of the map Y is just that of the projection of $P \cap Q$ to T^*B. Now, the function constructed above pulls back under χ to a solution of the same problem for the model case, with of course some particular normalization. Together these two observations show that the odd part of Φ, restricted to the boundary, is actually independent of both the normalization and the reduction to normal form chosen. In particular it is the same as that of the solution to the model problem obtained in (4.0.6), (4.0.7). This completes the proof of the Proposition since (4.2.7) implies in particular that ζ is a defining function for the fold.

§4.3: Phase functions

Since ζ is a defining function for the fold of the map Y used in the definition of θ and ζ, we have succeeded in constructing solutions of the form (4.0.4) to the eikonal equations in the region $\{\zeta \geq 0\}$, a region of the type (4.2.6). We next extend these functions outside this domain in such a fashion that the eikonal equations continue to hold to infinite order at the boundary. It is not in general possible to solve the eikonal equations exactly across this hypersurface because the problem becomes elliptic.

Proposition 4.3.1. *Let p_2 be the (real) principal symbol of a differential operator defined and with $\mathcal{C}^\infty$ coefficients in some neighborhood of a hypersurface $B \subset \mathbb{R}^{n+}$. If* (4.1.1)–(4.1.5) *hold at a point $m \in P \cap Q \subset T^*\mathbb{R}^{n+1}$ then there exist real functions θ and ζ which are $\mathcal{C}^\infty$ in a conic neighborhood Σ of $\pi(m) \times (1, 0, \dots, 0) \in \mathbb{R}^{n+1} \times \mathbb{R}^n$, are homogeneous of degrees* 1 *and* 2/3, *respectively, and have the properties:*

$$\zeta = \xi_1^{-1/3} \xi_n \text{ on } \Sigma \cap (B \times \mathbb{R}^n) \tag{4.3.2}$$

$$d_x \left(\frac{\partial \theta}{\partial \xi_j} \right), \ j = 1, \dots, n \text{ are linearly independent on } \Sigma \tag{4.3.3}$$

(4.3.4) *equations* (4.0.1), (4.0.2) *hold in $\zeta \geq 0$ and in Taylor series on B.*

Proof. We start with the results of Proposition 4.2.8. As noted above it is only necessary to extend the ϕ and ζ so obtained from the region (4.2.6) so that (4.0.1), (4.0.2) continue to hold in the sense of formal power series at B and (4.3.6) holds on B. Thus, we first specify θ and ζ as Taylor series on the boundary, then choose appropriate functions with these Taylor series.

Applying the Malgrange preparation theorem to p allows it to be written:

$$p = p'[(\xi_{n+1} - a(x, \xi'))^2 - b(x, \xi')],$$

where $\xi' = (\xi_1, \ldots, \xi_n)$ and a and b are real, $p' \neq 0$ near the base point m. Thus in solving (4.1.13) the term p' can be dropped. The conditions (4.1.11), (4.1.4) and (4.1.5) imply that

$$\xi_{n+1} = a, \quad b = 0, \quad d_{\xi'} b \neq 0 \ \text{ at } \ m,$$

with $b = 0,\ x_{n+1} = 0$ being the glancing surface. The differential equation (4.0.5) then becomes:

$$\frac{\partial \phi^{\pm}}{\partial x_{n+1}} - a\Big(x, \frac{\partial \phi^{\pm}}{\partial x'}\Big) = \pm b\Big(x, \frac{\partial \phi^{\pm}}{\partial x'}\Big)^{1/2} = 0, \text{ in } \zeta \geq 0. \tag{4.3.5}$$

If θ and ζ are simply extended as smooth, real-valued and homogeneous functions across $\zeta = 0$, i.e. into $\zeta \leq 0$, then (4.3.5) continues to hold for the extended $\phi^{\pm}$, which we write as $\phi^{\pm\prime}$, with a smooth error term which vanishes to all orders at $\zeta = 0$.

$$\frac{\partial \phi^{\pm\prime}}{\partial x_{n+1}} - a\Big(x, \frac{\partial \phi^{\pm\prime}}{\partial x'}\Big) \mp \frac{1}{2} b\Big(x, \frac{\partial \phi^{\pm\prime}}{\partial x'}\Big)^{1/2} = e_{\pm}; \text{ with } e_{\pm} = 0 \text{ in } \zeta \geq 0. \tag{4.3.6}$$

To solve (4.3.5) in the sense of formal power series at $\{x_{n+1} = 0\}$ it is only necessary to add to ϕ' a formal power series:

$$\phi'' = \sum_{k=1}^{\infty} x_{n+1}^k \, g_k(x', \xi).$$

In view of (4.3.6) one can solve for the g_k successively, uniquely as $\mathcal{C}^\infty$ functions vanishing in $\xi_n > 0$, so that (4.3.5) holds formally for $\phi = \phi' + \phi''$. Clearly this gives extension of ϕ and ζ with the properties desired.

Exercise 4.3.7: Show that you can arrange that $\theta|_B$ generates the canonical transformation (4.1.9).

§4.4: Transport Equations

We next proceed to the discussion of the transport equations, derived in Chapter 1 for the case of a second order hyperbolic equation. These are all of the form:

$$2\langle d\theta, dg\rangle + 2\zeta\langle d\zeta, dh\rangle + \langle d\zeta, d\zeta\rangle h + Ag + \zeta B h = F_1, \tag{4.4.1}$$

$$2\langle d\zeta, dg\rangle + 2\langle d\theta, dh\rangle + Bg + Ah = F_2, \tag{4.4.2}$$

(see (1.5.13)). Here, θ and ζ are the solutions of the eikonal equations, (4.0.1) and (4.0.2), just constructed. The forcing terms F_1 and F_2 and the coefficients A, B are given (in general complex–valued) $\mathcal{C}^\infty$ functions. Just as for the eikonal equation, (4.4.1) and (4.4.2) can be reduced to the usual type of transport equations by introducing the singular dependent variables:

$$a^{\pm} = g \pm (-\zeta)^{1/2} h, \text{ in } \zeta \leq 0,$$

and the functions:

$$G^{\pm} = (A \pm (-\zeta)^{1/2} B), \quad F^{\pm} = (F_1 \pm (-\zeta)^{1/2} F_2).$$

Indeed, if g and h satisfy (4.4.1) and (4.4.2) then:

$$2\langle d_x\phi^{\pm}, d_x a^{\pm}\rangle + G^{\pm} a^{\pm} = F^{\pm}. \tag{4.4.3}$$

Conversely we can recover (4.4.1) and (4.4.2), in $\zeta \geq 0$ by taking linear combinations of (4.4.3) for the two signs.

To interpret (4.4.3) note that $a^{\pm}$, $G^{\pm}$ and $F^{\pm}$ are $\mathcal{C}^{\infty}$ functions of x, ξ and $\zeta^{1/2}$ in $\zeta \geq 0$. Referring back to the construction of the phase function $\phi^{\pm}$, this means that these functions pull back under Y in (4.1.13) to smooth functions on the characteristic surface P. Thus, writing a, G, F for these smooth lifts, (4.4.3) becomes:

$$2\langle d\Phi, da\rangle + Ga = F. \tag{4.4.4}$$

Here the differential is still with respect to x alone, but this can be reinterpreted as being on each of the Lagrangian submanifolds Λ_ξ used in the definition of Φ. In fact the vector field $2\langle d\Phi, \cdot\rangle$ is just of the Hamilton vector field of p. Indeed since the projection from Λ_ξ to the base is a diffeomorphism, and this is used to introduce the x_j as coordinates in Λ_ξ, the Hamilton vector field in these coordinates is just given by the projection of H_p to the base, i.e., precisely

$$\sum_{j=1}^{n} \frac{\partial p}{\partial \Xi_j}(x, \Xi).$$

Thus (4.4.4) becomes:

$$H_p a + Ga = F, \tag{4.4.5}$$

Now (4.4.5) is the usual form of the transport equation.

There is no difficulty in solving (4.4.5), but returning to the source of the transport equations (4.4.1), (4.4.2) we find that a solution should be sought with:

$$h = 0 \text{ on } B, \; g = 1 \text{ at the base point } m. \tag{4.4.6}$$

In terms of the function a obtained by lifting $a^{\pm}$ to P, these can be written

$$a \text{ on } P \cap Q \text{ is the lift of a } \mathcal{C}^{\infty} \text{ function on } T^*B, \; a(m) = 1. \tag{4.4.7}$$

Before tackling this problem directly let us simplify (4.4.5) by removing some of the data. Certainly the equation:

$$H_p a_1 + G' a_1 = F'$$

has a solution with $a_1(m) = 0$, since this just involves integration of a real non–vanishing vector field. The solution can be chosen homogeneous of degree $m-1+r$ if F' is of degree r and p of degree m. Subtracting a_1 from a in (4.4.5) reduces us to consideration of the homogeneous equation. Similarly the term of order zero can be removed by solving:

$$H_p b = G', \quad b(m) = 0,$$

and setting

$$a - a_1 = \exp(-b)u.$$

The equation has thereby been simplified to:

$$H_p u = 0, \tag{4.4.8}$$

at the expense however of complicating the boundary condition (4.4.7) to:

$$[\exp(-b)u]_O = e, \quad v_O = (\mathcal{I}_Q^* v - v)\tau^{-1}, \quad u(m) = 1. \tag{4.4.9}$$

where τ is some primitve $\mathcal{I}_Q$-odd function. Thus $v = v_E + \tau v_O$ where v_E and v_O are both even under $\mathcal{I}_Q$. The error, e, in (4.4.9) is also some $\mathcal{I}_Q$-even function coming from a_1, and $\mathcal{I}_Q$ is the involution on $P \cap Q$ arising from projection to T^*B. Observe that the differential equation (4.4.8) just reduces to the evenness of u, on $P \cap Q$ under the involution corresponding to the projection into the manifold of bicharacteristics on P. Since the canonical transformation χ reduces these two involutions to normal form, equivalent to (2.3.8) and (2.3.9), this problem has already been solved in §2.8. One only needs to observe that if

$$u = u_E + \tau u_O,$$

then (4.4.9) can be written:

$$u_O = c u_E + f, \tag{4.4.10}$$

where c and f are given $\mathcal{I}_Q$-even functions. This can be solved using Proposition 2.8.2.

This solves the transport equations in the region $\zeta \geq 0$. The extension of the smooth functions g, h into the region $\zeta < 0$ in such a way that the transport equations continue to hold to all orders at the boundary B can be accomplished directly, as in §4.3.

Proposition 4.4.11. *Suppose P and Q satisfy the hypotheses of Proposition 4.3.1 and θ, ζ are as constructed there. Then for any $\mathcal{C}^\infty$ functions A, B, F_1, F_2 homogeneous of degrees 1, $2/3$, r and $r-1/3$, and for any real-valued $\mathcal{C}^\infty$ functions c, d on the boundary and homogeneous of degrees $-1/3$ and $r-4/3$ there is a $\mathcal{C}^\infty$ solution g, h (in $\zeta \geq 0$ and to all orders at the boundary) to* (4.4.1)–(4.4.2) *satisfying the boundary condition:*

$$h = cg + d, \quad g(m) = 1, \tag{4.4.12}$$

and homogeneous of degrees $r-1$ and $r-4/3$, respectively.

Proof. This proposition just summarizes the discussion above, except that the boundary condition has been generalized. Following the discussion of the boundary conditions beginning at (4.4.6), gives in place of (4.4.9) the condition:

$$[e^{-b}u]_O = c[e^{-b}u]_E + e', \tag{4.4.13}$$

in terms of the $\mathcal{I}_Q$-odd and even parts. Since

$$[\alpha\beta]_E = [\alpha]_E \cdot [\beta]_E + \tau^2[\alpha]_O \cdot [\beta]_O,$$

one again arrives at (4.4.10), with different functions c, and f even under $\mathcal{I}_0$. Thus the proof above suffices to give the more general result as stated.

If we replace appeal to Proposition 2.8.2 by a use of the result of Exercise 2.8.14, we obtain the following useful variant of Proposition 4.4.11.

Proposition 4.4.14. *Under the hypotheses of Proposition 4.4.11, given b, d, homogeneous of degrees $-1/3$ and $r-1$, respectively, one can obtain solutions g, h (on $\zeta \geq 0$ and to all orders at the boundary) to (4.4.1)–(4.4.2), satisfying the boundary condition*

$$g = b\zeta h + d, \tag{4.4.15}$$

and homogeneous of degrees $r-1$ and $r-4/3$, respectively.

Chapter 5: Airy multipliers

In this chapter we shall consider the operators which are given by Fourier multiplication or division by one of the functions

$$\begin{gathered}\Phi_{\pm}(\zeta_0) = \frac{A'_{\pm}(\zeta_0)}{A_{\pm}(\zeta_0)}, \quad \Phi i(\zeta_0) = \frac{Ai'(\zeta_0)}{Ai(\zeta_0)}, \quad Ai(\zeta_0)A_{\pm}(\zeta_0), \\ Ai'(\zeta_0)A_{\pm}(\zeta_0), \quad Ai(\zeta_0)A'_{\pm}(\zeta_0), \quad Ai'(\zeta_0)A'_{\pm}(\zeta_0),\end{gathered} \tag{5.0.1}$$

and certain others, where ζ_0 is the function

$$\zeta_0(\xi) = \xi_1^{-1/3}(\xi_n + iT). \tag{5.0.2}$$

The corresponding Fourier multipliers will be denoted, respectively, by

$$\begin{gathered}\Phi_{\pm}, \quad \Phi i_T, \quad (\mathcal{A}i\,\mathcal{A}_{\pm})_T, \\ (\mathcal{A}i'\mathcal{A}_{\pm})_T, \quad (\mathcal{A}i\mathcal{A}'_{\pm})_T, \quad (\mathcal{A}i'\mathcal{A}'_{\pm})_T.\end{gathered} \tag{5.0.3}$$

Of the Fourier multipliers in (5.0.3) the first two are particularly important since they constitute the 'non-classical' parts of the Neumann operator in the diffractive and gliding cases respectively, as we will see in Chapter 8.

We begin in §5.1 with the even simpler Airy multipliers $\mathcal{A}i$ and $\mathcal{A}i'$, which define Fourier integral operators with folding canonical relations. The main result of this section is that these operators give rise to microlocal models for arbitrary Fourier integral operators with folding canonical relations. In §5.2 we gather certain estimates on Airy functions, established in Appendix A, which are then brought to bear in §5.3 to establish mapping properties of the operators listed in (5.0.3) and their inverses on Sobolev spaces. Finally, in §5.4 we show how these operators, applied to distributions, move wave front sets around.

Of course our study is microlocal, so we typically compose the operators in (5.0.3) and their inverses with a microlocal cutoff $\phi(D)$, where $\phi(\xi) \in S^0_{1,0}(\mathbb{R}^n)$ satisfies

$$\begin{aligned}\phi(\xi) = 1 \quad &\text{for} \quad |\xi_n| \leq \frac{1}{2}\xi_1, \quad |\xi| \geq 1, \\ 0 \quad &\text{for} \quad |\xi_n| \geq \xi_1.\end{aligned} \tag{5.0.4}$$

§5.1: Fourier integral operators with folding canonical relations

A simple example of a Fourier integral operator associated to a folding canonical relation is the convolution operator with Schwartz' kernel:

$$\alpha = \delta(x_1 - \frac{1}{3}x_n^3)\delta(x_2)\cdots\delta(x_{n-1}).$$

The Fourier multiplier corresponding to this convolution operator is:

$$
\begin{aligned}
F(\xi) &= \int \delta\Big(x_1 - \frac{1}{3}x_n^3\Big)e^{-i\xi_1 x_1 - i\xi_n x_n}\,dx_1\,dx_n \\
&= \int \exp\Big(-\frac{1}{3}i\xi_1 x_n^3 - i\xi_n x_n\Big)\,dx_n \\
&= \xi_1^{-1/3}Ai(\xi_1^{-1/3}\xi_n),
\end{aligned}
\tag{5.1.1}
$$

where the formula (A.0.1):

$$
Ai(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(ist + \frac{1}{3}it^3)\,dt
$$

has been used. Acting on distributions with wavefront set in a suitably small conic neighborhood of $\{\xi_n = 0\}$ this operator differs from the operator $\mathcal{A}i$ defined by (5.0.3) with the Fourier multiplier $Ai(\zeta)$ by an elliptic factor and an additive smoothing operator:

$$
\alpha * \equiv |D_1|^{-1/3}\mathcal{A}i, \text{ near } \{\xi_n = 0, \xi_1 = 1\}. \tag{5.1.2}
$$

From the form (5.1.1) of α this shows that in the notation of Hörmander ([H1]),

$$
\mathcal{A}i \in I^{-1/6}(\mathbb{R}^n, \mathbb{R}^n; C_0),
$$

where C_0 is the canonical relation associated to the phase function

$$
\phi(x, y, \xi, \tau) = (x - y)\cdot\xi + \xi_n\tau\xi_1^{-1} - \frac{1}{3}\tau^3\xi_1^{-2}.
$$

Namely, the (twisted) conormal bundle to the surface

$$
S = \Big\{(x, y) \in \mathbb{R}^{2n}; x_1 - y_1 - \frac{1}{3}(x_n - y_n)^3 = 0, x_2 = y_2, \ldots, x_{n-1} = y_{n-1}\Big\}.
$$

That is,

$$
N^*S = \{(x, y, \xi, \eta) \in \mathbb{R}^{4n}; (x, y) \in S, \eta + \xi = 0, \xi_n = (x_n - y_n)^2\xi_1\},
$$

so

$$
\begin{aligned}
C_0 = \Big\{(x, \xi, y, \eta);\ & \xi = \eta, x_j = y_j, 2 \le j \le n-1, \\
& x_1 - y_1 = \frac{1}{3}(x_n - y_n)^3, \xi_1(x_n - y_n)^2 = \xi_n, \Big\},
\end{aligned}
\tag{5.1.3}
$$

which is the relation considered in (3.5.3).

The same argument shows that

$$\mathcal{A}i' \in I^{1/6}(\mathbb{R}^n, \mathbb{R}^n; C_0),$$

since up to an elliptic factor and a smoothing error, $\mathcal{A}i'$, obtained from (5.0.3) with the multiplier Ai' is given by convolution with $x_n\alpha$.

Now, let C be a general folding canonical relation from X_1 to X_2, with fold set Σ. Consider the space $I^m(X_1, X_2; C)$ of Fourier integral operators associated with C. Of course, $I^m(X_1, X_2; C)$ is a module over the (properly supported) pseudodifferential operators:

$$OPS^0_P(X_2) \cdot I^m(X_1, X_2; C) \subset I^m(X_1, X_2; C), \tag{5.1.4}$$

and similarly from the other side. If C were microlocally a canonical transformation near a point λ then one elliptic element would generate $I^m(X_1, X_2; C)$ microlocally as a module (5.1.4). In the case of a folding canonical transformation two elements are needed. Naturally one of them, A_1, should be elliptic but this does not suffice. Indeed the symbol of $P \cdot A_1$ is $\sigma(P)\sigma(A_1)$, with $\sigma(P)$ necessarily even under the involution $\mathcal{J}_2$ of C corresponding to the points identified by projection into T^*X_2. Thus the symbol of $P \cdot A$ cannot be an arbitrary symbol on the relation C.

Proposition 5.1.5. *Suppose that $\lambda \in \Sigma$, the fold of the folding canonical relation C from X_1 to X_2, and $A_1 \in I^m(X_1, X_2; C)$ is elliptic at λ. If $A_2 \in I^m(X_1, X_2; C)$ is such that on C, $\sigma(A_2) = \beta\sigma(A_1)$ where $\beta - \mathcal{J}_2^*\beta$ vanishes to precisely first order on Σ near λ, then any $A \in I^M(X_1, X_2; C)$ can be written microlocally near λ:*

$$A = P_1 \cdot A_1 + P_2 \cdot A_2, \text{ with } P_i \in OPS^{M-m}(X_2),\ i = 1, 2. \tag{5.1.6}$$

Proof. The basic result in the theory of Fourier integral operators (or Lagrangian distributions) is that modulo $I^{M-1}(X_1, X_2; C)$, an element $A \in I^M(X_1, X_2; C)$ is determined by its symbol $\sigma(A)$, a section of the Keller-Maslov (line) bundle over C. Since A_1 is elliptic,

$$\sigma(A) = q\sigma(A_1),$$

with q a symbol of order $M - m$ on C. From the remarks above it suffices to decompose q as:

$$q = p_1 + p_2\beta, \tag{5.1.7}$$

where p_1 and p_2 are even under $\mathcal{J}_2$, and β is as in the statement of the proposition. Decomposing β into its $\mathcal{J}_2$-odd and even parts gives

$$\beta = \beta_e + \beta_o,$$

where by hypothesis β_o is a defining function for the fold. The decomposition of q into $\mathcal{J}_2$-odd and even parts can therefore be written

$$q = p_1' + p_2\beta_o,$$

with p_1' and p_2 $\mathcal{J}_2$-even. The (5.1.7) holds with $p_1 = p_1' - p_2\beta_e$. This gives (5.1.6) modulo $I^{m-1}(X_1, X_2; C)$ and a standard inductive argument over the symbol filtration completes the proof of the proposition.

Theorem 5.1.8. *Let $A \in I^m(X_1, X_2; C)$ be a Fourier integral operator of order m associated to a folding canonical relation C from X_1 to X_2. Given any point $\lambda = (m_2, m_1) \in \Sigma$, the fold of C, there exist elliptic Fourier integral operators $J_1 \in I^0(X_1, \mathbb{R}^n; \chi_1), J_2 \in I^0(\mathbb{R}^n, X_2; \chi_2)$ associated to canonical transformations and pseudodifferential operators $P_1 \in OPS^{m+1/6}(\mathbb{R}^n),\ P_2 \in OPS^{m-1/6}(\mathbb{R}^n)$ such that:*

$$A \equiv J_2 \cdot (P_1 \mathcal{A}i + P_2 \mathcal{A}i') \cdot J_1 \text{ microlocally near } \lambda. \tag{5.1.9}$$

Proof. We take the canonical transformations χ_1 and χ_2 from Theorem 3.5.7, reducing the folding canonical relation C to the normal form Λ_h near λ. Let $J_1 \in I^0(X_1, \mathbb{R}^n; \chi_1)$ and $J_2 \in I^0(\mathbb{R}^n, X_2; \chi_2)$ be any properly supported Fourier integral operators elliptic at the appropriate base points, with small essential supports and microlocal inverses J_i^{-1}. Then,

$$A' = J_2^{-1} \cdot A \cdot J_1^{-1} \in I^m(\mathbb{R}^n, \mathbb{R}^n; C_0).$$

Applying Proposition 5.1.5, with $A_1 = |D|^{1/6}\mathcal{A}i$ and $A_2 = |D|^{-1/6}\mathcal{A}i'$, which clearly satisfy the hypotheses, the decomposition (5.1.6) immediately gives (5.1.9) microlocally near λ.

As a simple application of this reduction result one can give the sharp order of continuity of such operators on Sobolev spaces.

Corollary 5.1.10. *If C is a folding canonical relation from X_1 to X_2 and $A \in I^m(X_1, X_2; C)$ has essential support near some point λ in the fold of C then*

$$A : H^s_c(X_1) \longrightarrow H^{s-m-1/6}_{loc}(X_2), \quad \forall\ s. \tag{5.1.11}$$

Furthermore, if A has a homogeneous principal symbol of degree m then

$$A : H^s_c(X_1) \longrightarrow H^{s-m}_{loc}(X_2), \tag{5.1.12}$$

if and only if $\sigma(A)$ vanishes on the fold of C.

Proof. The continuity statement (5.1.11) follows directly from the decomposition (5.1.9), standard continuity results for pseudodifferential operators and Fourier integral operators associated to canonical transformations and the bounds:

$$Ai(\zeta) \in L^\infty(\mathbb{R}^n), \quad (1 + |\xi|)^{-1/6} Ai'(\zeta) \in L^\infty(\mathbb{R}^n),$$

which follow from Appendix A.

Moreover the second term in (5.1.9) always gives an operator bounded as in (5.1.12). Since the Fourier integral operators are elliptic, (5.1.12) holds precisely when

$$P_1 \cdot \mathcal{A}i : H^s_c \longrightarrow H^{m-s}_{\text{loc}}. \tag{5.1.13}$$

Now, if $\sigma(A)$ restricted to the fold set is not zero then P_1 is elliptic at some point on the fold. Then (5.1.13) implies that $|D|^{1/6}\mathcal{A}i$ is itself microlocally bounded on H^s. This is clearly not the case, proving the Corollary.

Exercise 5.1.14: Prove the analogue of Theorem 5.1.8 for the action of pseudodifferential operators as a right module, i.e., that in place of (5.1.9) one can obtain a microlocal decomposition

$$A \equiv J_2 \cdot (\mathcal{A}iP_1 + \mathcal{A}i'P_2) \cdot J_1. \tag{5.1.15}$$

§5.2: Estimates on the multipliers

All the functions $b(\zeta)$ in (5.0.1) are meromorphic in the entire complex plane. We note here basic estimates on these multipliers in three regions, a strip around the positive real axis extending into a neighborhood of the origin:

$$\mathcal{P}_T = \{z \in \mathbb{C}; \mathrm{Re}(z) \geq -\gamma(1 + |\operatorname{Im}(z)|),\ |\operatorname{Im}(z)| \leq T\}, \tag{5.2.1}$$

a similar strip around the negative real axis:

$$\mathcal{T}_T = \{z \in \mathbb{C}; \mathrm{Re}(z) \leq -\gamma(1 + |\operatorname{Im}(z)|),\ |\operatorname{Im}(z)| \leq T\}, \tag{5.2.2}$$

and a smaller neighborhood of the negative real axis:

$$\mathcal{D}_T = \{z \in \mathbb{C}; \mathrm{Re}(z) \leq -\gamma(1 + |\operatorname{Im}(z)|),\ |\operatorname{Im}(z)| \leq T(1 + |z|)^{-1/2}\}. \tag{5.2.3}$$

In all cases $\gamma > 0$. Of course for $T > 0$ the region $\mathcal{D}_T$ is contained in $\mathcal{T}_T$, but the estimates in $\mathcal{D}_T$ will be stronger than those in $\mathcal{T}_T$. We shall also use the notation:

$$\mathcal{D}_T^{\pm} = \{z \in \mathcal{D}_T; \pm\operatorname{Im}(z) > 0\}, \quad \mathcal{T}_T^{\pm} = \{z \in \mathcal{T}_T; \pm\operatorname{Im}(z) > 0\}. \tag{5.2.4}$$

Figure 5.1

The proof of each of the following Lemmas can be found in Appendix A, where a detailed study of the behaviour of the Airy functions is carried out.

Lemma 5.2.5. *For $\gamma > 0$ sufficiently small and all $T > 0$ the functions in* (5.0.1) *are holomorphic in $\mathcal{P}_T$ and satisfy symbol estimates there:*

$$|b^{(j)}(z)| \leq C_j(1+|z|)^{m-j} \text{ in } \mathcal{P}_T \tag{5.2.6}$$

where

$$m = \begin{cases} \dfrac{1}{2} & \text{for } b = \Phi_\pm,\ \Phi i,\ A'_\pm Ai', (A_\pm Ai)^{-1} \\ 0 & \text{for } b = A'_\pm Ai,\ A_\pm Ai',\ (A'_\pm Ai)^{-1},\ (A_\pm Ai')^{-1} \\ -\dfrac{1}{2} & \text{for } b = \Phi_\pm^{-1},\ \Phi i^{-1},\ A_\pm Ai,\ (A'_\pm Ai')^{-1} \end{cases}. \tag{5.2.7}$$

The region $\mathcal{D}_T$ is important first because the branch of the fractional power $(-z)^{3/2}$ that is real and positive on the negative real axis is defined and analytic in $\mathcal{D}_T$. More importantly:

$$-CT \leq \operatorname{Im}(-z)^{3/2} \leq CT \text{ on } \mathcal{D}_T, \tag{5.2.8}$$

and because of this the exponential factor $\exp[\pm(2/3)(-z)^{3/2}]$ is bounded on $\mathcal{D}_T$.

Lemma 5.2.9. *For $\gamma > 0$ sufficiently small and each $T > 0$ there are constants $c, C > 0$ such that in $\mathcal{D}_T$*

$$|A_\pm(z)|,\ |Ai(z)| \leq C(1+|z|)^{-1/4}, \tag{5.2.10}$$

$$c|\operatorname{Im}(z)|(1+|z|) \leq |\Phi i(z)| \leq C|\operatorname{Im}(z)|^{-1}, \tag{5.2.11}$$

$$c|\operatorname{Im}(z)| \leq |A_\pm(z)Ai(z)| \leq C(1+|z|)^{-1/2}, \tag{5.2.12}$$

$$c|\operatorname{Im}(z)|(1+|z|)^{1/2} \leq |A'_\pm(z)Ai(z)|,\ |A_\pm Ai'| \leq C, \tag{5.2.13}$$

$$c|\operatorname{Im}(z)|(1+|z|) \leq |A'_\pm(z)Ai'(z)| \leq C(1+|z|)^{1/2}. \tag{5.2.14}$$

In the larger region $\mathcal{T}_T$ similar, but in some cases weaker estimates hold:

Lemma 5.2.15. *If $\gamma > 0$ is sufficiently small then, for each $T > 0$, $\Phi_\pm$ and $\Phi_\pm^{-1}$ satisfy the symbol estimates* (5.2.6) *in $\mathcal{T}_T$, with $m = 1/2$ and $m = -1/2$, respectively. Moreover,*

$$c|\operatorname{Im}(z)|(1+|z|)^{1/2} \leq |\Phi i(z)| \leq C|\operatorname{Im}(z)|^{-1}(1+|z|)^{1/2}, \tag{5.2.16}$$

and for the remaining cases of $b_\pm$ and m as in (5.2.7)

$$c|\operatorname{Im}(z)|(1+|z|)^m \leq |b_\pm(z)| \leq C|\operatorname{Im}(z)|^{-1}(1+|z|)^m \text{ in } \mathcal{T}_T^\pm \tag{5.2.17}$$

and

$$\begin{aligned}c|\operatorname{Im}(z)|(1+|z|)^m \exp[\pm 2\operatorname{Re}(-z)^{1/2}\operatorname{Im}(z)] \le |b_\pm(z)| \\ \le C|\operatorname{Im}(z)|^{-1}(1+|z|)^m \exp[\pm 2\operatorname{Re}(-z)^{1/2}\operatorname{Im}(z)] \quad \text{in } \mathcal{T}_T^\mp.\end{aligned} \tag{5.2.18}$$

In the region $\mathcal{D}_T$ there are more refined decompositions which capture the asymptotic behaviour of the various functions in (5.0.1). The simplest case after $\Phi_\pm$ is for the products of two Airy functions.

Lemma 5.2.19. *If $T > 0$ is sufficiently small then for each choice of*

$$b = A_\pm Ai,\ A'_\pm Ai,\ A_\pm Ai',\ A'_\pm Ai', \tag{5.2.20}$$

there are functions a_1, a_2 holomorphic in $\mathcal{D}_T$, satisfying the symbol estimates (5.2.6) *there with the order given by* (5.2.7) *and such that*

$$b_\pm = a_1 \exp\Big[\pm\frac{4}{3}i(-\operatorname{Re}(z))^{3/2}\Big] + a_2 \ \text{ in } \mathcal{D}_T^\pm. \tag{5.2.21}$$

The behaviour of Φi, or of any of the functions involving division by $\mathcal{A}i$ or $\mathcal{A}i'$ is similar but more subtle, as it involves an infinite series expansion of this type. For any open set $\mathcal{V} \subset \mathbf{C}$ let $\|b\|_{j,m,\mathcal{V}}$ denote the infimum over the constants for which (5.2.6) is valid.

Lemma 5.2.22. *Given $T > 0$ there exist $C_T > 0$ and functions $a_{k,\delta}$, $k = 0, 1, \dots,$ $\delta = \pm$, holomorphic in the region:*

$$\mathcal{V}^\delta = (\mathcal{D}_T^\delta \setminus \mathcal{D}_{T/2}^\delta) \cap \{\operatorname{Re}(z) < -C_T\}, \tag{5.2.23}$$

such that, for $m = 1/2$

$$\sum_{k=0}^\infty \|a_{k,\delta}\|_{m,j,\mathcal{V}^\delta} < \infty \quad j = 0, 1, \dots \tag{5.2.24}$$

and

$$\Phi i(z) = \sum_{k=0}^\infty a_{k,\delta} \exp\Big[-\delta\frac{4}{3}ki(-\operatorname{Re}(z))^{3/2}\Big] \ \text{ in } \mathcal{V}^\delta, \quad \delta = \pm. \tag{5.2.25}$$

More generally if Φi is replaced by Φi^{-1} the same result holds with $m = -1/2$. If Φi is replaced by $b_\pm^{-1}$, the inverse of one of the functions in (5.2.20)*, then a similar result holds with m given as in* (5.2.7) *and the corresponding series* (5.2.25) *convergent in the region $\mathcal{V}^\mp$.*

§5.3: Airy multiplier operator estimates

We proceed to examine the (local) boundedness on Sobolev spaces of the operators listed in (5.0.3), and their inverses.

The first case we consider is the basic one for diffractive problems, with the multipliers $\Phi_\pm$ in (5.0.3). The operators $\Phi_\pm$ are analytically and geometrically simpler even than $\mathcal{A}i$ of §5.1. Although representing essentially the same geometry they are in fact pseudodifferential operators.

Proposition 5.3.1. *For any cut-off ϕ as in* (5.0.4),

$$\Phi_\pm \in OPS^{1/3}_{1/3,0}, \quad \Phi_\pm^{-1} \in OPS^{0}_{1/3,0}. \tag{5.3.2}$$

Proof. Consider a general region:

$$\mathcal{U} = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n; |\xi_n| < c\xi_1, \xi_1 > 0, |\xi| > 1\}, \tag{5.3.3}$$

which includes the area in which the cut-off function in (5.0.4) is essentially supported. Since ϕ is a classical symbol of order 0, the first part of (5.3.2) follows from symbol estimates of type $(1/3, 0)$ and order $1/3$ for $\Phi_\pm(\zeta)$ in the region $\mathcal{U}$. These in turn follow from the chain rule and the estimates on $\Phi_\pm(z)$ in Lemma 5.2.5 and Lemma 5.2.15. Similarly for Φ^{-1}.

In fact these symbols satisfy somewhat more precise estimates:

$$|D^\gamma_{\xi_n} D^\alpha_{\xi'} \Phi_\pm(\zeta_0)| \le C_{\alpha\gamma} |\xi_1|^{1/3-|\alpha|} (\xi_1^{1/3} + |\xi_n|)^{-|\gamma|}, \tag{5.3.4}$$

and similarly:

$$|D^\gamma_{\xi_n} D^\alpha_{\xi'} \Phi_\pm^{-1}(\zeta_0)| \le C_{\alpha\gamma} |\xi_1|^{1/6-|\alpha|} (|\xi_1|^{1/3} + |\xi_n|)^{-1/2-|\gamma|}. \tag{5.3.5}$$

Some consequence of these estimates will be used in later discussions.

Next we consider the Fourier multipliers defined by Φi. This case is more complicated than either $\mathcal{A}i$ or $\Phi_\pm$, since the multiplier is not well-defined because of singularities on the real axis. To overcome this it is only necessary to complexify the argument. Thus consider for any $T \in \mathbb{R} \setminus 0$ sufficiently small

$$\Phi i(\xi_1^{-1/3}(\xi_n + iT)). \tag{5.3.6}$$

The estimates (5.2.11) show that these are of tempered growth in $\mathcal{U}$. If a cut-off function ϕ, as in (5.0.4), or more generally

$$\phi \in S^0(\mathbb{R}^n), \quad \operatorname{supp}(\phi) \subset \left\{\xi_1 > \frac{1}{3}(|\xi| + 1)\right\} \tag{5.3.7}$$

is inserted this gives smooth functions of tempered growth on $\mathbb{R}^n$. Thus convolution operators

$$\Phi i_T, \quad \Phi i_T^{-1} \tag{5.3.8}$$

are well-defined.

Similarly the other functions in (5.0.1) all lead to tempered functions on the region $\mathcal{U}$ with slow growth as $|\operatorname{Re}(\xi_n)| \to \infty$ so the corresponding operators

$$\begin{gathered}(\mathcal{A}_\pm \cdot \mathcal{A}i)_T, \quad (\mathcal{A}_\pm \cdot \mathcal{A}i')_T, \quad (\mathcal{A}'_\pm \cdot \mathcal{A}i)_T, \quad (\mathcal{A}'_\pm \cdot \mathcal{A}i')_T, \\ (\mathcal{A}_\pm \mathcal{A}i)_T^{-1}, \quad (\mathcal{A}_\pm \cdot \mathcal{A}i')_T^{-1}, \quad (\mathcal{A}'_\pm \cdot \mathcal{A}i)_T^{-1}, \quad (\mathcal{A}'_\pm \cdot \mathcal{A}i')_T^{-1},\end{gathered} \tag{5.3.9}$$

are all well-defined for $T \neq 0$.

Proposition 5.3.10. *For any $T \neq 0$ sufficiently small and any $s \in \mathbb{R}$, the operators in* (5.3.8) *and* (5.3.9) *are bounded from $H^s(\mathbb{R}^n)$ to $H^{s-q}(\mathbb{R}^n)$, where corresponding to the values of m in* (5.2.7) *$q = 1/3$ if $m = 1/2$ and $q = 0$ if $m = 0$ or $m = -1/2$.*

Proof. These continuity statements follow from the estimates in §5.2 above. Setting $z = \xi_1^{-1/3}\xi_n + iT\xi_1^{-1/3}$, the region $\mathcal{D}_T$ in (5.2.4) contains the points with $\xi_1 \geq c(T), \xi_n \leq 0$ for each $T \neq 0$. From (5.2.11) and (5.2.12) it follows that

$$|\Phi i(\xi_1^{-1/3}(\xi_n + iT))| \leq CT^{-1}\xi_1^{1/3}, \tag{5.3.11}$$

$$|\Phi i^{-1}(\xi_1^{-1/3}(\xi_n + iT))| \leq CT^{-1}\xi_1^{1/3}(1 + \xi_1^{-1/3}|\xi_n|)^{-1}, \tag{5.3.12}$$

$$|(A_\pm \cdot Ai)(\xi_1^{-1/3}(\xi_n + iT))| \leq C(1 + \xi_1^{-1/3}|\xi_n|)^{-1/2}, \tag{5.3.13}$$

and

$$|(A_\pm \cdot Ai)^{-1}(\xi_1^{-1/3}(\xi_n + iT))| \leq CT^{-1}\xi_1^{1/3}. \tag{5.3.14}$$

Estimates on the complementary region $\xi_n \geq 0$ are more elementary and follow directly from Lemma 5.2.5. Thus in the region $\xi_1 \geq C$, $\xi_n \geq 0$, $0 < |T| < T_0$,

$$|\Phi i(\xi_1^{-1/3}(\xi_n + iT))| \leq C(1 + \xi_1^{-1/3}|\xi_n|)^{1/2} \leq C(1 + |\xi|)^{1/3}, \tag{5.3.15}$$

$$|\Phi i^{-1}(\xi_1^{-1/3}(\xi_n + iT))| \leq C(1 + \xi_1^{-1/3}|\xi_n|)^{-1/2}, \tag{5.3.16}$$

$$|(A_\pm \cdot Ai)(\xi_1^{-1/3}(\xi_n + iT))| \leq C(1 + \xi_1^{-1/3}|\xi_n|)^{-1/2}, \tag{5.3.17}$$

$$|(A_\pm \cdot Ai)^{-1}(\xi_1^{-1/3}(\xi_n + iT))| \leq (1 + \xi_1^{-1/3}|\xi_n|)^{1/2} \leq C(1 + |\xi|)^{1/3}. \tag{5.3.18}$$

The assertions of the proposition now follow directly from (5.3.11)–(5.3.14) and (5.3.15)–(5.3.18).

§5.4: Wavefront relations

To start with we shall consider the 'classical' region, away from the fold $\xi_n = 0$. Thus let $\psi_1 \in S^0(\mathbb{R}^n)$ be a conic cut-off function such that for some $\varepsilon > 0$

$$\psi_1(\xi) = 0 \ \text{ if } \ |\xi_n| \le \varepsilon|\xi|, \tag{5.4.1}$$

$$\psi_1(\xi) = 1 \ \text{ if } \ |\xi_n| \ge 2\varepsilon|\xi|,\ |\xi| \ge 1. \tag{5.4.2}$$

We shall consider the convolution operators, denoted for example $\psi_1\Phi i_T$, obtained by inserting ψ_1 into (5.0.3) when $\mathcal{B}$ is one of the multipliers considered above. Recall the form of the canonical relation C_0, given in (5.1.3) above. Away from the fold set:

$$\Sigma = \{\xi_n = 0\},$$

C_0 is the graph of a pair of canonical transformations, the billiard ball maps $\delta_\pm$ discussed in Chapter 3. Away from Σ these maps have no recurrent points, since under iteration,

$$x_n(\delta_\pm^k) \longrightarrow \infty \text{ as } k \longrightarrow \infty.$$

The composite relation with k factors:

$$C_0 \cdot C_0 \cdots C_0$$

has, always away from Σ, $k+1$ components, namely the graphs of the iterates,

$$\delta_+^k, \delta_+^{k-2}, \ldots, \delta_-^k,$$

where in case the identity, as $\delta_\pm^0$, occurs the graph is restricted to $\xi_n < 0$. We shall denote by

$$\mathcal{C}_{k,\mathrm{reg}} = C_0 \cdots C_0 \cup [\mathrm{graph}(Id) \cap \{\xi_n > 0, \xi_1 > 0\}] \text{ over } \xi_1 > 0, \xi_n \ne 0, \tag{5.4.3}$$

the relation consisting of the k-fold iterate of C_0, over $\xi_1 > 0$, $\xi_n < 0$ and the identity over $\xi_1 > 0$, $\xi_n > 0$.

All these graphs, of the powers of the $\delta_\pm$, are disjoint away from Σ and locally finite in the sense that only a finite number of components meet any compact subset of $\xi_n > 0$, $\xi_1 > 0$. This allows us to define the formally infinite composite:

$$\mathcal{C}_{\infty,\mathrm{reg}} = \left[\bigcup_{k\in\mathbf{N}} (\mathrm{graph}(\delta_\pm^k)\right] \cup [\mathrm{graph}(Id) \cap \{\xi_n > 0, \xi_1 > 0\}] = \bigcup_k \mathcal{C}_{k,\mathrm{reg}}, \tag{5.4.4}$$

as a $\mathcal{C}^\infty$ locally embedded canonical relation. The subrelations:

$$\mathcal{C}^\pm_{\infty,\mathrm{reg}} = \left[\bigcup_{\pm k\ge 0} \mathrm{graph}(\delta_\pm^k)\right] \cup [\mathrm{graph}(Id) \cap \{\xi_n > 0, \xi_1 > 0\}]\,. \tag{5.4.5}$$

are of particular importance. Notice that in $\xi_1 > 0$, in which region the $\delta_\pm$ are defined,

$$\pm[\delta_\pm^* x_n - x_n] \geq 0.$$

Thus $\mathcal{C}^\pm_{\infty,\mathrm{reg}}$ are the parts of $\mathcal{C}_{\infty,\mathrm{reg}}$ where $x_n \geq y_n$ and $x_n \leq y_n$ respectively. We shall also use the further notation:

$$\mathcal{C}_{\pm,\mathrm{reg}} = [\mathrm{graph}(Id) \cap \{\xi_1 > 0, \xi_n \neq 0\}] \cup [\mathrm{graph}(\delta_\pm)]\,,$$
$$\delta_\pm \mathcal{C}^\pm_{\infty,\mathrm{reg}} = \left[\bigcup_{\pm k \geq 1} \mathrm{graph}(\delta_\pm^k)\right] [\mathrm{graph}(Id) \cap \{\xi_1 > 0, \xi_n \neq 0\}]\,.$$

Let us briefly consider the structure of Fourier integral operators associated with these various relations. Since they are all immersed canonical relations it is only necessary to find a parametrizations of each to get at least microlocal representations of the associated operators. In fact from the discussion above of Fourier integral operators associated to $\mathcal{C}_0$ it is apparent that

$$\tau_\pm = (x - y)\cdot\xi \mp \frac{2}{3}(-\xi_n)^{3/2}\xi_1^{-1/2}$$

are parametrizations of the graphs of $\delta_\pm$ in $\xi_n < 0$, $\xi_1 > 0$. More generally then the powers, δ_+^k, are parametrized by the phase functions:

$$\tau_k = (x - y)\cdot\xi - \frac{2}{3}k(-\xi_n)^{3/2}\xi_1^{-1/2}.$$

Fourier integral operators in, for example, the space $I^m(\mathbb{R}^n, \mathcal{C}^+_\infty)$ associated with $\mathcal{C}^+_\infty$ just consists of those operators with Schwartz' kernels which can be written microlocally as finite sums of oscillatory integrals:

$$I_k = \int \exp(i\tau_k) a_k(x, y, \xi)\, d\xi,$$

where a_k is a symbol of appropriate type and $k \geq 0$.

Proposition 5.4.6. *If ψ_1 is a conic cut-off as in* (5.4.1)–(5.4.2) *then*

$$\psi_1 \Phi i_T \in I^{1/3}(\mathbb{R}^n, \mathcal{C}^{\mathrm{sgn}(T)}_{\infty,reg}), \tag{5.4.7}$$

$$\psi_1 (\Phi i)_T^{-1} \in I^{-1/3}(\mathbb{R}^n, \mathcal{C}^{\mathrm{sgn}(T)}_{\infty,reg}), \tag{5.4.8}$$

$$\begin{gathered}\psi_1(\mathcal{A}_{-\,\mathrm{sgn}(T)}\mathcal{A}i)_T^{-1} \in I^{1/3}(\mathbb{R}^n, \mathcal{C}^{\mathrm{sgn}(T)}_{\infty,reg}),\\ \psi_1(\mathcal{A}_{\mathrm{sgn}(T)}\mathcal{A}i)_T^{-1} \in I^{1/3}(\mathbb{R}^n, \delta^2_{\mathrm{sgn}(T)}\mathcal{C}^{\mathrm{sgn}(T)}_{\infty,reg}),\end{gathered} \tag{5.4.9}$$

$$\psi_1(\mathcal{A}_{-\operatorname{sgn}(T)}\mathcal{A}i')_T^{-1} \in I^0(\mathbb{R}^n, \mathcal{C}_{\infty,reg}^{\operatorname{sgn}(T)}),$$
$$\psi_1(\mathcal{A}_{\operatorname{sgn}(T)}\mathcal{A}i')_T^{-1} \in I^0(\mathbb{R}^n, \delta^2_{\operatorname{sgn}(T)}\mathcal{C}_{\infty,reg}^{\operatorname{sgn}(T)}), \tag{5.4.10}$$

and

$$\psi_1\mathcal{A}_\pm\mathcal{A}i \in I^0(\mathbb{R}^n, \mathcal{C}_{\pm,reg}), \quad \psi_1\mathcal{A}_\pm\mathcal{A}i' \in I^0(\mathbb{R}^n, \mathcal{C}_{\pm,reg}) \tag{5.4.11}$$

are all classical Fourier integral operators.

Proof. These results follow from the asymptotic expansions derived in Appendix A. Since ψ_1 has essential support in two disconnected cones, one in $\xi_n > 0$ and the other in $\xi_n < 0$, we can consider separately the two regions, by replacing ψ_1 by

$$\psi_1 = \psi_1' + \psi_1'', \tag{5.4.12}$$

where ψ_1' has support in $\xi_n > 0$ and ψ_1'' has support in $\xi_n < 0$ outside some large ball. Now, $\psi_1'\Phi i_T$, $\psi_1'(\Phi i_T)^{-1}$ and $\psi_1'(\mathcal{A}_\pm\mathcal{A}i_T)^{-1}$ are classical pseudodifferential operators.

As for $\psi_1''\Phi i_T$, etc., write

$$\zeta_0 = \xi_1^{-1/3}(\xi_n + iT) = \omega\xi_1^{2/3} + iT\xi_1^{-1/3}, \quad \omega = \frac{\xi_n}{\xi_1},$$

and note that on the support of $\psi_1''(\xi)$, ζ_0 runs over a subset of $\mathbb{C}$ to which Lemma 5.2.22 applies. The results of Proposition 5.4.6 follow from this.

Next we consider the behaviour of these various convolution operators microlocally near the non-classical region Σ. If fact the simplest possible extension of the result in Proposition 5.4.6 is valid. Take the closures of the various relations defined above:

(5.4.13)
$$\mathcal{C}^\pm = \operatorname{cl}\left[\bigcup_{k\geq 0}\operatorname{graph}(\delta_\pm^k)\right] \cup \left[\operatorname{graph}(Id) \cap \{\xi_n \geq 0, \xi_1 > 0\}\right],$$

(5.4.14)
$$\mathcal{C}_\pm = \left[\operatorname{graph}(Id) \cap \{\xi_1 > 0\}\right] \cup \operatorname{cl}\left[\operatorname{graph}(\delta_\pm)\right],$$

(5.4.15)
$$\delta_\pm\mathcal{C}_\infty^\pm = \operatorname{cl}\left[\bigcup_{\pm k\geq 1}\operatorname{graph}(\delta_\pm^k)\right]\left[\operatorname{graph}(Id) \cap \{\xi_1 > 0, \xi_n \neq 0\}\right].$$

The form of C_0 is such that the additional points obtained in this way are easily described.

Lemma 5.4.16. *We have*

$$\mathcal{C}_\infty^{\mathrm{sgn}(T)} \setminus \mathcal{C}_{\infty,reg}^{\mathrm{sgn}(T)} = \mathcal{C}_\infty^{\mathrm{sgn}(T)} \cap \{\xi_n = 0\}$$

(5.4.17)

$$= \{(x,y,\xi,\eta); \xi_n = \eta_n = 0, \xi = \eta, x_j = y_j \text{ for } j < n, \mathrm{sgn}(T)(x_j - y_j) \geq 0\}.$$

Proof. The first equality is clear from the discreteness of $\mathcal{C}_{\infty,\mathrm{reg}}^{\pm}$ in $\xi_n \neq 0$. The second follows immediately from the form of the iterated billiard ball maps:

$$\delta_{\pm}^{k}(x,\xi) = \Big(x_1 \pm \frac{k}{3}\Big(-\frac{\xi_n}{\xi_1}\Big)^{3/2}, x'', x_n \pm k\Big(-\frac{\xi_n}{\xi_1}\Big)^{1/2}, \xi\Big), \quad \xi_n < 0,\ \xi_1 \geq 0.$$

Thus if a sequence in $\mathcal{C}_{\infty,\mathrm{reg}}^{\pm}$ is to converge as $\xi_n \to 0$ then $k(-\xi_n/\xi_1)^{1/2}$ must converge. The only restriction on the limit is that it must be positive. Necessarily, $k \to \infty$ and $x_1 - y_1 \to 0$, giving (5.4.17).

Similar remarks apply to the other relation, so

$$\delta_{\pm}\mathcal{C}_\infty^{\pm} \setminus \delta_{\pm}\mathcal{C}_{\infty,\mathrm{reg}}^{\pm}$$

is also given by the same formula (5.4.17). One the other hand,

$$\mathcal{C}_{\pm} \setminus \mathcal{C}_{\pm,\mathrm{reg}} = \mathrm{graph}(Id) \cap \{\xi_n = 0, \xi_1 > 0\}. \tag{5.4.18}$$

Theorem 5.4.19. *If ϕ is a cut-off as in* (5.0.6) *then the operators in* (5.0.3) *satisfy:*

$$WF'(\Phi i_T),\ WF'((\Phi i)_T^{-1}) \subset \mathcal{C}_\infty^{\mathrm{sgn}(T)}, \tag{5.4.20}$$

$$WF'((\mathcal{A}_{\mp\,\mathrm{sgn}(T)}\mathcal{A}i)_T^{-1}),\ WF'((\mathcal{A}_{\mp\,\mathrm{sgn}(T)}\mathcal{A}i')_T^{-1}) \subset \mathcal{C}_\infty^{\pm\,\mathrm{sgn}(T)}, \tag{5.4.21}$$

and

$$WF'(\mathcal{A}_{\pm}\mathcal{A}i),\ WF'(\mathcal{A}_{\pm}\mathcal{A}i') \subset \mathcal{C}_{\pm}. \tag{5.4.22}$$

Proof. Consider first (5.4.20). Let us assume for simplicity that $T > 0$. After Proposition 5.4.6 it remains only to show that if u is a distribution of compact support with wavefront set contained in a cone Γ, and if

$$\Gamma_e^{+} = \{(x,y,\xi,\eta); x_n \geq y_n,\ x_j = y_j,\ 1 \leq j < n, \xi = \eta, \xi_n = \eta_n = 0\}$$

then

$$\mathrm{WF}(\Phi i_T(u)) \cap \{\xi_n = 0\} \subset \Gamma_e^{+} \cdot \mathrm{WF}(u), \quad T > 0, \tag{5.4.23}$$

obtained by flowing out from Γ in the positive x_n direction within the fold surface. First we show this without the sign condition.

Consider the vector field

$$V_1 = \frac{\partial}{\partial \xi_1} + \frac{1}{3}\xi_1^{-1}(\xi_n + iT)\frac{\partial}{\partial \xi_n}.$$

With $\zeta_T = \xi_1^{-1/3}(\xi_n + iT)$,

$$V_1 \zeta_T = 0. \tag{5.4.24}$$

Thus,

$$V_1 \left[\phi(\xi)\Phi i(\zeta_T)\right] = V_1\phi(\xi)\Phi i(\zeta_T). \tag{5.4.25}$$

It follows that if b is the kernel of the convolution operator Φi_T then

$$f = \phi(D)\left[x_1 + \frac{1}{3}|D_1|^{-1}(D_n + iT)x_n\right] b \tag{5.4.26}$$

is the kernel of a convolution operator, F, of the form (5.0.3) except that ϕ is replaced by a symbol of order minus one supported away from $\{\xi_n = 0\}$. Thus f has no wavefront near $\{\xi_n\}$ and F has no wavefront relation there. Since the operator applied to b in (5.4.26) is elliptic on the fold set, away from $\{x_1 = 0\}$

$$WF(b) \cap \{\xi_n = 0\} \subset \{x_1 = 0\}.$$

Similarly the simpler vector fields

$$V_j = \frac{\partial}{\partial \xi_j}, \quad j = 2, \dots, n-1,$$

can be used to show that

$$WF(b) \cap \{\xi_n = 0\} \subset \{x_1 = 0, x_2 = 0, \dots, x_{n-1} = 0\}.$$

This proves (5.4.23) except for the sign condition on x_n.

The sign condition follows from the observation that

$$\begin{aligned} T > 0 &\Rightarrow \Phi i_T \delta(x) = 0, \quad \text{for} \quad x_n < 0, \\ T < 0 &\Rightarrow \Phi i_T \delta(x) = 0, \quad \text{for} \quad x_n > 0, \end{aligned} \tag{5.4.27}$$

and similarly for the other Airy multipliers. This follows from the holomorphy of the function $\Phi i(\xi_1^{-1/3}(\xi_n + iT))$ for ξ_n in a half-plane ($\operatorname{Im}\xi_n > 0$, for $T > 0$; $\operatorname{Im}\xi_n < 0$, for $T < 0$), the estimate (A.4.35), and the Paley-Wiener theorem.

Chapter 6: Fourier-Airy Operators

This chapter contains a discussion of some of the properties of the operators that will be used in the next chapter to give parametrices for boundary problems. There are three related types of Fourier-Airy integral operators we will consider. These can be written symbolically:

$$Av = \int \big[gA_\pm(\zeta) + ihA'_\pm(\zeta)\big]A_\pm(\zeta_0)^{-1}e^{i\theta}\hat{v}(\xi)\,d\xi, \tag{6.0.1}$$

$$Bv = \int \big[gAi(\zeta) + ihAi'(\zeta)\big]A_\pm(\zeta_0)e^{i\theta}\hat{v}(\xi)\,d\xi, \tag{6.0.2}$$

and

$$Cv = \int \big[gAi(\zeta) + ihAi'(\zeta)\big]Ai(\zeta_0)^{-1}e^{i\theta}\hat{v}(\xi)\,d\xi. \tag{6.0.3}$$

Note that the operator C in (6.0.3) can be written as the composite:

$$C = B \circ (\mathcal{A}_\pm\mathcal{A}i)_T^{-1}, \tag{6.0.4}$$

where $(\mathcal{A}_\pm\mathcal{A}i)_T$ is an Airy multiplier studied in Chapter 5. For this reason we concentrate our attention on the two operators A, B given by (6.0.1), (6.0.2).

In §6.1 we lay down our basic hypotheses on the phase functions ζ and θ, and on the amplitudes g and h. These hypotheses are satisfied in the case of operators that will be proposed in Chapter 7 as parametrices for solutions to wave equations with grazing and gliding rays, as consequences of the material developed in Chapter 4. Section 6.2 gives a brief justification that A and B are well defined on smooth functions with compact support. Sections 6.3 and 6.4 make a more detailed study of the operators B and A, respectively, on distributions. These operators are decomposed into various pieces, which can be treated as Fourier integral operators with singular phase functions. These results are applied in §6.5 to the study of the microlocal singularities of Av, Bv, and Cv.

§6.1: Basic hypotheses

Here we set conditions on the various ingredients in (6.0.1)–(6.0.3). First, we take

$$\zeta_0 = \xi_1^{-1/3}(\xi_n + iT). \tag{6.1.1}$$

The phase functions ζ, θ and amplitudes g, h are defined and smooth for x is some neighborhood $\overline{U}$ in Ω of a boundary point $x_0 \in \partial\Omega$ and for real ξ in $\Gamma = \{|\xi_n| \leq$

$C_0\xi_1,\ \xi_1 > 0\}$, and in a complex neighborhood $\widetilde{\Gamma}$ in the ξ_n variable. The two phase functions are

$$\theta = \theta(x, \xi', \xi_n + iT) \tag{6.1.2}$$

and

$$\zeta = \zeta(x, \xi', \xi_n + iT), \tag{6.1.3}$$

which were constructed in Chapter 4, and the amplitudes are

$$g = g(x, \xi', \xi_n + iT), \tag{6.1.4}$$

$$h = h(x, \xi', \xi_n + iT). \tag{6.1.5 }$$

These of course are classical symbols. In (6.0.1) we can take $T = 0$, but in (6.0.2)–(6.0.3) we want to take $T \neq 0$. We make the following hypotheses on the phase functions and amplitudes, which, as we have seen in Chapter 4, can be arranged to hold:

(6.1.6)

$$\begin{gathered}\zeta(x,\xi),\ \theta(x,\xi)\ \text{ real valued for } \xi \in \mathbb{R}^n,\ \text{almost analytic in } \xi_n, \\ \text{and homogeneous of degree } 2/3 \ \text{ and } \ 1,\ \text{respectively, in } \xi,\end{gathered}$$

$$d_x\theta(x,\xi) \neq 0 \ \text{ in } \ \overline{U} \times (\Gamma \setminus 0), \tag{6.1.7}$$

$$\zeta(x,\xi) = \xi_1^{-1/3}\xi_n \ \text{ for } \ x \in \partial\Omega, \tag{6.1.8}$$

$$\frac{\partial\zeta}{\partial\nu} < 0 \ \text{ in } \ (6.0.1); \quad \frac{\partial\zeta}{\partial\nu} > 0 \ \text{ in } \ (6.0.2), \quad x \in \partial\Omega. \tag{6.1.9}$$

$$g \in S^m(\overline{U} \times \Gamma), \quad h \in S^{m-1/3}(\overline{U} \times \Gamma), \tag{6.1.10}$$

$$F \in \mathcal{E}'(\mathbb{R}^n), \quad WF(F) \subset \{\xi : |\xi_n| \leq C_0\xi_1/2,\ \xi_1 > 0\}. \tag{6.1.11}$$

§6.2: Action of A and B on $\mathcal{C}_c^\infty$

Our first task is to give $A(F)$ and $B(F)$ a meaning for $F \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. In such a case, the integrals (6.0.1) and (6.0.2) are absolutely convergent, but even this fact requires an argument. Indeed, we have the following.

Lemma 6.2.1. *We have the estimates, for* $|T| \le T_0$*,*

$$|D_x^\beta A_\pm(\zeta)A_\pm(\zeta_0)^{-1}| \le C_\beta(1+|\xi|)^{1/6+|\beta|}, \tag{6.2.2}$$

and

$$|D_x^\beta Ai(\zeta)A_\pm(\zeta_0)| \le C_\beta(1+|\xi|)^{|\beta|}, \tag{6.2.3}$$

with similar estimates holding for x*-derivatives of* $A'_\pm(\zeta)/A_\pm(\zeta_0)$ *and for* $Ai'(\zeta)A_\pm(\zeta_0)$*.* ∎

Proof. Use of the chain rule gives

$$D_x^\beta A_\pm(\zeta) = \sum_{|\mu|\le|\beta|} a_\mu(x,\xi)A_\pm^{(\mu)}(\zeta), \tag{6.2.4}$$

with $A_\mu \in S^{2\mu/3}$. The Airy equation implies

$$A_\pm^{(\mu)}(z) = p_\mu(z)A_\pm(z) + q_\mu(z)A'_\pm(z), \tag{6.2.5}$$

where $p_\mu(z)$ and $q_\mu(z)$ are polynomials, of the following orders:

$$\begin{aligned} p_{2k}(z) &= O(|z|^k), \quad p_{2k+1}(z) = O(|z|^{k-1}), \\ q_{2k}(z) &= O(|z|^{k-2}), \quad q_{2k+1}(z) = O(|z|^k). \end{aligned} \tag{6.2.6}$$

These formulas (and analogues for Ai) reduce the proof of the lemma to establishing (6.2.2)–(6.2.3) for $\beta = 0$, as well as obtaining the estimates

$$|A'_\pm(\zeta)A_\pm(\zeta_0)^{-1}| \le C(1+|\xi|)^{1/3}, \quad |Ai'(\zeta)A_\pm(\zeta_0)| \le C(1+|\xi|)^{1/6}. \tag{6.2.7}$$

Now to establish (6.2.2) with $\beta = 0$, use the asymptotic expansion

$$A_\pm(z) = \Psi(\omega^{\mp 2}z)e^{\mp(2/3)i(-z)^{3/2}},$$

derived in Appendix A; see (A.1.5). We have

$$\frac{A_\pm(\zeta)}{A_\pm(\zeta_0)} = \frac{\Psi(\omega^{\mp 2}\zeta)}{\Psi(\omega^{\mp 2}\zeta_0)}e^{\mp(2i/3)[(-\zeta)^{3/2}-(-\zeta_0)^{3/2}]}. \tag{6.2.8}$$

The asymptotic behavior of $\Psi(z)$ given by (A.1.4) implies

$$\left|\frac{\Psi(\omega^{\mp 2}\zeta)}{\Psi(\omega^{\mp 2}\zeta_0)}\right| \le C(1+|\xi|)^{1/6}. \tag{6.2.9}$$

Furthermore the hypothesis (6.1.9) implies the exponential factor in (6.2.8) is bounded. This gives the estimate (6.2.2), for $\beta = 0$, and the other estimates are similarly established. This proves the lemma.

From Lemma 6.2.1 it follows immediately that, if $F \in C_0^\infty(\mathbb{R}^n)$, then $A(F)$ and $B(F)$ belong to $C^\infty(\overline{U})$. The main task of the rest of this chapter is to define $A(F)$ and $B(F)$ for $F \in \mathcal{E}'(\mathbb{R}^n)$ satisfying (6.1.11) and to analyze the singularities of these distributions on $\overline{U}$.

§6.3: Analysis of B on distributions

Our finer investigation of the operators A and B will start with B, given by (6.0.2). First, we separate ζ and ζ_0 into their real and imaginary parts:

$$\zeta = \omega + i\sigma, \quad \zeta_0 = \omega_0 + i\sigma_0. \tag{6.3.1}$$

Note that $\sigma, \sigma_0 \in S^{-1/3}$ and $\omega, \omega_0 \in S^{2/3}$; $\omega_0 = \xi_1^{-1/3}\xi_n$ and $\sigma_0 = T\xi_1^{-1/3}$. We introduce a cut-off:

$$\chi_1(\omega_0) + \chi_2(\omega_0) = 1, \tag{6.3.2}$$

where χ_j are smooth with $\chi_1(s) = 0$ for $s \le -1$, $\chi_2(s) = 0$ for $s \ge 1$, so $\chi_1(s) = 1$ for $s \ge 1$ and $\chi_2(s) = 1$ for $s \le -1$. Extend to complex argument by

$$\chi_j(s) = \chi_j(\operatorname{Re} s). \tag{6.3.3}$$

Consider now

$$B_1(F) = \int \big[g\, Ai(\zeta) + ih\, Ai'(\zeta)\big] A_\pm(\zeta_0)\chi_1(\zeta_0) e^{i\theta} \hat{F}(\xi)\, d\xi. \tag{6.3.4}$$

We claim B_1 behaves like a Poisson integral. We introduce a class of symbols as follows. Suppose $\overline{U} = [0, a) \times \mathcal{O}$, $\mathcal{O} \subset \partial\Omega$, with coordinates $x = (y, x')$.

Definition 6.3.5. We say $p(y, x', \xi) \in S^m_{\rho,\delta,\nu}(\overline{U} \times \Gamma)$ if, on $\overline{U} \times \Gamma$, we have

$$|D_y^k D_{x'}^\beta D_\xi^\alpha p(y, x', \xi)| \le C_{\alpha\beta k}(1 + |\xi|)^{m - \rho|\alpha| + \delta|\beta| + \nu k}. \tag{6.3.6}$$

The key to the behavior of (6.3.4) is given by the following result.

Lemma 6.3.7. *We have*

$$Ai(\zeta)A_\pm(\zeta_0)\chi_1(\zeta_0) \in S^0_{1/3,2/3,1}(\overline{U} \times \Gamma), \tag{6.3.8}$$

$$Ai'(\zeta)A_\pm(\zeta_0)\chi_1(\zeta_0) \in S^{1/6}_{1/3,2/3,1}(\overline{U} \times \Gamma). \tag{6.3.9}$$

Furthermore, if the left sides are multiplied by y^j, the orders on the right sides can be reduced by $2j/3$.

Proof. Note that, by assumption (6.1.9),

$$\operatorname{Re}\zeta \ge \operatorname{Re}\zeta_0 + C_1 y|\xi|^{2/3}, \quad C_1 > 0. \tag{6.3.10}$$

Now it is easy to establish

$$Ai(\zeta)\chi_3(\zeta_0) \in S^0_{1/3,2/3,1}(\overline{U} \times \Gamma), \tag{6.3.11}$$

for any $\chi \in C_0^\infty(\mathbb{R})$ (with $\chi_3(\zeta_0) = \chi_3(\mathrm{Re}\,\zeta_0)$), with a similar result for $Ai'(\zeta)\chi_3(\zeta_0)$, so without loss of generality we can replace $\chi_1(\zeta_0)$ by $\chi_1'(\zeta_0)$, supported on $\mathrm{Re}\,\zeta_0 \geq 1/2$. Use the decompositions

$$A_\pm(\zeta_0) = \Gamma_\pm(\zeta_0) e^{(2/3)\zeta_0^{3/2}}, \quad \mathrm{Re}\,\zeta_0 > 0, \tag{6.3.12}$$

$$Ai(\zeta) = \Psi(\zeta) e^{-(2/3)\zeta^{3/2}}, \quad \mathrm{Re}\,\zeta > 0, \tag{6.3.13}$$

derivable from (A.1.3)–(A.1.4), where

$$\Gamma_\pm(\zeta_0) \sim \zeta_0^{-1/4} \sum_{j\geq 0} \gamma_j^\pm \, \zeta_0^{-3j/2}. \tag{6.3.14}$$

We have

$$Ai(\zeta)A_\pm(\zeta_0)\chi_1'(\zeta_0) = \chi_1'(\zeta_0)\Gamma_\pm(\zeta_0)\Psi(\zeta)e^{-(2/3)(\zeta^{3/2}-\zeta_0^{3/2})}. \tag{6.3.15}$$

We can replace $\chi_1'(\zeta_0)$ by

$$\chi_1'(\zeta_0) = \chi_1^\#(\zeta_0)^2 = \chi_1^\#(\zeta_0)^2 \chi_1^b(\zeta)^2, \tag{6.3.16}$$

with $\chi_1^\#(s)$, $\chi_1^b(s)$ supported on $\mathrm{Re}\,s \geq 1/4$, equal to 1 for $s \geq 1$, and in order to prove (6.30), it suffices to show

$$\chi_1^\#(\zeta_0)\Gamma_\pm(\zeta_0) \in S^0_{1/3,0}(\Gamma), \tag{6.3.17}$$

$$\chi_1^b(\zeta)\Psi(\zeta) \in S^0_{1/3,2/3,2/3}(\overline{U} \times \Gamma), \tag{6.3.18}$$

$$\chi_1^\#(\zeta_0)\chi_1^b(\zeta)e^{-(2/3)(\zeta^{3/2}-\zeta_0^{3/2})} \in S^0_{1/3,2/3,1}(\overline{U} \times \Gamma). \tag{6.3.19}$$

The containments (6.3.17) and (6.3.18) are equivalent to routine estimates on the derivatives of these functions, so we concentrate on the details of (6.3.19). The chain rule gives

(6.3.20)
$$\begin{aligned} &D_y^k D_{x'}^\beta D_\xi^\alpha e^{-(2/3)(\zeta^{3/2}-\zeta_0^{3/2})} \\ &\quad = \sum C\, D_y^{k_1} D_{x'}^{\beta_1} D_\xi^{\alpha_1}(\zeta^{3/2}-\zeta_0^{3/2}) \cdots D_y^{k_\mu} D_{x'}^{\beta_\mu} D_\xi^{\alpha_\mu}(\zeta^{3/2}-\zeta_0^{3/2})\, e^{-(2/3)(\zeta^{3/2}-\zeta_0^{3/2})}. \end{aligned}$$

Here the sum is over

$$\alpha_1 + \cdots + \alpha_\mu = \alpha, \quad \beta_1 + \cdots + \beta_\mu = \beta, \quad k_1 + \cdots + k_\mu = k.$$

Note that (6.1.9) implies

$$\operatorname{Re}(\zeta^{3/2}-\zeta_0^{3/2})\ge C' y^{3/2}|\xi|,\quad \text{for } \zeta_0\ge 0,\ |\xi| \text{ large.} \tag{6.3.21}$$

Elementary estimates give, on the support of $\chi_1^{\#}(\zeta_0)\chi_1^b(\zeta)$,

(6.3.22)
$$|D_y^k D_{x'}^\beta D_\xi^\alpha(\zeta^{3/2}-\zeta_0^{3/2})|\, e^{-cy^{3/2}|\xi|}\le C|\zeta|^{1/2}|\xi|^{2k/3-|\alpha|/3},\quad k\ge 1,$$

(6.3.23)
$$|D_{x'}^\beta D_\xi^\alpha(\zeta^{3/2}-\zeta_0^{3/2})|\, e^{-cy^{3/2}|\xi|}\le C(|\xi|^{-|\alpha|/3}+|\zeta|^{1/2}|\xi|^{-|\alpha|}),\quad |\beta|\ge 1,$$

(6.3.24)
$$|D_\xi^\alpha(\zeta^{3/2}-\zeta_0^{3/2})|\, e^{-c(\zeta^{3/2}-\zeta_0^{3/2})}\le C|\xi|^{-|\alpha|/3},\quad |\alpha|\ge 1.$$

From these estimates, (6.3.19) is a simple consequence. This gives (6.3.8), and the assertion (6.3.9) is similarly established. The last assertion of Lemma 6.3.7 comes about because

$$y^j e^{-cy^{2/3}|\xi|}\le C_j|\xi|^{-2j/3}.$$

This proves the lemma.

We turn next to the analysis of

$$B_2(F)=\int\big[g\,Ai(\zeta)+ih\,Ai'(\zeta)\big]A_\pm(\zeta_0)\chi_2(\zeta_0)e^{i\theta}\hat F(\xi)\,d\xi. \tag{6.3.25}$$

Indeed, the operators $\mathfrak{A}_\pm$ defined by

$$(\mathfrak{A}_\pm F)\hat{}(\xi)=A_\pm(\zeta_0)\chi_2(\zeta_0)\hat F(\xi) \tag{6.3.26}$$

are easily analyzed as Fourier integral operators with singular phase, and we have

$$B_2(F)=B_2^{\#}(\mathfrak{A}_\pm F), \tag{6.3.27}$$

with

$$B_2^{\#}(G)=\int\big[g\,Ai(\zeta)+ih\,Ai'(\zeta)\big]e^{i\theta}\hat G(\xi)\,d\xi. \tag{6.3.28}$$

One way to analyze (6.3.28) is to use the partition of unity

$$\chi_1(\zeta)+\chi_2(\zeta)=1, \tag{6.3.29}$$

with $\chi_j(s)$ as in (6.3.2). Call the resulting decomposition of $B_2^{\#}$

$$B_2^{\#}=B_{21}+B_{22}. \tag{6.3.30}$$

An analysis similar to that of (6.3.4), even a little simpler, exhibits B_{21} as a Poisson-like integral. Indeed, an argument parallel to but a little simpler than the proof of Lemma 6.3.7 gives:

Lemma 6.3.31. *We have*

$$Ai(\zeta)\chi_1(\zeta) \in S^0_{1/3,2/3,2/3}(\overline{U} \times \Gamma), \tag{6.3.32}$$

$$Ai'(\zeta)\chi_1(\zeta) \in S^{1/6}_{1/3,2/3,2/3}(\overline{U} \times \Gamma). \tag{6.3.33}$$

Furthermore, if the left sides are multiplied by $y^j \chi_2^{\#}(\zeta_0)$, where $\chi_2^{\#}(\zeta_0)$ is smooth and equal to 0 *for $\zeta_0 > 2$, equal to* 1 *for $\zeta_0 < 1$, the orders are reduced by $2j/3$.*

Thus, in order to understand $B_2(F)$, given by (6.3.25), we are reduced to understanding

$$B_{22}(G) = \int \bigl[g\, Ai(\zeta) + ih\, Ai'(\zeta)\bigr]\chi_2(\zeta) e^{i\theta} \hat{G}(\xi)\, d\xi. \tag{6.3.34}$$

Our analysis is simplified if we make the following (permissible) hypothesis:

$$\chi_2(s) = 0 \quad \text{for} \quad \operatorname{Re} s \geq -\frac{1}{2}. \tag{6.3.35}$$

We will write (6.3.34) as a sum of two Fourier integral operators with singular phase functions, using the decomposition

$$\begin{aligned} Ai(\zeta) &= \overline{\omega} A_+(\zeta) + \omega A_-(\zeta) \\ &= \overline{\omega}\Psi_+(\zeta) e^{-(2i/3)(-\zeta)^{3/2}} + \omega \Psi_-(\zeta) e^{(2i/3)(-\zeta)^{3/2}}, \end{aligned} \tag{6.3.36}$$

for $\operatorname{Re}\zeta < 0$, which is given by (A.0.4) and (A.1.5), where we have set

$$\Psi_\pm(\zeta) = \Psi(\omega^{\mp 2}\zeta). \tag{6.3.37}$$

The functions $\Psi_\pm(\zeta)$ have asymptotic expansions derived from (A.1.4), so

$$\Psi_\pm(\zeta) \sim \zeta^{-1/4} \sum_{j \geq 0} a_j^\pm \, (-\zeta)^{-3j/2}, \quad -\zeta \to \infty. \tag{6.3.38}$$

Similarly we have

$$Ai'(\zeta) = \tilde{\Psi}_+(\zeta) e^{-(2i/3)(-\zeta)^{3/2}} + \tilde{\Psi}_-(\zeta) e^{(2i/3)(-\zeta)^{3/2}}, \quad \operatorname{Re}\zeta < 0, \tag{6.3.39}$$

with

$$\tilde{\Psi}_\pm(\zeta) \sim \zeta^{1/4} \sum_{j \geq 0} b_j^\pm \, (-\zeta)^{-3j/2}. \tag{6.3.40}$$

It follows that

$$B_{22} = B_{22}^+ + B_{22}^-, \tag{6.3.41}$$

with

$$B_{22}^\pm(G) = \omega^{\mp 1} \int \bigl[g\Psi_\pm(\zeta) + ih\tilde{\Psi}_\pm(\zeta)\bigr] e^{i(\theta \mp (2/3)(-\zeta)^{3/2})} \chi_2(\zeta)\hat{G}(\xi)\, d\xi. \tag{6.3.42}$$

From the support condition (6.3.35) on $\chi_2(\zeta)$ and the fact that

$$\begin{aligned} \Psi_\pm(\zeta) &\in S^0_{1/3,2/3,2/3}(\overline{U} \times \Gamma), \\ \tilde{\Psi}_\pm(\zeta) &\in S^{1/6}_{1/3,2/3,2/3}(\overline{U} \times \Gamma), \end{aligned} \tag{6.3.43}$$

it follows that (6.3.42) defines a pair of Fourier integral operators with singular phase function, such as treated in Appendix D.

§6.4: Analysis of A on distributions

The operator A, given by (6.0.1), can be analyzed by a process similar to that applied to B above. We start with the partition (6.3.29) and write

$$A = A_1 + A_2, \tag{6.4.1}$$

with

$$A_j(F) = \int \big[g\, A_\pm(\zeta) + ih\, A'_\pm(\zeta)\big]\chi_j(\zeta)A_\pm(\zeta_0)^{-1}e^{i\theta}\hat{F}(\xi)\,d\xi. \tag{6.4.2}$$

The same reasoning used in the proof of Lemma 6.3.7 gives:

Lemma 6.4.3. *We have*

$$A_\pm(\zeta)\chi_1(\zeta)A_\pm(\zeta_0)^{-1} \in S^{1/6}_{1/3,2/3,1}(\overline{U}\times\Gamma), \tag{6.4.4}$$

$$A'_\pm(\zeta)\chi_1(\zeta)A_\pm(\zeta_0)^{-1} \in S^{1/3}_{1/3,2/3,1}(\overline{U}\times\Gamma), \tag{6.4.5}$$

multiplication by y^j decreasing the orders by $2j/3$. Thus A_1 is a Poisson-like integral.

We rewrite A_2 as

$$A_2(F) = D_2(\mathcal{A}_\pm^{-1}F), \tag{6.4.6}$$

where

$$(\mathcal{A}_\pm^{-1}F)\hat{}\,(\xi) = A_\pm(\zeta_0)^{-1}\hat{F}(\xi) \tag{6.4.7}$$

is treated in Chapter 5. The operator D_2 is defined by

$$D_2(G) = \int \big[g\, A_\pm(\zeta) + ih\, A'_\pm(\zeta)\big]\chi_2(\zeta)e^{i\theta}\hat{G}(\xi)\,d\xi. \tag{6.4.8}$$

The analysis of D_2 as a Fourier integral operator with singular phase proceeds exactly as that of C_{22}, given by (6.3.34). We have

$$D_2(G) = \int \big[g\Psi_\pm(\zeta) + ih\tilde{\Psi}_\pm(\zeta)\big]e^{i(\theta\mp(2/3)(-\zeta)^{3/2})}\chi_2(\zeta)\hat{G}(\xi)\,d\xi, \tag{6.4.9}$$

assuming, as we might, that $\chi_2(s) = 0$ for $s \geq -1/2$.

§6.5: Microlocal behavior of A, B, and C

Here we describe the singularities of AF, BF, and CF for distributions F. We start with BF, considering the two pieces that arose in §6.3:

$$BF = B_1 F + B_2^{\#} \mathfrak{A}_{\pm} F, \tag{6.5.1}$$

with B_1 given by (6.3.4) and $B_2^{\#}$ given by (6.3.28).

Proposition 6.5.2. *On the interior region* Ω,

$$B_1 F \in \mathcal{C}^{\infty}(\Omega). \tag{6.5.3}$$

Proof. Indeed, given $\ell \in \mathbb{Z}^+$, we can pick k such that

$$y^k B_1 F \in \mathcal{C}^{\ell}(\overline{\Omega}),$$

by the results of Lemma 6.3.7.

To state a result on the singularity of $B_1 F$ at the boundary of Ω, we bring in the Fourier integral operator K, given by

$$KF(x') = \int e^{i\theta_0} \hat{F}(\xi)\, d\xi, \quad \theta_0 = \theta\big|_{\partial\Omega}. \tag{6.5.4}$$

We will augment the hypothesis (6.1.7) as follows:

$$\theta_0 \;\; \text{generates a (locally bijective) canonical transformation.} \tag{6.5.5}$$

Hence K is an elliptic Fourier integral operator, which is microlocally invertible.

Proposition 6.5.6. *Assume $WF(KF)$ is disjoint from an open conic set $\Gamma \subset T^*\partial\Omega \setminus 0$. Let $P \in OPS^0(\partial\Omega)$ have symbol supported in Γ. Then*

$$P(x', D_{x'}) B_1 F \in \mathcal{C}^{\infty}(\overline{\Omega}).$$

Recall from §6.3 that

$$\mathfrak{A}_{\pm} = \mathcal{A}_{\pm} \chi_2(\zeta_0(D)), \tag{6.5.7}$$

and

$$B_2^{\#} G = \int \big[g Ai(\zeta) + ih Ai'(\zeta) \big] e^{i\theta} \hat{G}(\xi)\, d\xi. \tag{6.5.8}$$

One way to analyze $B_2^{\#} \mathfrak{A}_{\pm} F$ is to expand it out as

$$B_2^{\#} \mathfrak{A}_{\pm} F = B_{21} \chi_2^{\#}(\zeta_0(D)) \mathfrak{A}_{\pm} F + \overline{\omega} B_{22}^{+} \mathfrak{A}_{\pm} F + \omega B_{22}^{-} \mathfrak{A}_{\pm} F, \tag{6.5.9}$$

with B_{21} defined by (6.3.29)–(6.3.30) and $B_{22}^{\pm}$ by (6.3.42). Straightforward analogues of Proposition 6.5.2 and Proposition 6.5.6 give

$$B_{21}\chi_2^{\#}(\zeta_0(D))\mathfrak{A}_{\pm}F \in \mathcal{C}^\infty(\Omega), \tag{6.5.10}$$

and, when $P \in OPS^0(\partial\Omega)$ has symbol supported in $\Gamma \subset T^*\partial\Omega \setminus 0$ and $\Gamma \cap WF(KG) = \emptyset$,

$$P(x', D_{x'})B_{21}\chi_2^{\#}(\zeta_0(D))G \in \mathcal{C}^\infty(\overline{\Omega}). \tag{6.5.11}$$

This is applicable to (6.5.9) with $G = \mathfrak{A}_{\pm}F$, and the formula (6.5.7) for $\mathfrak{A}_{\pm}$ yields

$$WF(\mathfrak{A}_{\pm}F) = \delta_h^{\pm 1/2}\ WF(F) \cap \{\xi_n \le 0\}, \tag{6.5.12}$$

assuming $WF(F) \subset \{\xi_1 > 0\}$, where

$$\delta_h^{\pm 1/2}(x,\xi) = \Big(x_1 \pm \frac{1}{3}\Big(-\frac{\xi_n}{\xi_1}\Big)^{3/2}, x'', x_n \pm \Big(-\frac{\xi_n}{\xi_1}\Big)^{1/2}, \xi\Big). \tag{6.5.13}$$

As for the last two terms in (6.5.9), we can specify the singularities in the interior region Ω as follows. Recall that

$$B_{22}^{\pm}G = \int e(x,\xi)e^{i(\theta \mp (2/3)(-\zeta)^{3/2})}\hat{G}(\xi)\, d\xi, \tag{6.5.14}$$

with

$$e(x,\xi) = \big[g\Psi_{\pm}(\zeta) + ih\widetilde{\Psi}_{\pm}(\zeta)\big]\chi_2(\zeta) \in S^m_{1/3,2/3,2/3}(\overline{U} \times \Gamma). \tag{6.5.15}$$

(We assume $\chi_2(s) = 0$ for $\operatorname{Re} s \ge -1/2$.) The operators $B_{22}^{\pm}$ act as Fourier integral operators with singular phase, and we have

$$\begin{gathered} WF\ B_{22}^{\pm}G\big|_{\Omega} \subset \text{ image of } \ WF(G) \text{ under} \\ (\nabla_\xi\varphi^{\pm}, \xi) \mapsto (x, \nabla_x\varphi^{\pm}), \end{gathered} \tag{6.5.16}$$

where

$$\varphi^{\pm}(x,\xi) = \theta \mp \frac{2}{3}(-\zeta)^{3/2}, \tag{6.5.17}$$

and the wave front relation is supported in the region where $\zeta(x,\xi) \le 0$.

A couple of remarks on the operator $B_2^{\#}$ are in order. One is that the integrand in (6.5.8) can be extended to x in a two-sided neighborhood $\widetilde{\Omega}$ of $\partial\Omega$, and then

(6.5.9)–(6.5.17) hold on $\widetilde{\Omega}$. Furthermore, the two pieces in (6.5.17) actually fit together to form a smooth canonical relation, and

$$B_2^\# : \mathcal{C}^{-\infty}(\mathbb{R}^n) \longrightarrow \mathcal{C}^{-\infty}(\widetilde{\Omega}) \tag{6.5.18}$$

is a classical Fourier integral operator, at least microlocally on a conic neighborhood of $\{\xi_n = 0\}$. To see this, one can plug the integral representation (A.0.1) of $Ai(\zeta)$ and its analogue for $Ai'(\zeta)$ into (6.5.8) and make a change of variable to get (modulo a smoothing operator)

$$B_2^\# G(x) = \iiint a(x,\xi,\tau)e^{i\psi(x,y,\xi,\tau)}G(y)\,dy\,d\tau\,d\xi, \tag{6.5.19}$$

with

$$a(x,\xi,\tau) = \xi_1^{-2/3}\big[g(x,\xi) + i\xi_1^{-2/3}\tau h(x,\xi)\big]b(\xi,\tau), \tag{6.5.20}$$

and

$$\psi(x,y,\xi,\tau) = \theta(x,\xi) - y\cdot\xi + (\xi_1^{-1}\tau)\xi_1^{1/3}\zeta(x,\xi) + \frac{1}{3}(\xi_1^{-1}\tau)^2\tau. \tag{6.5.21}$$

Here $b(\xi,\tau) \in S^0$ is a cut-off supported in $\{(\xi,\tau) : |\tau| < c\xi_1\}$, equal to 1 on a conic neighborhood of $\tau = 0$. The representation (6.5.19) as a Fourier integral operator yields a description of the microlocal singularities of $B_2^\#\mathfrak{A}_\pm F = B_2^\# G$ consistent with that given above, but in a somewhat neater form. Note that the canonical relation for $B_2^\#$ is given by

$$\Lambda'_\psi = \{((x,\nabla_x\psi),(y,-\nabla_y\psi)) : \nabla_{\xi,\tau}\psi(x,y,\xi,\tau) = 0\}, \tag{6.5.22}$$

and $\nabla_{\xi,\tau}\psi = 0$ if and only if

$$y = \nabla_\xi\Big(\theta + \xi_1^{-2/3}\tau\zeta + \frac{1}{3}\xi_1^{-2}\tau^3\Big), \quad \tau^2 = -\xi_1^{4/3}\zeta(x,\xi). \tag{6.5.23}$$

The constraint $|\xi_n| \le C'\xi_1$ forces $|\zeta(x,\xi)| \le C\xi_1^{2/3}$ when x is close to $\partial\Omega$, and hence forces $|\tau| \le C\xi_1$ when (6.5.23) holds.

Next we consider the microlocal behavior of AF, which, following §6.4, we decompose as

$$AF = A_1F + A_2F, \tag{6.5.24}$$

with A_j given by (6.4.2). Using Lemma 6.4.3, we have the following analogue of Proposition 6.5.2 and Proposition 6.5.4.

Proposition 6.5.25. *On the interior region* Ω*,*

$$A_1 F \in \mathcal{C}^\infty(\Omega). \tag{6.5.26}$$

Furthermore, if $WF(KF)$ *is disjoint from an open conic set* $\Gamma \subset T^*\partial\Omega \setminus 0$ *and* $P \in OPS^0(\partial\Omega)$ *has symbol supported in* Γ*, then*

$$P(x', D_{x'}) A_1 F \in \mathcal{C}^\infty(\overline{\Omega}). \tag{6.5.27}$$

We could analyze the second term in (6.5.24) using (6.4.6), i.e., $A_2 = D_2 \mathcal{A}_\pm^{-1}$. Alternatively, we can decompose it further:

$$A_2 F = A_{21} F + A_{22} F, \tag{6.5.28}$$

with

$$A_{2j} F = \int \big[g A_\pm(\zeta) + ih A'_\pm(\zeta) \big] \chi_2(\zeta) \chi_j(\zeta_0) A_\pm(\zeta_0)^{-1} e^{i\theta} \hat{F}(\xi)\, d\xi. \tag{6.5.29}$$

Parallel to Lemma 6.4.3 we have:

$$\begin{aligned} A_\pm(\zeta)\chi_2(\zeta)\chi_1(\zeta_0) A_\pm(\zeta_0)^{-1} &\in S^{1/6}_{1/3,2/3,1}(\overline{U} \times \Gamma), \\ A'_\pm(\zeta)\chi_2(\zeta)\chi_1(\zeta_0) A_\pm(\zeta_0)^{-1} &\in S^{1/3}_{1/3,2/3,1}(\overline{U} \times \Gamma), \end{aligned} \tag{6.5.30}$$

and multiplication by y^j decreases the order by $2j/3$. Thus:

Proposition 6.5.31. *On the interior region* Ω*,*

$$A_{21} F \in \mathcal{C}^\infty(\Omega). \tag{6.5.32}$$

Furthermore, if $WF(KF)$ *is disjoint from an open conic set* $\Gamma \subset T^*\partial\Omega \setminus 0$*, and* $P \in OPS^0(\partial\Omega)$ *has symbol supported in* Γ*, then*

$$P(x', D_{x'}) A_{21} F \in \mathcal{C}^\infty(\overline{\Omega}). \tag{6.5.33}$$

This leaves $A_{22}F$, which we write as

$$A_{22} F = \int e(x,\xi) e^{i(\theta \mp (2/3)[(-\zeta)^{3/2} - (-\zeta_0)^{3/2}])} \hat{F}(\xi)\, d\xi, \tag{6.5.34}$$

with

$$\begin{aligned} e(x,\xi) &= \big[g\Psi_\pm(\zeta) + ih\widetilde{\Psi}_\pm(\zeta) \big] \chi_2(\zeta)\chi_2(\zeta_0)\Psi_\pm(\zeta_0)^{-1} \\ &\quad \in S^{m+1/6}_{1/3,2/3,1}(\overline{U} \times \Gamma). \end{aligned} \tag{6.5.35}$$

From this we can deduce that

$$\begin{aligned} WF\, A_{22}F\big|_\Omega \subset \ &\text{image of } WF(F) \cap \{\xi_n \le 0\} \text{ under} \\ &(\nabla_\xi \psi^\pm, \xi) \mapsto (x, \nabla_x \psi^\pm), \end{aligned} \tag{6.5.36}$$

where

$$\psi^\pm(x,\xi) = \theta \mp \frac{2}{3}\big[(-\zeta)^{3/2} - (-\zeta_0)^{3/2}\big]. \tag{6.5.37}$$

As for the behavior at the boundary, we have the following.

Proposition 6.5.38. *Let* $(p_0, \eta_0) \in T^*\partial\Omega \setminus 0$ *and assume* $(p_0, \eta_0) \notin WF(KF)$. *Then there exists a conic neighborhood* Γ *of* (p_0, η_0) *in* $T^*\partial\Omega \setminus 0$ *and a neighborhood* $\overline{U}$ *of* p_0 *in* $\overline{\Omega}$ *such that if* $P \in OPS^0(\partial\Omega)$ *has symbol supported in* Γ, *then*

$$P(x', D_{x'})A_{22}F \in \mathcal{C}^\infty(\overline{U}). \tag{6.5.39}$$

As for the microlocal behavior of CF, this follows directly from the formula (6.0.4), the analysis of B given above, and the analysis of $(\mathcal{A}_\pm \mathcal{A}i)_T^{-1}$ given in Chapter 5.

Chapter 7: Parametrices for the Dirichlet problem

The material developed in the preceding six chapters will now be brought together in order to construct microlocal parametrices for the Dirichlet problem:

$$P(x,D)u = 0 \text{ in } \Omega, \tag{7.0.1}$$

$$u\big|_{\partial\Omega} = f. \tag{7.0.2}$$

Here we shall suppose that P is a second order differential operator of real principal type (this is later generalized in Chapter 12) with bicharacteristics which have simple tangency to the boundary, $\partial\Omega$, of the region of interest, Ω. We shall assume that Ω is either bicharacteristically convex, or else bicharacteristically concave, for P. This condition on the principal symbol p, of P, was analyzed in §4.1. Since the conditions imposed are all coordinate independent we can suppose that $\Omega \subset Z = [0,\infty) \times \mathbb{R}^n$ and that $\partial\Omega \subset \mathbb{R}^n$.

By a parametrix for (7.0.1)–(7.0.2) we mean a map

$$E : \mathcal{C}^{-\infty}(\partial\Omega) \longrightarrow \mathcal{C}^{-\infty}(\Omega), \tag{7.0.3}$$

with the property that, for all f,

$$P(x,D)Ef \in \mathcal{C}^{\infty}(\overline{\Omega}), \quad Ef\big|_{\partial\Omega} - f \in \mathcal{C}^{\infty}(\partial\Omega). \tag{7.0.4}$$

In §7.1 we bring to fruition the method first described in §1.5, yielding a pair of parametrices $E_{\pm}$, in the diffractive case. Implementing this method is at this point a matter of using the results of Chapter 4 to solve the eikonal and transport equations and the results of Chapters 5–6 to deal with the Fourier-Airy operators that arise. In §7.2 we treat the analogous construction when gliding rays are present.

The parametrices E_+ and E_- send singularities in either of two directions. It is significant that the parametrices constructed in §§7.1–7.2 are unique, up to smoothing operators. Let us expand on this in the case $\Omega = \mathbb{R} \times \mathcal{O}$, with coordinates (t,x), and P hyperbolic with respect to t. Say $f \in \mathcal{C}_c^{-\infty}(\mathbb{R} \times \mathcal{O})$. Then the hpyerbolic PDE (7.0.1)–(7.0.2) has a unique outgoing solution u^+, satisfying $u^+(t,x) = 0$ for $t << 0$, and a unique incoming solution u^-, satisfying $u^-(t,x) = 0$ for $t >> 0$. In this case the microlocal uniqueness is the statement that

$$E_+ f - u^+ \in \mathcal{C}^{\infty}(\overline{\Omega}), \quad E_- f - u^- \in \mathcal{C}^{\infty}(\overline{\Omega}). \tag{7.0.5}$$

This can be established using well known *global* energy estimates for hyperbolic equations, as we briefly discuss in §7.3. We follow this up in §7.4 with a discussion of the propagation of singularities for $u^{\pm}$, which follows via (7.0.5) and the analysis of the Fourier-Airy operators used to produce $E_{\pm}$.

In §7.5 we apply the construction of diffractive parametrices of §7.1 together with other geometrical results established in Chapters 3–4 to the production of a microlocal model, die to M. Farris, of the solution operator to the wave equation on a region with diffractive boundary.

§7.1: Diffractive points

We briefly recall the hypotheses we are placing on the differential operator in the diffractive case and then recall the constructions of the preceding sections, leading to microlocal parametrices for the Dirichlet problem (7.0.1), (7.0.2). Thus, P is a second order differential operator defined and with $\mathcal{C}^\infty$ coefficients in some region:

$$\Omega = \{(x,y) \in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1};\ 0 \le x \le \varepsilon,\ |y| \le \varepsilon\},\ \varepsilon > 0. \tag{7.1.1}$$

$$P = \sum_{k+|\alpha|\le 2} p_{k,\alpha}(x,y) D_x^k D_y^\alpha,\ p_{k,\alpha} \in \mathcal{C}^\infty(\Omega). \tag{7.1.2}$$

The first thing we require is that the coefficients of the second order part be real-valued, or equivalently that the principal symbol:

$$p(x,y,\xi,\eta) = \sum_{k+|\alpha|=2} p_{k,\alpha}(x,y)\xi^k \eta^\alpha \tag{7.1.3}$$

be real-valued.

Now we shall construct a parametrix to solve (7.0.1) and (7.0.2), modulo $\mathcal{C}^\infty$ errors, when $\mathrm{WF}(f)$ is concentrated near some point $\bar\rho = (\bar y, \bar\eta) \in T^*\mathbb{R}^n \setminus 0$. This point is a glancing point for p if and only if:

$$\exists\ \xi' \in \mathbb{R},\ \text{such that } p(0,\bar y,\xi',\bar\eta) = d_\xi p(0,\bar y,\xi',\bar\eta) = 0. \tag{7.1.4}$$

In §4.1 it is noted that this condition can be written in terms of the Hamilton vector field

$$H_p = \frac{\partial p}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial}{\partial \xi} + \sum_{j=1}^n \Big[\frac{\partial p}{\partial \eta_j}\frac{\partial}{\partial y_j} - \frac{\partial p}{\partial y_j}\frac{\partial}{\partial \eta_j}\Big],$$

as

$$H_p x = 0 \text{ at } (0,\bar y,\xi',\bar\eta). \tag{7.1.5}$$

The additional conditions we impose are that $\partial\Omega$ be a non-characteristic hypersurface for P at $(0,\bar y)$:

$$H_x^2 p = \frac{\partial^2 p}{\partial \xi^2}(0,\bar y,\xi',\bar\eta) = p_{2,0}(0,\bar y) \ne 0, \tag{7.1.6}$$

and that $\bar\rho$ be a diffractive (i.e., grazing) point for P:

$$H_p^2 x > 0 \text{ at } (0,\bar y,\xi',\bar\eta). \tag{7.1.7}$$

We also require, at least initially, the non-degeneracy condition:

$$d_{(\xi,\eta)} p(0,\bar y,\xi',\bar\eta) \ne 0. \tag{7.1.7A}$$

The parametrix will be constructed in the form of a Fourier-Airy integral operator (6.0.1), i.e.,

$$\begin{aligned} E_\pm(f)(x,y) &= \mathcal{L}_\pm F(x,y) \\ &= \int \big[gA_\pm(\zeta) + ihA'_\pm(\zeta)\big] A_\pm(\zeta_0)^{-1} e^{i\theta(x,y,\xi)} \hat{F}(\xi)\, d\xi, \end{aligned} \tag{7.1.8}$$

where F and f are related below. The phase functions θ, ζ and amplitudes g, h are as introduced in Chapter 1; their construction will also be recalled below. In particular, we arrange that g be elliptic and $h|_{x=0} = 0$, and $\zeta|_{x=0} = \zeta_0 = \xi_1^{-1/3}\xi_n$. Thus

$$\mathcal{L}F(0,y) = \int g_0 e^{i\theta_0(y,\xi)} \hat{F}(\xi)\, d\xi = JF(y), \tag{7.1.8A}$$

where $\theta_0 = \theta|_{x=0}$, $g_0 = g|_{x=0}$. Thus J is microlocally an elliptic Fourier integral operator, and to have $E_\pm f(0,y) = f(y)$ mod $\mathcal{C}^\infty$, we take

$$F = J^{-1} f, \tag{7.1.8B}$$

where J^{-1} denotes a microlocal parametrix for J.

Proposition 7.1.9. *Let P be a second order operator of real principal type as in* (7.1.1)–(7.1.3) *and suppose $\bar\rho = (\bar y, \bar\eta)$ is a glancing point for P in the sense that* (7.1.4)–(7.1.7) *hold. Then there exist phase functions θ and ζ and symbols g and h satisfying* (6.1.6)–(6.1.8)*, the first condition in* (6.1.9)*, and* (6.1.10) *such that $E_\pm$ in* (7.1.8) *give microlocal parametrices for the Dirichlet problem* (7.0.1)–(7.0.2) *at $\bar\rho$, for some open conic neighbourhood γ of $\overline{\rho}$ in $T^*\partial\Omega \setminus 0$.*

Proof. The conditions (6.1.6)–(6.1.8), the first condition in (6.1.9), and (6.1.10) should hold so that $E_\pm$ is defined and the results of Chapter 6 apply. Assuming this for the moment, the composite operator $P \cdot E$ is also of the type (6.0.1), with the formula (1.5.9)–(1.5.11) valid for the symbols a, b of $P \cdot E$. Indeed, this is immediate when g and h are rapidly decreasing and follows in general from the continuity of E in the symbol topology. This relationship can be written:

$$a = Q_{11} g + Q_{12} h, \quad b = Q_{21} g + Q_{22} h, \tag{7.1.10}$$

with the Q_{ij} differential operators.

As we will show below, in order that $P \cdot E$ be a smoothing operator, it suffices for the symbols a and b to satisfy:

$$a, b \in S^{-\infty} \text{ in } \zeta < 0 \text{ and in Taylor series at } x = 0 \text{ over } \Omega'. \tag{7.1.11}$$

As is briefly explained in Chapter 1, these conditions will be arranged by taking g and h to be asymptotic sums of symbols:

$$g \sim \sum_{j=1}^{\infty} g_j, \quad h \sim \sum_{j=1}^{\infty} h_j, \tag{7.1.12}$$

where the g_j and h_j are symbols of orders $-j$ and $-j-1/3$ respectively. Then the symbols a and b given by (7.1.10) have corresponding asymptotic expansions:

$$a \sim \sum_{j=1}^{\infty} a_j, \quad b \sim \sum_{j=1}^{\infty} b_j \tag{7.1.13}$$

with a_j and b_j of orders $-j+2$ and $-j+5/3$ obtained by decomposing the operators in (7.1.10) according to homogeneity. The leading terms in Q in (7.1.10) in terms of homogeneity are:

$$\begin{aligned} Q^0_{11} &= \langle d\theta, d\theta\rangle + \zeta\langle d\zeta, d\zeta\rangle, & Q^0_{12} &= 2\zeta\langle d\theta, d\zeta\rangle, \\ Q^0_{22} &= -i(\langle d\theta, d\theta\rangle + \zeta\langle d\zeta, d\zeta\rangle), & Q^0_{21} &= 2i\langle d\theta, d\zeta\rangle, \end{aligned} \tag{7.1.14}$$

where $\langle \cdot, \cdot\rangle$ is the bilinear form obtained by polarization from the quadratic form which is the principal symbol of P. Thus the eikonal equations:

$$\langle d\theta, d\theta\rangle + \zeta\langle d\zeta, d\zeta\rangle = \langle d\theta, d\zeta\rangle = 0, \text{ in } \zeta < 0 \text{ and in Taylor series at } x = 0, \tag{7.1.15}$$

ensure that these terms of top homogeneity in (7.1.10) make no contribution to the singularities. Equating the lower order terms in (7.1.10) formally to zero gives the transport equations:

$$2\langle d\theta, dg_j\rangle - 2\zeta\langle d\zeta, dh_j\rangle + (P_2\theta)g_j - (\langle d\zeta, d\zeta\rangle + P_2\zeta)h_j = iPg_{j-1}, \tag{7.1.16}$$

$$2\langle d\zeta, dg_j\rangle - 2\langle d\theta, dh_j\rangle + (P_2\zeta)g_j - (P_2\theta)h_j = -iPh_{j-1}. \tag{7.1.17}$$

Here $g_{-1} = h_{-1} = 0$, by convention, and these equations also only need hold in $\zeta < 0$ and in Taylor series at $x = 0$; P_2 is the operator P with the term of order zero dropped.

With these preliminaries the construction of E can proceed easily. The eikonal equations (7.1.15) are solved in §4.2, with θ and ζ satisfying the conditions (6.1.3), (6.1.4), (6.1.5) and (6.1.6) in a small cone satisfying (6.1.2). The successive transport equations (7.1.16) and (7.1.17) are solved in §4.3. In fact the solutions obtained there allow the additional constraints:

$$h\big|_{\{x=0\}} = 0, \tag{7.1.18}$$

and

$$g\big|_{\{x=0\}} \in S^0 \text{ is elliptic at } \bar\rho \text{ with essential support in } \gamma', \tag{7.1.19}$$

for any preassigned conic neighborhood γ' of $\bar\rho$, to be imposed. Here g and h are functions of (x, y, η) only. Moreover the condition (7.0.3) allows the transport equations to be solved in a neighborhood of $\bar y$, with the symbols having support within a closed conic subset of the open cone in which the phase functions are defined. Thus the Fourier integral operator J defined by (7.1.8A) is elliptic, and we can define F by (7.1.8B).

At this point we have

$$\begin{aligned} PE_{\pm}f &= \int \big[aA_{\pm}(\zeta) + bA'_{\pm}(\zeta)\big] A_{\pm}(\zeta_0)^{-1} e^{i\theta} \hat{F}(\xi)\, d\xi \\ &= \int G(x,y,\xi) e^{i\theta} \hat{F}(\xi)\, d\xi. \end{aligned} \tag{7.1.20}$$

The result (7.1.11) implies the estimates

$$\begin{aligned} &|G(x,y,\xi)| \le C_N |\xi|^{-N}, \quad \xi_n \le 0,\ (x,y) \in \overline{\Omega}, \\ &|G(x,y,\xi)| \le C_N (|\xi|^{-N} + x^N) \big| e^{-(2/3)(\zeta_0^{3/2} - \zeta^{3/2})} \big| \cdot |\xi|^m, \quad \xi_n \ge 0. \end{aligned} \tag{7.1.21}$$

Recall that x is a defining coordinate for $\partial\Omega$, $x > 0$ in Ω. In both cases we have

$$|G(x,y,\xi)| \le C_N |\xi|^{-N}, \tag{7.1.22}$$

and such estimates also hold for all x, y-derivatives, so $PE_{\pm}f \in \mathcal{C}^\infty(\overline{\Omega})$.

This shows that $E_{\pm}$ give microlocal parametrices for the Dirichlet problem for each choice of sign, proving the proposition.

Remark. The Fourier integral operator J has the following geometrical significance. Namely, its canonical transformation χ_J conjugates the billiard ball map $\delta^{\pm}$ on $T^*\partial\Omega$ to the normal form $\delta_h^{\pm}$ given in (4.1.10). This geometrical fact will be useful in §7.5.

§7.2: Gliding points

The construction of microlocal parametrices for the Dirichlet problem near a gliding point on the boundary is very similar to the diffractive case analyzed above. Indeed, the only difference in terms of the hypotheses on P is that we are working 'on the other side' of the boundary, so that (7.1.7) is replaced by:

$$H_p^2 x < 0 \ \text{ at } \ (0, \bar{y}, \xi', \bar{\eta}). \tag{7.2.1}$$

The microlocal parametrices are now of the form (6.0.3):

$$\begin{aligned} E_{\pm}(f)(x,y) &= \mathcal{L}F(x,y) \\ &= \int \big[Ai(\zeta) + ihAi'(\zeta)\big] Ai(\zeta_0)^{-1} e^{i\theta} \hat{F}(\xi)\, d\xi, \end{aligned} \tag{7.2.2}$$

In this case we evaluate the phase functions and amplitudes at $(x, y, \xi', \xi_n + iT)$. In particular,

$$\zeta_0 = \xi_1^{-1/3}(\xi_n + iT). \tag{7.2.3}$$

There are two issues that make this construction a bit more complicated than the case treated in §7.1. One is the fact that we are evaluating at $\xi_n + iT$ rather than at real ξ_n. Now the eikonal and transport equations hold (for $\zeta < 0$) at real ξ_n. However, the almost-analytic continuation yields phase functions and amplitudes that satisfy the appropriate eikonal and transport equations (on $\zeta < 0$) modulo $O(|\xi|^{-\infty})$.

The other issue is that the eikonal and transport equations, which hold on $\zeta < 0$, do not hold on all of Ω for $\zeta < 0$, in the gliding case, but only on

$$\Omega_\xi = \{(x, y) : \zeta(x, y, \xi) \le 0\}. \tag{7.2.4}$$

The upshot of this is the following more elaborate analysis of (7.1.20)–(7.1.21). We have

$$\begin{aligned} PE_\pm f &= \int \big[aAi(\zeta) + bAi'(\zeta)\big] A_\pm(\zeta_0) e^{i\theta} [A_\pm(\zeta_0) Ai(\zeta_0)]^{-1} \hat{F}(\xi)\, d\xi \\ &= \int G(x, y, \xi) e^{i\theta} [A_\pm(\zeta_0) Ai(\zeta_0)]^{-1} \hat{F}(\xi)\, d\xi, \end{aligned} \tag{7.2.5}$$

and this time

$$\begin{aligned} &|G(x, y, \xi)| \le C_N |\xi|^{-N}, \quad \xi_n \le 0,\ (x, y) \in \overline{\Omega}_\xi, \\ &|G(x, y, \xi)| \le C_N (|\xi|^{-N} + \gamma^N) e^{-c\gamma^{3/2}|\xi|} |\xi|^m, \quad \xi_n \le 0,\ (x, y) \in \Omega \setminus \overline{\Omega}_\xi, \\ &|G(x, y, \xi)| \le C_N (|\xi|^{-N} + x^N) \big| e^{-(2/3)(\zeta_0^{3/2} - \zeta^{3/2})} \big| \cdot |\xi|^m, \quad \xi_n \ge 0. \end{aligned} \tag{7.2.6}$$

Here $\gamma = |\xi|^{-2/3}\zeta(x, y, \xi)$. In all cases

$$|G(x, y, \xi)| \le C_N |\xi|^{-N}, \tag{7.2.7}$$

and such estimates also hold for all x, y-derivatives, so again $PE_\pm f \in \mathcal{C}^\infty(\overline{\Omega})$. Hence we have:

Proposition 7.2.8. *If the diffractive hypothesis,* (7.1.7)*, is replaced by the gliding hypothesis* (7.2.1) *then the conclusions of Proposition 7.1.9 still hold with* (7.1.8) *replaced by* (7.2.2).

§7.3: Justification in the hyperbolic case

Here we justify the parametrix construction in the case when P is hyperbolic, using global energy estimates. For simplicity we take

$$P = \frac{\partial^2}{\partial t^2} - \Delta, \tag{7.3.1}$$

on $\Omega = \mathbb{R} \times \mathcal{O}$, where Δ is the Laplace operator on the Riemannian manifold with boundary $\overline{\mathcal{O}}$. We allow either grazing or gliding rays, and in either case let E_+ be the appropriate parametrix constructed in §7.1 or §7.2.

Proposition 7.3.2. *Let $f \in \mathcal{C}^{-\infty}(\partial\Omega)$ be supported in $\{t \geq 0\}$. Assume u^+ satisfies*

$$Pu^+ = 0 \ \ \textit{on} \ \ \Omega, \quad u^+\big|_{\partial\Omega} = f, \tag{7.3.3}$$

and

$$u^+(t,x) = 0 \ \ \textit{for} \ \ t < 0. \tag{7.3.4}$$

Then

$$u^+ - E_+ f \in \mathcal{C}^\infty(\overline{\Omega}). \tag{7.3.5}$$

The analysis of Fourier-Airy operators done in §6.5 shows that $E_+ f$ is smooth on $(-\infty, 0) \times \overline{\mathcal{O}}$, so, altering it by an element of $\mathcal{C}^\infty(\overline{\Omega})$ if necessary, we can assume that

$$E_+ f(t,x) = 0 \ \ \text{for} \ \ t < -1.$$

Since we also have (7.0.4) for $E_+ f$, we see that $v = u^+ - E_+ f$ satisfies the hypotheses of the following global regularity result.

Proposition 7.3.6. *Assume v satisfies*

$$Pv \in \mathcal{C}^\infty(\overline{\Omega}), \quad v\big|_{\partial\Omega} \in \mathcal{C}^\infty(\partial\Omega), \tag{7.3.7}$$

and

$$v(t,x) = 0 \ \ \textit{for} \ \ t << 0. \tag{7.3.8}$$

Then

$$v \in \mathcal{C}^\infty(\overline{\Omega}). \tag{7.3.9}$$

Proofs of such a result in a rather general context can be found in [ChP], [RaM], and [Sak]. We sketch a very simple demonstration for the case (7.3.1). First, a finite propagation speed argument allows us to assume $\overline{\mathcal{O}}$ is compact. Next, a formal power series construction yields

$$v = v_0 + v_1 \tag{7.3.10}$$

with

$$v_0 \in \mathcal{C}^\infty(\overline{\Omega}), \quad v_j(t,x) = 0 \ \text{ for } \ t << 0, \tag{7.3.11}$$

and

$$Pv_1 = g, \quad v_1\big|_{\partial\Omega} = 0, \tag{7.3.12}$$

where $g \in \mathcal{C}^\infty(\overline{\Omega})$ vanishes to *infinite order* at $\partial\Omega$. We desire to prove that $v_1 \in \mathcal{C}^\infty(\overline{\Omega})$. To this end, write

$$v_1(t,x) = \int_{-T}^{t} \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}}\, g(s)\, ds, \tag{7.3.13}$$

where T is fixed so that $v_1(t,x) = 0$ for $t \le -T$. Now we have

$$g(s) \in \mathcal{D}(\Delta^k), \quad \forall\ k, \tag{7.3.14}$$

since g vanishes to infinite order at $\partial\Omega$, so (7.3.13) displays $v_1(t)$ as a continuous function of t with values in $\mathcal{D}(\Delta^k)$, for all k. Similarly $\partial_t^j v_1(t)$ has this property, so indeed $v_1 \in \mathcal{C}^\infty(\overline{\Omega})$.

§7.4: Propagation of singularities

The second order operator P has a generalized bicharacteristic flow on ${}^bT^*\overline{\Omega}\setminus 0 = (T^*\Omega \setminus 0) \cup (T^*\partial\Omega \setminus 0)$, characterized as follows. If $\rho \in T^*\Omega \setminus 0$ is characteristic for P, the bicharacteristic through ρ is the standard integral curve of H_p through ρ. If $\rho \in T^*\partial\Omega \setminus 0$, the bicharacteristic through ρ is defined if $\rho \in \mathcal{H} \cup \mathcal{G}_d \cup \mathcal{G}_g$. (Throughout this section we assume the glancing set is exhausted by $\mathcal{G}_d \cup \mathcal{G}_g$.) If $\rho \in \mathcal{H}$, two null bicharacteristics of P in $T^*\Omega \setminus 0$ pass over ρ, one reflecting into the other. If $\rho \in \mathcal{G}_d$, a grazing ray passes through ρ. If $\rho \in \mathcal{G}_g$, a gliding ray passes through ρ.

Given $f \in \mathcal{C}_c^{-\infty}(\partial\Omega)$, we see that $E_+ f$ and $E_- f$ send singularities off along bicharacteristics lying over $WF\, f$. (For simplicity take $\Omega = \mathbb{R} \times \mathcal{O}$ and P given by (7.3.1).) We have from §7.3 that the outgoing solution u^+ to (7.3.3)–(7.3.4) has the property that

$$WF_b\, u^+ \subset \mathcal{F}_+(WF\, f), \tag{7.4.1}$$

where $\mathcal{F}_+(WF\,f)$ denotes the union of generalized bicharacteristic curves of P starting at points in $WF\,f \subset T^*\partial\Omega \setminus 0$ and going forward in time, reflecting off or travelling along the boundary as indicated above. Similarly, with obvious notation,

$$WF_b\,u^- \subset \mathcal{F}_-(WF\,f). \tag{7.4.2}$$

Actually, what we know at this point is a local version of these results, since we have available from §§7.1–7.2 local parametrices $E_\pm$. Nevertheless, these results are globally valid. To see this, first note that we can write any $f \in \mathcal{C}_c^{-\infty}(\partial\Omega)$ as a finite sum of terms (which we relabel f) for which local parametrices are available. Then (7.4.1)–(7.4.2) hold on a small neighborhood in $\overline{\Omega}$ of the support of each such f. That they hold globally is then a consequence of the following result on propagation of singularities.

Proposition 7.4.3. *(Assume the glancing set of $T^*\partial\Omega\setminus 0$ is exhausted by $\mathcal{G}_d\cup\mathcal{G}_g$.) Suppose $u \in \mathcal{C}^{-\infty}(\Omega)$ satisfies*

$$Pu = F \in \mathcal{C}^\infty(\overline{\Omega}), \quad u\big|_{\partial\Omega} = G \in C^\infty(\partial\Omega). \tag{7.4.4}$$

Then $WF_b\,u \subset {}^bT^*\overline{\Omega} \setminus 0$ is invariant under the generalized bicharacteristic flow of P.

Indeed, the local validity of (7.4.1)–(7.4.2) can be used to establish Proposition 7.4.3, as we now show. For simplicity we continue to assume P is given by (7.3.1), and $\Omega = \mathbb{R}\times\mathcal{O}$, and we will also assume $u \in C(\mathbb{R}, H^1(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega))$. Also, using well known existence results, we can assume $F = 0$ and $G = 0$ in (7.4.4).

Assume $\rho \in {}^bT^*\overline{\Omega}\setminus 0$ and $\rho \notin WF_b\,u$. Let γ be the generalized bicharacteristic curve through ρ. We want to show that $\gamma \cap WF_b\,u = \emptyset$. Suppose to the contrary that there exists $\rho_1 \in \gamma$ such that $\rho_1 \in WF_b\,u$. Say for the sake of argument that the t-coordinate (call it t_1) of ρ_1 exceeds that of ρ. In such a case, we can pick $\rho_1 \in \gamma$ having *minimal* t-coordinate greater than that of ρ, with $\rho_1 \in WF_b\,u$. Now we will obtain a contradiction.

We only worry about the cases $\rho_1 \in \mathcal{G}_d$ and $\rho_1 \in \mathcal{G}_g$. Pick $\rho_0 \in \gamma$, close to ρ_1, with t-coordinate t_0 slightly less than t_1 (so a local parametrix construction will be applicable). Note that if $\rho_1 \in \mathcal{G}_d$ then $\rho_0 \in T^*\Omega\setminus 0$ and if $\rho_1 \in \mathcal{G}_g$ then $\rho_0 \in \mathcal{G}_g$. By out set-up, $\rho_0 \notin WF_b\,u$.

Assume $\overline{\mathcal{O}}$ is contained in a complete Riemannian manifold $\widetilde{\mathcal{O}}$ without boundary. Let $f(x) = u(t_0,x)$ and $g(x) = u_t(t_0,x)$, and extend f and g by 0 on $\widetilde{\mathcal{O}}\setminus\mathcal{O}$. For $t > t_0$, write $u = v + w$ where v satisfies

$$Pv = 0 \ \text{ on } \ I\times\widetilde{O}, \quad v(t_0,x) = f(x), \quad v_t(t_0,x) = g(x),\ x \in \widetilde{\mathcal{O}}, \tag{7.4.5}$$

(with $I = (t_0-\varepsilon, t_1+\varepsilon)$) and, on $I\times\mathcal{O}$, w satisfies

$$Pw = 0, \quad w\big|_{I\times\partial\mathcal{O}} = -v\chi_{t\ge t_0}(t), \quad w(t,x) = 0 \ \text{ for } \ t < t_0. \tag{7.4.6}$$

Now it is clear that $\rho_1 \notin WF_b\, v|_{\overline{\Omega}}$. Finally, the constructions of §§7.1–7.3 show that $\rho_1 \notin WF_b\, w$, providing the desired contradiction, and completing the proof of Proposition 7.4.3.

Remark. The propagation of singularities result given in Proposition 7.4.3 has been proven under general conditions, not requiring that all glancing points be either diffractive of gliding, in [MeS1] and [MeS2]; see also [Iv].

§7.5: A representation for the wave evolution operator in the diffractive case

In this section we give a microlocal model for the solution operator $e^{iT\Lambda}$ where $\Lambda = (-\Delta)^{1/2}$ and Δ is the Laplace operator on a Riemannian manifold Ω with diffractive boundary. Let $\mathcal{O}$ be an open set bounded away from $\partial\Omega$, and assume the following property: any geodesic segment of length T issuing from a point of $\mathcal{O}$, reflecting off $\partial\Omega$ by the usual rules of geometrical optics, hits $\partial\Omega$ at most once, and ends away from $\partial\Omega$, in an open set U. It follows that, for any $u \in \mathcal{E}'(\mathcal{O})$, $e^{iT\Lambda}u$ is $\mathcal{C}^\infty$near $\partial\Omega$, having singular support inside U.

Any $u \in \mathcal{E}(\mathcal{O})$ can be written as a sum $u = u_1 + u_2$ where the rays issuing from $WF(u_1)$ avoid glancing intersection with $\partial\Omega$ and those issuing from $WF(u_2)$ intersect $\partial\Omega$ at or near glancing. The analysis of $e^{iT\Lambda}u_1$ as a Fourier integral operator acting on u_1 is elementary, so we can restrict attention to $u = u_2$. Also note there is no loss of generality in suppposing the closure of U is close to $\partial\Omega$ (though disjoint from it), since further propagating by $e^{it\Lambda}$ just acts as a Fourier integral operator.

Our goal here is to produce the microlocal form

$$e^{iT\Lambda}u = K_2\Big(\frac{\mathcal{A}_-}{\mathcal{A}_+}\Big)K_1 u \tag{7.5.1}$$

for such u, where K_j are elliptic Fourier integral operators. This result was obtained by Farris [Fa]. The operator $\mathcal{A}_-/\mathcal{A}_+$ is Fourier multiplication:

$$\mathcal{F}\Big(\frac{\mathcal{A}_-}{\mathcal{A}_+}\Big)f(\xi) = \frac{A_-}{A_+}(\zeta_0)\hat{f}(\xi). \tag{7.5.2}$$

We will assume for convenience that Ω is compact, though the considerations here are purely local, so this hypothesis can easily be dropped. The results aply in particular to the case $\Omega = \mathbb{R}^n \setminus K$, where K is compact and strictly convex.

To start, let Δ_0 be the Laplace operator on a compact Riemannian manifold M, containing Ω in its interior, so $\partial\Omega$ is a smooth hypersurface. Let $\mathcal{F}_T = e^{iT\Lambda_0}$ where $\Lambda_0 = (-\Delta_0)^{1/2}$, and let $\mathcal{R} : \mathcal{E}'(\mathcal{O}) \longrightarrow \mathcal{D}'(\mathbb{R}\times\partial\Omega)$ be given by

$$\mathcal{R}u = e^{iT\Lambda_0}u\big|_{\mathbb{R}\times\partial\Omega}. \tag{7.5.3}$$

Also let $\mathcal{E}_T : \mathcal{E}'(\mathbb{R} \times \partial\Omega) \longrightarrow \mathcal{D}'(\Omega)$ be defined as follows. $\mathcal{E}_T f$ is the value at $t = T$ of the outgoing solution w to the wave equation on $\mathbb{R} \times \Omega, (w = 0$ for $t << 0)$ with boundary condition $w_{|\mathbb{R}\times\partial\Omega} = f$. In other words, $\mathcal{E}_T f = E_+ f|_{t=T}$. Let us modify $\mathcal{R}$ by multiplying by a cutoff $\psi(t)$, equal to 1 for $0 \le t \le t$ and supported on a small neighborhoood of $[0, T]$; keep the notation $\mathcal{R}$ for the altered operator. Then $\mathcal{E}_T \mathcal{R}$ is well defined, and

$$e^{iT\Lambda} = \mathcal{F}_T - \mathcal{E}_T \mathcal{R}, \tag{7.5.4}$$

modulo a smoothing operator, when aplied to elements u such as specified above.

Note that for such $u, \mathcal{R}u$ has wave front set near glancing. Thus $F = J^{-1} f = J^{-1}(\mathcal{R}u)$ has wave front set near $\xi_n = 0$, where J is the Fourier integral operator defined by (7.1.8A). Making a smooth perturbation we can suppose $\hat{F}(\xi)$ is supported on a small conic neighborhoood Γ of $\xi_n = 0$. For $\xi \in \Gamma$ and x away from $\partial\Omega, \zeta$ is bounded away from 0, so we can replace $A(\zeta)$ and $A'(\zeta)$ by their asymptotic expansions, and write

$$\mathcal{E}_T = L\mathcal{A}_+^{-1} J^{-1}, \tag{7.5.5}$$

where (for $\hat{F}$ supported in Γ)

$$LF = \int [gA_+(\zeta) + ihA'_+(\zeta)] e^{i\theta} \hat{F}(\xi)\, d\xi, \tag{7.5.6}$$

the integral being evaluated at $t = T$ and restricted to $x \in U$. Thus L is an elliptic Fourier integral operator.

The map $\mathcal{R}$ is a Fourier integral operator with folding canonical relation, whose boundary maps $\delta^{\pm}$ on $T^*(\mathbb{R} \times \partial\Omega) \setminus 0$ are seen to coincide with the billiard ball map. Thus, by Theorem 5.1.8, one can write

$$J^{-1}\mathcal{R}K = \mathcal{A}i\tilde{P}_1 + \mathcal{A}i'\tilde{P}_2, \tag{7.5.7}$$

for certain pseudodifferential operators $\tilde{P}_j$. Here J is the elliptic Fourier integral operator used above, and K is an ellliptic Fourier integral operator, which can be taken to be of the form

$$KF = \int [g\overline{A}_+(\zeta) + ih\overline{A}'_+(\zeta)] e^{i\theta} \hat{F}(\xi)\, d\xi, \tag{7.5.8}$$

(for $\hat{F}$ supported in Γ), the integral being evaluated at $t = 0$. Note that $\overline{A}_+ = A_-$ (for real arguments). From the form of (7.5.8) it is clear that

$$K\mathcal{A}_-^{-1} J^{-1} = \mathcal{E}_0^-, \tag{7.5.9}$$

where $\mathcal{E}_0^- f$ is the value at $t = 0$ of the incoming solution $\tilde{w}$ to the wave equation on $\mathbb{R} \times \Omega$ ($\tilde{w} = 0$ for $t >> 0$) with boundary condition $\tilde{w} = f$ on $\mathbb{R} \times \partial\Omega$, i.e.,

$\mathcal{E}_0^- f = E_- f|_{t=0}$. The fact that K, given by (7.5.8), works in (7.5.7), follows from the fact that $J^{-1}\mathcal{R}K$ is a Fourier integral operator whose (folding) canonical relation coincides with that of $\mathcal{A}i$, which in turn is a simple consequence of (7.5.9). Combining (7.5.5) and (7.5.7) gives

$$\mathcal{E}_T\mathcal{R} = L\mathcal{A}_+^{-1}(\mathcal{A}i\tilde{P}_1 + \mathcal{A}i'\tilde{P}_2)K^{-1}. \tag{7.5.10}$$

The following simple consequence of the geometry of the various Fourier integral operators will be useful.

Lemma 7.5.11. *$L^{-1}\mathcal{F}_T K$ and its microlocal inverse $K^{-1}\mathcal{F}_T^{-1}L$ are (elliptic) pseudodifferential operators on $\xi_n \leq 0$.*

Proof. It suffices to give the proof for $\xi_n < 0$. Use the representations

$$\mathcal{F}_T K = \mathcal{F}_T\mathcal{E}_0^- J\mathcal{A}_-, \quad \text{Ł} = \mathcal{E}_T J\mathcal{A}_+, \tag{7.5.12}$$

(on $\xi_n < 0$). Each of these operators is an elliptic Fourier integral operator in this region, and to see that they move wave front sets in the same fashion, it suffices to note that

$$J\,\frac{\mathcal{A}_-}{\mathcal{A}_+}\,J^{-1} \tag{7.5.13}$$

has, in $\chi_J(\xi_n < 0)$, the canonical transformation equal to δ^-, i.e., the '-' half of the billiard ball map, since $\mathcal{A}_-/\mathcal{A}_+$ has canonical transformation δ_h^- (half of $\delta_h^\pm$ of (4.1.10)).

Given this lemma, we have, in addition to (7.5.7),

$$J^{-1}\mathcal{R}(\mathcal{F}_T^{-1}L) = \mathcal{A}iP_1 + \mathcal{A}i'P_2, \tag{7.5.14}$$

for certain pseudodifferential operators P_j, of respective orders 0 and $-1/3$. Hence, as a convenient modification of (7.5.10), we have

$$\mathcal{E}_T\mathcal{R} = L\mathcal{A}_+^{-1}(\mathcal{A}iP_1 + \mathcal{A}i'P_2)L^{-1}\mathcal{F}_T. \tag{7.5.15}$$

Returning to (7.5.4) we see that, microlocally,

$$e^{iT\Lambda} = L[1 - \mathcal{A}_+^{-1}(\mathcal{A}iP_1 + \mathcal{A}i'P_2)]L^{-1}\mathcal{F}_T. \tag{7.5.16}$$

We are well on the way to proving (7.5.1), with $K_2 = L$ and $K_1 = L^{-1}\mathcal{F}_T$ (up to a pseudodifferential factor). The rest of the argument wil consist of simplifying the operator in brackets in (7.5.16), and showing how $\mathcal{A}_-/\mathcal{A}_+$ arises. Note that, by virtue of the known propagation of singularities for the operator $e^{iT\Lambda}$, the operator in brackets above must move wave front sets the same way that $\mathcal{A}_-/\mathcal{A}_+$ does. This observation will give rise to a cancellation effect below that will be instrumental in yielding the normal form.

To start working on this operator, use

$$\mathcal{A}i = \overline{\omega}\mathcal{A}_+ + \omega\mathcal{A}_-, \quad \omega = e^{-\pi i/3},$$

to write $1 = (\omega\mathcal{A}i - \omega^2\mathcal{A}_-)\mathcal{A}_+^{-1}$ and hence

$$\text{(7.5.17)} \quad 1 - \mathcal{A}_+^{-1}(\mathcal{A}iP_1 + \mathcal{A}i'P_2) = \frac{\mathcal{A}_-}{\mathcal{A}_+}[-\omega^2 - \mathcal{A}_-^{-1}(\mathcal{A}i(P_1+\omega) + \mathcal{A}i'P_2)] = \frac{\mathcal{A}_-}{\mathcal{A}_+}W.$$

Using the Wronskian relation

$$A_+'Ai - A_+Ai' = \alpha, \quad \alpha =$$

we rewrite the operator W in (7.5.16) as

$$\text{(7.5.18)} \qquad W = -\omega^2 - \alpha\mathcal{A}_-^{-1}\mathcal{A}_+^{-1}P_2 - \mathcal{A}_-^{-1}\mathcal{A}i(P_1 + \omega - \Phi_+P_2),$$

recalling $\Phi_+ = \mathcal{A}_+'/\mathcal{A}_+ \in OPS^{1/3}_{1/3,0}$. By the observation in the last paragraph, W must preserve wave front sets. In particular the last term, involving $\mathcal{A}_-^{-1}\mathcal{A}i$, must preserve wave front sets. This implies

$$\text{(7.5.19)} \qquad P_1 + \omega - \Phi_+P_2 \in OPS^{-\infty} \text{ on } \xi_n < 0.$$

Taking adjoints gives the same result for $P_1^* + \overline{\omega} - P_2^*\Phi_+$. The following result gives the cancellation effect mentioned above.

Lemma 7.5.20. *Let $A \in OPS^m, B \in OPS^{m-1/3}$. Supppose*

$$\text{(7.5.21)} \qquad T = A + B\Phi \in OPS^{-\infty} \text{ on } \xi_n < 0.$$

Then all the terms in the asymptotic expansion of the symbols of A and B must vanish to infinite order at $\xi_n = 0$. The same is true for the case

$$\text{(7.5.22)} \qquad T = A + \Phi B \in OPS^{-\infty} \text{ on } \xi_n < 0.$$

In either case, we have

$$\text{(7.5.23)} \qquad T\mathcal{A}_\pm^{-1},\ T\mathcal{A}i,\ \mathcal{A}_\pm^{-1}T,\ \mathcal{A}iT \in OPS^{-\infty}.$$

Proof. Replacing $\Phi(\zeta_0)$ by its asymptotic expansion gives an infinite set of identities from (7.5.21), a priori satisfied for $\xi_n < 0$ only, but, by continuity, satisfied for $\xi_n \le 0$. For the principal symbols one gets

$$a_0 + b_0|\xi_1^{1/3}\xi_n|^{1/2} = 0,$$

which implies a_0 and b_0 must each vanish to infinite order at $\xi_n = 0$. Such vanishing of lower order terms follows inductively, so case (7.5.21) is done. The treatment of the case (7.5.22) follows from the standard formulas for the complete symbols of A^* and B^*, and (7.5.23) is an elementary consequence.

Applying the lemma to (7.5.19), we conclude that

$$W = -\omega^2 - \alpha(\mathcal{A}_+\mathcal{A}_-)^{-1}P_2, \tag{7.5.24}$$

modulo a smoothing operator and that each term in the symbol expansion of P_2 vanishes to infinite order at $\xi_n = 0$. Consequently, even though we only have

$$(\mathcal{A}_+\mathcal{A}_-)^{-1} \in OPS^{1/3}_{1/3,0},$$

this yields

$$\alpha(\mathcal{A}_+\mathcal{A}_-)^{-1}P_2 = P_3 \in OPS^0. \tag{7.5.25}$$

Our analysis of (7.5.16) has hence yielded the main result of this section:

Theorem 7.5.26. *If Ω has diffractive boundary, then a microlocal representation for $e^{iT\Lambda}$, acting on $u \in \mathcal{E}'(\Omega)$ such that $e^{iT\Lambda}u$ has singular support away from $\partial\Omega$, is given by*

$$e^{iT\Lambda} = K_2\,\frac{\mathcal{A}_-}{\mathcal{A}_+}\,K_1,$$

where K_j are elliptic Fourier integral operators. With L given by (7.5.6)*, we can take $K_2 = L$ and $L_1 = -(\omega^2 + P_3)L^{-1}\mathcal{F}_T$.*

Chapter 8: Neumann problem

Associated to a second order differential operator with real principal symbol on a region, Ω, with smooth boundary non-characteristic for P, there are two geometrically natural boundary problems. First is the Dirichlet problem, discussed above. Secondly, and somewhat less well-behaved, there is always a natural Neumann problem in which the value of the normal derivative of the solution is specified at the boundary.

Let p be the principal symbol of the operator. In any local coordinates (x, y, ξ, η) dual to local coordinates in Ω in which $\partial\Omega = \{x = 0\}$ the coefficient of ξ^2 in p does not vanish at $x = 0$, since the boundary is non-characteristic. Thus completing the square:

$$p(0, y, \xi, \eta) = \pm\Big(a_0(y)\xi + \sum_{i=1}^{n} a_i(y)\eta_i\Big)^2 + \sum_{i,j=1}^{n} b_{ij}(y)\eta_i\eta_j, \quad a_0(y) > 0. \tag{8.0.1}$$

This decomposition is unique, with the quadratic form only involving η. In fact under a change of coordinates to a new system (x', y', ξ', η') dual to coordinates in which the boundary is $\{x' = 0\}$ the new variables η', at $x = x' = 0$ depend linearly on the variable η, not on ξ. Thus the decomposition (8.0.1) is actually coordinate independent and the linear form

$$l = a_0(y)\xi + \sum_{i=1}^{n} a_i(y)\eta_i \tag{8.0.2}$$

is well-defined on $T^*_{\partial\Omega}\Omega$. This defines a vector field, ∂_ν, on Ω at $\partial\Omega$ by:

$$\sigma_1(\partial_\nu) = l. \tag{8.0.3}$$

If $\Omega = \mathbb{R}_t \times \Omega'_x \subset \mathbb{R}^{n+1}$ is a product, with $P = \partial_t^2 - \Delta$ the standard wave operator it is easy to check that ∂_ν is then the inward unit normal to Ω' with respect to the Euclidean structure.

Thus the Neumann problem,

$$Pu = 0, \quad \partial_\nu u\big|_{\partial\Omega} = g, \tag{8.0.4}$$

is always well-defined. It can be treated by methods similar to those used above to examine the Dirichlet problem (7.0.1), (7.0.2). This is not necessary however since the solution of the Dirichlet problem allows the Neumann problem (8.0.4), or any other boundary problem, to be replaced by the discussion of the invertibility of the appropriate operator on the boundary; i.e., to be reduced to the boundary.

§8.1: Neumann operator: diffractive case

The Neumann operators are microlocally defined via the parametrices $E_\pm$ constructed in Chapter 7, as:

$$N_\pm(f) = \partial_\nu E_\pm(f)\big|_{\partial\Omega}. \tag{8.1.1}$$

To solve (8.0.4) it is only necessary to solve the equation in the boundary:

$$N_\pm(f) = g \text{ at } \rho, \tag{8.1.2}$$

since then from the definition (8.1.1),

$$u = E_\pm(f) \Longrightarrow \partial_\nu u\big|_{\partial\Omega} \equiv g \text{ at } \rho. \tag{8.1.3}$$

This gives a microlocal solution to the Neumann problem (8.0.1). The microlocal invertibility properties of the Neumann operators are easily deduced from their properties as Airy operators.

Proposition 8.1.4. *The microlocal Neumann operators at $\bar\rho \in \mathcal{G}_d$ can be written in the form*

$$N_\pm \equiv J \cdot [A \cdot \Phi_\pm + B] \cdot J^{-1} \quad \textit{at } (\bar\rho, \bar\rho) \tag{8.1.5}$$

where J is an elliptic Fourier integral operator associated to a canonical diffeomorphism from a conic neighbourhood of $\bar\rho$ to a conic neighbourhood of $(0, \dots, 0, 1) \in T^*\mathbb{R}^n$, J^{-1} is a microlocal inverse for J and A, B are pseudodifferential operators $A \in \mathrm{OPS}^{2/3}(\mathbb{R}^n)$, $B \in \mathrm{OPS}^0(\mathbb{R}^n)$, and A is elliptic.

Proof. We apply ∂_ν to the representation (7.1.8) of $E_\pm$, obtaining

$$\partial_\nu E_\pm(f) = \frac{1}{(2\pi)^n} \int \Big[g' \frac{A_\pm(\zeta)}{A_\pm(\zeta_0)} + ih' \frac{A'_\pm(\zeta)}{A_\pm(\zeta_0)}\Big] e^{i\theta} \hat F \, d\xi, \tag{8.1.6}$$

with the new symbols satisfying:

$$\begin{aligned} g' &= \partial_\nu g + i[\partial_\nu\theta]g - i\zeta[\partial_\nu\zeta]h, \\ h' &= \partial_\nu h + [\partial_\nu\zeta]g + i[\partial_\nu\theta]h. \end{aligned} \tag{8.1.7}$$

By construction the symbol h vanishes at the boundary, $x = 0$. Moreover, from (4.2.5) $\partial_\nu\zeta\big|_{x=0} \neq 0$ so the second equation in (7.1.15) shows that

$$\partial_\nu\theta = 0 \text{ at } x = 0. \tag{8.1.8}$$

Thus from (8.1.7)

$$g' = \partial_\nu g, \quad h' = [\partial_\nu\zeta]g + \partial_\nu h \text{ on } \partial\Omega \tag{8.1.9}$$

and hence

$$g'\big|_{x=0} \in S^0, \quad h'\big|_{x=0} \in S^{2/3}, \quad h' \text{ elliptic.} \tag{8.1.10}$$

This gives the representation of the Neumann operator (8.1.5) by taking J to be the Fourier integral operator defined by (7.1.8A), J^{-1} a microlocal inverse for J near the base point and then setting

$$\begin{aligned} \int g' e^{i\theta_0} \hat{F}(\xi)\, d\xi &= JB(F), \\ i \int h' e^{i\theta_0} \hat{F}(\xi)\, d\xi &= JA(F), \end{aligned} \tag{8.1.11}$$

uniquely defining pseudodifferential operators B and A, microlocally near ρ. The remaining properties of A and B stated in the Proposition follow from (8.1.10).

Returning to the microlocal solvability of the equation (8.1.2) we have:

Proposition 8.1.12. *Near a diffractive point for a second order differential operator with real principal symbol the Neumann operator is microlocal and microlocally hypoelliptic.*

Proof. As shown in Chapter 5, $\Phi_\pm = \mathcal{A}'_\pm / \mathcal{A}_\pm$ belong to $\mathrm{OPS}^{1/3}_{1/3,0}$ and hence

$$A\Phi_\pm + B \in \mathrm{OPS}^{1}_{1/3,0}\,. \tag{8.1.13}$$

This operator is hypoelliptic; indeed $\Phi_\pm^{-1} \in \mathrm{OPS}^{0}_{1/3,0}$, so

$$A + B\Phi_\pm^{-1} \in \mathrm{OPS}^{2/3}_{1/3,0} \quad \text{is elliptic,} \tag{8.1.14}$$

and therefore

$$(A\Phi_\pm + B)^{-1} = \Phi_\pm^{-1}(A + B\Phi_\pm^{-1})^{-1} \in \mathrm{OPS}^{-2/3}_{1/3,0}\,. \tag{8.1.15}$$

Consequently, for N given by (8.1.5), we have that

$$N^{-1} \equiv J\Phi_\pm^{-1}(A + B\Phi_\pm^{-1})^{-1} J^{-1} \tag{8.1.16}$$

is a microlocal inverse for the Neumann operator (8.1.5) near the point $\bar{\rho}$.

This microlocal invertibility enables us to construct a parametrix for the Neumann Problem under the same hypotheses as in Proposition 7.1.9.

§8.2: Neumann operator: gliding case

Near a gliding point for a second order differential operator the Neumann operator takes a form similar to (8.1.5) but involving the oscillatory Airy operators discussed in Chapter 5.

Proposition 8.2.1. *The microlocal Neumann operators of P at a gliding point $\bar{\rho} \in \mathcal{G}_d$ are Airy operators:*

$$N_\pm \equiv J^{-1} \cdot [A \cdot \Phi i_\pm + B] \cdot J \text{ at } (\bar{\rho}, \bar{\rho}), \tag{8.2.2}$$

where J is an elliptic Fourier integral operator associated to a canonical diffeomorphism from a conic neighbourhood of $\bar{\rho}$ to a conic neighbourhood of $(0, \dots, 0, 1) \in T^*\mathbb{R}^n$, J^{-1} is a microlocal inverse for J and A, B are pseudodifferential operators $A \in \mathrm{OPS}^{2/3}(\mathbb{R}^n)$, $B \in \mathrm{OPS}^0(\mathbb{R}^n)$, with A elliptic.

Proof. The derivation of this formula is precisely the same as for Proposition 8.1.4 only starting with the formula (7.2.2) for the parametrix in the gliding case.

As distinct from the diffractive case considered above, the Neumann operator is not a pseudodifferential operator near a gliding point, nor is it microlocal. The propagation of singularities for such operators is discussed extensively in Chapter 5. Fortunately the microlocal invertibility is still easily established.

Proposition 8.2.3. *The Neumann operators* (8.2.3) *at a gliding point have unique microlocal parametrices with the same wavefront relation as given in* (5.9.20) *for Φi.*

Proof. From Proposition 5.3.10, it follows that, microlocally near the point $\bar{\rho}$,

$$\Phi i_\pm^{-1} A^{-1} B : H^s \longrightarrow H^{s+1/3}. \tag{8.2.4}$$

Now write, at least formally,

$$A\Phi i_\pm + B = A\Phi i_\pm (I + \Phi i_\pm^{-1} A^{-1} B). \tag{8.2.5}$$

By (8.2.4) we can asymptotically expand this formal inverse modulo microlocally smoothing operators as

$$(A\Phi i_\pm + B)^{-1} \sim \Phi i_\pm^{-1} A^{-1} \sum_{j \geq 0} (-1)^j [\Phi i_\pm^{-1} A^{-1} B]^j. \tag{8.2.6}$$

Consequently, the Neumann operator can be inverted microlocally by

$$N_\pm^{-1} \sim J \Phi i_\pm^{-1} A^{-1} \sum_{j \geq 0} (-1)^j [\Phi i_\pm^{-1} A^{-1} B]^j J^{-1}. \tag{8.2.7}$$

Since each of the terms in (8.2.7) has wavefront set in the same relation as the Neumann operator itself, and the Sobolev regularity of these terms increases to infinity, it follows that the microlocal inverse has this wavefront relation. From this the uniqueness follows directly, since operators with such an estimate on the wavefront relation microlocally form a group under composition.

§8.3: Microlocal energy estimates

Note that (8.2.4) and the standard continuity properties of Fourier integral operators give, for (8.2.7),

$$N^{-1} : H^s \longrightarrow H^{s+1/3}. \tag{8.3.1}$$

This is a weaker result than in the diffractive case, where (8.1.16) implies

$$N^{-1} : H^s \longrightarrow H^{s+2/3}. \tag{8.3.2}$$

The difference between (8.3.1) and (8.3.2) reflects a difference in estimates for solutions to the Neumann problem (8.0.4) in the gliding and grazing cases, respectively. We note that the solution to (8.1.16) with Dirichlet boundary condition

$$u\big|_{\mathbb{R}\times\partial K} = f \in \mathcal{C}_c^{-\infty}(\mathbb{R}\times\partial K) \tag{8.3.3}$$

satisfies the strong estimates

$$\|u\|_{H^s([-T,T]\times B)} \le C_{T,B}\|f\|_{H^s}, \tag{8.3.4}$$

for any bounded $B \subset \mathbb{R}^n \setminus K$, f supported in $-T \le t \le T$. This is a special case of estimates of Kreiss [Kr] and Sakamoto [Sak]. Now if the boundary is diffractive, (8.3.2) implies, for solutions of the Neumann problem (8.0.4), that

$$\|u\|_{H^s([-T,T]\times B)} \le C_{T,B}\|g\|_{H^{s-2/3}}. \tag{8.3.5}$$

In the gliding case, (8.3.1) implies

$$\|u\|_{H^s([-T,T]\times B)} \le C_{T,B}\|g\|_{H^{s-1/3}}. \tag{8.3.6}$$

This last estimate is not sharp. In fact, for a general boundary, elementary considerations show that we have an estimate

$$\|u\|_{H^s([-T,T]\times B)} \le C_{T,B}\|g\|_{H^{s-1/2}}. \tag{8.3.7}$$

Indeed, choose G on $\mathbb{R}\times B$ such that $G \in H^{s+1}$ and $(\partial/\partial\nu)G\big|_{\mathbb{R}\times\partial K} = g$. Then write $u = G + v$ where v solves

$$\begin{gathered}\Big(\frac{\partial^2}{\partial t^2} - \Delta\Big)v = -\Big(\frac{\partial^2}{\partial t^2} - \Delta\Big)G = H \text{ on } K^{\complement},\\ \frac{\partial v}{\partial \nu} = 0 \text{ on } \mathbb{R}\times\partial K, \quad v = 0 \text{ for } t << 0.\end{gathered}$$

The formula

$$v(t) = \int_{-\infty}^{t} (-\Delta)^{-1/2}\sin[(t-s)(-\Delta)^{1/2}]H(s)\,ds$$

shows

$$\|v\|_{H^1([-T,T]\times B)} \le C_{T,B}\|H\|_{L^2} \le C'_{T,B}\|G\|_{H^2} \le C''_{T,B}\|g\|_{H^{1/2}},$$

which gives (8.3.7) for $s = 1$. We refer to [RaMa] for the general argument.

Now the general estimate (8.3.7) is stronger than the estimate (8.3.6). On the other hand, applying the trace theorem to (8.3.7) yields only the very weak result

$$\|u\|_{H^{s-1/2}([-T,T]\times\partial K)} \le C_T\|g\|_{H^{s-1/2}}, \tag{8.3.8}$$

or

$$N^{-1} : H^s \longrightarrow H^s, \tag{8.3.9}$$

which is weaker than the result (8.3.1) on N^{-1} in the gliding case.

Remark. Estimates on solutions to the Neumann problem stronger than (8.3.5)–(8.3.7) have been established by D. Tataru, [Tat].

§8.4: Neumann operator identities

The forms (8.1.5) and (8.2.2) for the Neumann operator N used the prescription $h|_{\partial\Omega} = 0$. We now show how other sets of solutions g', h' to the transport equations yield other forms of the Neumann operator. Comparing these forms for N yields some non-obvious identities among Airy operators, which will be further discussed in Chapters 9 and 10. We give details for the gliding case.

If we take (7.2.2) as the parametrix for the gliding ray problem, with g, h replaced by g', h', other solutions to the transport equations, and do not require $h'|_{\partial\Omega} = 0$, but still require $\theta_\nu|_{\partial\Omega} = 0$, we get, modulo $\mathcal{C}^\infty$,

$$u\big|_{\partial\Omega} = \int \big(g' + ih'\Phi i(\zeta_0)\big)e^{i\theta}\hat{F}_1(\xi)\,d\xi, \tag{8.4.1}$$

for a certain F_1, hence

$$\partial_\nu u\big|_{\partial\Omega} = \int (\zeta_\nu g' + ih'_\nu)e^{i\theta}\Phi i_T\hat{F}_1\,d\xi + \int (ih'\zeta_\nu\zeta + g'_\nu)e^{i\theta}\hat{F}_1\,d\xi. \tag{8.4.2}$$

If we set

$$J'G = \int g' e^{i\theta}\hat{G}(\xi)\,d\xi, \tag{8.4.3}$$

then

$$u\big|_{\partial\Omega} = J'(I + R\Phi i_T)F_1 \tag{8.4.4}$$

and

$$\partial_\nu u\big|_{\partial\Omega} = (J_1'\Phi i_T + K_0')F_1 \tag{8.4.5}$$

with

$$J_1' = J'A', \quad K_0' = J'B'. \tag{8.4.6}$$

Here

$$A' \in \mathrm{OPS}^{2/3} \ \text{ has principal symbol } \zeta_\nu, \tag{8.4.7}$$

and, if g_0' and h_0' are the principal symbols of g' and h',

$$B' \in \mathrm{OPS}^1 \ \text{ has principal symbol } i(h_0'/g_0')\zeta_\nu\zeta_0. \tag{8.4.8}$$

We suppose $g_0' \neq 0$ on $\partial\Omega$. The operator R occurring in (8.4.4) satisfies

$$R \in \mathrm{OPS}^{-1/3} \ \text{ has principal symbol } h_0'/g_0'. \tag{8.4.9}$$

Now we see that

$$\partial_\nu u\big|_{\partial\Omega} = J'(A'\Phi i_T + B')F_1, \tag{8.4.10}$$

which implies

$$N = J'(A'\Phi i_T + B')(I + R\Phi i_T)^{-1}(J')^{-1}, \tag{8.4.11}$$

granted the invertibility of $I + R\Phi i_T$, a point that will be established in the next chapter.

We can compare (8.4.11) with the form (8.2.2) for N. Note that

$$J' = JD, \tag{8.4.12}$$

where

$$D \in \mathrm{OPS}^0 \ \text{ has principal symbol } g_0'/g_0. \tag{8.4.13}$$

Thus identity of the two expressions for N is equivalent to the identity:

$$A\Phi i_T + B = D(A'\Phi i_T + B')(I + R\Phi i_T)^{-1}D^{-1}. \tag{8.4.14}$$

We remark on some aspects of the general form (8.4.11) for the Neumann operator (in the gliding case). First, the principal symbol of A', ζ_ν, is independent of the choice of solutions to the transport equations. Also, the principal symbol of B' is a smooth multiple of ξ_n (assuming $g' \neq 0$), so it always vanishes at $\xi_n = 0$. As for R, whose principal symbol is h_0'/g_0', as shown in Proposition 4.4.11, this can be taken to be an arbitrary smooth function, homogeneous of degree $-1/3$. This freedom to specify the principal symbol of R will be utilized in the next two chapters. Here we remark it allows the principal symbol of B' to be an arbitrary multiple of ξ_n.

Of course, we have similar results for the Neumann operator in the diffractive case. There is no need to write down the details.

Chapter 9: Airy operator calculus – the diffractive case

As discussed in Chapter 8, to construct microlocal parametrices for more general boundary problems near diffractive, or gliding, points there are two obvious methods. The first is to construct the parametrix directly, as for the Dirichlet problem, and to prove the solvability of the transport equations, with the appropriate boundary conditions. The second method, adopted from now on, is to reduce the problem to the boundary. The reduction of quite general boundary problems to the boundary yields an operator equation involving the Neumann operator, an Airy operator. In this chapter the solvability of such equations:

$$(A \cdot N_+ + B)u = f \tag{9.0.1}$$

is considered microlocally near diffractive points. The same type of analysis near gliding points will be given in Chapter 10.

In fact these two methods for analyzing general boundary problems are closely related. Thus to solve the equation (9.0.1) in the simplest case, when B is elliptic, we use identities for Airy operators established in §9.1, exploiting the Neumann operator identities of §8.4. We note that, if B in (9.0.1) is elliptic the microlocal solvability of the equation is trivial, i.e. is a consequence of the usual pseudodifferential calculus. However the investigation of the algebraic properties of Airy operators carried out below gives a quite simple form for the inverse in this and other cases considered in subsequent chapters. It also serves as a simple introduction to the somewhat more subtle calculus of Airy operators in the gliding case.

§9.1: Simple Airy operators

Working microlocally near the usual base point $\bar{\xi} = (0; 0, \ldots, 1) \in T^*\mathbb{R}^n$ consider the two-sided module generated by the basic convolution operator $\Phi_+ \in \mathrm{OPS}^{1/3}_{1/3,0}$. This is spanned by the simple Airy operators:

$$\begin{gathered} Q = A \cdot \Phi_+ \cdot A' + B \in \mathcal{A}^{m,+}_S \iff A \in \mathrm{OPS}^{\mu}, A' \in \mathrm{OPS}^{\mu'}, \\ \mu + \mu' = m - \frac{1}{3},\ B \in \mathrm{OPS}^m . \end{gathered} \tag{9.1.1}$$

In particular

$$\mathrm{OPS}^m \subset \mathcal{A}^{m,+}_S, \quad \forall\ m \in \mathbb{R}. \tag{9.1.2}$$

The suffix 'S' here corresponds to the fact that there is only one factor and one term involving Φ in the definition (9.1.1), as opposed to the more general spaces considered below. From the computations in Chapter 5:

$$\mathcal{A}^{m,+}_S \subset \mathrm{OPS}^m_{1/3,0} . \tag{9.1.3}$$

These sets $\mathcal{A}^{m,+}_S$ are not even linear spaces, but are of prime importance here since the Neumann operator is certainly of this type. An important part is played in the consideration of these operators by the following identity.

Proposition 9.1.4. *Given* $H \in \mathrm{OPS}^{m-2/3}$ *there exist* $G_i \in \mathrm{OPS}^{m-1/3}, i = 1, 2$ *with equal principal symbols and* $L \in \mathrm{OPS}^m$ *such that*

$$\Phi_+ \cdot H \cdot \Phi_+ = G_1 \cdot \Phi_+ - \Phi_+ \cdot G_2 + L. \tag{9.1.5}$$

Proof. It suffices to consider the case $m = 1/3$ since an elliptic operator, such as $\langle D\rangle^m$, which commutes with Φ_+ can be used to change the order. Also, given H, it suffices to prove (9.1.5) for some operator H' with the same principal symbol as H, since then an inductive argument yields a proof of the full statement.

We will obtain (9.1.5) from the Neumann operator identity (8.4.14), or rather its Φ_+ analogue:

$$(A\Phi_+ + B)D(I + R\Phi_+) = D(A'\Phi_+ + B') \tag{9.1.6}$$

where

$$\sigma_{2/3}(A) = \sigma_{2/3}(A') = \zeta_\nu, \ \sigma_0(D) = \frac{g'}{g}, \text{ and } \sigma_{-1/3}(R) = \frac{h'}{g'}. \tag{9.1.7}$$

and furthermore $\sigma_1(B') = i(h'/g')\zeta_\nu\zeta$, while $B \in OPS^0$. Hence, with $H' = DR$,

$$\Phi_+ H' \Phi_+ = (A^{-1}DA' - A^{-1}BDR)\Phi_+ - \Phi_+ D + A^{-1}(DB' - BD), \tag{9.1.8}$$

where A^{-1} denotes a microlocal parametrix of the elliptic operator A. This identity is of the form (9.1.5). Note that

$$\sigma_{-1/3}(H') = \frac{h'}{g}. \tag{9.1.9}$$

Recall that g, h solve the transport equations with $h\big|_{\partial\Omega} = 0, g\big|_{\partial\Omega}$ elliptic. By Proposition 4.4.11, given g, we can solve the transport equations for g', h', with $h'\big|_{\partial\Omega}$ specified to be equal to $\sigma_{-1/3}(H')g$, for any given symbol $\sigma_{-1/3}(H') \in S^{-1/3}$, so the Proposition is proved.

Since the identity (9.1.5) is rather useful in what follows it is important to note that the principal symbols of H, G_j, and L are related by (9.1.9) and

$$\sigma_0(G_j) = \frac{g'}{g}, \quad \sigma_{1/3}(L) = i\zeta\frac{h'}{g}. \tag{9.1.10}$$

Let us note the following elementary special case of (9.1.5):

$$\Phi_+^2 = (D_1^{1/3}x_n)\Phi_+ - \Phi_+(D_1^{1/3}x_n) - D_1^{-1/3}D_n, \tag{9.1.11}$$

which follows immediately from the ODE

$$\Phi'(\zeta) = \Phi(\zeta)^2 - \zeta. \tag{9.1.12}$$

We further remark that any choice of second order operator P on a region with a diffractive boundary could have been made. One convenient choice would be the Friedlander example (3.6.7), for which

$$g = 1,\ \zeta_\nu = \xi_1^{2/3} \text{ on } \{x = 0\},$$

by the solution (4.0.7) to the eikonal equations in this case.

One immediate consequence of the identity (9.1.5) is that the linear space of operators spanned by the simple Airy operators is a filtered algebra. Set

$$\mathcal{A}_F^{m,+} = \mathrm{sp}(\mathcal{A}_S^{m,+}). \tag{9.1.13}$$

Then:

Proposition 9.1.14. *For any $m, m' \in \mathbb{R}$ with composition defined microlocally:*

$$\mathcal{A}_F^{m,+} \cdot \mathcal{A}_F^{m',+} \subset \mathcal{A}_F^{m+m',+}. \tag{9.1.15}$$

Proof. Clearly it suffices to show that

$$\mathcal{A}_S^{m,+} \cdot \mathcal{A}_S^{m',+} \subset \mathcal{A}_F^{m+m',+}. \tag{9.1.16}$$

Expanding a product gives immediately:

$$(A_1 \cdot \Phi_+ \cdot A_1' + B_1) \cdot (A_2 \cdot \Phi_+ \cdot A_2' + B_2) \equiv A_1 \cdot \Phi_+ \cdot H \cdot \Phi_+ \cdot A_2' \mod \mathcal{A}_F^{m+m',+},$$

with

$$H = A_1' \cdot A_2.$$

Then the identity (9.1.5) allows the remaining term to be expanded into a finite sum of simple Airy operators.

As is shown below the 'natural' space of Airy operators $\mathcal{A}^{m,+}$ is slightly larger than this finite span.

We have only described here the composition properties of the pseudodifferential operators associated to Φ_+. Those associated to Φ_- are simply the adjoints of these operators, as is immediately apparent:

Proposition 9.1.17. *If Φ_+ is replaced by Φ_- in* (9.1.1) *and the resulting space of operators is denoted $\mathcal{A}_S^{m,-}$ then:*

$$Q \in \mathcal{A}_S^{m,+} \Longleftrightarrow Q^* \in \mathcal{A}_S^{m,-}. \tag{9.1.18}$$

§9.2: Ellipticity in $\mathcal{A}_S^{*,\pm}$

Here we will examine parametrices near $\{\xi_n = 0\}$ for operators of the form

$$Q = A\Phi + B \in \mathcal{A}_S^{m,\pm}, \text{ with } B \in OPS^m \text{ elliptic.} \tag{9.2.1}$$

We set $\Phi = \Phi_\pm$. Part of the interest in (9.2.1) is its natural appearance in boundary problems satisfying a Lopatinski condition, as will be seen in Chapter 12. First, it is clear that the hypothesis implies

$$Q \in OPS^m_{1/3,0} \text{ is elliptic near } \{\xi_n = 0\}, \tag{9.2.2}$$

so we have a parametrix

$$Q^{-1} \in OPS^{-m}_{1/3,0}, \tag{9.2.3}$$

with a symbol expansion of a straightforward, though complicated form. We will use Airy operator identities to give a special form for the parametrix here, a special case of more general results to be presented in §9.7.

Composing Q on the right with a parametrix $B^{-1} \in OPS^{-m}$ for B, we can reduce our study to that of

$$Q = I + A\Phi, \quad A \in OPS^{-1/3}. \tag{9.2.4}$$

Note that, for any elliptic $D \in OPS^0$,

$$(I + A\Phi)(I - AD\Phi D^{-1}) = I + A\Phi - AD\Phi D^{-1} - A\Phi A D \Phi D^{-1}. \tag{9.2.5}$$

Now exploit the identity (9.1.5), with $H = AD$, choosing $D = -G_2$, which by (9.1.10) we can suppose to be elliptic. We obtain

$$\begin{aligned} (I + A\Phi)(I - AD\Phi D^{-1}) &= I + A(G_2 - G_1)D\Phi D^{-1} \\ &= (I - ALD^{-1}) + A(G_2 - G_1)\Phi D^{-1} \\ &= E + F\Phi D^{-1}, \end{aligned} \tag{9.2.6}$$

with $E \in OPS^0$ elliptic near $\{\xi_n = 0\}$, $D^{-1} \in OPS^0$, $F \in OPS^{-4/3}$, the ellipticity of E following from the vanishing of the principal symbol of L on $\{\xi_n = 0\}$. Thus, setting $C = E^{-1}F$, we have:

Proposition 9.2.7. *For $A \in OPS^{-1/3}, I + A\Phi$ has near $\{\xi_n = 0\}$ a microlocal parametrix*

$$(I + A\Phi)^{-1} = (I - AD\Phi D^{-1})(I + C\Phi D^{-1})^{-1}E^{-1}, \tag{9.2.8}$$

with $C \in OPS^{-4/3}$, and E, D elliptic in OPS^0.

Note that $C\Phi D^{-1} \in OPS^{-1}_{1/3,0}$, so the Neumann expansion

$$(I + C\Phi D^{-1})^{-1} \sim I + \sum_{j\geq 1} (-C\Phi D^{-1})^j \tag{9.2.9}$$

is asymptotic. Note that, by Proposition 9.1.14, the j^{th} term in this sum belongs to $\mathcal{A}_F^{-j,\pm}$.

Let us note the result of a slightly cruder argument. Namely, we have

$$\begin{aligned}(I + A\Phi)(I - A\Phi) &= I - A\Phi^2 A + A[A, \Phi]A \\ &= (I + AD_1^{-1/3} D_n A) - A[\Phi, D_1^{1/3} x_n]A + A[A, \Phi]A \\ &= E_1 + \mathcal{R},\end{aligned} \tag{9.2.10}$$

with $E_1 \in OPS^0$ elliptic near $\{\xi_n = 0\}$ and $\mathcal{R} \in OPS^{-2/3}_{1/3,0}$. The ellipticity of this operator in $OPS^0_{1/3,0}$, microlocally near $\{\xi_n = 0\}$, is clear, but (9.2.6) has a neater form than this identity.

§9.3: Hypoellipticity in $\mathcal{A}_S^{*,\pm}$

The ellipticity of simple Airy operators discussed in §9.2 corresponds precisely to ellipticity in $\mathrm{OPS}^m_{1/3,0}$. There is another condition on the principal symbol which implies microlocal invertibility. Namely, consider the operator $Q = A\Phi_+ + B \in \mathcal{A}_S^{m,+}$, with

$$A \in OPS^{m-1/3} \text{ elliptic and } B \in OPS^m \text{ with } \sigma_m(B) = 0 \text{ on } \{\xi_n = 0\}. \tag{9.3.1}$$

Under this hypothesis we can write $Q = (A + B\Phi_+^{-1})\Phi_+$, with $A + B\Phi_+^{-1}$ elliptic in $OPS^{m-1/3}_{1/3,0}$, and apply the $OPS^*_{1/3,0}$ pseudodifferential operator calculus to construct a parametrix. As before, we will exploit Airy operator identities to produce a neater form for the parametrix.

Proposition 9.3.2. *If $Q = A\Phi_+ + B \in \mathcal{A}_S^{m,+}$ satisfies* (9.3.1)*, then there exists an operator $Q' \in \mathcal{A}_S^{-m,+}$ such that*

$$Q \cdot Q' = R\Phi_+ + E, \quad R \in \mathrm{OPS}^{-1/3} \text{ elliptic and } E \in \mathrm{OPS}^{-1}. \tag{9.3.3}$$

In fact, $Q' = C\Phi_+ + D$, with D elliptic in OPS^{-m}.

Proof. Taking $Q' = C\Phi_+ + D$, we apply (9.1.5) with $H = C$ to obtain

$$(A\Phi_+ + B)\cdot(C\Phi_+ + D) = (A\cdot G_1 + B\cdot C)\cdot\Phi_+ + A\cdot\Phi_+\cdot(D - G_2) + (L + B\cdot D). \tag{9.3.4}$$

First we choose D to make the second term trivial:

$$D = G_2. \tag{9.3.5}$$

Here G_2 is determined by the choice of C. The symbolic equation to be satisfied is therefore:

$$\sigma_0(L + B \cdot D) = \zeta\Big(h + b'\frac{g'}{g}\Big) = 0, \tag{9.3.6}$$

by (9.1.9)–(9.1.10). Here the hypothesis on B is used to write its principal symbol as $\zeta b'$. Again, as shown in Proposition 4.4.11 there is a solution of this equation making g, and hence $A \cdot G_1 + B \cdot C$ elliptic. This completes the proof of the Proposition.

Notice that Proposition 9.3.2 allows the microlocal inverse to Q, under the hypotheses (9.3.1), to be written in the form:

$$Q^{-1} \sim (C + D\Phi_+^{-1})D'\Big[\sum_{j=0}^{\infty}(-1)^j(E \cdot \Phi_+^{-1} \cdot D')^j\Big], \tag{9.3.7}$$

where D' is a parametrix for R. Since

$$E \cdot \Phi_+^{-1} \cdot D' \in \mathrm{OPS}^{-2/3}_{1/3,0}, \tag{9.3.8}$$

the sum in (9.3.7) is asymptotic, i.e., consists of progressively lower order operators. The leading part of the inverse therefore determines its regularity:

$$Q^{-1} \in \mathrm{OPS}^{-m+1/3}_{1/3,0}. \tag{9.3.9}$$

Since there is no possibility of better regularity we see:

$$Q \text{ of the form (9.3.1) is hypoelliptic with loss of } \frac{1}{3} \text{ derivatives.} \tag{9.3.10}$$

§9.4: The operator classes $\mathcal{A}^{*,\pm}$

Next we consider a microlocal algebra of operators containing both the finite span $\mathcal{A}^{*,+}_F$ and the inverses of the elliptic elements discussed in Proposition 9.2.7. For each $j \in \mathbb{N}$, let $\Phi^{(j)}$ represent the operator with symbol

$$\Phi^{(j)}(\xi_1^{-1/3}\xi_n) = \xi_1^{j/3}\Big(\frac{\partial}{\partial\xi_n}\Big)^j\Phi(\xi_1^{-1/3}\xi_n). \tag{9.4.1}$$

Then set

$$\begin{aligned}\mathcal{A}^{m,\pm} = \Big\{Q \in \mathrm{OPS}^m_{1/3,0};\, Q - B \sim \sum_{j\ge 0} A_j \cdot \Phi^{(j)}_\pm,\\ \text{with } B \in \mathrm{OPS}^m,\, A_j \in \mathrm{OPS}^{m-(j+1)/3}\Big\}.\end{aligned} \tag{9.4.2}$$

As usual this definition is microlocal, in a small conic neighborhood of $\bar\xi = (0; 1, 0, \ldots, 0)$. Each operator in $\mathcal{A}^{m,\pm}$ can be represented in terms of an oscillatory integral involving only the Airy quotient.

Proposition 9.4.3. *An operator Q belongs to $\mathcal{A}^{m,\pm}$ if and only if it can be written in the form*

$$Qu(x) = \int [a(x,y,\xi)\Phi_\pm(\xi_1^{-1/3}\xi_n) + b(x,y,\xi)]u(y)e^{i(x-y)\cdot\xi}\,dy\,d\xi, \tag{9.4.4}$$
$$a(x,y,\xi) \in S^{m-1/3},\ b(x,y,\xi) \in S^m.$$

Proof. Since we can always compose with some elliptic operator such as D_1^m it suffices to consider the case $m = 0$. Suppose Q has the form (9.4.2). Take the symbol of a particular term:

$$Q_j(x,\xi) = A_j(x,\xi)\Phi^{(j)}(\xi_1^{-1/3}\xi_n) \in S^{-(j+1)/3}, \quad j \geq 1. \tag{9.4.5}$$

We see that, with $B_j(x,\xi) = A_j(x,\xi)\xi_1^{j/3}$,

$$\begin{aligned} Q_j u &= \int B_j(x,\xi)\left[\Big(\frac{\partial}{\partial\xi_n}\Big)^j \Phi_\pm\right] e^{i(x-y)\cdot\xi} u(y)\,dy\,d\xi \\ &= \int \Phi_\pm\Big(-\frac{\partial}{\partial\xi_n}\Big)^j \left[B_j(x,\xi)e^{i(x-y)\cdot\xi}\right] u(y)\,dy\,d\xi \\ &= (-1)^j \sum_{l=0}^{j} i^l \binom{j}{l} \int \left[\partial_{\xi_n}^{j-l} B_j(x,\xi)\right](x_n - y_n)^l\ \Phi\ e^{i(x-y)\cdot\xi} u(y)\,dy\,d\xi. \end{aligned} \tag{9.4.6}$$

As j or $l \to \infty$ the symbol in (9.4.6) converges in the sense of formal power series at $x = y$ or asymptotically. Clearly then it is possible to choose $a(x,y,\xi)$ in (9.4.4) to expand to this double series with errors either vanishing to high order at $x = y$ or of low symbolic order. Note that we can take the leading term $a_0(x,y,\xi)$ of $a(x,y,\xi)$ to be

$$a_0(x,y,\xi) = A_0(x,\xi). \tag{9.4.7}$$

and we can set $b(x,y,\xi) = B(x,\xi)$. This shows that every operator in $\mathcal{A}^{0,\pm}$ is of the form (9.4.4). The converse statement follows from the standard method of obtaining the symbol of a pseudodifferential operator in $\mathrm{OPS}^0_{1/3,0}$ from a multisymbol representation such as (9.4.4), so the proposition is proved.

The following simple observation will be useful.

Lemma 9.4.8. *For any $m \in \mathbb{R}$, $\mathcal{A}^{m,\pm}$ is a two-sided OPS^0-module. An operator Q belongs to $\mathcal{A}^{m,+}$ if and only if $Q^* \in \mathcal{A}^{m,-}$.*

Proof. It is clear from the definition that

$$A \in OPS^0,\ Q \in \mathcal{A}^{m,\pm} \Longrightarrow AQ \in \mathcal{A}^{m,\pm}.$$

The behavior under adjoints follows from the representation (9.4.4), and then the implication $QA \in \mathcal{A}^{m,\pm}$ follows by taking adjoints.

Knowing that $\mathcal{A}^{m,\pm}$ is a two-sided OPS^0-module, we see that

$$\mathcal{A}_F^{m,\pm} \subset \mathcal{A}^{m,\pm}. \tag{9.4.9}$$

The following is a partial converse.

Proposition 9.4.10. *Any finite sum*

$$B + \sum_{j=0}^{N} A_j \Phi^{(j)}, \quad B \in OPS^m,\ A_j \in OPS^{m-(j+1)/3}$$

belongs to $\mathcal{A}_F^{m,\pm}$.

Proof. It suffices to show that

$$\Phi_\pm^{(j)} \in \mathcal{A}_F^{(j+1)/3,\pm},\ \text{for } j \geq 1. \tag{9.4.11}$$

(Note that $\Phi_\pm^{(j)} \in OPS^0_{1/3,0}$ for $j \geq 1$.) This can be established by exploiting the identity

$$\Phi'(\zeta) = \zeta - \Phi(\zeta)^2, \tag{9.4.12}$$

which implies, by induction,

$$\Phi^{(j-1)}(\zeta) = \sum_{l=0}^{j} p_{jl}(\zeta)\Phi(\zeta)^l, \tag{9.4.13}$$

where p_{jj} is a nonzero constant and more generally $p_{jk}(\zeta)$ is a polynomial belonging to the space $\mathcal{P}_{j-k}$, where $\mathcal{P}_i$ is the space of polynomials in ζ spanned by monomials ζ^l with

$$l \leq \frac{1}{2}i, \quad 2l \equiv i (\text{mod } 3). \tag{9.4.14}$$

The inclusion (9.4.11) then follows from (9.4.13), since, by Proposition 9.1.14, we have

$$\Phi_\pm^j \in \mathcal{A}_F^{j/3,\pm}. \tag{9.4.15}$$

Another way to see (9.4.11) is to note that

$$\Phi_\pm^{(j)} = \left(\text{ad } x_n D_1^{1/3}\right)^j\ \Phi_\pm, \tag{9.4.16}$$

since both left and right multiplication by $B \in OPS^{1/3}$ map $\mathcal{A}_F^{m,\pm}$ to $\mathcal{A}_F^{m+1/3,\pm}$.

We now show that $\mathcal{A}^{*,\pm}$ contains the parametrices constructed in §9.2.

Proposition 9.4.17. *If* $Q = A\Phi_\pm + B \in \mathcal{A}_S^{m,\pm}$ *with* $B \in OPS^m$ *elliptic, then we have* $Q^{-1} \in \mathcal{A}^{-m,\pm}$.

Proof. Following the proof of Proposition 9.2.7, it suffices to show that the series

$$\sum_{j\geq 0}(-C\Phi_\pm B^{-1})^j = \sum_{j\geq 0} S_j \tag{9.4.18}$$

with $C \in OPS^{-4/3},\ B^{-1} \in OPS^0$, is asymptotic to an element of $\mathcal{A}^{0,\pm}$. Note that $S_j \in \mathcal{A}^{-j,\pm}$. Thus we can write

$$S_j = A_j \Phi_\pm B_j + C_j, \tag{9.4.19}$$

with

$$A_j, C_j \in OPS^{-j}, \quad B_j \in OPS^{-1/3}. \tag{9.4.20}$$

Applying the $S^*_{1/3,0}$ symbol calculus to (9.4.19) yields asymptotic series

$$S_j \sim C_j + \sum_{l\geq 0} A_{jl}\Phi_\pm^{(l)}, \quad A_{jl} \in OPS^{-j-(l+1)/3}, \tag{9.4.21}$$

and the sum over j of this clearly yields an asymptotic series of the form (9.4.2).

§9.5: Microlocal completeness of $\mathcal{A}^{m,\pm}$

A linear subspace $\mathcal{L}$ of $OPS^m_{\rho,\delta}$ is said to be microlocally complete if

$$P_j \in \mathcal{L},\ P_j \sim P \in OPS^m_{\rho,\delta} \Longrightarrow P \in \mathcal{L}, \tag{9.5.1}$$

where $P_j \sim P$ means $P - P_j \in OPS^{m_j}_{\rho,\delta}$ with $m_j \to -\infty$. It is clear from (9.4.9) and Proposition 9.4.10 that $\mathcal{A}^{m,\pm}$ and $\mathcal{A}_S^{m,\pm}$ have the same microlocal completion. The goal of this section is to prove:

Proposition 9.5.2. *For each* $m \in \mathbb{R},\ \mathcal{A}^{m,\pm}$ *is microlocally complete.*

This result is not entirely trivial for the following reason. Suppose $Q \in \mathcal{A}^{m,\pm}$ has an expansion of the form (9.4.2) and that $Q \in OPS^\mu_{1/3,0}$ with $\mu < m$. It does not follow that $Q \in \mathcal{A}^{\mu,\pm}$. It will be useful to record some of the properties that such operators Q do have. To begin, we characterize those $T \in \mathcal{A}^{0,\pm}$ which belong to $\mathrm{OPS}^{-2/3}_{1/3,0}$:

Lemma 9.5.3. $T \in \mathcal{A}^{0,\pm} \cap \mathrm{OPS}^{-2/3}_{1/3,0}$ *if and only if it has an expansion of the form* (9.4.2)*, with* $B \in S^{-1}$ *and* $A_0 \in S^{-4/3}$.

Proof. Certainly if T has such an expansion it is in $\mathrm{OPS}^{-2/3}_{1/3,0}$. Thus suppose that $T \in \mathcal{A}^{0,\pm} \cap \mathrm{OPS}^{-2/3}_{1/3,0}$. Let $b_0 \in S^0$, $a_{00} \in S^{-1/3}$ be the principal symbols of B and A_0 in (9.4.2). Due to the asymptotic expansion

$$\Phi(\zeta) \sim c\sqrt{\zeta} + \cdots,$$

we see that $T \in \mathrm{OPS}^{-2/3}_{1/3,0}$ implies the identity

$$b_0 = C\xi_1^{1/3} a_{00} \Big(\frac{\xi_n}{\xi_1}\Big)^{1/2}. \tag{9.5.4}$$

This in turn implies that both b_0 and a_{00} must vanish to infinite order at $\xi_n = 0$. Thus (for A_{00} with full symbol a_{00}),

$$B + A_{00}\Phi \in \mathrm{OPS}^0,$$

so we can replace B in (9.5.2) by $B + A_{00}\Phi$ and suppose without loss of generality that $a_{00} = 0$. Then (9.5.4) implies $b_0 = 0$, which proves the lemma.

One might think that Lemma 9.5.3 has the following generalization, namely if $T \in \mathcal{A}^{m,\pm} \cap OPS^{\mu}_{1/3,0}$ with $\mu < m$, then it has an expansion of the form (9.4.2) with $B \in OPS^{\mu}$ and $A_j \in OPS^{\mu_j}$, $\mu_j =$ min $(\mu, m-(j+1)/3)$, but this does not hold for $\mu < m - 2/3$. To see an example of this, start with $\Phi'_+ \in \mathcal{A}^{2/3,+} \cap OPS^0_{1/3,0}$, to which Lemma 9.5.3 applies, and square it. Making use of the ODE (9.1.12) for Φ, we can establish

$$(\Phi'_+)^2 = -\Phi_+ + 2\zeta\Phi'_+ - \frac{1}{2}\Phi'''_+ \in \mathcal{A}^{4/3,+} \cap OPS^0_{1/3,0}; \tag{9.5.5}$$

cf. (9.6.14), in the next section. The low order of the right side of (9.5.5) results from a partial cancellation of Φ_+ and $2\zeta\Phi'_+$, each of which is in $S^{1/3}_{1/3,0}$. Note that Lemma 9.5.3 does apply to (9.5.5). We now present the appropriate generalization of this Lemma.

Proposition 9.5.6. *Let $Q \in \mathcal{A}^{m,\pm} \cap OPS^{\mu}_{1/3,0}$. Fix $N < \infty$. Then there exists $K < \infty$ such that, if $\mu < m - K$, then Q has an expansion of the form* (9.4.2) *with the following property. All the terms homogeneous of degree $m, m-1, \cdots, m-N$ in the expansion of the symbol of B and all the terms homogeneous of degree $m - 2/3, \cdots, m - 2/3 - N$ in the expansion of the symbols of the A_j vanish to order at least N at $\{\xi_n = 0\}$.*

The proof of this Proposition will be postponed until §9.7, as it forms part of the argument proving Proposition 9.7.1.

We now prove Proposition 9.5.2. Thus let $Q_j \in \mathcal{A}^{m,\pm}$, $Q_j \sim Q \in OPS^{m}_{1/3,0}$. Consider the differences $R_j = Q_{j+1} - Q_j$, which have large negative order as $j \to \infty$, so Proposition 9.5.6 applies to each R_j. We can substitute the expansion of the form (9.4.2) so described, into each term of the asymptotic series

$$Q_1 + R_1 + R_2 + \cdots + R_j + \cdots ;$$

grouping together the terms homogeneous of fixed order which are coefficients of each $\Phi^{(l)}$, one obtains a series

$$B' + \sum_l A'_l \Phi^{(l)},$$

asymptotically summing to a term $Q' \in \mathcal{A}^{m,\pm}$. It is easy to verify that $Q - Q' \in OPS^m_{cl}$, so indeed $Q \in \mathcal{A}^{m,\pm}$.

§9.6: Algebraic properties of $\mathcal{A}^{m,\pm}$

Using the algebraic properties of $\mathcal{A}^{*,\pm}_F$ established before plus the microlocal completeness result of the last section, we will be able to derive fairly easily the major algebraic properties of $\mathcal{A}^{*,\pm}$.

Proposition 9.6.1. *If $Q_k \in \mathcal{A}^{m_k,\pm}$, then*

$$Q_1 Q_2 \in \mathcal{A}^{m_1+m_2,\pm}, \tag{9.6.2}$$

and

$$[Q_1, Q_2] \in \mathcal{A}^{m_1+m_2,\pm} \cap OPS^{m_1+m_2-2/3}_{1/3,0} \tag{9.6.3}$$

Proof. If $Q_{kj} \sim Q_k$ with each $Q_{kj} \in \mathcal{A}^{m_k,\pm}_F$, then for each $j, Q_{1j}Q_{2j} \in \mathcal{A}^{m_1+m_2,\pm}_F$, while as $j \to \infty, Q_{1j}Q_{2j} \sim Q_1Q_2$, so (9.6.2) follows from microlocal completeness.

To see that the order of the commutator is $m_1 + m_2 - 2/3$, it suffices to note that

$$\partial_{\xi_j} \Phi(\xi_1^{-1/3}\xi_n) \in S^{-1/3}_{1/3,0},$$

which has been established in Chapter 5.

Next we turn to the construction of parametrices for elliptic operators in $\mathcal{A}^{*,\pm}$. The following result is a generalization of Proposition 9.2.7.

Proposition 9.6.4. *If $Q \in \mathcal{A}^{0,+}$ has the form* (9.4.2) *with $B \in \mathrm{OPS}^0$ elliptic, then $Q^{-1} \in \mathcal{A}^{0,+}$.*

Proof. We know that $Q^{-1} \in \mathrm{OPS}^0_{1/3,0}$. To verify the proposition, factor out B and write

$$Q = B(I + A\Phi + R), \tag{9.6.5}$$

with $A \in \mathrm{OPS}^{-1/3}$, $R \in \mathcal{A}^{0,+} \cap \mathrm{OPS}^{-2/3}_{1/3,0}$. By Proposition 9.4.17, the operator $I + A\Phi$ has a parametrix $S \in \mathcal{A}^{0,+}$. Now

$$(I + A\Phi + R)S = I + RS, \tag{9.6.6}$$

with $RS \in \mathcal{A}^{0,+} \cap OPS^{-2/3}_{1/3,0}$, and then

$$Q^{-1} = S(I + RS)^{-1}B^{-1}. \tag{9.6.7}$$

By Proposition 9.5.2, the Neumann series

$$(I + RS)^{-1} \sim \sum_{k \geq 0} (-RS)^k \tag{9.6.8}$$

produces an element

$$(I + RS)^{-1} \in \mathcal{A}^{0,+}, \tag{9.6.9}$$

so $Q^{-1} \in \mathcal{A}^{0,+}$, as asserted.

Here's another approach to products of Airy operators. Suppose $A, B \in \mathcal{A}^{0,+}$. We have, with $E \in \mathcal{A}^{0,+}$,

$$\begin{aligned} AB &\sim \sum_{j,k} A_j \Phi^{(j)} B_k \Phi^{(k)} + E, \ A_j \in \mathrm{OPS}^{-(j+1)/3}, \ B_k \in \mathrm{OPS}^{-(k+1)/3} \\ &\sim \sum_{j,k} C_{jk} \Phi^{(j)} \Phi^{(k)} + E, \qquad C_{jk} \in \mathrm{OPS}^{-(j+k+2)/3}. \end{aligned} \tag{9.6.10}$$

We want to show AB can be asymptotically represented in the form

$$AB \sim \sum_j D_j \Phi^{(j)} + F, \ D_j \in \mathrm{OPS}^{-(j+1)/3}, \ F \in \mathrm{OPS}^0. \tag{9.6.11}$$

In order to achieve this, we use the identity (9.4.13), with coefficients p_{jl} described there; recall p_{jj} is a nonzero constant. Thus we can invert the triangular system (9.4.13) to get

$$\Phi^j(\zeta) = r_{j0}(\zeta) + \sum_{k=1}^{j} r_{jk}(\zeta) \Phi^{(k-1)}, \quad r_{jk}(\zeta) \in \mathcal{P}_{j-k}, \ r_{jj} = 1/p_{jj} \neq 0. \tag{9.6.12}$$

Passing to (9.4.13) and back via (9.6.12) enables us to write

$$\Phi^{(j)} \Phi^{(k)} = \alpha_{0,j,k}(\zeta) + \sum_{\ell=1}^{j+k+2} \alpha_{\ell,j,k}(\zeta) \Phi^{(l-1)}(\zeta) \tag{9.6.13}$$

where $\alpha_{\ell,j,k}(\zeta) \in \mathcal{P}_{j+k-2-\ell}$ is a polynomial of degree at most $(j+k-\ell+2)/2$; in particular $\alpha_{j+k+1,j,k}$ is a non-zero constant. Note that, even though $\Phi^{(j)}\Phi^{(k)}$ is of order $1-j-k$ in ζ, terms on the right side of (9.6.13) can be of order up to $(j+k+2)/2$ in ζ. Particular examples of (9.6.13) are:

$$\begin{gathered} \Phi\Phi = \zeta - \Phi', \;\; \Phi\Phi' = \frac{1}{2} + \frac{1}{2}\Phi'', \\ \Phi'\Phi' = -\Phi + 2\zeta\Phi' - \frac{1}{2}\Phi''', \;\; \Phi\Phi'' = \Phi - 2\zeta\Phi' + \Phi'''. \end{gathered} \tag{9.6.14}$$

If we substitute (9.6.13) into (9.6.10), rearrangement produces a formal sum

$$\sum_{j,k} F_{jk}\Phi^{(j)} + E', \tag{9.6.15}$$

with $E' \in \mathcal{A}^{0,+}, F_{jk} \in \mathrm{OPS}^{-(j+1)/3}$. However, for each j, there are infinitely many terms F_{jk}, and $\sum_k F_{jk}$ is not asymptotic in the usual sense. But as $k \to \infty$, terms homogeneous of a fixed degree vanish to increasingly high order at $\zeta = 0$ (i.e., $\xi_n = 0$.) Thus we can find $F_j \in \mathrm{OPS}^{-(j+1)/3}$ such that $\sum_{k=1}^{N} F_{jk} - F_j$ vanishes to arbitrarily high order at $\zeta = 0$ for N large. Now form

$$T \sim \sum_{j\ge 0} F_j \Phi^{(j)}. \tag{9.6.16}$$

We see that $T \in \mathcal{A}^{0,+}$ and

$$AB - T \in \mathrm{OPS}^0. \tag{9.6.17}$$

This proves again that $AB \in \mathcal{A}^{0,+}$.

§9.7: A cancellation effect

In §7.5 we found it convenient to know that, if an operator of the form $P = A_0 + A_1\Phi, A_0 \in \mathrm{OPS}^m, A_1 \in \mathrm{OPS}^{m-1/3}$, has the property that it is of order $-\infty$ on the open cone $\xi_n < 0$, then in fact the complete symbols of A_0 and of A_1 vanish to infinite order at $\xi_n = 0, P \in \mathrm{OPS}^m$, and $P\mathcal{A}_{\pm}^{-1}$ and related operators belong to $\mathrm{OPS}^{-\infty}$. Here, we prove a more general result. We will draw further conclusions from this in Section 9.8. Our general result is the following.

Proposition 9.7.1. *Let $P \in \mathcal{A}^{m,+}$; say*

$$P \sim A_0 + \sum_{k\ge 0} A_{k+1}\Phi^{(k)}, \tag{9.7.2}$$

with $\Phi^{(0)} = \Phi, \Phi^{(1)} = \Phi'$, etc., $A_0 \in \mathrm{OPS}^m, A_{k+1} \in \mathrm{OPS}^{m-(k+1)/3}$. Suppose P belongs to $\mathrm{OPS}^{-\infty}$ in the open cone $\xi_n < 0$. Then each $A_k, k \ge 0$, has complete *symbol vanishing to infinite order at* $\xi_n = 0$. *Consequently,* $P \in \mathrm{OPS}^m$, *and the operators*

$$P\mathcal{A}i,\ P\mathcal{A}i',\ P\mathcal{A}_\pm^{-1}$$

are all infinitely smoothing.

Say the symbols of the operators A_k in (9.7.2) have asymptotic expansions

$$a_k(x,\xi) \sim \sum_{j\geq 0} a_{kj}(x,\xi), \tag{9.7.3}$$

with $a_{0j}(x,\xi)$ homogeneous of degree $m-j$, and generally $a_{k+1,j}(x,\xi)$ homogeneous of degree $m-(k+1)/3-j$ in ξ. Recall that $\Phi^{(k)}(\zeta)$ has the asymptotic expansion

$$\Phi^{(k)}(\zeta) \sim \sum_{j\geq 0} \alpha_{kj}(-\zeta)^{1/2-k-3j/2}, \quad \zeta \to -\infty. \tag{9.7.4}$$

The elementary argument given in the proof of Lemma 7.5.20 shows that the principal symbols $a_{00}(x,\xi)$ and $a_{10}(x,\xi)$ of A_0 and A_1 both vanish to infinite order at $\xi_n = 0$. That type of argument also shows that $a_{01}(x,\xi)$ vanishes to infinite order at $\xi_n = 0$ and that, modulo an infinitely flat term,

$$a_{20}(x,\xi) = \Big(\frac{\alpha_{00}}{\alpha_{10}}\Big)\zeta\, a_{11}(x,\xi). \tag{9.7.5}$$

Then looking at the coefficients of $(-\zeta)^{-1}$ and $(-\zeta)^{-2}$ in $a_{11}(x,\xi)\Phi(\zeta)$ and $a_{20}(x,\xi)\Phi'(\zeta)$, respectively, we see that, modulo an infinitely flat term,

$$a_{11}(x,\xi) = \Big(\alpha_{01} + \frac{\alpha_{00}}{\alpha_{10}}\Big)^{-1}\zeta\, a_{02}(x,\xi). \tag{9.7.6}$$

The strategy of groping ahead in this fashion rapidly runs out of steam, and a more systematic strategy is called for. If we set $\xi_1 = 1$, $\xi_n = s$, and suppress the variables x and $(\xi_2, ..., \xi_{n-1})$, set

$$b_{kj}(s) = a_{kj}(x,\xi)|_{\xi_1=1,\xi_n=s}. \tag{9.7.7}$$

Then, via (9.7.4), the hypothesis of Proposition 9.7.1 is equivalent to the sequence of identities

$$b_{0j}(s) + \sum_{k\geq 1, k+r+p=j+1} \alpha_{k-1,p} b_{kr}(s)(-s)^{3/2-k-3p/2} = 0, \tag{9.7.8}$$

for $s > 0$, and for $j = 0, 1, 2, ...$ Since $b_{kj}(s)$ are smooth in s, these identities must also hold in the sense of formal power series in $(-s)^{1/2}$, and one has, for $j \geq 0, \ell \geq 0$,

$$(\ell!)^{-1}\partial_s^\ell b_{0j}(0) + \sum_{k\geq 1}(q!)^{-1}\partial_s^q b_{kr}(0)\alpha_{k-1,p} = 0, \tag{9.7.9}$$

where the sum runs over (k, r, q, p), all non-negative, also satisfying $k+r+p = j+1$ and $q + 3/2 - k - 3p/2 = \ell$. From this it follows that, for any integer $q \geq 0$, as $t \to \infty$,

$$\sum_{\ell} (\ell!)^{-1} \partial_s^\ell b_{0j}(0)(-t)^\ell + \sum_{k \geq 1} (\ell!)^{-1} \partial_s^\ell b_{kj}(0)(-t)^\ell \Phi^{(k-1)}(-t) \sim 0, \tag{9.7.10}$$

where the first sum is over (ℓ, j), non-negative integers, such that $3j + 2\ell = q$, and the second sum is over (ℓ, j, k), non-negative, such that $3j + 2\ell + k = q$. Indeed, if one substitutes the expansion (9.7.4) into (9.7.10), one sees that the coefficients of all powers of t vanish, by virtue of (9.7.9). With the following lemma, we will see that all the coefficients $\partial_s^\ell b_{kj}(0)$ that appear in any expression (9.7.10) must vanish; since for any (j, k, ℓ) there is a q such that such a coefficient does appear, this will provide the proof that all the $b_{kj}(s)$ vanish to infinite order at $s = 0$, which yields all the assertions of Proposition 9.7.1. Thus it remains only to establish the following.

Lemma 9.7.11. *If $p_k(t)$ are polynomials in t such that*

$$\sum_{k=0}^{N} p_k(t) \Phi^k(t) \sim 0, \quad \textit{as} \quad t \to -\infty, \tag{9.7.12}$$

then all $p_k(t)$ are identically zero. Consequently, if $q_k(t)$ are polynomials in t such that

$$q_0(t) + \sum_{k=0}^{N-1} q_{k+1}(t) \Phi^{(k)} \sim 0, \quad \textit{as} \quad t \to -\infty, \tag{9.7.13}$$

then all $q_k(t)$ are identically zero.

Proof. Assuming (9.7.12), there exists a function $\Psi(t)$ algebraic in t^{-1}, near $t = \infty$, such that

$$\sum_{k=0}^{N} p_k(t) \Psi^k(t) = 0$$

for large negative t, and such that $\Psi(t)$ has the same asymptotic behavior as $\Phi(t)$ as $t \to -\infty$. It follows that the power series of $\Psi(t) - \alpha_{00}(-t)^{1/2}$ in $(-t)^{-1/2}$ converges near $t = \infty$, and consequently $\Psi(t)$ satisfies the ODE equivalent to Airy's equation,

$$\Psi'(z) = z - \Psi(z)^2.$$

From this it follows that

$$A(t) = \exp\Bigl(\int^t \Psi(s) \; ds \Bigr)$$

is an Airy function. Hence $\Psi = \Phi$, i.e., Φ must be algebraic near infinity. This contradicts the fact that $\Phi(z)$ has an infinite number of poles tending to ∞ in $\mathbb{C}$. Thus we have that (9.7.12) implies all $p_k(t) = 0$. As for (9.7.13), since Airy's equation also gives

$$\Phi^{(k)}(t) = \sum_{\ell \le k} p_{k\ell}(t)\Phi^{\ell}(t), \tag{9.7.14}$$

with $p_{kk}(t) = p_{kk}$ a nonzero constant, this case is reduced to (9.7.12), and the proof of the lemma is complete.

Note that, if we assume that P belongs to OPS^{m-K} on the open cone $\{\xi_n < 0\}$, the argument above gives $\partial_s^l b_{kj}(0) = 0$ for $k + j + l \le N$, with $N \to \infty$ as $K \to \infty$, thus proving Proposition 9.5.6.

Lemma 9.7.11 is also valid if $t \to -\infty$ is replaced by $t \to +\infty$, and one obtains the following variant of Proposition 9.7.1.

Proposition 9.7.15. *Let $P \in \mathcal{A}^{m,+}$ be as in* (9.7.2)*. If P belongs to* $\mathrm{OPS}^{-\infty}$ *on the open cone $\{\xi_n > 0\}$ then each A_k, $k \ge 0$, has complete symbol vanishing to infinite order at $\xi_n = 0$. Consequently $P \in \mathrm{OPS}^m$, with complete symbol vanishing on $\{\xi_n \ge 0\}$.*

§9.8: Some Egorov-type theorems

Egorov's theorem on conjugating a pseudodifferential operator by an elliptic Fourier integral operator plays an important role in linear PDE. The appearance of the operator $\mathcal{A}_+/\mathcal{A}_-$ as a microlocal model for the solution operator to a diffractive boundary problem, as discussed in §7.5, makes it of interest to conjugate a pseudodifferential operator by this operator. The fact that $\mathcal{A}_+/\mathcal{A}_-$ behaves like a Fourier integral operator with a singular phase makes it plausible that the conjugated operator will be of non-classical type, and indeed Airy operators of the sort considered in this chapter arise naturally, as well as one further sort of pseudodifferential operator, as we will see. The results in this section refine some of the results of [Fa].

The tools used in our analysis here will include results on Fourier integral operators with folding canonical relations obtained in Chapter 5, and the cancellation effect established in Section 9.7. We begin with the following simpler result.

Proposition 9.8.1. *If $P \in \mathrm{OPS}^m$, then*

$$\mathcal{A}_-^{-1}P = (P_1 + \Phi_- P_2)\mathcal{A}_-^{-1} \ \mathit{mod}\ \mathrm{OPS}^{-\infty}, \tag{9.8.2}$$

with $P_1 \in \mathrm{OPS}^m, P_2 \in \mathrm{OPS}^{m-1/3}$. The principal symbols of P and P_1 agree on $\{\xi_n = 0\}$.

Proof. The Wronskian relation (cf. (A.2.2))

$$A_-'Ai - A_-Ai' = c, \tag{9.8.3}$$

implies the operator identity

$$c\mathcal{A}_-^{-1} = -\mathcal{A}i' - \Phi_-\mathcal{A}i. \tag{9.8.4}$$

Now, since $\mathcal{A}i'P$ and $\mathcal{A}iP$ are Fourier integral operators associated to the folding canonical relation $\mathcal{C}_0$, by Theorem 5.1.8 we have

$$\mathcal{A}i'P = P_1^{\#}\mathcal{A}i + P_2^{\#}\mathcal{A}i', \quad \mathcal{A}iP = P_1^b + P_2^b\mathcal{A}i', \tag{9.8.5}$$

where $P_j^{\#}$ and P_j^b are classical pseudodifferential operators, of order $m+1/3$, m, m, and $m-1/3$, respectively. Consequently, we have

$$\begin{aligned} c\mathcal{A}_-^{-1}P &= -\mathcal{A}i'P - \Phi_-\mathcal{A}iP \\ &= -P_1^{\#}\mathcal{A}i - P_2^{\#}\mathcal{A}i' - \Phi_-P_1^b\mathcal{A}i - \Phi_-P_2^b\mathcal{A}i' \\ &= (-P_1^{\#} + P_2^{\#}\Phi_- + \Phi_-P_1^b + \Phi_-P_2^b\Phi_-)\mathcal{A}i + c(P_2^{\#} + \Phi_-P_2^b)\mathcal{A}_-^{-1} \end{aligned} \tag{9.8.6}$$

where (9.8.4) is again used in the last identity. Now, since $\mathcal{A}i$ is, on the open cone $\xi_n < 0$, a sum of two elliptic Fourier integral operators, one with canonical relation coinciding with that of $\mathcal{A}_-^{-1}$ and one with canonical relation disjoint from that one, we see that the factor $T = -P_1^{\#} + P_2^{\#}\Phi_- - \Phi_-P_1^b + \Phi_-P_2^b\Phi_-$ belongs to $\mathrm{OPS}^{-\infty}$ on the open cone $\xi_n < 0$. By the cancellation effect given in Proposition 9.7.1, this implies $T\mathcal{A}i \in \mathrm{OPS}^{-\infty}$, so we have (9.8.2), with $P_1 = P_2^{\#}, P_2 = P_2^b$.

Via the identity

$$Ai = \omega A_+ + \bar{\omega} A_-, \tag{9.8.7}$$

which gives the operator identity

$$\omega\frac{\mathcal{A}_+}{\mathcal{A}_-} = -\bar{\omega} + \mathcal{A}i\mathcal{A}_-^{-1}, \tag{9.8.8}$$

we can use (9.8.2) to establish our main result of this section, which is the following.

Proposition 9.8.9. *If* $P \in \mathrm{OPS}^m$, *then, modulo* $\mathrm{OPS}^{-\infty}$,

$$\Big(\frac{\mathcal{A}_+}{\mathcal{A}_-}\Big)P = \big[P + (P_1 + \Phi P_2)(\mathcal{A}_+\mathcal{A}_-)^{-1}\big]\Big(\frac{\mathcal{A}_+}{\mathcal{A}_-}\Big), \tag{9.8.10}$$

with $P_1,\ P_2$ *classical of order* $m-1/3$ *and* $m-2/3$ *respectively, and* $\Phi = \Phi_-$.

Proof. Using (9.8.8) and (9.8.2) gives

$$\begin{aligned} \omega\Big(\frac{\mathcal{A}_+}{\mathcal{A}_-}\Big)P &= -\bar{\omega}P + \mathcal{A}i\mathcal{A}_-^{-1}P \\ &= -\bar{\omega}P + \mathcal{A}i(\tilde{P}_1 + \Phi\tilde{P}_2)\mathcal{A}_-^{-1}, \end{aligned} \tag{9.8.11}$$

with $\tilde{P}_1 \in \mathrm{OPS}^m,\ \tilde{P}_2 \in \mathrm{OPS}^{m-1/3}$. If we apply Proposition 5.1.5, and the Wronskian identity (9.8.3), we get

$$
\begin{aligned}
&\omega\Big(\frac{\mathcal{A}_+}{\mathcal{A}_-}\Big)P \\
&= -\bar{\omega}P + (P_1^{\#}\mathcal{A}i + P_1^{b}\mathcal{A}i')\mathcal{A}_-^{-1} + \Phi(P_2^{\#}\mathcal{A}i + P_2^{b}\mathcal{A}i')\mathcal{A}_-^{-1} \\
&= -\bar{\omega}P + (P_1^{\#} - P_1^{b}\Phi + \Phi P_2^{\#} - \Phi P_2^{b}\Phi)\mathcal{A}i\mathcal{A}_-^{-1} - c(P_1^{b} + \Phi P_2^{b})\mathcal{A}_-^{-2}.
\end{aligned}
\tag{9.8.12}
$$

If we once more apply the operator identity (9.8.8), and also write the operator $\mathcal{A}_-^{-2}$ as $\mathcal{A}_-^{-2} = (\mathcal{A}_+\mathcal{A}_-)^{-1}(\mathcal{A}_+/\mathcal{A}_-)$, and note that, on a conic neighborhood of $\{\xi_n = 0\}$, we have

$$
(\mathcal{A}_+\mathcal{A}_-)^{-1} \in \mathrm{OPS}^{1/3}_{1/3,0},
\tag{9.8.13}
$$

then (9.8.12) becomes

$$
\begin{aligned}
\omega\Big(\frac{\mathcal{A}_+}{\mathcal{A}_-}\Big)P = \Big[&(P_1^{\#} - P_1^{b}\Phi + \Phi P_2^{\#} - \Phi P_2^{b}\Phi) \\
&- \frac{c}{\omega}(P_1^{b} + \Phi P_2^{b})(\mathcal{A}_+\mathcal{A}_-)^{-1}\Big]\omega\Big(\frac{\mathcal{A}_+}{\mathcal{A}_-}\Big) + T,
\end{aligned}
\tag{9.8.14}
$$

where

$$
T = -\bar{\omega}P + \bar{\omega}(P_1^{\#} - P_1^{b}\Phi + \Phi P_2^{\#} - \Phi P_2^{b}\Phi).
\tag{9.8.15}
$$

Again noting the wavefront relation of $\mathcal{A}_+/\mathcal{A}_-$ on $\{\xi_n < 0\}$, we see that the Airy operator T is of order $-\infty$ on the open cone $\{\xi_n < 0\}$, so, by Proposition 9.7.1, we have

$$
T \in \mathrm{OPS}^m,
\tag{9.8.16}
$$

with the complete symbol of T vanishing for $\{\xi_n \le 0\}$. Consequently, by (9.8.8), we can write, modulo $\mathrm{OPS}^{-\infty}$,

$$
T = -\omega^2 T\,\frac{\mathcal{A}_+}{\mathcal{A}_-},
\tag{9.8.17}
$$

and thereby absorb T into the term in brackets in (9.8.14), producing the identity

$$
\Big(\frac{\mathcal{A}_+}{\mathcal{A}_-}\Big)P = [P - (c/\omega)(P_1^{b} + \Phi P_2^{b})(\mathcal{A}_+\mathcal{A}_-)^{-1}]\Big(\frac{\mathcal{A}_+}{\mathcal{A}_-}\Big).
\tag{9.8.18}
$$

Relabelling $(c/\omega)P_j^b$ gives (9.8.10) and completes the proof.

Chapter 10: Airy operator calculus – gliding points

We next consider the oscillatory Airy operators, which occur as the Neumann operator at a gliding point for a second order boundary problem. The basic questions considered here are the same as in Chapter 9. The main difference arises from the geometric complexity of the gliding case. As explained in Chapter 8, the Neumann operator at a gliding point has wavefront relation consisting of all the positive powers of the billiard ball map together with the limit of this relation on the gliding surface, namely the (forward part of the) Hamilton foliation. Analytically this is reflected in the fact that a series such as (9.4.2) is not microlocally convergent if Φ_+ is replaced by Φi. Nevertheless, with some additional work a reasonably complete theory of such Airy operators is described here.

In §10.1, we derive the basic Airy operator identities; as in §9.1, we use the Neumann operator identities of §8.4. We treat elliptic operators in §10.2. In the diffractive case, we pointed out two approaches to the elliptic case, one using the $S^*_{1/3,0}$ pseudodifferential operator calculus, the other using Airy operator identities to give neater results. In the gliding case we also present two approaches, one (the second) again using Airy operator identities. The other approach we present is not at all symbolic, but rather exploits energy estimates, thus leading to a somewhat less detailed description of the parametrices than one would like. Thus the method of Airy operator identities provides an even more substantial advantage in the gliding case than it did in Chapter 9. We use this method in §10.3 to treat a class of hypoelliptic operators. We study some general classes of Airy operators in §10.4.

§10.1: The classes $\mathcal{A}i_S^{m,\pm}$

Recall that the Airy multipliers Φi_T are well-defined, provided $T \neq 0$, see Chapter 5. The simple Airy operators are defined by analogy with (9.1.1):

$$Q = A \cdot \Phi i_T \cdot A' + B \in \mathcal{A}i_S^{m,\pm} \iff \tag{10.1.1}$$

$$A \in \mathrm{OPS}^{\mu},\ A' \in \mathrm{OPS}^{\mu'},\ \mu + \mu' = m - \frac{1}{3}, B \in \mathrm{OPS}^m, sgn(T) = \pm.$$

From the results in §5.8 it is immediately clear that

$$WF'(Q) \subset WF'(A) \circ \mathcal{C}_\infty^{sgn(T)} \circ WF'(A'). \tag{10.1.2}$$

The first thing to justify here is the notation, that the class of simple Airy operators only depends on the sign of T and not its value. Indeed, the value of T can be changed by conjugation with an exponential, which can be absorbed into the pseudodifferential factors.

Again the elementary treatment of these Airy operators is based on the identity (8.4.14). It is only necessary to be a little more careful about the notion of microlocal equality of operators. Recall from Appendix C that two operators, mapping functions on one manifold, X, to another, Y, are microlocally equal at $(\bar{\xi}, \bar{\eta}) \in T^*X \times T^*Y \setminus 0$ if $\rho_1 \cdot (A - B) \cdot \rho_2$ is a smoothing operator provided ρ_1 and ρ_2 are pseudodifferential operators with compact supports and essential supports sufficiently near the points $\bar{\xi}$ and $\bar{\eta}$. This relation is written:

$$A \equiv B \text{ at } (\bar{\xi}, \bar{\eta}).$$

Proposition 10.1.3. *There exists a conic neighborhood* Γ *of* $(\bar{\xi}, \bar{\xi})$ *such that, given* $H \in \mathrm{OPS}^{m-2/3}$ *with essential support in* Γ, *there exist* $G_i \in \mathrm{OPS}^{m-1/3}, i = 1, 2$ *and* $L \in \mathrm{OPS}^m$ *such that*

$$\Phi i_T \cdot H \cdot \Phi i_T \equiv G_1 \cdot \Phi i_T - \Phi i_T \cdot G_2 + L \text{ in } \Gamma. \tag{10.1.4}$$

Proof. The proof of Proposition 9.1.4 carries over essentially unchanged. Of course the results from Chapter 8, including the microlocal uniqueness of the Neumann operator in the gliding case, need to be used.

Similarly, Proposition 9.1.14 carries over directly, provided a little care is taken with the microlocality:

Proposition 10.1.5. *If* Γ *is a sufficiently small conic neighborhood of* $(\bar{\xi}, \bar{\xi})$ *then the span:*

$$\mathcal{A}i_F^{m,+} = \Big\{ Q = B + \sum_{\text{finite}} A_j \cdot \Phi i_T \cdot A_j' \Big\} \subset \ sp(\mathcal{A}i_S^{m,+}), \tag{10.1.6}$$

taken with pseudodifferential operators with proper support and essential support in Γ, *forms an algebra with*

$$\mathcal{A}i_F^{m,+} \cdot \mathcal{A}i_F^{m',+} \subset \mathcal{A}i_F^{m+m',+}. \tag{10.1.7}$$

Proof. Again the proof of Proposition 9.1.14 carries over essentially unchanged.

Parallel to (9.1.11), we note the special case

$$\Phi i_T^2 = D_1^{1/3} x_n \Phi i_T - \Phi i_T D_1^{1/3} x_n - D_1^{-1/3}(D_n + iT), \tag{10.1.7A}$$

again a consequence of the identity $\Phi i'(\zeta) = \Phi i(\zeta)^2 - \zeta$.

Notice also that Proposition 9.1.17 can be immediately generalized:

$$Q \in \mathcal{A}i_K^{m,+} \Longleftrightarrow Q^* \in \mathcal{A}i_K^{m,-}, \quad K = S \text{ or } F. \tag{10.1.8}$$

§10.2: Ellipticity in $\mathcal{A}i_S^{m,+}$

We seek to construct a microlocal inverse for $(A + B\Phi i_T)$, with $A \in \mathrm{OPS}^m$ elliptic, $B \in OPS^{m-1/3}$. Composing by A^{-1}, we consider the equation

$$(I + F\Phi i_T)u = g, \quad F \in OPS^{-1/3}. \tag{10.2.1}$$

By adjusting F by a smoothing operator, we can suppose that, on any given range $|\sigma| \le M$, there is a constant K with

$$F : H^{\sigma-1/3} \longrightarrow H^\sigma \tag{10.2.2}$$

having operator norm less than K.

We are assuming g has wave front set in a cone Γ on which $|\xi_n| \le C_1|\xi_1|$. Since we anticipate obtaining a solution u, mod C^∞, to (10.2.1), with wave front set also in Γ, we will alter (10.2.1) to

$$(I + G\Phi i_T)u = g, \tag{10.2.3}$$

where

$$G = F\chi(D), \tag{10.2.4}$$

with $\chi(\xi) \in S^0$ supported in $|\xi_n| \le 2C_1|\xi_1|$ and equal to 1 on a conic neighborhood of the set Γ. If we obtain a solution mod $\mathcal{C}^\infty, u$, to (10.2.3) and show that such u has wave front set in Γ, it will follow that u is a solution mod C^∞ to (10.2.3). Our actual argument will be a little more subtle than this.

We claim that, if the constant C_1 defining Γ is small enough and T is large enough, then, for $|\sigma| \le M$

$$\|G\Phi i_T\|_{\mathcal{L}(H^\sigma, H^\sigma)} \le \frac{1}{8}. \tag{10.2.5}$$

Indeed, this is a simple consequence of the estimates (5.3.11) and (5.3.15):

$$\big|\Phi i\big(\xi_1^{-1/3}(\xi_n + iT)\big)\big| \le C\big(T^{-1} + |\xi_1^{-1}\xi_n|^{1/2}\big)\xi_1^{1/3}, \tag{10.2.6}$$

for large ξ_1, because the coefficient of $\xi_1^{1/3}$ in (10.2.6) is small if T is large and $|\xi_1^{-1}\xi_n|$ is small. Granted (10.2.5), if $g \in H^\sigma \cap \mathcal{E}'$ for some $|\sigma| < M$, the Neumann series expansion gives

$$u = \sum_{j=0}^{\infty} (-G\Phi i_T)^j g, \tag{10.2.7}$$

as the solution to (10.2.3).

We want to compare u with $\psi(D)u$ where $\psi(D)$ is 1 on a conic neighborhood of Γ, so $\psi(D)g = g$ mod C^∞. This implies

$$[\psi(D) + \psi(D)G\Phi i_T]u = g \mod \mathcal{C}^\infty, \tag{10.2.8}$$

so

$$(I + G\Phi i_T)(\psi(D)u) = g - [\psi(D), G]\Phi i_T u \ (\mathrm{mod}\ \mathcal{C}^\infty) = g - h, \tag{10.2.9}$$

with

$$h \in H^{\sigma+1}. \tag{10.2.10}$$

Thus, if $|\sigma + 1| \leq M$, we have

$$u - \psi(D)u \in H^{\sigma+1}. \tag{10.2.11}$$

This implies that

$$(I + F\Phi i_T)u = g + \tilde{h}, \tag{10.2.12}$$

with

$$\tilde{h} \in H^{\sigma+1}. \tag{10.2.13}$$

Applying this argument again we obtain v with

$$(I + F\Phi i_T)v = -\tilde{h} \text{ mod } H^{\sigma+2} \tag{10.2.14}$$

if $|\sigma + 2| \leq M$, and continuing and summing we obtain $\tilde{u}_S \in H^\sigma$ with

$$(I + F\Phi i_T)\tilde{u} = g \text{ mod } H^M. \tag{10.2.15}$$

Indeed, replace u by $\psi(D)u$ in (10.2.12), obtaining a modified $\tilde{h}$, still satisfying (10.2.13), and with wave front set in the conic support of $\psi(\zeta)$, well within the set where $\chi(\zeta) = 1$. Repeating this, we can suppose $\tilde{u}$ solving (10.2.15) has wave front set inside the region where $\chi(\zeta) = 1$. Thus

$$(I + G\Phi i_T)\tilde{u} = g \text{ mod } H^M, \tag{10.2.16}$$

which, by uniqueness of solutions to (10.2.3) implies

$$\tilde{u} = u \text{ mod } H^M. \tag{10.2.17}$$

Here M can be taken arbitrarily large. We conclude that u, given by (10.2.7), is a solution mod C^∞ to (10.2.1), with wave front set in Γ.

The next point we shall strive to understand is that, if, in (10.2.3), g is smooth for $t < t_o$, then the solution u is also smooth for $t < t_o$.

We can take u to be given by (10.2.7). It suffices to show that if $g \in H^\sigma$, $|\sigma| \leq M$, and g vanishes for $x_n < 0$, then u belongs to H^M for $x_n < -a$, given any $a > 0$. We will rewrite the term $(-G\Phi i_T)^j$ in (10.2.7) by decomposing $-G$. Write

$$-G = A_j + B_j, \tag{10.2.18}$$

where we specify A_j and B_j to satisfy the following conditions.

Lemma 10.2.19. *Given* $G \in \mathrm{OPS}^{-1/3}$ *constructed above, we can pick* $A_j \in \mathrm{OPS}^{-1/3}, B_j \in \mathrm{OPS}^{-\infty}$, *so that* (10.2.18) *holds, with the kernel* $K_{A_j}(x, x')$ *of* A_j *supported on*

$$|x_n - x'_n| \leq \frac{a}{j},$$

and such that, for $|\sigma| \leq M$

$$\|A_j \Phi i_T\|_{\mathcal{L}(H^\sigma, H^\sigma)} \leq \frac{1}{4}, \tag{10.2.20}$$

$$\|B_j \Phi i_T\|_{\mathcal{L}(H^\sigma, H^\sigma)} \leq \frac{1}{4}, \tag{10.2.21}$$

$$\|B_j\|_{\mathcal{L}(H^{\sigma-1/3}, H^M)} \leq C_1 j^\mu, \tag{10.2.22}$$

for some constants C_1, μ.

Proof. If K_G is the kernel of G, we define K_{A_j} by

$$K_{A_j}(x, x') = \chi_j(x_n - x'_n) K_G(x, x'). \tag{10.2.23}$$

Here $\chi_j(s) = \chi\big((j+1)s\big)$ where $\chi(s) \in C_0^\infty(-a, a)$ is equal to 1 for $|s| \leq a/2$. Now, if $G = G(x, D)$, then $A_j = A_j(x, D)$ with

$$A_j(x, \xi) = \int \hat{\chi}_j(\xi_n - \xi'_n) G(x, \xi_1, ..., \xi_{n-1}, \xi'_n)\, d\xi'_n. \tag{10.2.24}$$

Note that the behavior as a function of ξ_1 is not affected. In particular, we can suppose all functions are supported on $\xi_1 \geq B$, a large number. This makes it easy to obtain (10.2.20) and (10.2.21). Finally, the estimate (10.2.22) is a simple consequence of the basic estimates on the kernel of a pseudodifferential operator.

Now we have

$$u = \sum_{j=0}^{\infty} T_j g \tag{10.2.25}$$

with

$$T_j = (A_j \Phi i_T + B_j \Phi i_T)^j. \tag{10.2.26}$$

We will estimate the H^M norm of $T_j g$ on the set $x_n < -a$, assuming $g \in H^\sigma$ is supported on $x_n \geq 0$. Note that Φi_T is the operation of convolution by a distribution supported in $x_n \geq 0$. Now T_j is a sum of 2^j terms, each term being a product of j factors, each factor being either $A_j \Phi i_T$ or $B_j \Phi i_T$. Let us write

$$T_j = \sum_{k=1}^{2^j} P_{jk}. \tag{10.2.27}$$

If a factor of the form $B_j \Phi i_T$ occurs in P_{jk}, we have

(10.2.28) $$\|P_{jk} g\|_{H^M} \leq C_1 j^\mu 4^{-j+1}.$$

The sum of all such terms is thus an operator T'_j with

(10.2.29) $$\|T'_j\|_{\mathcal{L}(H^\sigma, H^M)} \leq 4c_1 j^\mu 2^{-j}.$$

meanwhile, if all the factors in P_{jk} are of the form $A_j \Phi i_T$, then

(10.2.30) $$P_{jk} g = 0 \ \text{ for } \ x_n < -a.$$

by the support conditions on $g, \Phi i_T$, and A_j. It follows that

(10.2.31) $$\|u\|_{H^M(\Omega_a)} \leq \|g_S\|_{H^\sigma} \cdot 4c_1 \cdot \sum_{j=0}^{\infty} j^\mu 2^{-j},$$

where $\Omega_a = \{x : x_n < -a\}$. This proves our contention that u belongs to H^M for $x_n < -a$.

We now want to investigate the singularities of u for $x_n \geq 0$. Since the equation $(I + F\Phi i_T)u = g$ has elementary nature away from $\xi_n = 0$, we need only show that, if $WF(g)$ is contained in a small neighborhood of a point where $\xi_n = 0$, then $WF(u)$ is contained in a small conic neighborhood of the gliding ray through this point. Now if $Q \in \mathrm{OPS}^0$ is any operator commuting with $\chi(D)$, which has symbol $\xi_1^{-1/3}(\xi_n + iT)$, we have

(10.2.32) $$[Q, \Phi i_T] = 0.$$

Such operators exist in great profusion. In particular given any gliding ray γ we can find such Q with symbol equal to 1 on a small conic neighborhood of γ and vanishing outside a slightly larger conic neighborhood, such that (10.2.32) holds, at least modulo a smoothing operator. Now, with g as above, $g \in H^s$, and $u \in H^s$, smooth for $x_n < 0$, solving

(10.2.33) $$(I + F\Phi i_T)u = g, \quad F \in \mathrm{OPS}^{-1/3},$$

we have, for $Q \in \mathrm{OPS}^0$ satisfying (10.2.32), the relation

(10.2.34) $$(I + F\Phi i_T)Qu = Qg - [Q, F]\Phi i_T u.$$

Note that $[Q, F] \in \mathrm{OPS}^{-4/3}$, so $[Q, F]\Phi i_T$ has order -1. Thus

(10.2.35) $$Qg \in C^\infty \Longrightarrow Qu \in H^{s+1}.$$

Picking a sequence Q_ν satisfying (10.2.32), with the symbol of $Q_{\nu-1}$ equal to 1 on the conic support of the symbol of Q_ν, we can iterate this argument, to obtain

$$Q_1 g \in C^\infty \Longrightarrow Q_\nu u \in H^{s+\nu}. \tag{10.2.36}$$

This controls the singularities of the solution to (10.2.33).

We now give another analysis of solutions to (10.2.1) which will make the singularities of u manifest. Parallel to (9.2.5)–(9.2.6), as a consequence of Proposition 10.1.3, we have

$$(I + F\Phi i_T)(I - FD\Phi i_T D^{-1}) = E(I + C\Phi i_T D^{-1}), \tag{10.2.37}$$

for certain eliptic $D,\ E \in OPS^0$, and

$$C \in OPS^{-4/3}. \tag{10.2.38}$$

Thus

$$C\Phi i_T D^{-1} : H^s \longrightarrow H^{s-1} \tag{10.2.39}$$

for all s, so the Neumann expansion (parallel to (9.2.9))

$$(I + C\Phi i_T D^{-1})^{-1} \sim I + \sum_{j\geq 1} (-C\Phi i_T D^{-1})^j \tag{10.2.40}$$

is asymptotic, in the sense of consisting of operators which are smoothing to progressively higher degrees. We hence have a right parametrix to $A + B\Phi i_T$:

$$S = (I - FD\Phi i_T D^{-1})(I + C\Phi i_T D^{-1})^{-1} E^{-1} A^{-1} \tag{10.2.41}$$

A similar construction produces a left parametrix, and the standard argument shows the two parametrices must agree. We have the following result.

Theorem 10.2.42. *Let $A \in OPS^m$ be elliptic, $B \in OPS^{m-1/3}$. Then $A + B\Phi i_T$ has a two sided parametrix S of the form* (10.2.41). *Furthermore, the wave front relation of S is contained in $\mathcal{C}_\infty^{sgn(T)}$, describing the wave front relation of Φi_T.*

Having constructed a two sided parametrix, we could dispense with the arguments based on operator norm estimates in the first part of this section.

We remark that A and B in Theorem 10.2.42 need not be scalar. It is also useful to understand the inverse of

$$A + B_1 \Phi i_T B_2, \tag{10.2.43}$$

with $A \in \mathrm{OPS}^m$ elliptic and $B_j \in \mathrm{OPS}^{m_j}, m_1 + m_2 = m - 1/3$. Multiplying by A^{-1}, we may as well consider the inverse of

$$I + S_1 \Phi i_T S_2, \quad S_j \in \mathrm{OPS}^{m_j},\ m_1 + m_2 = -\frac{1}{3}. \tag{10.2.44}$$

Note that comparing Neumann series

$$(I+S_1\Phi i_T S_2)^{-1} = I - S_1\Phi i_T S_2 + \cdots + (-1)^k S_1\Phi i_T (S_2S_1)\cdots(S_2S_1)\Phi i_T S_2 + \cdots,$$

and

$$(I+S_2S_1\Phi i_T)^{-1} = I - S_2S_1\Phi i_T + S_2S_1\Phi i_T S_2S_1\Phi i_T - \cdots$$

gives

$$(I+S_1\Phi i_T S_2)^{-1} = I - S_1\Phi i_T (I+S_2S_1\Phi i_T)^{-1} S_2, \tag{10.2.45}$$

so Theorem 10.2.42 can be applied to the study of $(I+S_1\Phi i_T S_2)^{-1}$.

§10.3: Hypoellipticity in $\mathcal{A}i_S^{m,\pm}$

Inverses of operators of the form $A + B\Phi i_T$ with $A \in \mathrm{OPS}^m$ elliptic and $B \in \mathrm{OPS}^{m-1/3}$ as above arise in the treatment of strongly well posed problems, for which a Lopatinski condition is satisfied; see Chapter 12. As we have seen, treating the Neumann boundary condition involves inverting Φi_T. Other situations lead one to invert $A + B\Phi i_T$ under the hypothesis

(10.3.1)

$B \in \mathrm{OPS}^{m-1/3}$ elliptic; $A \in \mathrm{OPS}^m$ with principal symbol vanishing at $\xi_n = 0$.

Note that

$$A + B\Phi i_T = (A + B\Phi i_T^{-1})\Phi i_T.$$

Thus we may as well study invertibility of $A + B\Phi i_T^{-1}$. That this operator is microlocally invertible, at least for T large and on $|\xi_n| \le \varepsilon\xi_1$, ε small, is a consequence of the following estimate, analogous to the estimate (5.3.12), valid for ξ_1 large

$$\big|(\xi_n + iT)\Phi i\big((\xi_n+iT)\xi_1^{-1/3}\big)^{-1}\big| \le C\big(T^{-1} + |\xi_1^{-1}\xi_n|^{1/2}\big)\xi_1^{1/3}. \tag{10.3.2}$$

As in (5.4.12), this estimate follows from

$$|z\Phi i(z)^{-1}| \le C\big(|\operatorname{Im} z|^{-1} + |z|^{1/2}\big) \ \text{ for } 0 \le \operatorname{Im} z \le C_1, \tag{10.3.3}$$

and (10.3.3) is proved the same way (5.2.11) is.

To understand the singularities of $(A + B\Phi i_T)^{-1}$ in this case we look for an identity so we can parallel the argument of Theorem 10.2.42. Such an identity is readily obtained, in parallel with the proof of Proposition 9.3.2. Thus, parallel to (9.3.3), we obtain

$$C \in OPS^{-m} \text{ elliptic}, \quad D \in OPS^{-m-1/3}, \tag{10.3.4}$$

such that

$$(A + B\Phi i_T)(C + D\Phi i_T) = R\Phi i_T + E, \tag{10.3.5}$$

with

$$R \in OPS^{-1/3} \text{ elliptic}, \quad E \in OPS^{-1}. \tag{10.3.6}$$

Writing

$$R\Phi i_T + E = (I + E\Phi i_T^{-1} R^{-1}) R\Phi i_T, \tag{10.3.7}$$

we have a right parametrix of $A + B\Phi i_T$ of the form

$$S = (C\Phi i_T^{-1} + D) R^{-1} (I + E\Phi i_T^{-1} R^{-1})^{-1} \tag{10.3.8}$$

where, since

$$E\Phi i_T^{-1} R^{-1} : H^s \longrightarrow H^{s+1/3} \tag{10.3.9}$$

for all s, the Neumann series

$$(I + E\Phi i_T^{-1} R^{-1})^{-1} \sim \sum_{j \geq 0} (-C\Phi i_T^{-1} R^{-1})^j \tag{10.3.10}$$

is asymptotic. We note parenthetically one difference from the proof of Proposition 9.3.1, namely that $E\Phi_+^{-1} R^{-1}$ has order $-2/3$ on Sobolev spaces, rather than merely order $-1/3$, as in (10.3.9). In any event, (10.3.9) is strong enough to produce the right parametrix (10.3.8). Similarly a left parametrix is constructed, yielding the desired result:

Theorem 10.3.11. *Under the hypothesis* (10.3.1), $A + B\Phi i_T$ *has a two-sided parametrix* S *of the form* (10.3.8)–(10.3.10), *with wave front relation contained in* $\mathcal{C}_\infty^{sgn(T)}$.

§10.4: The operator classes $\mathcal{A}i^{m,\pm}$ and $\mathcal{A}i_\sigma^{m,\pm}$

It is desirable to have an equally precise theory of an algebra of operators containing those of the form

$$A\Phi i + B, \quad A \in \mathrm{OPS}^{-1/3}, \ B \in \mathrm{OPS}^0. \tag{10.4.1}$$

The first difference with the case of Φ is that, in formal sums analogous to (9.4.2):

$$B + \sum_{j \geq 0} A_j(x,\xi) \ \Phi i^{(j)}, \quad A_j(x,\xi) \in S^{-(j+1)/3}, \tag{10.4.2}$$

the j^{th} term has order zero on Sobolev spaces, so (10.4.2) could not be asymptotic in the usual sense. We can still make use of the following concept.

Definition 10.4.3. *$\mathcal{A}i^{m,+}$ consists of operators of the form*

$$Cu(x) = \int \big[a(x,y,\xi)\Phi i(\zeta_0) + b(x,y,\xi)\big]u(y)e^{i(x-y)\cdot\xi}\,dy\,d\xi, \tag{10.4.4}$$

with $a(x,y,\xi) \in S^{m-1/3}$, $b(x,y,\xi) \in S^m$.

Note that adjoints of operators in $\mathcal{A}i^{m,+}$ are characterized as being given by (10.4.4) with Φi replaced by $\overline{\Phi i}$. If $P \in \mathrm{OPS}^\mu$ is applied to (10.4.4), it can be applied under the integral on the right to give:

Proposition 10.4.5. *If $P \in \mathrm{OPS}^\mu$ and $C \in \mathcal{A}i^{m,+}$, then*

$$PC \in \mathcal{A}i^{m+\mu,+}. \tag{10.4.6}$$

If we use $(CP)^* = P^*C^*$ we have:

Corollary 10.4.7. *Under the hypotheses above, also*

$$CP \in \mathcal{A}i^{m+\mu,+}. \tag{10.4.8}$$

Since the operator Φi clearly belongs to $\mathcal{A}i^{1/3,+}$, we have

$$A \in \mathrm{OPS}^m \Longrightarrow A\Phi i \in \mathcal{A}i^{m+1/3,+}, \tag{10.4.9}$$

and furthermore, if $A' \in \mathrm{OPS}^\mu$,

$$A\Phi i A' \in \mathcal{A}i^{m+\mu+1/3,+}. \tag{10.4.10}$$

We can also analyze the operator Φi^2, using the identity

$$\Phi i^2(\zeta_0) = \zeta_0 - \Phi i'(\zeta_0). \tag{10.4.11}$$

Of course, $\zeta_0 = \xi_1^{-1/3}(\xi_n + iT) \in S^{2/3}$. As for the operator $\Phi i'$, we have

$$\begin{aligned} \Phi i' u(x) &= \int \xi_1^{1/3}[\partial_{\xi_n}\Phi i(\zeta_0)]e^{i(x-y)\cdot\xi}u(y)\ dy\ d\xi \\ &= i\int \xi_1^{1/3}(x_n - y_n)\Phi i(\zeta_0)e^{i(x-y)\cdot\xi}u(y)\,dy\,d\xi, \end{aligned} \tag{10.4.12}$$

integrating by parts. Thus we see that

$$\Phi i' \in \mathcal{A}i^{2/3,+}, \tag{10.4.13}$$

and hence, by (10.4.11),

$$\Phi i^2 \in \mathcal{A}i^{2/3,+}. \tag{10.4.14}$$

It is difficult to parallel for $\mathcal{A}i^{m,+}$ the development of $\mathcal{A}^{m,+}$ given in Chapter 9. The failure of (10.4.2) to be asymptotic is largely behind this. In particular, we do not establish that $\mathcal{A}i^{*,+}$ is an algebra. However, we will obtain good algebraic properties for a subclass $\mathcal{A}i^{*,+}_{\sigma}$, which we will define shortly, and we will show that this class contains parametrices of elliptic elements in $\mathcal{A}i^{*,+}_{F}$.

Proposition 10.1.3 allows us to replace a product involving two factors of Φi by terms linear in Φi. Consequently, any product of the form

$$A_1\,\Phi i\,A_2\,\Phi i\cdots A_k\,\Phi i,\quad A_j\in \mathrm{OPS}^{m_j}, \tag{10.4.15}$$

can be rewritten as

$$\sum_{j=1}^{l} B_j\Phi i C_j + G_0, \tag{10.4.16}$$

where $B_j\in \mathrm{OPS}^{\mu_j}, C_j\in\mathrm{OPS}^{\lambda_j}$, with

$$\mu_j+\lambda_j+\frac{1}{3}=\frac{k}{3}+\sum_{j=1}^{k} m_j=\sigma,\quad G_0\in\mathrm{OPS}^{\sigma}\,.$$

This makes it natural to consider the following class of operators.

Definition 10.4.17. *$\mathcal{A}i^{m,+}_{\sigma}$ consists of operators with asymptotic expansion of the form*

$$T\sim B+\sum_{j\ge 0} A_j\Phi i C_j, \tag{10.4.18}$$

with

$$B\in\mathrm{OPS}^{m}, \tag{10.4.19}$$

and

$$A_j\in\mathrm{OPS}^{\mu_j},\quad C_j\in\mathrm{OPS}^{\lambda_j},\quad \mu_j+\lambda_j+\frac{1}{3}=m-l_j, \tag{10.4.20}$$

where

$$l_j\ge 0 \text{ is an integer},\quad l_j\to\infty \text{ as } j\to\infty. \tag{10.4.21}$$

The result (10.1.7) proves the following.

Proposition 10.4.22. *If* $T_j \in \mathcal{A}i_\sigma^{m_j,+}$ *for* $j = 1, 2$, *then*

$$T_1 T_2 \in \mathcal{A}i_\sigma^{m_1+m_2,+}. \tag{10.4.23}$$

Meanwhile, (10.4.6) *and* (10.4.8) *imply*

$$\mathcal{A}i_\sigma^{m,+} \subset \mathcal{A}i^{m,+}. \tag{10.4.24}$$

We can now obtain the analogue of Proposition 9.4.17 for these Airy operators.

Proposition 10.4.25. *Suppose*

$$T = B + A\Phi i, \tag{10.4.26}$$

with

$$B \in \mathrm{OPS}^0 \ \ \textit{elliptic}, \quad A \in \mathrm{OPS}^{-1/3}. \tag{10.4.27}$$

Then

$$T^{-1} \in \mathcal{A}i_\sigma^{0,+} \subset \mathcal{A}i^{0,+}. \tag{10.4.28}$$

Proof. By Theorem 10.2.42, we have

$$T^{-1} = D(I + FD\Phi i D^{-1})(I + C\Phi i D^{-1})^{-1} E^{-1} B^{-1}, \tag{10.4.29}$$

with $F \in OPS^{-1/3}$, E and D elliptic in OPS^0, and

$$C \in \mathrm{OPS}^{-4/3}.$$

Thus the Neumann expansion

$$(I + C\Phi i D^{-1})^{-1} \sim \sum_{j \geq 0} (-C\Phi i D^{-1})^j \tag{10.4.30}$$

is asymptotic, with $(-C\Phi i_T B^{-1})^j \in \mathcal{A}i_F^{-j,+}$, by Proposition 10.4.22, so we get

$$(I + C\Phi i D^{-1})^{-1} \in \mathcal{A}i_\sigma^{0,+}. \tag{10.4.31}$$

Applying Proposition 10.4.22 once more, to (10.4.29), we complete the proof.

It is useful to record the symbolic nature of the parametrix (10.4.28). Factoring out B, we may as well look at

$$T = I + S\Phi i, \quad S \in \mathrm{OPS}^{-1/3}. \tag{10.4.32}$$

A straightforward check of the symbols of the operators in the fundamental Airy operator identity (10.1.4) (cf. (9.1.7)–(9.1.9)) establishes the following.

Corollary 10.4.33. *Let $S \in \mathrm{OPS}^{-1/3}$ be given. Let (g', h') be solutions to the transport equations, for Friedlander's model, such that*

$$\frac{h'}{g'} = -\sigma(S) \text{ on } \partial\Omega. \tag{10.4.34}$$

Let $D \in \mathrm{OPS}^0$ have principal symbol

$$\sigma(D) = g'\big|_{\partial\Omega}. \tag{10.4.35}$$

Then

$$(I + S\Phi i)(I - SD\Phi i D^{-1}) = I + B' \text{ mod } \mathcal{A}i_\sigma^{-1,+}, \tag{10.4.36}$$

where $B' \in \mathrm{OPS}^0$ has principal symbol

$$\sigma(B') = \sigma(S)(h'/g')\zeta_0 = -\sigma(S)^2 \xi_1^{-1/3} \xi_n. \tag{10.4.37}$$

We remark that Proposition 10.4.25 has a natural extension:

Proposition 10.4.38. *Let $T \in \mathcal{A}i_\sigma^{0,+}$ have the form* (10.4.18) *with $B \in \mathrm{OPS}^0$ elliptic. Then $T^{-1} \in \mathcal{A}i_\sigma^{0,+}$.*

Proof. We will apply the formula (10.2.45). We may as well suppose

$$T = I + \sum_1^K A_j \Phi i C_j,$$

since after a finite number of terms in (10.4.18) the rest have lower order and can be handled by the usual Neumann expansion. Now if A_j, C_j are $k \times k$ matrices of operators (we may as well suppose $A_j \in \mathrm{OPS}^{-1/3}, C_j \in \mathrm{OPS}^0$), define

$$C^\infty(\Omega, \mathbb{C}^k) \xrightarrow{S_2} C^\infty(\Omega, \mathbb{C}^{kK}) \xrightarrow{S_1} C^\infty(\Omega, \mathbb{C}^k)$$

by

$$S_2 u = (C_1 u, \dots, C_K u), \quad S_1(v_1, \dots, v_K) = A_1 v_1 + \cdots + A_K v_K.$$

We have, by (10.2.45),

$$T^{-1} = I - S_1 \Phi i (I + S_2 S_1 \Phi i)^{-1} S_2,$$

and by Proposition 10.4.25, in conjunction with Proposition 10.4.22, we have $T^{-1} \in \mathcal{A}i_\sigma^{0,+}$, as asserted.

We now obtain a result complementary to Proposition 10.1.3, which will be useful in the study of certain classes of Airy operator equations. We rewrite (10.1.4) as

$$D\Phi i - \Phi i D = \Phi i H \Phi i - L + \Phi i \delta_1. \tag{10.4.39}$$

Here, the symbols of $D = G_1, H$, and L are, respectively, the restrictions to $\partial\Omega$ of

$$g', \quad h', \quad h'\zeta \tag{10.4.40}$$

where (g', h') solves the transport equations (for Friedlander's model), and $\delta_1 = G_2 - G_1$. In Theorem 10.1.3, we exploited our ability to pick $h'|_{\partial\Omega}$ arbitrarily (homogeneous of order $-1/3$), which was proved in Proposition 4.4.11. Exploiting instead Proposition 4.4.14, we can solve the transport equations for g', h' and specify arbitrarily

$$g'\big|_{\partial\Omega} = d. \tag{10.4.41}$$

We can then prove the following.

Theorem 10.4.42. *Let $D \in \mathrm{OPS}^0$ be given. Then there exist $H \in \mathrm{OPS}^{-1/3}, L \in \mathrm{OPS}^{1/3}$, and $\delta_1 \in \mathrm{OPS}^{-1}$ such that*

$$D\Phi i - \Phi i D = \Phi i H \Phi i - L + \Phi i \delta_1. \tag{10.4.43}$$

The principal symbols of D, H, and L are given as in (10.4.40).

A useful corollary is obtained by multiplying (10.4.43) by Φi^{-1}. We obtain

$$\Phi i^{-1} D \Phi i - D = H \Phi i - \Phi i^{-1} L + \delta_1. \tag{10.4.44}$$

This permits the following boundedness result.

Corollary 10.4.45. *For any $D \in \mathrm{OPS}^0$, we have the continuous map*

$$\Phi i^{-1} D \Phi i : H^s \longrightarrow H^s.$$

Proof. By (10.4.44), we have

$$\Phi i^{-1} D \Phi i = (D + \delta_1) + H \Phi i - \Phi i^{-1} L. \tag{10.4.46}$$

It suffices to show the right side of (10.4.46) is continuous on H^s. Clearly $D + \delta_1$ is. Since $\Phi i : H^s \to H^{s-1/3}$ and $\Phi i^{-1} : H^s \to H^{s-1/3}$, and since $H \in \mathrm{OPS}^{-1/3}$, we also see that $H\Phi i$ is appropriately bounded. It remains to check $\Phi i^{-1} L$. Now $L \in \mathrm{OPS}^{1/3}$ has principal symbol $h'\zeta_0$, which vanishes on $\xi_n = 0$. Thus the boundedness of $\Phi^{-1}L$ would follow from the boundedness of the Fourier multiplier $\langle\xi\rangle^{-1/3}\zeta_0\Phi i(\zeta_0)^{-1}$, with $\zeta_0 = \xi_1^{-1/3}(\xi_n + iT)$. Indeed, (5.4.12) says

$$|\Phi i(\zeta_0)|^{-1} \le C_T\langle\xi\rangle^{1/3}\langle\zeta_0\rangle^{-1} \quad \text{on} \quad \xi_n \le 0,$$

so

$$\big|\langle\xi\rangle^{-1/3}\zeta_0\Phi i(\zeta_0)^{-1}\big| \le C_T' \quad \text{on} \quad \xi_n \le 0. \tag{10.4.47}$$

Meanwhile (5.4.16) says

$$|\Phi i(\zeta_0)|^{-1} \le C\langle\zeta_0\rangle^{-1/2} \quad \text{on} \quad \xi_n \ge 0,$$

so

$$\big|\langle\xi\rangle^{-1/3}\zeta_0\Phi i(\zeta_0)^{-1}\big| \le C' \quad \text{on} \quad \xi_n \ge 0. \tag{10.4.48}$$

This completes the proof.

Chapter 11: Transmission problems

In this section we consider various phenomena which arise in the study of the transmission of waves from one medium to another, across an interface. Thus we suppose we are working on a smooth manifold $\mathcal{O}$, divided into two parts $\mathcal{O}_1$ and $\mathcal{O}_2$ by a smooth surface Σ. Suppose that the $\mathcal{O}_j$ are given separate Riemannian metrics, smooth up to the common boundary Σ from either side. Let A_j be the Laplace operator on $\mathcal{O}_j$, perhaps with some first order terms added on:

$$A_j = \mathit{\Delta}_j + L_j(x, D). \tag{11.0.1}$$

We consider solutions u_j to the wave equations

$$\left(\frac{\partial^2}{\partial t^2} - A_j\right) u_j = 0 \quad \text{in} \;\; \Omega_j = \mathbb{R} \times \mathcal{O}_j \tag{11.0.2}$$

with transmission conditions at $\Gamma = \mathbb{R} \times \Sigma$:

$$u_1 - u_2 = f, \tag{11.0.3}$$

$$\frac{\partial u_1}{\partial \nu_1} + a(x) \frac{\partial u_2}{\partial \nu_2} + \sum_{j=1}^{2} b_j(x) u_j = g. \tag{11.0.4}$$

Here u_j is the limiting value at Γ of u_j in Ω_j, and $\partial/\partial\nu_j$ is the normal vector field to Γ, with respect to the metric in Γ_j, pointing into Ω_j. We assume the factor $a(x)$ in (11.0.4) to be positive. In particular (11.0.3), (11.0.4) then include the trivial case where the two metrics are simply the restrictions of one $\mathcal{C}^\infty$ metric and the transmission conditions represent continuity of the solution and its first derivative across the (illusory) interface; namely

$$u_1 = u_2, \quad \frac{\partial u_1}{\partial \nu_1} = -\frac{\partial u_2}{\partial \nu_2}. \tag{11.0.5}$$

In general we shall suppose that f, g in (11.0.3)–(11.0.4) are distributions with compact supports and require u_j to be outgoing, i.e.,

$$u_j = 0 \;\; \text{for} \;\; t << 0. \tag{11.0.6}$$

Using a superposition argument we may also suppose that f and g have wave front set in a small conic neighborhood $U \times V$ of a point $(y_0, \eta_0) \in T^*\Gamma \setminus 0$. Now we impose the basic hypothesis that, from either side, Γ is globally either bicharacteristically convex or concave. Thus at the point (y_0, η_0) there are four possibilities corresponding to the disjoint decomposition:

$$(y_0, \eta_0) \in T^*\Gamma \setminus 0 = \mathcal{H}^{(i)} \cup \mathcal{E}^{(i)} \cup \mathcal{G}_d^{(i)} \cup \mathcal{G}_g^{(i)} \quad i = 1, 2. \tag{11.0.7}$$

These four cases for each side give a total of sixteen overall possibilities, or ten when the freedom to renumber the sides is taken into account. Some of these are elementary, especially the three cases where (y_0, η_0) is not glancing from either side. For example, when no rays pass over (y_0, η_0) from either side, one has microlocally an elliptic boundary value problem. When there is a ray passing over (y_0, η_0) from one or both sides, and all such are transversal to Γ, the construction of parametrices is a special case of a general construction for strongly well posed hyperbolic systems, going back to [Nir] as pointed out in [Tay1]; in these cases Nosmas [Nos] gives a detailed construction.

The cases where there is a tangential ray over (y_0, η_0) from just one side are also readily analysed in terms of the methods already used above. This is examined in §11.1. For the case where (y_0, η_0) is diffractive with respect to both sides a parametrix was constructed in [Tay3]. This is recalled briefly in §11.2. A situation more likely to arise physically is that (y_0, η_0) is diffractive from one side and gliding from the other. This is one of the cases which will be considered in some detail, in §11.3.

§11.1: Glancing and transversal

We consider first transmission problem when (y_0, η_0) is a glancing point from Ω_1 but from Ω_2 is either hyperbolic or elliptic, i.e., either from Ω_2 there passes over (y_0, η_0) a transversal ray or no ray. This case can be resolved using the calculus developed above.

We know that in all cases the solution to (11.0.2) and (11.0.6) is given, at least microlocally, by the parametrix for the Dirichlet problem in each Ω_j. Thus it suffices to understand the restrictions of the u_j to Γ, which we also denote u_j. Thus we need to solve the boundary equations

$$u_1 - u_2 = f, \tag{11.1.1}$$

$$N_1 u_1 + a(x) N_2 u_2 + \sum b_j(x) u_j = g. \tag{11.1.2}$$

Here the N_j are the forward Neumann operators for Ω_j, which we studied in Chapter 8. Our problem therefore is to produce a solution (u_1, u_2) modulo $\mathcal{C}^\infty$ to the system (9.5), (9.6). Using (11.1.1) we can eliminate u_2 and obtain the single equation

$$[N_1 + a(x) N_2 + b(x)] u_1 = h, \tag{11.1.3}$$

where

$$h = g + af - b_2 f, \tag{11.1.4}$$

and

$$b(x) = b_1(x) + b_2(x). \tag{11.1.5}$$

Now if (y_0, η_0) is a gliding point for Ω_1, then, by the results of Chapter 8, we know that microlocally near (y_0, η_0) the operator N_1 takes the form

$$N_1 = J_1(A_1 \Phi i_T + B_1)J_1^{-1}. \tag{11.1.6}$$

In this decomposition J_1 is an elliptic Fourier integral operator associated to a canonical tranformation which puts the billiard ball map of Ω_1 into normal form, $B_1 \in OPS^0$, and $A_1 \in OPS^{1/3}$ is elliptic. The principal symbol of given by A_1 is $(\partial/\partial\nu_1)\zeta_1|_\Gamma$ if (7.6.4) is the parametrix in Ω_1. Thus, by (4.2.7), the principal symbol of A_1 is positive near (y_0, η_0).

If (y_0, η_0) is an elliptic or a hyperbolic point for Ω_2, then

$$N_2 \in OPS^1 \text{ is elliptic, at } (y_0, \eta_0), \tag{11.1.7}$$

and the principal symbol of N_2 is either purely imaginary with positive imaginary part (hyperbolic point) or real and positive (elliptic point). Thus we can transform equation (11.1.3) to

$$[A_1 \Phi i_T + J_1^{-1}(aN_2 + b)J_1 + B_1]v = J_1^{-1}h \tag{11.1.8}$$

where

$$v = J_1^{-1}u_1. \tag{11.1.9}$$

Finally therefore we obtain as a microlocal equation

$$(\Phi i_T + E)v = \tilde{h}, \tag{11.1.10}$$

where

$$\tilde{h} = A_1^{-1}J_1^{-1}h, \tag{11.1.11}$$

and

$$E = A_1^{-1}[J_1^{-1}(aN_2 + b)J_1 + B_1] \in OPS^{1/3}. \tag{11.1.12}$$

Note that E is elliptic, and its principal symbol is either $\sqrt{-1}$ times a positive function (hyperbolic point) or real and positive (elliptic point).

Thus the equation (11.1.10) is a special case of a class of equations treated in Chapter 10. The operator $\Phi i_T + E$ is an elliptic element of $\mathcal{A}i_S^{1,+}$, so Theorem 10.2.42 describes a parametrix for the transmission problem in this case. This problem has also been considered by Petkov [Pet1].

Similarly in case (y_0, η_0) is a diffractive point with espect to Ω_1 we obtain in place of (11.1.6)

$$N_1 = J_1(A_1 \Phi_\pm + B_1)J_1^{-1}, \tag{11.1.13}$$

where J_1 is again an elliptic Fourier integral operator associated to a canonical tranformation reducing the billiard ball map on Ω_1 (which only exists as a Taylor series) to normal form. The same discussion reduces the boundary equation to

$$(\Phi_\pm + E)v = \tilde{h}, \tag{11.1.14}$$

with E and $\tilde{h}$ still given by (11.1.11) and (11.1.12). In this case the problem is reduced to one discussed in Chapter 9.

Summarizing we have therefore shown:

Proposition 11.1.15. *If γ is a sufficiently small conic neighborhood of $(y_0, \eta_0) \in T^*\Gamma$, where $(y_0, \eta_0) \in \mathcal{H}^{(2)} \cup \mathcal{E}^{(2)}$, and $f,\ g \in C_c^{-\infty}(\Gamma)$ have $\mathrm{WF}(f),\ \mathrm{WF}(g) \subset \gamma$ then the solution (u_1, u_2) to (11.0.2)–(11.0.6) satisfies*

$$\mathrm{WF}(u_i\big|_\Gamma) \cap \gamma \subset \mathrm{WF}(f) \cup \mathrm{WF}(g) \text{ if } (y_0, \eta_0) \in \mathcal{G}_d^{(1)}, \tag{11.1.16}$$

(11.1.17)

$$\mathrm{WF}(u_i\big|_\Gamma) \cap \gamma \subset \mathrm{WF}(f) \cup \mathrm{WF}(g) \cup \bigcup_{j \geq 0}^{\infty} \delta_+^j\big(\mathrm{WF}(f) \cup \mathrm{WF}(g)\big) \text{ if } (y_0, \eta_0) \in \mathcal{G}_g^{(1)},$$

for $i = 1, 2$, where δ_+ is the forward billiard ball map, i.e., $\delta_+^*(t) \geq t$.

§11.2: Diffractive and diffractive

We now consider the boundary equation (11.1.3) microlocally near a point $(y_0, \eta_0) \in T^*\Gamma \setminus 0$ which is diffractive from each side. In this case we have

$$N_j = J_j(A_j \Phi_\pm + B_j) J_j^{-1} \tag{11.2.1}$$

and each canonical transformation $\chi_{J_j}^{-1}$ maps (y_0, η_0) to a point in $T^*\mathbb{R}^n \setminus 0$ on which $\xi_n = 0$.

If we set $v = J_1^{-1} u_1$, the equation (11.1.3) becomes

$$\big[(A_1\Phi_\pm + B_1) + K^*(A_3\Phi_\pm + B_3)K\big]v = J^{-1}h = \tilde{h}, \tag{11.2.2}$$

where

$$K = J_2^{-1} J_1 \tag{11.2.3}$$

is an elliptic Fourier integral operator, whose principal symbol we can suppose to be unitary,

$$A_3 = (J_2^{-1} a J_2) A_2 \tag{11.2.4}$$

is, like A_1, an elliptic operator in $OPS^{2/3}$ with positive principal symbol, and, like $B_1,\ B_3 \in OPS^0$.

Denote by $\mathcal{P}$ the operator in brackets in (11.2.2):

$$\mathcal{P} = Q_1 + K^* Q_3 K, \tag{11.2.5}$$

where $Q_j = A_j \Phi_\pm + B_j$ are Airy operators described above. Without extra geometrical restrictions, $K^* Q_3 K$ is not contained in a class of pseudodifferential operators with a good symbol calculus; it can be shown to belong to $OPS^1_{1/3,2/3}$, but that is not particularly useful. Therefore we will not use a symbolic construction of a parametrix to show that $\mathcal{P}$ is microlocally hypoelliptic. Rather we will use energy estimates, based on the fact that

$$\lambda \in \mathbb{R} \Longrightarrow \mathrm{Re}\ \Phi_+(\lambda) > 0 \text{ and } \mathrm{Im}\ \Phi_+(\lambda) > 0. \tag{11.2.6}$$

This is established in Appendix A; see (A.3.13)–(A.3.14). This leads to the following energy estimate.

Proposition 11.2.7. *Set $\beta = e^{-\pi i/4}$, and let $\sigma < 1/3$. Then, for $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$,*

$$\text{(11.2.7A)} \qquad Re\ (\beta \mathcal{P} u, u) \geq C_1 \|u\|^2_{H^{1/3}} - C_2 \|u\|^2_{H^\sigma},$$

provided $\Phi = \Phi_+$; for $\Phi = \Phi_-$, use $\beta = e^{\pi i/4}$.

Proof. Such an estimate is valid for each of the two terms in (11.2.5); Gårding's inequality implies

$$\text{Re}\ (\beta A_j \Phi_+ u, u) \geq C\ \text{Re}\ (\beta \Phi_+ \Lambda^{1/3} u, \Lambda^{1/3} u) - C' \|u\|^2_{H^\sigma}.$$

Also one can replace u by Ku in such an estimate. Then use (11.2.6). This yields (11.2.7A) for $\sigma = 0$, which suffices.

From (11.2.7A) it follows that, for $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$,

$$\text{(11.2.8)} \qquad \|u\|_{H^{1/3}} \leq C \|\mathcal{P} u\|_{H^{-1/3}} + C \|u\|_{H^\sigma}.$$

To pass from such an estimate to a regularity result, we need to microlocalize this estimate. This can be done once we take into account estimates for commutators of $\mathcal{P}$ with operators in $OPS^m_{1,0}$. The fact that

$$\nabla_\xi \Phi \in S^{-1/3}_{1/3,0}$$

implies that

$$\text{(11.2.9)} \qquad \text{Ad}\ Q_j : OPS^m_{1,0} \longrightarrow OPS^{m+1/3}_{1/3,0},$$

where $(\text{Ad}\ Q_j)P = [Q_j, P]$ is the commutator. Thus, if $\mathcal{O}(\rho)$ denotes the class of operators with the property

$$\text{(11.2.10)} \qquad T \in \mathcal{O}(\rho) \Longleftrightarrow T : H^s \longrightarrow H^{s-\rho}, \quad \forall\ s,$$

we see that

$$\text{(11.2.11)} \qquad \text{Ad}\ \mathcal{P} : OPS^m_{1,0} \longrightarrow \mathcal{O}\big(m + \frac{1}{3}\big).$$

Note that $\mathcal{P} \in \mathcal{O}(1)$. Replacing u be $\Lambda^{s-1/3} u$ in (11.2.8), we see that one immediate consequence of (11.2.11) is the more general a priori estimate

$$\text{(11.2.12)} \qquad \|u\|_{H^s} \leq C \|\mathcal{P} u\|_{H^{s-1/3}} + C \|u\|_{H^\sigma}, \quad \sigma << s,$$

for all $s \in \mathbb{R}$.

Suppose γ is an open conic subset of $T^*\mathbb{R}^n \setminus 0$, and $E(x,\xi) \in S^0$ has support in a closed subcone γ_1, and is elliptic on a smaller subcone γ_2. If we apply (11.2.12) to $E(x,D)u$, we obtain

$$\text{(11.2.13)} \qquad \|Eu\|_{H^s} \leq C \|E\mathcal{P} u\|_{H^{s-2/3}} + C \|[\mathcal{P}, E]u\|_{H^{s-2/3}} + C \|u\|_{H^\sigma}.$$

To proceed, we need the following.

Lemma 11.2.14. *Let $F(x,\xi) \in S^0$ be supported in γ and equal to 1 on γ_1. Then*

$$[\mathcal{P}, E] = [\mathcal{P}, E]F \ mod\ \mathcal{O}(-\infty). \tag{11.2.15}$$

Proof. $[Q_1, E] = [Q_1, E]F \bmod OPS^{-\infty}$ since $[Q_1, E] \in OPS^{-2/3}_{1/3,0}$ has complete symbol supported in γ_1. That $[K^*Q_3K, E](I - F) \in \mathcal{O}(-\infty)$ follows from the identity

$$[K^*Q_3K, E]F = K^*[Q_3, KEK^*](KFK^*)K \bmod \mathcal{O}(-\infty),$$

and the fact that $KFK^* \in OPS^0$ has complete symbol equal to 1 on the conic support of the symbol of $[Q_3, KEK^*]$, by Egorov's theorem.

Thus we obtain from (11.2.13) the estimate

$$\|Eu\|_{H^s} \le C\|E\mathcal{P}u\|_{H^{s-2/3}} + C\|Fu\|_{H^{s-1/3}} + C\|u\|_{H^\sigma}. \tag{11.2.16}$$

The principal result of this section is that the corresponding microlocal regularity result holds.

Proposition 11.2.17. *If $u \in \mathcal{C}^{-\infty}$ and if $\mathcal{P}u$ belongs to $H^{s-2/3}$ microlocally on γ, then u belongs to H^s microlocally on γ.*

Proof. Let $J_\varepsilon = \varphi(\varepsilon D)$ be a Friedrichs mollifier, and apply (11.2.16) to $J_\varepsilon u$, obtaining

$$\begin{aligned}\|EJ_\varepsilon u\|_{H^s} \le C\|J_\varepsilon \mathcal{P}u\|_{H^{s-2/3}} &+ C\|E[J_\varepsilon, \mathcal{P}]u\|_{H^{s-2/3}} \\ &+ C\|FJ_\varepsilon u\|_{H^{s-1/3}} + C\|u\|_{H^\sigma},\end{aligned} \tag{11.2.18}$$

Since $\{J_\varepsilon : \varepsilon \in (0,1]\}$ is bounded in $OPS^0_{1,0}$, we see that

$$\{[J_\varepsilon, \mathcal{P}] : \varepsilon \in (0,1]\} \text{ is bounded in } \mathcal{O}(\frac{1}{3}). \tag{11.2.19}$$

Furthermore, in analogy with (11.2.15), we have

$$\{[J_\varepsilon, \mathcal{P}](I - F) : \varepsilon \in (0,1]\} \text{ is bounded in } \mathcal{O}(-\infty). \tag{11.2.20}$$

Therefore (11.2.18) yields

$$\|EJ_\varepsilon u\|_{H^s} \le C\|EJ_\varepsilon \mathcal{P}u\|_{H^{s-2/3}} + C\|F^\# u\|_{H^{s-1/3}} + C\|u\|_{H^\sigma}, \tag{11.2.21}$$

with C independent of $\varepsilon \in (0,1]$, where $F^\#$ has symbol supported in γ and equal to 1 on a conic neighborhood of supp $F(x,\xi)$. Taking $\varepsilon \to 0$ and applying a standard inductive argument proves the Proposition.

Corollary 11.2.22. *If γ is a small conic neighborhood of $(y_0, \eta_0) \in T^*\Gamma \setminus 0$, which is diffractive from each side, then the solutions u_1, u_2 to (11.0.2)–(11.0.6) satisfy*

$$WF(u_i\big|_\Gamma) \cap \gamma \subset WF(f) \cup WF(g), \quad i = 1, 2. \tag{11.2.23}$$

§11.3: Diffractive and gliding

The next transmission problem we consider is the case where gliding points in $T^*\Gamma \setminus 0$ for Ω_1 exactly coincide with diffractive points for Ω_2 :

$$\mathcal{G}_g^{(1)} = \mathcal{G}_d^{(2)} \text{ in } T^*\Gamma \setminus 0 \text{ near } (y_0, \eta_0). \tag{11.3.1}$$

Since these glancing surfaces $\mathcal{G}^{(i)}$ are the null surfaces of the Lorentzian metric induced on Γ by the two Riemannian metrics induced on Σ from the two sides, the equality (11.3.1) automatically extends to the lift of a neighborhood of y_0 in the base. In fact (11.3.1) is equivalent to:

(11.3.2) the two induced metrics on Γ are equal near y_0.

The case we are most interested in here corresponds to the strengthening of (11.3.1). We always have the freedom to make a coordinate change on the two sides of Γ separately, provided the transformations agree on Γ itself. The condition should be independent of such a tranformation. For example simple equality of the metrics on the fibres above Γ :

(11.3.3) the two metrics are equal at Γ in a neighborhood of y_0

can always be assumed, i.e., arranged by choice of coordinates. Indeed this equality always holds in normal Riemannian coordinates, obtained from the Collar Neighborhood Theorem. The first geometric invariants at Γ are the principal, or directional curvatures. The equality of these curvatures from the two sides can be stated in the obviously intrinsic form:

(11.3.4) Any curve in Γ has the same curvature measured from the two sides.

Recall from Chapter 3 that in the reduction of the billiard ball map to normal form the Taylor series of the function ζ was shown to be well-defined at the glancing surface. Of course the glancing surface is just $\zeta = 0$. This function gives a useful reinterpretation of the geometric assumption (11.3.4).

Lemma 11.3.5. *Given* (11.3.1) *the condition* (11.3.4) *on the metrics on the two sides of* Γ *is equivalent to requiring*

$$d(\zeta_1) = d(\zeta_2) \quad at \quad \mathcal{G} = \mathcal{G}^{(1)} = \mathcal{G}^{(2)}; \tag{11.3.6}$$

it is also equivalent to the existence of two canoncial transformations $\mathcal{J}_1$, $\mathcal{J}_2$ *on* $T^*\Gamma$ *reducing the two billiard ball maps to the same normal form and such that*

$$\mathcal{J}_1^* = \mathcal{J}_2^* \ on\ T_\lambda^*\Gamma, \quad \forall\ \lambda \in \mathcal{G}. \tag{11.3.7}$$

Proof. In §4.5 the particular properties of ζ and the reduction to normal form for the wave operator were discussed. It follows directly from (4.5.xx) that the curvature condition (11.3.4) is equivalent to (11.3.6). Moreover, since a canonical tranformation reducing the billiard ball map $\delta_\pm$ to normal form only needs to satisfy

$$\mathcal{J}^*\zeta = \zeta_0, \tag{11.3.8}$$

it is clear that the $\mathcal{J}_i$ can be chosen to satisfy (11.3.7) if (11.3.6) holds. Conversely (11.3.7) certainly implies (11.3.6) so the lemma is proved.

In the case of second order scalar operators A_j that we are considering these conditions amount to supposing the coefficients of the principal parts of A_1 and A_2 have the same limiting values on Γ, although perhaps derivatives of these quantities could jump across Γ, and lower order terms could also have jumps. In this case, the boundary equation (11.1.3) is to be analyzed. The Neumann operator N_1 is given by (11.1.6), and the operator N_2 is given by

$$N_2 = J_2(A_2\Phi_T + B_2)J_2^{-1}. \tag{11.3.9}$$

We can write

$$a(x)N_2 = J_2(\alpha A_2\Phi_T + \tilde{B}_2)J_2^{-1}. \tag{11.3.10}$$

Thus equation (11.1.3) becomes

$$\big[J_1(A_1\Phi i_T + B_1)J_1^{-1} + J_2(\alpha A_2\Phi_T + \tilde{B}_2)J_2^{-1} + b(x)\big]u_1 = h. \tag{11.3.11}$$

If we set

$$J = J_1^{-1}J_2, \tag{11.3.12}$$

we can rewrite this as

$$\big[A_1\Phi i_T + B_1 + J(\alpha A_2\Phi_T + \tilde{B}_2)J^{-1} + \tilde{b}\big]v = J_1^{-1}h, \tag{11.3.13}$$

where $\tilde{b} = J_1^{-1}b(x)J_1$ and

$$v = J_1^{-1}u_1. \tag{11.3.14}$$

Here the $J_i,\ i = 1,2$ are associated to the two canoncial tranformations in (11.3.7), thus J is a Fourier integral operator associated to a canonical tranformation which is the identity on $\mathcal{G}$ and hence at $\mathcal{G}$.

Now to analyze $J\Phi_T J^{-1}$, we write J in the form

$$Ju(x) = \int a(x,\xi)e^{i\phi(x,\xi)-iy\cdot\xi}u(y)\,dy\,d\xi, \tag{11.3.15}$$

and write J^{-1} as

$$J^{-1}u(x) = \int b(y,\xi)e^{i\phi(y,\xi)+ix\cdot\xi}u(y)\,dy\,d\xi. \tag{11.3.16}$$

Thus, since Φ_T is a convolution operator with multiplier $\Phi_T(\xi)$, (11.3.15) gives:

$$J\Phi_T J^{-1}u(x) = \int a(x,\xi)\Phi_T(\xi)b(y,\xi)e^{i[\phi(x,\xi)-\phi(y,\xi)]}u(y)\,dy\,d\xi. \tag{11.3.17}$$

Now set

$$\phi(x,\xi) - \phi(y,\xi) = (x-y)\cdot\eta(x,y,\xi), \tag{11.3.18}$$

with $\eta(x,y,\xi)$ homogeneous of degree 1 in ξ, and smooth. Furthermore, this can be inverted locally to give

$$\xi = \Xi(x,y,\eta). \tag{11.3.19}$$

If a factor $\alpha(\xi)$ with small conic support is inserted in (11.3.17) (this can be removed by a partition of unity argument), we get, upon changing variables of integration

$$J\Phi_T J^{-1}u(x) = \int q\big(x,y,\Xi(x,y,\eta)\big)\ e^{i(x-y)\cdot\eta}\ u(y)D(x,y,\eta)\,dy\,d\eta, \tag{11.3.20}$$

where

$$q(x,y,\xi) = a(x,\xi)\Phi_T(\xi)b(y,\xi), \tag{11.3.21}$$

and $D(x,y,\eta)$ is the absolute value of the Jacobian determinant:

$$D(x,y,\eta) = \left|\det\Big(\frac{\partial\Xi}{\partial\eta}\Big)\right|. \tag{11.3.22}$$

Thus, if we set

$$a(x,y,\eta) = q\big(x,y,\Xi(x,y,\eta)\big)\ D(x,y,\eta), \tag{11.3.23}$$

we have

$$J\Phi_T J^{-1}u(x) = \int a(x,y,\eta)e^{i(x-y)\cdot\eta}u(y)\,dy\,d\eta. \tag{11.3.24}$$

It is easy to verify that, for a general elliptic Fourier integral operator of the form (11.3.15), for which (11.3.19) holds, we have realized $J\Phi_T J^{-1}$ as a pseudodifferential operator with multiple symbol

$$J\Phi_T J^{-1} \in OPS^{1/3}_{1/3,2/3,2/3}, \tag{11.3.25}$$

where we say an operator whose form is given by the right side of (11.3.24) belongs to $OPS^m_{\rho,\delta_1,\delta_2}$ provided

$$|D_y^\gamma D_x^\beta D_\eta^\alpha a(x,y,\eta)| \le C_{\alpha\beta\gamma}(1+|\eta|)^{m-\rho|\alpha|+\delta_1|\beta|+\delta_2|\gamma|}. \tag{11.3.26}$$

But in fact, if J satisfies the conclusion of Lemma 11.3.5, so its associated canonical transformation $\mathcal{J}$ is the identity on $\xi_n = 0$, we can do better than (11.3.25). The transformation $\mathcal{J}$ is related to Φ by

$$\mathcal{J}\big(\nabla_\xi \phi(x,\xi),\xi\big) = \big(x,\nabla_x\phi(x,\xi)\big). \tag{11.3.27}$$

That $\mathcal{J}$ is the identity on $\{\xi_n = 0\}$ is equivalent to

$$\nabla_{x,\xi}\big[\phi(x,\xi) - x\cdot\xi\big] = 0 \text{ for } \xi_n = 0, \tag{11.3.28}$$

which implies

$$\phi(x,\xi) = x\cdot\xi + \xi_n^2\gamma(x,\xi), \tag{11.3.29}$$

for a smooth $\gamma(x,\xi)$, homogeneous of degree -1 in ξ. Thus, in this case

$$\eta(x,y,\xi) = \xi + \xi_n^2\sigma(x,y,\xi), \tag{11.3.30}$$

and hence

$$\Xi(x,y,\eta) = \eta + h_n^2\tilde{\sigma}(x,y,\eta), \tag{11.3.31}$$

with $\sigma,\tilde{\sigma}$ smooth and homogeneous of degree -1 in their Greek variables. Thus

$$\Phi_T\big(\Xi(x,y,\eta)\big) = \Phi\big(\eta_1^{-1/3}(\alpha_0(x,y,\eta)\eta_n + i\beta_0(x,y,\eta)T)\big), \tag{11.3.32}$$

with

$$\begin{aligned} \alpha_0(x,y,\eta) &= 1 + \eta_n\alpha_{-1}(x,y,\eta), \\ \beta_0(x,y,\eta) &= 1 + \eta_n^2\beta_{-2}(x,y,\eta). \end{aligned} \tag{11.3.33}$$

A simple application of the chain rule gives

$$\Phi_T\big(\Xi(x,y,\eta)\big) \in S^{1/3}_{1/3,0,0} \tag{11.3.34}$$

and hence the amplitude $a(x,y,\eta)$ given by (11.3.23) belongs to $S^{1/3}_{1/3,0,0}$, which improves (11.3.25) to

$$J\Phi_T J^{-1} \in OPS^{1/3}_{1/3,0,0}. \tag{11.3.35}$$

Now we can apply the standard result on reducing pseudodifferential operators with multiple symbols (see [Ho5], vol.3, or [Tay7], Chapter II, Theorem 3.8) and conclude:

Proposition 11.3.36. *If the canonical transformation associated with J is the identity on $\{\xi_n = 0\}$, then we have*

$$J\Phi_T J^{-1} = \Phi_{\mathcal{J}}(x, D) \in OPS^{1/3}_{1/3,0} \tag{11.3.36A}$$

with

$$\begin{aligned} \Phi_{\mathcal{J}}(x,\xi) &= \Phi_T\big(\Xi(x,x,\xi)\big) \mod S^{-1/3}_{1/3,0} \\ &= \Phi\big(\Xi_1^{-1/3}(\Xi_n + iT)\big). \end{aligned} \tag{11.3.37}$$

The remainder term has order $2/3$ below the principal term because first order derivatives of $\Phi_T(\xi)$ belong to $S^{-1/3}_{1/3,0}$.

Now we return to the analysis of the equation (11.3.13). We will suppose for the rest of this section that $a(x) \equiv 1$, so $\alpha = I$ in (11.3.13), and $\tilde{B}_2 = B_2$. Multiplying by A_1^{-1} gives

$$[\Phi i_T + P\Phi_{\mathcal{J}}(x, D) + B_3]v = A_1^{-1} J_1^{-1} h \tag{11.3.38}$$

where

$$P = A_1^{-1}(JA_2J^{-1}) \in OPS^0, \tag{11.3.39}$$

$$B_3 = A_1^{-1}B_1 + A_1^{-1}(JB_2J^{-1}) + A_1^{-1}\tilde{b} \in OPS^{-2/3}. \tag{11.3.40}$$

From (11.3.37) it is straightforward to check that

$$\Phi_{\mathcal{J}}(x,\xi) = \Phi(\zeta_0^*), \ \mod OPS^{-1/3}_{1/3,0}, \tag{11.3.41}$$

where

$$\zeta_0^* = \mathcal{J}^*\zeta_0, \tag{11.3.42}$$

with, as usual

$$\zeta_0 = \xi_1^{-1/3}(\xi_n + iT). \tag{11.3.43}$$

It is useful to have the following information on the symbol $P(x,\xi)$ of P, defined by (11.3.39). Let $\zeta_{00} = \xi_1^{-1/3}\xi_n$ and $\zeta_{00}^* = \mathcal{J}^*\zeta_{00}$.

Lemma 11.3.44. *We have*

$$P(x,\xi) = -\Big(\frac{\zeta_{00}}{\zeta_{00}^*}\Big)^{1/2}, \quad \operatorname{mod} S^{-1}.$$

Proof. Since we know that $P \in OPS^0$, it suffices to prove the result off $\{\xi_n = 0\}$. There it follows by comparing the two Neumann operators N_1 and N_2, which, off any conic neighborhood of $\{\xi_n = 0\}$ are both (locally) pseudodifferential operators of classical type. The result is therefore straightforward. We note that the principal symbol of P in $\{\xi_n < 0\}$ is uniquely determined by the condition that, for transversally intersecting rays, the reflected wave is smoother than the incident wave or the refracted wave, which holds for solutions to the transmission problem considered here, assuming $a(x) = 1$ in (11.0.4).

From Lemma 11.3.44 we derive the following result which will make the analysis of (11.3.38) straightforward.

Lemma 11.3.45. *We have*

$$P\Phi_{\mathcal{J}}(x,D) = -\Phi_T + R, \tag{11.3.46}$$

with

$$R \in OPS^{-1/3}_{1/3,0}. \tag{11.3.47}$$

Proof. It is clear from Proposition 11.3.36 and Lemma 11.3.44 that it suffices to show that

$$s(x,\xi) = \Big(\frac{\zeta_{00}}{\zeta_{00}^*}\Big)^{1/2}\Phi(\zeta_{00}^*) - \Phi(\zeta_{00}) \in S^{-2/3}_{1/3,0}. \tag{11.3.48}$$

Now recall that

$$\Phi(\lambda) \in S^{1/2}_{1,0}(R), \quad \Phi'(\lambda) \in S^{-1/2}_{1,0}(\mathbb{R}), \tag{11.3.49}$$

and

$$\Phi(\lambda)^2 = \lambda - \Phi'(\lambda).$$

Thus

$$\frac{\lambda}{\Phi(\lambda)^2} = 1 + \frac{\Phi'(\lambda)}{\Phi^2(\lambda)} = 1 + \beta(\lambda); \quad \beta(\lambda) \in S^{-3/2}_{1,0}(\mathbb{R}), \tag{11.3.50}$$

so

$$\frac{\zeta_{00}}{\zeta_{00}^*}\frac{\Phi(\zeta_{00}^*)^2}{\Phi(\zeta_{00})^2} = \frac{1+\beta(\zeta_{00})}{1+\beta(\zeta_{00}^*)}. \tag{11.3.51}$$

Note that this is bounded, and bounded away from zero. Thus to estimate the difference between (11.3.51) and 1, we can take logs:

$$\log\Big[\frac{1+\beta(\zeta_{00})}{1+\beta(\zeta_{00}^*)}\Big] = \big[\beta(\zeta_{00})-\beta(\zeta_{00}^*)\big]F\big(\beta(\zeta_{00}),\beta(\zeta_{00}^*)\big), \tag{11.3.52}$$

with F smooth, so

$$F\big(\beta(\zeta_{00}),\beta(\zeta_{00}^*)\big) \in S^0_{1/3,0}. \tag{11.3.53}$$

Now, if $\mu = \zeta_{00}^*/\zeta_{00}$, then

$$\begin{aligned} \beta(\zeta_{00}^*)-\beta(\zeta_{00}) &= \zeta_{00}\int_1^\mu \beta'(\tau\zeta_{00})d\tau \\ &= \zeta_{00}(\mu-1)G(\mu,\zeta_{00}) \\ &= (\mu-1)H(\mu,\zeta_{00}), \end{aligned}$$

where $G(\mu,\lambda)\in S^{-5/2}$ and $H(\mu,\lambda)\in S^{-3/2}$. Thus

$$\beta(\zeta_{00}^*)-\beta(\zeta_{00}) = \left(\frac{\zeta_{00}^*}{\zeta_{00}}-1\right)H(\zeta_{00}^*/\zeta_{00},\zeta_{00}).$$

Now

$$\frac{\zeta_{00}^*}{\zeta_{00}}-1 = \tilde{C}\zeta_{00}, \quad \tilde{C}\in S^{-2/3},$$

so

$$\beta(\zeta_{00}^*)-\beta(\zeta_{00}) = \tilde{C}\tilde{H}(\zeta_{00}^*/\zeta_{00},\zeta_{00}), \tag{11.3.54}$$

with

$$\tilde{H}(\mu,\lambda)\in S^{-1/2}. \tag{11.3.55}$$

Consequently, (11.3.51) and (11.3.52) yield

$$\Big(\frac{\zeta_{00}}{\zeta_{00}^*}\Big)^{1/2}\frac{\Phi(\zeta_{00}^*)}{\Phi(\zeta_{00})} = 1+\xi_1^{-2/3}H^\#, \tag{11.3.56}$$

with

$$H^\#\in S^0_{1/3,0}, \quad |H^\#|\le C(1+|\zeta_{00}|)^{-1/2}. \tag{11.3.57}$$

This shows that

$$\Big(\frac{\zeta_{00}}{\zeta_{00}^*}\Big)^{1/2}\Phi(\zeta_{00}^*)-\Phi(\zeta_{00}) = \xi_1^{-2/3}H^\#\Phi(\zeta_{00}) = \xi^{-2/3}K^\#, \tag{11.3.58}$$

with $K^{\#} \in S^0_{2/3,0}$. This proves the Lemma.

We proceed with our analysis of the equation (11.3.38). We have

$$\Phi i_T + P\Phi_{\mathcal{J}}(x,D) + B_3 = \Phi i_T - \Phi_T + B_4, \tag{11.3.59}$$

with

$$B_4 \in OPS^{-1/3}_{1/3,0}. \tag{11.3.60}$$

We can invert the operator (11.3.59) using the Wronskian relation

$$\frac{A'}{A} - \frac{Ai'}{Ai} = \frac{c}{AAi},$$

which implies

$$\Big(\frac{A'}{A} - \frac{Ai'}{Ai}\Big)^{-1} = c^{-1}AAi,$$

so

$$(\Phi i_T - \Phi_T)^{-1} = c^{-1}\mathcal{A}_T\mathcal{A}i_T. \tag{11.3.61}$$

Recall that, on a conic neighborhood of $\xi_n = 0$, $\mathcal{A}_T\mathcal{A}i_T : H^s \longrightarrow H^s$. Now write (11.3.59) as

$$\Phi i_T - \Phi_T + B_4 = [I + c^{-1}B_4\mathcal{A}_T\mathcal{A}i_T](\Phi i_T - \Phi_T). \tag{11.3.62}$$

Hence the inverse is given by

$$\begin{aligned}(\Phi i_T - \Phi_T + B_4)^{-1} &= c^{-1}\mathcal{A}_T\mathcal{A}i_T[I + c^{-1}B_4\mathcal{A}_T\mathcal{A}i_T]^{-1} \\ &= c^{-1}\mathcal{A}_T\mathcal{A}i_T\sum_{k=0}^{\infty}\big[-c^{-1}B_4\mathcal{A}_T\mathcal{A}i_T\big]^k.\end{aligned} \tag{11.3.63}$$

Since

$$B_4\mathcal{A}_T\mathcal{A}i_T : H^s \longrightarrow H^{s+1/3}, \tag{11.3.64}$$

the terms in the Neumann expansion above are progressively smoother. This implies that multiply reflected waves are very smooth, uniformly as one approaches gliding, and in particular the gliding rays are “infinitely weak” in this case.

One special case of the transmission problem we have just considered is the following. We suppose $A = \Delta$ is the Laplace operator on $\Omega_1 \cup \Omega_2$, with metric tensor smooth across Γ, and consider solutions u_1, u_2 to

$$\Big(\frac{\partial^2}{\partial t^2} - \Delta\Big)u_j = 0 \text{ in } \Omega_j,\ u_1 - u_2 = 0 \text{ on } \Gamma, \quad \Big(\frac{\partial u_1}{\partial \nu_1} + \frac{\partial u_2}{\partial \nu_2}\Big) = \alpha u_1 \text{ on } \Gamma.$$

This transmission problem can be realized as the ideal limit of a sequence of Dirichlet problems where $\Gamma = \mathbb{R} \times \Sigma$, and Σ is replaced by a sequence of boundaries B_k consisting of a many of small balls with centers lying on Σ , with an appropriate density. See Rauch and Taylor [RaT2]. In this case, the operator J defined by (11.3.12) is the identity, and we can bypass Lemma 11.3.5 and Proposition 11.3.36.

Chapter 12: First order systems

In this chapter we consider parametrices for a $k \times k$ first order system of differential equations of the form

$$G(x, D_x)u = 0 \tag{12.0.1}$$

in the half space $\Omega = \mathbb{R}^{n+1}_+ = \{x_{n+1} > 0\}$, with boundary condition at $x_{n+1} = 0$,

$$B(x', D_{x'})u(0, x') = f, \tag{12.0.2}$$

where $x = (x', x_{n+1})$. Here $B \in \mathrm{OPS}^0$ is a $k \times k'$ matrix. The typical case to keep in mind is a boundary value problem for a hyperbolic system, with the variable x_1 representing time. We suppose $p(x, \xi) = \det G_1(x, \xi)$ is real with simple characteristics, G_1 denoting the principal symbol of G, and more precisely we suppose

$$p(x, \xi) = 0 \Longrightarrow \frac{\partial}{\partial \xi_1}\, p(x, \xi) \neq 0, \quad (\xi \neq 0). \tag{12.0.3}$$

We also suppose the boundary $\partial\Omega$ to be non-characteristic for G.

We present our geometrical hypotheses in the next section and set down the parametrices, in analogy with the construction in Chapter 7. The eikonal equations for the phase functions are treated in much the same fashion as before. The transport equations involve a little bit of linear algebra; they are treated in §12.2. Then we study respectively boundary problems of coercive type in §12.3 and boundary problems of Neumann type in §12.4, making use of results on elliptic and hypoelliptic Airy operators developed in Chapters 9 and 10. In §12.5 we study Maxwell's equations in a region bounded by a perfect conductor. These present a mixture of coercive and Neumann-type behavior.

§12.1: Ray geometry

Our goal is to construct a microlocal parametrix for solutions to (12.0.1), (12.0.2), satisfying the "outgoing" condition that u is zero for $x_1 << 0$, given $f \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$, under the diffractive (grazing) or gliding hypothesis on $\partial\Omega$ with respect to $p(x, \xi)$, which we now make precise.

We suppose that over a point $(x_0', \xi_0') \in T^*(\mathbb{R}^n) = T^*(\partial\Omega)$, there pass ℓ transversal rays and one tangential ray, i.e., null bicharacteristic strips of $p(x, \xi)$. There are then $\ell + 1$ points $\zeta_\nu \in T^*_{(x_0',0)}(\mathbb{R}^{n+1})$ belonging to the characteristic set of $p(x, \xi)$, such that if $\pi : T^*_{(x_0',0)}(\mathbb{R}^{n+1})$ is the natural projection, then $\pi(\zeta_\nu) = \xi_0'$, $\nu = 1, \ldots, \ell + 1$. Through these pass the null bicharacteristic strips γ_ν, with $\gamma_{\ell+1}$ glancing and the rest transversal to $\partial\Omega$. We suppose the tangential ray $\gamma_{\ell+1}$ makes only second order contact with $\partial\Omega$.

The equations for the bicharacteristic strips of p are

$$\dot{x}_j = \frac{\partial p}{\partial \xi_j}, \quad \dot{\xi}_j = -\frac{\partial p}{\partial x_j}.$$

In particular $\dot{x}_{n+1} = \partial p/\partial \xi_{n+1}$, so if $\gamma_{\ell+1}$ passes through $(x_0', 0, \zeta_{\ell+1}) = (x, \xi)$, we have

$$\frac{\partial}{\partial \xi_{n+1}} p(x, \xi) = 0. \tag{12.1.1}$$

We suppose $p(x, \xi) = 0$. The diffractive/gliding assumption given above is equivalent to

$$\{p, \{p, x_{n+1}\}\} \neq 0 \text{ at } (x, \xi).$$

As in Chapter 4, we make the further hypothesis to the effect that the hypersurfaces $\Sigma_p = \{p = 0\}$ and $T^*_{\partial\Omega}(\Omega)$ have glancing intersection, so $\{x_{n+1}, \{x_{n+1}, p\}\} \neq 0$ at (x, ξ), or

$$\frac{\partial^2}{\partial \xi_{n+1}^2} p(x, \xi) \neq 0 \text{ at } (x, \xi) = (x_0', 0, \zeta_{\ell+1}). \tag{12.1.2}$$

The "glancing variety" is the subset of $T^*(\partial\Omega) \setminus 0$ over which tangential null bicharacteristics of p pass, so $p(x, \xi) = 0$ and $(\partial/\partial \xi_{n+1}) p(x, \xi) = 0$. Since we assume (12.1.2), we can locally define a root $\xi_{n+1} = a(x, \xi')$ of

$$\frac{\partial}{\partial \xi_{n+1}} p(x, \xi', a) = 0.$$

Then the glancing variety Σ in $T^*(\partial\Omega) \setminus 0$ is defined by

$$A(x, \xi') = p(x, \xi', a(x, \xi')) = 0 \quad (x = (x', 0)). \tag{12.1.3}$$

Note that

$$\frac{\partial}{\partial \xi_1} A(x, \xi') = \frac{\partial}{\partial \xi_1} p\big(x, \xi', a(x, \xi')\big) + \frac{\partial a}{\partial \xi_1} \frac{\partial}{\partial \xi_{n+1}} p\big(x, \xi', a(x, \xi)\big),$$

but the latter term vanishes, so by (12.0.3) we have

$$\frac{\partial}{\partial \xi_1} A(x, \xi') = \frac{\partial}{\partial \xi_1} p\big(x, \xi', a(x, \xi')\big) \neq 0 \quad \text{at } A(x, \xi') = 0. \tag{12.1.4}$$

Thus, near $(x_0', \xi_0') \in T^*(\partial\Omega)$, the glancing variety is a smooth conic hypersurface σ, and $\Sigma \cap T^*_{x'}(\partial\Omega)$ is a hypersurface in each fiber, for x' close to x_0'.

We shall begin our construction of parametrices with the construction of solutions mod $\mathcal{C}^\infty$ to (12.0.1) of the form (for $x_{n+1} \geq 0$)

$$u = AF = \int \big[gA(\zeta) + ihA'(\zeta)\big] A(\zeta_0)^{-1} e^{i\theta} \hat{F}(\xi)\, d\xi, \tag{12.1.5}$$

or

$$u = BF = \int \big[gAi(\zeta) + ihAi'(\zeta)\big] A(\zeta_0) e^{i\theta} \hat{F}(\xi)\, d\xi, \tag{12.1.6}$$

in the grazing and gliding cases, respectively. Here $\xi \in \mathbb{R}^n$. The phase functions θ, ζ are constructed as in Chapter 4, homogeneous of degree 1 and 2/3, respectively. The amplitudes g, h will belong to S^0 and $S^{-1/3}$, as before, but this time they will take values in $\mathbb{C}^k$. These functions will be constructed as solutions to certain eikonal and transport equations. Put

$$g \sim \sum_j g^{(j)}, \quad h \sim \sum_j h^{(j)},$$

with $g^{(j)}$ homogeneous of degree $-j$ and $h^{(j)}$ homogeneous of degree $-1/3 - j$. Applying $G(x, D_x)$ to (12.1.5) or (12.1.6) and setting highest order terms equal to zero yields

$$\begin{gathered} G_1(x, d_x\theta) g^{(0)} + G_1(x, \zeta d_x\zeta) h^{(0)} = 0, \\ G_1(x, d_x\zeta) g^{(0)} + G_1(x, d_x\theta) h^{(0)} = 0. \end{gathered} \tag{12.1.7}$$

If we set

$$\phi^{\pm} = \theta \pm \frac{2}{3}\zeta^{3/2}, \tag{12.1.8}$$

since $G_1(x, \xi)$ is linear in ξ, (12.1.7) is equivalent to

$$G_1(x, d_x\phi^{\pm})(g^{(0)} \pm \zeta^{1/2} h^{(0)}) = 0. \tag{12.1.9}$$

In particular, we get the characteristic equation

$$p(x, d_x\phi^{\pm}) = 0. \tag{12.1.10}$$

This equation was treated in Chapter 4. We saw that functions θ, ζ could be obtained, smooth on $\bar{\Omega}$, solving (12.1.10) for $\xi_n \leq 0$ and solving it to infinite order on $\partial\Omega$ for $\xi_n \geq 0$, such that

$$\zeta\big|_{\partial\Omega} = \xi_1^{-1/3} \xi_n, \tag{12.1.11}$$

and such that $\theta|_{\partial\Omega\times\mathbb{R}^n}$ is the generating function of a canonical transformation taking the glancing variety to $\{\xi_n = 0\}$ and taking the billiard ball map on $\partial\Omega$ associated with H_p, near $\gamma_{\ell+1}$, to standard form. Of course, in (12.1.6), we take $\xi_n \mapsto \xi_n + iT$, as usual.

§12.2: Transport equations

For the next step in solving (12.1.7), we must construct the amplitudes $g^{(0)}$ and $h^{(0)}$. We tackle this along the lines of the transport equations treated in Chapter 4 also, but some linear algebra will be required. We retain the notation of Section 4, particularly as regards the Lagrangian manifold $S_s \subset T^*\Omega$, whose projection $\pi : S_v \longrightarrow \Omega$, recall, is a simple fold, with associated involution $j : S_v \longrightarrow S_v$. Note that, by the hypothesis that G has simple characteristics, i.e., by (12.0.3), ker $G_1(x,\xi)$ is a smoothly varying family of one dimensional vector spaces on $\{p = 0\}$. Since S_v is imbedded smoothly in $\{p = 0\}$, there is a smooth $C_k -$valued function R on $S_v, R(x,\xi) \in$ ker$G_1(x,\xi)$. Break up R into its even and odd parts with respect to the involution j on S_v, to write (with $\pi^{-1} : \mathbb{R}^{n+1} \longrightarrow S_v$ double valued, domain $\pi(S_v)$),

$$R \circ \pi^{-1} = \tilde{g} \pm \zeta^{1/2}\tilde{h} = R_\pm. \tag{12.2.1}$$

Make $\tilde{g}$ homogeneous of degree 0 and $\tilde{h}$ homogeneous of degree $-1/3$; $\tilde{g}, \tilde{h}$ are smooth on the closure of $\pi(S_v)$. We extend g, h smoothly across the caustic set $\partial\pi(S_v)$. To satisfy (12.1.8), we must find scalar functions $\sigma_\pm$ such that

$$g^{(0)} \pm \zeta^{1/2}h^{(0)} = \sigma_\pm(\tilde{g} \pm \zeta^{1/2}\tilde{h}). \tag{12.2.2}$$

We will look for $\sigma_\pm$ in the form

$$\sigma_\pm = \sigma_0 \pm \zeta^{1/2}\sigma_1, \tag{12.2.3}$$

with $\sigma_0 \in S^0, \sigma_1 \in S^{-1/3}$, homogeneous. Thus (12.2.2) becomes

$$g^{(0)} \pm \zeta^{1/2}h^{(0)} = \sigma_0\tilde{g} + \zeta\sigma_1\tilde{h} \pm \zeta^{1/2}(\sigma_1\tilde{g} + \sigma_0\tilde{h}). \tag{12.2.4}$$

To obtain the transport equations for σ_0, σ_1, we use the $j = 1$ equations from among the continuation of (12.1.7) to higher transport equations:

$$\begin{aligned} G_1(x, d\theta)g^{(j)} + G_1(x, \zeta d\zeta)h^{(j)} &= -G(g^{(j-1)}), \\ G_1(x, d\zeta)g^{(j)} + G_1(x, d\theta)h^{(j)} &= -G(h^{(j-1)}). \end{aligned} \tag{12.2.5}$$

In fact, let $L \in$ ker G_1^t, $L_\pm = L \circ \pi^{-1}$, so

$$L_\pm = \tilde{\tilde{g}} \pm \zeta^{1/2}\tilde{\tilde{h}} \in \text{ker } G_1^t(x, d\phi^\pm), \tag{12.2.6}$$

be constructed as in (12.2.1). With

$$E_j^\pm = g^{(j)} \pm \zeta^{1/2}h^{(j)}, \tag{12.2.7}$$

one gets from (12.2.5)

$$G_1(x, d\phi^\pm)E_j^\pm = -G(g^{(j-1)}) \mp \zeta^{1/2}G(h^{(j-1)}), \tag{12.2.8}$$

the right side being defined as 0 for $j = 0$ (so (12.1.9) is obtained). Equation (12.2.8) determines $E_j^\pm$ up to a multiple of $R_\pm$;

$$E_j^\pm = k_j^\pm + \sigma_\pm^{(j)}R_\pm, \tag{12.2.9}$$

where one sees that k_j can be written in the form

$$k_j^\pm = k_j^{(0)} \pm \zeta^{1/2}k_j^{(1)}, \tag{12.2.10}$$

with $k_j^{(m)}$ smooth, $m = 0, 1$, of degree $0, -1/3$. The transport equations for $\sigma_\pm = \sigma_\pm^{(0)}$ and the other $\sigma_\pm^{(j)} = \sigma_0^{(j)} \pm \zeta^{1/2}\sigma_1^{(j)}$ are obtained by taking the inner product of (12.2.8)with $L_\pm$:

$$L_\pm \cdot [G(g^{(j-1)}) \pm \zeta^{1/2}G(h^{(j-1)})] = 0. \tag{12.2.11}$$

Replacing $j-1$ by j in (12.2.11) and then replacing $g^{(j)}$ by $E_j \mp \zeta^{1/2}h^{(j)}$, we get

$$L_\pm \cdot [G(E_j) \mp G(\zeta^{1/2}h^{(j)}) \pm \zeta^{1/2}G(h^{(j)})] = 0, \tag{12.2.12}$$

which implies

$$L_\pm \cdot [G(E_j) \mp \tfrac{1}{2}\zeta^{-1/2}G_1(x, d\zeta^{1/2})(E_j^+ - E_j^-)] = 0. \tag{12.2.13}$$

Using the fact that

$$L_\pm \cdot G_1(x, d\zeta)R_\mp = \frac{1}{2}\zeta^{-1/2}L_\pm \cdot G_1(x, d\phi^+ - d\phi^-)R_\mp = 0, \tag{12.2.14}$$

we see that

$$\begin{aligned} &L_\pm \cdot G_1(x, d\zeta^{1/2})(E_j^+ - E_j^-) \\ &= 2\zeta^{1/2}L_\pm \cdot G_1(x, d\zeta^{1/2})k_j^{(1)} \pm L_\pm \cdot G_1(x, d\zeta^{1/2})\sigma_\pm^{(j)}(R_\pm - R_\mp) \\ &= 2\zeta^{1/2}L_\pm \cdot G_1(x, d\zeta^{1/2})k_j^{(1)} + 2\sigma_\pm^{(j)}\zeta^{1/2}L_\pm \cdot G_1(x, d\zeta^{1/2})\tilde{h}. \end{aligned} \tag{12.2.15}$$

Consequently (12.2.12) can be written as

$$\begin{aligned} L_\pm \cdot G(\sigma_\pm^{(j)}R_\pm) + L_\pm \cdot G(k_j^\pm) \mp L_\pm \cdot G_1(x, d\zeta^{1/2})k_j^{(1)}& \\ \mp \sigma_\pm^{(j)}L_\pm \cdot G_1(x, d\zeta^{1/2})\tilde{h} = 0.& \end{aligned} \tag{12.2.16}$$

This is a first order equation for $\sigma_\pm^{(j)}$. To analyze it, first note that

(12.2.17)
$$\begin{aligned} G(\sigma_\pm R_\pm) &= G_1(x, d\sigma_\pm)R_\pm + \sigma_\pm G(R_\pm) \\ &= G_1(x, d\sigma_\pm)R_\pm + \sigma_\pm G(\tilde{g}) \pm \sigma_\pm G(\zeta^{1/2}h) \\ &= G_1(x, d\sigma_\pm)R_\pm + \sigma_\pm G(\tilde{g}) \pm \zeta^{1/2}\sigma_\pm G(\tilde{h}) \pm \sigma_\pm G_1(x, d\zeta^{1/2})\tilde{h}. \end{aligned}$$

Similarly,

$$\begin{aligned} G(k_j^\pm) &= G(k_j^{(0)}) \pm G(\zeta^{1/2}k_j^{(1)}) \\ &= G(k_j^{(0)}) \pm \zeta^{1/2}G(k_j^{(1)}) \pm G_1(x, d\zeta^{1/2})k_j^{(1)}. \end{aligned} \tag{12.2.18}$$

Thus (12.2.16) is equivalent to

$$\begin{aligned} L_\pm \cdot G_1(x, d\sigma_\pm^{(j)})R_\pm + \sigma_\pm^{(j)} L_\pm \cdot G(\tilde{g}) \pm \zeta^{1/2}\sigma_\pm^{(j)} L_\pm \cdot G(\tilde{h}) & \\ + L_\pm \cdot [G(k_j^{(0)}) \pm \zeta^{1/2}G(k_j^{(1)})] &= 0. \end{aligned} \tag{12.2.19}$$

We thus want to understand the vector field $Z_\pm$ defined by

$$Z_\pm \sigma_\pm = L_\pm \cdot G_1(x, d\sigma_\pm)R_\pm. \tag{12.2.20}$$

To look into this, extend the 0-eigenvectors $R_\pm$ and $L_\pm$ off $\{p = 0\}$, to be C^∞ eigenvectors of $G_1(x, \xi)$ and $G_1(x, \xi)^t$, respectively, associated with the eigenvalue $\lambda_0(x, \xi)$ or $G_1(x, \xi)$ which is smooth and which vanishes on $\{p = 0\}$. Thus

$$\begin{aligned} G_1(x, \xi)R(x, \xi) &= \lambda_0(x, \xi)R(x, \xi), \\ L(x, \xi)G_1(x, \xi) &= \lambda_0(x, \xi)L(x, \xi). \end{aligned} \tag{12.2.21}$$

We can arrange that $L(x, \xi) \cdot R(x, \xi) \neq 0$. Note that

$$\begin{aligned} L(x, \xi) \cdot G_1(x, \xi)R(x, \xi) &= \lambda_0(x, \xi)L(x, \xi) \cdot R(x, \xi) \\ &= a(x, \xi)p(x, \xi) \end{aligned} \tag{12.2.22}$$

where $a(x, \xi)$ is C^∞ and non-vanishing. This follows from the assumption that the zeros of $p(x, \xi)$ are simple. Note that

$$L(x, \xi) \cdot G_1(x, \xi) = 0 = G_1(x, \xi)R(x, \xi) \text{ on } \{p = 0\}. \tag{12.2.23}$$

Also note that the Hamiltonian vector fields of p and $L \cdot G_1 R$ are parallel on $\{p = 0\}$:

$$H_{L \cdot G_1 R} = aH_p \text{ on } \{p = 0\}. \tag{12.2.24}$$

Since, for $f = f(x), H_p f = \sum_j \partial_{\xi_j} p \cdot \partial_{x_j} f$, we see that, for $f = f(x)$, on $\{p = 0\}$,

$$
\begin{aligned}
H_{L\cdot G_1 R} f &= \sum_j \partial_{\xi_j}(L \cdot G_1 R)\partial_{x_j} f(x) \\
&= \sum_j L \cdot \big(\partial_{\xi_j} G_1 \cdot \partial_{x_j} f(x)\big) R \quad \text{(by (12.2.24))} \\
&= L \cdot G_1\big(x, df(x)\big) R,
\end{aligned}
\tag{12.2.25}
$$

using the linearity of $G_1(x,\xi)$ in ξ. Thus, by (12.2.24),

$$
L(x,\xi) \cdot G_1\big(x, df(x)\big) R(x,\xi) = a H_p f \quad \text{on} \quad \{p = 0\}. \tag{12.2.26}
$$

Thus with

$$
\sigma^{(j)} = \sigma_{\pm}^{(j)} \circ \pi, \tag{12.2.27}
$$

the transport equation (12.2.19) is equivalent to

$$
a H_p \sigma^{(j)} + A\sigma^{(j)} = B, \tag{12.2.28}
$$

where

$$
\begin{aligned}
A &= L_{\pm} \cdot [G(\tilde{g}) \pm \zeta^{1/2} G(\tilde{h})] \circ \pi \in C^\infty(S_v), \\
B &= L_{\pm} \cdot [G(k_j^{(0)}) \pm \zeta^{1/2} G(k_j^{(1)})] \circ \pi \in C^\infty(S_v).
\end{aligned}
\tag{12.2.29}
$$

Obtaining smooth non-vanishing solutions of (12.2.28) is routine, and the discussion of the transport equations is easily completed, along the lines of Chapter 4.

Concerning the principal term $g^{(0)}$ of g, appearing in (12.1.5) or (12.1.6), we remark that we can certainly arrange that the term $\tilde{g}(x,\xi)$ be nonzero on $\partial\Omega$ at $\xi_n = 0$, and hence that $g^{(0)}$ be nonzero there. Depending on G_1, it may or may not happen that $\tilde{h}$ is linearly independent of $\tilde{g}$ at some point and hence by (12.3.5) it may or may not be the case that $h^{(0)}$ is linearly independent of $g^{(0)}$ at such a point.

§12.3: Coercive boundary conditions

We now have the amplitudes and phase functions for (12.1.5), (12.1.6). F is a scalar distribution to be determined by the boundary condition (12.0.2). Next, there are two more types of solutions of (12.0.1) to write down. The first comes from the l families of non-glancing rays hitting $\partial\Omega$ near (x_0', ξ_0'). These give rise to solutions of (12.0.1) of the form

$$
u_j = A_j F_j = \int a_j(x,\xi) e^{i\phi_j(x,\xi)} \hat{F}_j(\xi)\, d\xi, \quad 1 \le j \le l, \tag{12.3.1}
$$

determined by the usual methods of geometrical optics. The amplitudes $a_j(x,\xi)$ take values in $\mathbb{C}^k$ and the F_j are scalar valued distributions. The final term is the 'elliptic term,' of the form

$$u_- = A_- F_- = \int a_-(x,\xi) e^{ix'\cdot\xi} \hat{F}_-(\xi)\, d\xi, \tag{12.3.2}$$

where $a_-(x,\xi)$ is an amplitude of Poisson type, i.e.,

$$x_{n+1}^l D_{n+1}^j a_-(x,\xi) \text{ bounded in } S^{j-l}_{1,0}(\mathbb{R}^n) \ \text{ for } \ 0 \le x_{n+1} \le 1, \tag{12.3.3}$$

and F_- takes values in $\mathbb{C}^\mu, \mu$ being the dimension of the sum of the generalized eigenspaces of $H(x,\xi')$ with negative real part, for (x,ξ') near $(x_0', 0, \xi_0')$, where $G_1(x,\xi) = \xi_{n+1} G_{n+1} + G(x,\xi')$ and $H(x,\xi') = (1/i) G_{n+1}^{-1} G(x,\xi')$. The amplitude $a_-(x,\xi)$ takes values in $\mathcal{L}(\mathbb{C}^\mu, \mathbb{C}^k)$.

Our goal is to construct a solution mod C^∞ to (12.0.1), (12.0.2) which is smooth along those rays going in the negative x_1 direction. This choice fixes the choice $A = A_+$ or A_- in (12.1.5), in the grazing case, and selects λ terms of the form (12.3.1), say $u_1, \dots, u_\lambda$. We want to construct u, solving (12.0.1), (12.0.2), as a superposition of solutions of the form (12.1.5) or (12.1.6), of the form (12.3.1), $1 \le j \le \lambda$, and of the form (12.3.2). We get an equation for

$$F' = \big(F, F_j (1 \le j \le \lambda), F_-\big) \tag{12.3.4}$$

as follows. We will give the details for the gliding case, where (12.1.6) occurs.

Restricting (12.1.6) to $\partial\Omega = \{x_{n+1} = 0\}$, we get

$$\begin{aligned} BF\big|_{\partial\Omega} &= \int \big[g + ih\Phi i(\zeta_0)\big] e^{i\theta} Ai(\zeta_0) A(\xi_0) \hat{F}(\xi)\, d\xi \\ &= J(A_1^\# + A_2^\# \Phi i_T) \mathcal{A} i_T \mathcal{A}_T F, \end{aligned} \tag{12.3.5}$$

where J is an elliptic FIOP:

$$Jf = \int e^{i\theta} \hat{f}(\xi)\, d\xi, \tag{12.3.6}$$

with $A_1^\# \in \mathrm{OPS}^0,\ A_2^\# \in \mathrm{OPS}^{-1/3}$, the symbols taking values in $\mathcal{L}(\mathbb{C}, \mathbb{C}^k)$. Restricting (12.3.1) to $\partial\Omega$ gives

$$A_j F_j\big|_{\partial\Omega} = \int a_j(x', 0, \xi) e^{ix'\cdot\xi} \hat{F}_j(\xi)\, d\xi = A_j^o F_j \quad (1 \le j \le \lambda), \tag{12.3.7}$$

provided we prescribe as initial data for the phase function ϕ_j that $\phi_j(x', 0, \xi) = x' \cdot \xi$. Thus $A_j^o \in \mathrm{OPS}^0$, with symbol taking values in $\mathcal{L}(\mathbb{C}, \mathbb{C}^k)$. Finally

$$A_- F_-\big|_{\partial\Omega} = \int a_-(x', 0, \xi) e^{ix'\cdot\xi} \hat{F}_-(\xi)\, d\xi = A_-^o F_-. \tag{12.3.8}$$

Here $A^o_- \in \mathrm{OPS}^0$, with symbol taking values in $\mathcal{L}(\mathbb{C}^\mu, \mathbb{C}^k)$. If

$$u = BF + \sum_j A_j F_j + A_- F_-, \tag{12.3.9}$$

then the boundary condition (12.0.2) is

$$B(x', D_{x'})\Big(J(A_1^\# + A_2^\# \Phi i_T)\mathcal{A} i_T \mathcal{A}_T F + \sum_j A_j^o F_j + A_-^o F_-\Big) = f. \tag{12.3.10}$$

Here f takes values in $\mathbb{C}^{k'}$. We shall assume

$$k' = 1 + \lambda + \mu, \tag{12.3.11}$$

so (12.3.10) is a determined system. Let

$$B^\#(x', D_{x'}) = J^{-1} B(x', D_{x'}) J, \tag{12.3.12}$$

so (12.3.10) becomes

$$\begin{aligned} B^\#(x', D_{x'})\Big((A_1^\# + A_2^\# \Phi i_T)\mathcal{A} i_T \mathcal{A}_T F + \sum_j J^{-1} A_j^o J J^{-1} F_j \\ + J^{-1} A_-^o J J^{-1} F_-\Big) = J^{-1} f. \end{aligned} \tag{12.3.13}$$

Now if we define the $\mathbb{C}^{1+\lambda+\mu}$ -valued distribution G by

$$G = \big(G_0, G_j (1 \le j \le \lambda), G_-\big) = \big(\mathcal{A} i_T \mathcal{A}_T F, J^{-1} F_j (1 \le j \le \lambda), J^{-1} F_-\big), \tag{12.3.14}$$

then (12.3.13) becomes

$$B^\#(x', D_{x'})\Big((A_1^\# + A_2^\# \Phi i_T) G_0 + \sum_j A_j^b G_j + A_-^b G_-\Big) = J^{-1} f, \tag{12.3.15}$$

where the pseudodifferential operators $A_j^b, A_-^b \in \mathrm{OPS}^0$ are defined by

$$A_j^b = J^{-1} A_j^o J, \quad A_-^b = J^{-1} A_-^o J. \tag{12.3.16}$$

Set

$$B_0(x', D_{x'}) G = B^\#(x', D_{x'})\Big(A_1 G_0 + \sum_{j=1}^{\lambda} A_j^b G_j + A_-^b G_-\Big), \tag{12.3.17}$$

so $B_0 \in \mathrm{OPS}^0$, and (12.3.13) reads

$$\big[B_0(x', D_{x'}) + B\Phi i_T\big] G = J^{-1} f, \tag{12.3.18}$$

where

$$B = B^{\#}(x', D_{x'}) A_2 \pi_0 \in \mathrm{OPS}^{-1/3}, \tag{12.3.19}$$

and $\pi_0 \in \mathrm{OPS}^0$ is defined by

$$\pi_0 G = G_0. \tag{12.3.20}$$

At $\mathcal{J}^{-1}(x_0', \xi_0')$, the principal symbol of B_0 is the product of the principal symbol of $B^{\#}$ and the map

$$\Delta(y, \xi) : \mathbb{C}^{1+\mu} \longrightarrow \mathbb{C}^k \tag{12.3.21}$$

defined as follows. Say $\mathcal{J}(x', \eta) = (y, \xi)$. Let e_ν denote the standard basis of $\mathbb{C}^{1+\lambda+\mu}, 0 \le \nu \le \lambda + \mu$, and imbed $\mathbb{C}^\mu$ into $\mathbb{C}^{1+\lambda+\mu}$ as the last factor. Then $\Delta(y, \xi)$ takes e_0 to $g^{(0)}(x', \xi), e_\nu$ to $a_\nu(x', \xi)$ for $1 \le \nu \le j + \lambda$, and it coincides with the map $a_-(x', \xi)$ on $\mathbb{C}^\mu$. Thus $B_0(x', D_{x'})$ is elliptic if and only if the composite map $B^{\#}\Delta :$

$$\mathbb{C}^{k'} \xrightarrow{\Delta} \mathbb{C}^k \xrightarrow{B^{\#}} \mathbb{C}^{k'} \tag{12.3.22}$$

is an isomorphism. As long as $B_0(x', D_{x'})$ is elliptic, Theorem 10.2.42 applies, to express, mod C^∞, the solution G of (12.3.18). Since the map $(\mathcal{A} i_T \mathcal{A}_T)^{-1}$ is elucidated in Chapter 5, we can then solve (12.3.14) for F', and hence obtain a solution mod C^∞ to the boundary condition (12.3.10), i.e., to (12.0.2).

In the grazing case, one obtains, in place of (12.3.18)

$$\big[B_0(x', D_{x'}) + B\Phi\big] G = J^{-1} f, \tag{12.3.23}$$

with

$$\begin{aligned} G &= \big(G_0, G_j(l \le j \le \lambda), G_-\big) \\ &= \big(F, J^{-1} F_j(l \le j \le \lambda), J^{-1} F_-\big). \end{aligned} \tag{12.3.24}$$

Again ellipticity of $B_0(x', D_{x'}) \in \mathrm{OPS}^0$ implies solvability mod C^∞ of (12.3.23); this time, of course, the inverse operator $[B_0(x', D_{x'}) + B\Phi]^{-1}$ is a microlocal operator. This grazing case was discussed by Taylor, in [Tay3], and in further detail in Section 7 of [Tay8]. In these papers, it was allowed that several grazing rays pass over a point in $T^*\partial\Omega \setminus 0$.

We have proved the following result.

Theorem 12.3.25. *Consider the system* (12.0.1), (12.0.2), *with boundary $\partial\Omega$ being either grazing or gliding, under hypothesis* (12.1.2). *Assume at most one tangential ray passes over any point of $T^*\partial\Omega \setminus 0$. If the composite map* (12.3.22) *is bijective, then the construction* (12.1.5), (12.1.6), (12.3.1), (12.3.2) *produces a parametrix whose singularities obey the laws of geometrical optics.*

§12.4: Neumann type boundary conditions

We next consider a class of boundary conditions for which the invertibility of (12.3.22) is violated, a class related to the previous class as the Neumann boundary condition is related to the Dirichlet condition. The hypotheses we make on this new class of boundary conditions are the following

$$B^{\#}\Delta \text{ annihilates } e_0 \ \ (\text{i.e., } B^{\#} \text{ annihilates } g^{(0)}) \ \text{ on } \{\xi_n = 0\}, \tag{12.4.1}$$

$$B^{\#}\tilde{\Delta} \text{ is invertible,} \tag{12.4.2}$$

where $\tilde{\Delta}$ takes e_0 to $h^{(0)}$ and coincides with Δ on the complementary subspace. In this case, let

$$H = \big(H_0, (H_j), H_-\big) = \big(\Xi_1^{-1/3}\Phi i_T G_0, (G_j), G_-\big). \tag{12.4.3}$$

Then rewrite (12.3.15) as

$$B^{\#}(x', D_{x'})\Big((B_2^{\#} + B_1^{\#}\Phi i_T^{-1})H_0 + \sum_j A_j^b H_j + A_-^b H_-\Big) = J^{-1}f, \tag{12.4.4}$$

with

$$B_j^{\#} = A_j^{\#}\Xi_1^{1/3}, \tag{12.4.5}$$

and if we set

$$B_1(x', D_{x'})h = B^{\#}(x', D_{x'})\Big(B_2^{\#}H_0 + \sum_{j=1}^{\lambda} A_j^b H_j + A_-^b H_-\Big), \tag{12.4.6}$$

the equation (12.4.4) becomes

$$\big(B_1(x', D_{x'}) + B'\Phi i_T^{-1}\big)H = J^{-1}f \tag{12.4.7}$$

where

$$B' = B^{\#}(x', D_{x'})B_1^{\#}\pi_0 \in \mathrm{OPS}^{1/3}\,. \tag{12.4.8}$$

The hypothesis (12.4.1) implies

$$B' \text{ has vanishing principal symbol on } \{\xi_n = 0\}. \tag{12.4.9}$$

If we rewrite (12.4.7) as

$$\big(B_1(x', D_{x'})\Phi i_T + B'\big)\Phi i_T^{-1}H = J^{-1}f, \tag{12.4.10}$$

we see that, as long as $B_1(x', D_{x'})$ is *elliptic* in OPS^0, then, in view of (12.4.9), Theorem 10.3.11 applies to construct the inverse (mod C^∞)

$$[B_1(x', D_{x'})\Phi i_T + B']^{-1}.$$

On the other hand, the formula (12.4.6) makes it clear that (12.4.2) implies the ellipticity of $B_1(x', D_{x'})$. We have hence proved the following result, parallel to Theorem 12.3.25, at least in the gliding case, and the grazing case is handled similarly.

Theorem 12.4.11. *The results of Theorem 12.3.25 on parametrices for* (12.0.1), (12.0.2) *continue to hold if the condition* (12.3.22) *on invertibility of* $B^{\#}\Delta$ *is replaced by hypotheses* (12.4.1) *and* (12.4.2).

§12.5: Maxwell's equations

Before proceeding to the discussion of general boundary problems for first order systems we consider the important special case of Maxwell's equations in a vacuum bounded by a perfect conductor. Let Ω be a region in $\mathbb{R}^3$ bounded and with smooth boundary. In units in which the speed of light is 1, the equations of propagation are

$$\begin{aligned} &\frac{\partial E}{\partial t} - \operatorname{curl} E = 0, \quad \operatorname{div} E = 0 \\ &\frac{\partial B}{\partial t} + \operatorname{curl} E = 0, \quad \operatorname{div} B = 0. \end{aligned} \tag{12.5.1}$$

At the boundary of the perfect conductor, we have

$$\nu \times E = 0, \quad \nu \cdot B = 0, \tag{12.5.2}$$

with ν as usual denoting the unit normal vector field pointing into Ω. From (12.5.1) we can deduce that E and B each solve the scalar wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) E = 0, \tag{12.5.3}$$

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) B = 0. \tag{12.5.4}$$

The appropriate set of self-adjoint boundary conditions to impose on (12.5.3) and (12.5.4), corresponding to a perfectly conducting boundary, is:

$$\nu \times E = 0,\ \operatorname{div} E = 0 \quad \text{on } \partial\Omega, \tag{12.5.5}$$

$$\nu \cdot B = 0,\ \nu \times \operatorname{curl} B = 0 \quad \text{on } \partial\Omega. \tag{12.5.6}$$

We will construct a parametrix for the inhomogeneous boundary problem for the electric field E

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) E = 0, \tag{12.5.7}$$

$$\nu \times E = f,\ \operatorname{div} E = f_0 \text{ on } \partial\Omega, \tag{12.5.8}$$

where $f, f_0 \in \mathcal{C}_c^{-\infty}(\mathbb{R} \times \partial\Omega)$ are given, and we require

$$E = 0 \ \text{ for } \ t << 0, \tag{12.5.9}$$

and in particular describe the propagation of singularities under our standard assumption that the boundary is either diffractive or gliding. We concentrate on the gliding case here, noting the minor modifications needed to treat the somewhat simpler grazing case. Treatments of Maxwell's equations in the grazing case have been given in [Tay3], [Tay7], and [Yi].

In (12.5.8) f takes values in the complexified tangent bundle to $\partial\Omega$, identified as the annihilator of ν, $\nu \cdot f \equiv 0$ and f_0 is complex-valued. A parallel treatment of the magnetic field B is easily made; we omit the details.

To reduce the problem to the boundary we write the solution to (12.5.8) in terms of the Dirichlet data for E

$$E_0 = E\big|_{\mathbb{R}\times\partial\Omega}. \tag{12.5.10}$$

Thus the normal derivative is given by the Neumann operator applied component by component to E_0.

$$\partial_\nu E\big|_{\mathbb{R}\times\partial\Omega} = N \cdot E_0. \tag{12.5.11}$$

We use the parametrix for E :

$$E = \int \big[gAi(\zeta) + ihAi'(\zeta)\big] A(\zeta_0) e^{i\theta} \hat{F}(\xi)\, d\xi, \tag{12.5.12}$$

with F taking values in $\mathbb{C}^3$. We use the conditions

$$\zeta\big|_{\partial\Omega} = \zeta_0, \quad \frac{\partial}{\partial\nu}\theta\big|_{\partial\Omega} = 0, \quad h\big|_{\partial\Omega} = 0. \tag{12.5.13}$$

Now, as in Chapter 7

$$E_0 = \int g e^{i\theta} Ai(\zeta_0) A(\zeta_0) \hat{F}(\xi)\, d\xi = J(\mathcal{A}i_T \mathcal{A}_T F). \tag{12.5.14}$$

Meanwhile, we have

$$\nu \times E\big|_{\partial\Omega} = \nu \times J\mathcal{A}i_T\mathcal{A}_T F = \nu \times E_0, \tag{12.5.15}$$

and

$$\begin{aligned} \operatorname{div} E\big|_{\partial\Omega} = & \int (g\nabla\zeta + i\nabla h)\cdot e^{i\theta}(\Phi i_T \mathcal{A}i_T\mathcal{A}_T F)\widehat{\ }(\xi)\, d\xi \\ & + \int (ig\nabla\theta + \nabla g)\cdot e^{i\theta}(\mathcal{A}i_T\mathcal{A}_T F)\widehat{\ }(\xi)\, d\xi. \end{aligned} \tag{12.5.16}$$

By comparison with the Fourier integral operator

$$Ju = \int g e^{i\theta} \hat{u}(\xi)\, d\xi, \tag{12.5.17}$$

we have

$$
(12.5.18)\qquad
\begin{aligned}
&\int (g\nabla\zeta + i\nabla h)\cdot e^{i\theta}\hat{u}(\xi)\,d\xi = D_1 J u,\\
&\int (ig\nabla\theta + \nabla g)\cdot e^{i\theta}\hat{u}(\xi)\,d\xi = D_2 J u,
\end{aligned}
$$

where $D_1 \in \mathrm{OPS}^{2/3},\ D_2 \in \mathrm{OPS}^1$. The symbols of D_1 and D_2 take values in the vector space of linear maps of $\mathbb{C}^3$ to $\mathbb{C}$. The principal symbol of D_1 is $\nabla\zeta\cdot$, and that of D_2 is $i\nabla\theta\,\cdot$. Note that $\nabla\zeta$ is orthogonal to $\partial\Omega$ and $\nabla\theta$ is parallel to $\partial\Omega$. We have, from (12.5.16),

$$
(12.5.19)\qquad
\begin{aligned}
\operatorname{div} E\big|_{\partial\Omega} &= (D_1 J\Phi i_T J^{-1} + D_2)E_0\\
&= J(\tilde{D}_1\Phi i_T + \tilde{D}_2)J^{-1}E_0,
\end{aligned}
$$

where

$$
(12.5.20)\qquad \tilde{D}_j = J^{-1} D_j J.
$$

Thus, our boundary conditions (12.5.8) give rise to the system

$$
(12.5.21)\qquad \nu\times E_0 = f,
$$

$$
(12.5.22)\qquad J(D_1\Phi i_T + D_2)J^{-1}E_0 = f_0.
$$

We can reduce this to a scalar equation, setting

$$
(12.5.23)\qquad E_0 = f_1 + e_0\nu,
$$

where ν is the unit normal to $\partial\Omega$, and f_1 is the tangential field satisfying $\nu\times f_1 = f$. This gives the following equation for the single unknown scalar e_0 :

$$
(12.5.24)\qquad J(\tilde{D}_1\Phi i_T + \tilde{D}_2)J^{-1}(e_0\nu) = f_0 - J(\tilde{D}_1\Phi i_T + \tilde{D}_2)J^{-1}f_1 = f_2.
$$

Note that the grazing ray problem gives rise to the analogous equation

$$
(12.5.25)\qquad J(\tilde{D}_1\Phi + \tilde{D}_2)J^{-1}(e_0\nu) = f_2.
$$

In either case, since the principal symbol of $\tilde{D}_2$ is $i\nabla\theta\cdot$ and $\nabla\theta\cdot\nu = 0$, we see that $J\tilde{D}_2 J^{-1}\nu \in \mathrm{OPS}^0$. This makes the solvability of (12.5.25) rather easy, since, with $\tilde{\nu} = J^{-1}\nu J$

$$
(12.5.26)\qquad \Phi\tilde{\nu} = \tilde{\nu}\Phi + [\Phi,\tilde{\nu}],
$$

and

$$
(12.5.27)\qquad [\Phi,\tilde{\nu}] \in \mathrm{OPS}^{-1/3},
$$

while

$$S = \tilde{D}_1 \tilde{\nu} \in \mathrm{OPS}^{2/3} \ \text{ is elliptic}, \tag{12.5.28}$$

since the principal symbol of $\tilde{D}_1$ is $\nabla\zeta\,\cdot$. Thus we can solve (12.5.25), modulo $\mathcal{C}^\infty$, for e_0, with $\mathrm{WF}(e_0) = \mathrm{WF}(f_2)$.

Since (12.5.27) breaks down when Φ is replaced by Φi_T, the equation (12.5.24) requires more work.

The equation (12.5.24) is a special case of an equation of the form

$$\sum_{j=1}^{k} P_j^{\#} \Phi i_T P_j u = f, \tag{12.5.29}$$

where $P_j, P_j^{\#} \in \mathrm{OPS}_0$ and

$$P_j^{\#} = P_j^* \mod \mathrm{OPS}^{-1}, \ \sum P_j^{\#} P_j \ \text{elliptic}. \tag{12.5.30}$$

Working microlocally, we can suppose P_1 is elliptic. Multiplying $\Sigma P_j^{\#} \Phi i P_j$ on the left by $(P_1^{\#})^{-1}$ and on the right by P_1^{-1} and renotating, we reduce our problem to solving

$$\Big(\Phi i + \sum_{j=1}^{k} P_j^{\#} \Phi i_T P_j\Big) u = f. \tag{12.5.31}$$

We can estimate the $H^{s-1/3}$ norm of u in terms of the H^s norm of f, as follows. Take the inner product of both sides of (12.5.31) with u, to get

$$\mathrm{Im}\ (\Phi i\ u, u) + \sum \mathrm{Im}\ (\Phi i P_j u, P_j' u) = \mathrm{Im}\ (f, u), \tag{12.5.32}$$

where, by (12.5.30), $P_j' = P_j \mod \mathrm{OPS}^{-1}$. Now as proved in Appendix A,

$$\mathrm{Im}\, \Phi i(z) \geq C(1 + |z|)|\,\mathrm{Im}\, z|, \quad z \in \mathcal{D};$$

see (A.4.26). Together with simple estimates for $\mathrm{Re}\, z \geq 0$, we get

$$\mathrm{Im}\, \Phi i(\zeta) \geq C_T \langle \xi \rangle^{-1/3}, \tag{12.5.33}$$

and hence

$$\mathrm{Im}\ (\Phi i_T v, v) \geq C_T \|v\|^2_{H^{1/6}}. \tag{12.5.34}$$

Consequently (12.5.32) gives

$$\|u\|^2_{H^{-1/6}} \leq C\|f\|^2_{H^{1/6}} + C'\|u\|^2_{H^{-M}}. \tag{12.5.35}$$

From here, standard methods of functional analysis give a solution (modulo $\mathcal{C}^\infty$) to (12.5.31). We will obtain the solution by a different method, which will allow for an analysis of the singularities of u.

Applying Φi^{-1} to (12.5.31), we have that equation equivalent to

$$\Big(I + \sum_j \Phi i^{-1} P_j^{\#} \Phi i P_j\Big) u = \Phi i^{-1} f = f_1. \tag{12.5.36}$$

Now the Airy operator identity (10.4.65) implies

$$\Phi i^{-1} P_j^{\#} \Phi i = P_j^b + E_j \Phi i - \Phi i^{-1} C_j \tag{12.5.37}$$

where

(12.5.38)

$$E_j \in \mathrm{OPS}^{-1/3};\ C_j \in \mathrm{OPS}^{1/3} \ \text{ has principal symbol vanishing at } \xi_n = 0,$$

and

$$P_j^b = P_j^{\#} \quad \mathrm{mod}\ \ \mathrm{OPS}^{-1}\,. \tag{12.5.39}$$

Thus (12.5.36) becomes

$$\Big(A + \sum_j E_j \Phi i P_j - \Phi i^{-1} C^{\#}\Big) u = f_1 \tag{12.5.40}$$

where

$$A = I + \sum_j P_j^b P_j \in \mathrm{OPS}^0,\ \text{elliptic}, \tag{12.5.41}$$

and

$$C^{\#} = \sum_j C_j P_j \tag{12.5.42}$$

has the properties of C_j in (12.5.38).

If we denote a parametrix of A by $B \in \mathrm{OPS}^0$ and set $B_j = BE_j \in \mathrm{OPS}^{-1/3}$, we get

$$\Big(I + \sum B_j \Phi i_T P_j - B \Phi i_T^{-1} C^{\#}\Big) u = B f_1 = f_2. \tag{12.5.43}$$

Now, as shown in Appendix A, we have the estimates

$$|\Phi i(\zeta)| \le C(|\zeta|^{1/2} + T^{-1} \xi_1^{1/3}),$$

and

$$|\zeta \Phi i(\zeta)^{-1}| \leq C(|\zeta|^{1/2} + T^{-1}\xi_1^{1/3}).$$

Hence, if $B_j, P_j, B, C^{\#}$ are altered so their symbols are supported in a small conic neighborhood of $\xi_n = 0$, and are further appropriately altered by smoothing operators, we get arbitrarily small operator norm on H^s for

$$S = \sum_j B_j \Phi i_T P_j - B \Phi i_T^{-1} C^{\#},$$

provided also T is taken sufficiently large. Hence, as in the analysis of (10.2.1), the solution to (12.5.43) is given by the Neumann series expansion

$$u = \sum_{k=0}^{\infty} (-S)^k f_2.$$

The argument given in Chapter 10 (see (10.2.18)–(10.2.31)) shows that if f is smooth for $x_n < 0$ then so is u. From there, the same commutator argument as given in Chapter 10 (see (10.2.32)–(10.2.36)) shows that the wave front set of u has the same sort of relation to $\mathrm{WF}(f)$ as in the other classes of Airy operator equations we have encountered, involving Φi.

Appendix A: Airy functions and Airy quotients

We collect below some of the basic properties of the functions $Ai(z)$ and $A_{\pm}(z)$ which are used extensively in this work, and also results on various quotients of these functions and their derivatives. Most of the material on the Airy functions proper is contained in Olver [Ol2] and Miller [Mil].

For $s \in \mathbb{R}$, $Ai(s)$ is defined by:

$$Ai(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(st+t^3/3)}\, dt. \tag{A.0.1}$$

This integral is not absolutely convergent, but is well-defined as the Fourier transform of a tempered distribution. It follows directly that Ai satisfies the second order differential equation (Airy's equation)

$$Ai''(s) - sAi(s) = 0. \tag{A.0.2}$$

From (A.0.2) it follows that $Ai(z)$ extends to an entire holomorphic function on $\mathbb{C}$. Set

$$A_{\pm}(z) = Ai(e^{\mp 2\pi i/3} z). \tag{A.0.3}$$

Thus, $A_{\pm}(z)$ also satisfy the differential equation (A.0.2). In fact we have

$$Ai(z) = e^{\pi i/3} A_{+}(z) + e^{-\pi i/3} A_{-}(z), \tag{A.0.4}$$

as we proceed to show.

Note that $Ai(z)$ is real for real z, so (A.0.3) implies that:

$$A_{-}(z) = \overline{A_{+}(\bar{z})}. \tag{A.0.5}$$

Thus we must have

$$Ai(z) = cA_{+}(z) + \bar{c}A_{-}(z). \tag{A.0.6}$$

Evaluating $Ai(0)$ and $Ai'(0)$ in two ways each, using (A.0.6) and (A.0.3), gives

$$c + \bar{c} = 1, \quad c\omega^{-2} + \bar{c}\omega^{2} = 1,$$

where

$$\omega = e^{\pi i/3}, \tag{A.0.7}$$

and this in turn implies that $c = \omega^2/(1+\omega^2) = 1/(1-\omega) = \omega$, which proves (A.0.4).

§A.1: Asymptotic expansion

An integral formula for $Ai(z)$ which is convergent for all $z \in \mathbb{C}$ can easily be obtained. Replace t in (A.0.1) by iv and deform the contour so that for real z,

$$Ai(z) = \frac{1}{2\pi i} \int_L e^{v^3/3 - zv}\, dv, \tag{A.1.1}$$

where L is any contour that begins at a point at infinity in the sector $-\pi/2 \leq \arg(v) \leq -\pi/6$, and ends at infinity in the sector $\pi/6 \leq \arg(v) \leq \pi/2$. Since both sides of (A.1.1) are entire analytic, we have the identity for all $z \in \mathbb{C}$.

From (A.1.1) we can obtain a formula, valid in the region

$$\{z \in \mathbb{C}; |\arg(z)| \leq (1-\delta)\pi\}, \quad \delta > 0, \tag{A.1.2}$$

i.e., in the complex plane C with a small conic neighborhood of the closed negative real axis removed. Indeed, for $z \in \mathbb{R}^+$, set $v = z^{1/2} + it^{1/2}$ on the upper half of the path L in (A.1.1) and $v = z^{1/2} - it^{1/2}$ on the lower half to obtain:

$$\begin{aligned} Ai(z) &= \frac{1}{2\pi}\, e^{-(2/3)z^{3/2}} \int_0^\infty \cos\Big(\frac{1}{3}t^{3/2}\Big) \exp(-tz^{1/2}) t^{-1/2}\, dt \\ &= \Psi(z)\, e^{-(2/3)z^{3/2}}. \end{aligned} \tag{A.1.3}$$

Since the right side is clearly holomorphic in the region (A.1.2), there is identity in that region. Well-known asymptotic methods can now be applied, in particular the method of steepest descents, to the integral defining $\Psi(z)$, giving

$$\Psi(z) \sim z^{-1/4} \sum_{j=0}^{\infty} a_j z^{-3j/2}, \quad a_0 = \frac{1}{4}\pi^{-3/2}, \tag{A.1.4}$$

as $|z| \to \infty$ within the region (A.1.2). Formal term by term differentiation yields valid asymptotic expansions in this region for the derivatives of $\Psi(z)$, see [Ol2].

The asymptotic expansion (A.1.3), (A.1.4) implies

$$A_\pm(z) = \Psi(\omega^{\mp 2} z) \exp\Big(\mp \frac{2}{3} i (-z)^{3/2}\Big) \tag{A.1.5}$$

in the regions

$$\Big\{z \in \mathbb{C}; \Big|\arg(z) \mp \frac{2}{3}\pi\Big| \leq (1-\delta)\pi\Big\}, \quad \delta > 0, \tag{A.1.6}$$

and in these regions $\Psi(\omega^{\mp 2} z)$ has the same sort of asymptotic expansion as (A.1.4).

Another useful integral formula for $Ai(s), s > 0$, is obtained by writing the integral (A.0.1) as

$$Ai(s) = \frac{1}{\pi} \int_0^\infty \cos\Big(st + \frac{1}{3}t^3\Big)\, dt,$$

and making the change of variable $t = 2s^{1/2}\sinh(v/3)$. Since

$$4\sinh^3\Big(\frac{v}{3}\Big) + 3\sinh\Big(\frac{v}{3}\Big) = \sinh v,$$

it follows that:

$$Ai(z) = \frac{2}{\sqrt{3}\pi}\Big(\frac{z}{3}\Big)^{1/2} \int_0^\infty \cos\Big(\frac{2}{3}z^{3/2}\, \sinh v\Big) \cosh\Big(\frac{1}{3}v\Big)\, dv. \tag{A.1.7}$$

The integral on the right is a modified Hankel function. Generally, if $\xi > 0$ and $0 < \nu < 1$,

$$\begin{aligned} K_\nu(\xi) &= \frac{1}{\cos(\pi\nu/2)} \int_0^\infty \cos(\xi \sinh t)\cosh(\nu t)\, dt \\ &= \int_0^\infty e^{-\xi \cosh t} \cosh(\nu t)\; dt, \end{aligned} \tag{A.1.8}$$

the latter integral being convergent and holomorphic for $\mathrm{Re}(\xi) > 0$; see Erdelyi et al. [Er], Vol. 2, p. 82, or Lebedev, [Leb], pp. 119–140. Thus

$$Ai(z) = \frac{1}{\pi}\Big(\frac{z}{3}\Big)^{1/2} K_{1/3}\Big(\frac{2}{3}z^{3/2}\Big), \quad |\arg(z)| < \frac{1}{3}\pi. \tag{A.1.9}$$

Since $K_\nu(z)$ solves the modified Bessel equation

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} - \Big(1 + \frac{\nu^2}{z^2}\Big)w = 0, \tag{A.1.10}$$

it follows that $K_\nu(z)$ is holomorphic in $|\arg(z)| < \pi$, and (A.1.9) therefore holds in the larger region $|\arg(z)| < 2\pi/3$. In fact $K_\nu(z)$ can be continued to the logarithmic plane covering $\mathbb{C}\backslash 0$, and then (A.1.9) is valid globally.

The formula (A.1.8) implies that, for fixed $\nu > 0$, as $\xi \to 0$, $|\arg\ \xi| < \pi$,

$$K_\nu(\xi) \sim \frac{1}{2}\int_0^\infty e^{-(1/2)\xi e^t}\ e^{\nu t}\ dt \sim \frac{1}{2}\int_1^\infty e^{-\xi s/2}\ s^{\nu-1}\ ds \sim \frac{1}{2}\Gamma(\nu)\Big(\frac{2}{\xi}\Big)^\nu, \tag{A.1.11}$$

and hence the identity (A.1.9) implies

$$Ai(0) = \frac{1}{2\pi}\ 3^{-1/6}\ \Gamma\Big(\frac{1}{3}\Big) = \frac{3^{-2/3}}{\Gamma(2/3)}, \tag{A.1.12}$$

Figure A.1

the last identity in (A.1.12) following from $\Gamma(1/3)\Gamma(2/3) = \pi/(\sin \pi/3) = 2\pi/\sqrt{3}$. Further computation (cf. (A.2.12)) gives

$$Ai'(0) = -\frac{1}{2\pi}\, 3^{1/6}\, \Gamma\Big(\frac{2}{3}\Big) = -\frac{3^{-1/3}}{\Gamma(1/3)}. \tag{A.1.13}$$

Figure A.1 is a graph of $y = Ai(s)$, $s \in \mathbb{R}$, produced by numerically integrating (A.0.2), using the initial data (A.1.12)–(A.1.13).

§A.2: Zeroes of Ai

The formulæ (A.1.3), (A.1.4) show that for any $\delta > 0$, there is some finite $R(\delta)$ such that $Ai(z)$ has no zeroes in (A.1.2) for $|z| > R(\delta)$. In this section we show that all the zeroes of $Ai(z)$ and all those of $Ai'(z)$ are real and negative. First we give a proof of an important special case of this.

Proposition A.2.1. *$A_\pm(s)$, $A'_\pm(s)$ are not zero for any $s \in \mathbb{R}$.*

Proof. This is a simple consequence of the Wronskian relation:

$$A'_+(z)A_-(z) - A_+(z)A'_-(z) = c_0 i = \frac{1}{2\pi i}. \tag{A.2.2}$$

By (A.0.5) and the same equation for the derivatives, the real zeroes of A_+ and A_-, or of their derivatives, must coincide. The existence of one such common zero would imply $c_0 = 0$ in (A.2.2). Disregarding our explicit computation of c_0, we see that this would imply $A_+(z) = c' A_-(z)$. This is not possible, since it would contradict (A.1.5).

The next result implies that

$$Ai(z) \neq 0, \quad |\arg(z)| \leq \frac{1}{3}\pi. \tag{A.2.3}$$

Proposition A.2.4. *$K_\nu(z) \neq 0$ for $|\arg(z)| \leq \pi/2$, if $\nu \in \mathbb{R}^+$.*

Proof. By (A.1.8) $K_\nu(z)$ is real for real z, so it is enough to consider z in the fourth quadrant. We use the argument principle, and compute the change in the argument of $K_\nu(z)$ along a closed curve $ABCD$ as pictured in Fig. A.2. Along the piece AB the change in argument can be computed approximately from the asymptotic expansion:

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \sum_{k=0}^{\infty} a_k(\nu) z^{-k}, \quad |z| \to \infty,$$

which can be obtained from (A.1.8). Thus:

$$\text{(A.2.5)} \qquad \arg(K_\nu(B)) - \arg(K_\nu(A)) = -\frac{1}{4}\pi - iA + o(1) \text{ as } |A| = |B| \to \infty.$$

On BC there is no change of argument since $K_\nu(z)$ is real and positive, by (A.1.8). On CD, we use the asymptotic expansion (A.1.11) for $K_\nu(z)$, as $z \to 0$, and conclude

$$\text{(A.2.6)} \qquad \arg(K_\nu(D)) - \arg(K_\nu(C)) = \frac{1}{2}\nu\pi + o(1), \quad |C| = |D| \to 0.$$

Figure A.2

To find the change in argument from D to A we need to study $K_\nu(z)$ further. Consider the identity:

$$\text{(A.2.7)} \qquad K_\nu(-it) = \frac{\pi i}{2}\, e^{\pi\nu i/2} \left[J_\nu(t) + iY_\nu(t)\right],$$

which can be obtained from (A.1.8) by transformation of the integrals (see Olver [Ol2]). The Bessel functions $J_\nu(t)$ and $Y_\nu(t)$ satisfy Bessel's equation:

$$\frac{d^2w}{dt^2} + \frac{1}{t}\frac{dw}{dt} + \left(1 - \frac{\nu^2}{t^2}\right) w = 0. \tag{A.2.8}$$

Both are real for $t > 0$ real. Hence their positive real zeroes intertwine:

$$0 < y_{\nu,1} < j_{\nu,1} < y_{\nu,2} < j_{\nu,2} < \dots.$$

Now we need to show that the kth positive zero of $J_\nu(t)$ is given by:

$$j_{\nu,k} = \pi(k + \tfrac{1}{2}\nu - \tfrac{1}{4}) + o(1) \text{ as } k \to \infty \quad (\nu \text{ fixed}). \tag{A.2.9}$$

In fact the asymptotic expansion:

$$\begin{aligned} J_\nu(t) \sim \left(\frac{2}{\pi t}\right)^{1/2} &\Big[\cos\Big(z - \frac{1}{2}\pi\nu - \frac{1}{4}\pi\Big) \sum_{l=0}^{\infty} a_l(\nu) t^{-2l} \\ &- \sin\Big(z - \frac{1}{2}\pi\nu - \frac{1}{4}\pi\Big) \sum_{l=0}^{\infty} b_l(\nu) t^{-2l-1}\Big], \quad t \to \infty, \end{aligned} \tag{A.2.10}$$

which is readily obtained from an integral formula such as:

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \int_{-1}^{1} (1-t^2)^{\nu-1/2} \cos(zt)\, dt, \quad |\arg(z)| < \pi, \tag{A.2.11}$$

shows that $J_\nu(t)$ does have zeroes with the asymptotic behaviour (A.2.9), for large k. That the appropriate one is exactly the kth can be decided easily. For $\nu = 1/2$, $J_{1/2}(t) = \sqrt{(2/\pi t)}\sin t$, so (A.2.9) holds exactly in that case. For general ν, (A.2.9) follows from the analyticity in ν and and the argument principle, there being no zeroes near $t = 0$.

Returning to the analysis of $K_\nu(z)$ on DA, we see from (A.2.9) that, if $A = -iy_{\nu,k}$ then the change of argument of $K_\nu(z)$ on DA cancels out the change along the rest of the curve, up to a term which is $o(1)$ as $|A|, |B| \to \infty$, $|C|, |D| \to 0$. This proves Proposition A.2.4 and hence (A.2.3), since the change of argument must be an integer, hence zero.

In a fashion similar to (A.1.9) it can be shown that:

$$Ai'(z) = -\frac{z}{\sqrt{3}\pi}\; K_{2/3}\Big(\frac{2}{3} z^{3/2}\Big), \tag{A.2.12}$$

so Proposition A.2.4 also implies that:

$$Ai'(z) \neq 0, \quad |\arg(z)| \leq \frac{1}{3}\pi. \tag{A.2.13}$$

In order to show that all the zeroes of $Ai(z)$ and of $Ai'(z)$ are real it remains to demonstrate that

$$Ai(z), Ai'(z) \neq 0, \quad |\pi - \arg(z)| < \frac{2}{3}\pi, \quad z \notin \mathbb{R}^-. \tag{A.2.14}$$

To do this we follow the method of Lommel, as described by Olver [Ol2].

Pick $a, b \in \mathbb{C},\ a^3 \neq b^3$. From the identity:

$$\frac{d}{dz}\Big[bAi(az)Ai'(bz) - aAi(bz)Ai'(az)\Big] = z(b^3 - a^3)Ai(az)Ai(bz),$$

we conclude that:

$$\int_0^1 tAi(at)Ai(bt)\,dt$$
$$= \frac{1}{a^3 - b^3}\Big[bAi(a)Ai'(b) - aAi(b)Ai'(a)\Big] - \frac{b-a}{a^3-b^3}Ai(0)Ai'(0).$$

Similarly,

$$\int_0^1 Ai'(at)Ai'(bt)\,dt$$
$$= \frac{1}{a^3 - b^3}\Big[a^2Ai(a)Ai'(b) - b^2Ai(b)Ai'(a)\Big] - \frac{a^2-b^2}{a^3-b^3}Ai(0)Ai'(0).$$

Suppose that $a = re^{i\theta}$ is a nonreal zero of $Ai(z)$ or of $Ai'(z)$. Then so is $b = re^{-i\theta}$ and from these formulæ, we get:

$$\int_0^1 tAi(at)Ai(bt)\,dt = -r^{-2}\,\frac{\sin\theta}{\sin 3\theta}\,Ai(0)Ai'(0), \tag{A.2.15}$$

$$\int_0^1 Ai'(at)Ai'(bt)\,dt = -r^{-1}\,\frac{\sin 2\theta}{\sin 3\theta}\,Ai(0)Ai'(0). \tag{A.2.16}$$

The integrals on the left are positive and $Ai(0)Ai'(0)$ is negative. This implies that both $\sin\theta/\sin 3\theta$ and $\sin 2\theta/\sin 3\theta$ must be positive and finite. This is not possible in the range $|\pi - \arg(a)| < 2\pi/3,\ a \notin \mathbb{R}^-$, so (A.2.14) holds. Together with (A.2.3) this gives:

Theorem A.2.17. *All the zeroes of $Ai(z)$ and $Ai'(z)$ are real and negative.*

Given that all the zeroes of $Ai(z)$ are real and negative, say:

$$Ai(s_j) = 0, \quad 0 > s_0 > s_1 \cdots \to -\infty, \tag{A.2.18}$$

FIGURE A.3

we can write:

$$\chi(z) = (1/2i)\log\left(\frac{A_+(z)}{A_-(z)}\right) \tag{A.2.19}$$

for z in the plane $\mathbb{C}$ slit along the two rays starting from $e^{\pm 2\pi i/3}s_0$; see Figure A.3.

Also we shall denote by $\mathcal{K}$ the region:

$$\mathcal{K} = \Big\{z \in \mathbb{C}; \operatorname{Re}(z) \leq \frac{1}{2}\operatorname{Re}(e^{2\pi i/3}s_0)\Big\}.$$

Now, with

$$F(z) = [A_+(z)A_-(z)]^{1/2}, \tag{A.2.20}$$

we have

$$A_\pm(z) = F(z)e^{\pm i\chi(z)} \tag{A.2.21}$$

The asymptotic expansion (A.1.4), (A.1.5) gives:

$$F(z) \sim (-z)^{-1/4}\sum_{j=0}^{\infty} f_j(-z)^{-3j/2}, \quad z \in \mathcal{K},\ |z| \to \infty, \quad f_0 = \frac{1}{2\sqrt{\pi}} \tag{A.2.22}$$

and also for $z \in \mathcal{K}$,

$$\chi(z) \sim \frac{2}{3}(-z)^{3/2}\sum_{j=0}^{\infty} e_j(-z)^{-3j/2}, \quad e_0 = 1. \tag{A.2.23}$$

Thus (A.2.21), (A.2.22), (A.2.23) can be thought of as an asymptotic expansion for $A_\pm(z)$ which is in many ways more convenient than (A.1.4), (A.1.5). Note that (A.0.5) implies that

$$F(z) \text{ and } \chi(z) \text{ are real for } z \in \mathbb{R} \cap \mathcal{K}. \tag{A.2.24}$$

The definition (A.2.19) is equivalent to:

$$\frac{A_+(z)}{A_-(z)} = e^{2i\chi(z)}. \tag{A.2.25}$$

Differentiating and using the Wronskian relation (A.2.2) gives

$$2\chi'(z) = \frac{c_0}{F(z)^2}. \tag{A.2.26}$$

In terms of (A.2.21) a very convenient formula can be obtained for $Ai(z)$ for $z \in \mathcal{K}$ from (A.0.4). Namely,

$$Ai(z) = 2F(z)\cos\Big(\chi(z) - \frac{1}{3}\pi\Big) = 2F(z)\sin\Big(\chi(z) + \frac{1}{6}\pi\Big). \tag{A.2.27}$$

Since F is non-vanishing in $\mathcal{K}$ the zeroes of $Ai(z)$ must occur at the points where $\chi(s_j) + \pi/6$ is an integral multiple of π. In view of (A.2.23) and (A.2.24) this gives good asymptotic control over the behaviour of the zeroes of $Ai(z)$. Also, the asymptotic behaviour of $Ai(z)$ as $|z| \to \infty$ is elucidated by (A.2.27).

§A.3: Airy quotients

Next we record certain identities for Airy quotients. Formula (A.2.21) gives

$$\begin{aligned} \Phi_\pm(z) = \frac{A'_\pm(z)}{A_\pm(z)} &= \frac{F'(z)}{F(z)} \pm i\chi'(z) \\ &= \frac{F'(z)}{F(z)} \pm \frac{i}{2}\frac{c_0}{F(z)^2}. \end{aligned} \tag{A.3.1}$$

where the first equation is the definition of $\Phi_\pm(z)$. By (A.2.24) for real z this decomposes $\Phi_\pm(z)$ into its real and imaginary parts. Differentiating (A.2.27) leads to:

$$\begin{aligned} \Phi i(z) = \frac{Ai'(z)}{Ai(z)} &= \frac{F'(z)}{F(z)} + \chi'(z)\cot\Big(\chi(z) + \frac{1}{6}\pi\Big) \\ &= \frac{F'(z)}{F(z)} + \frac{1}{2}\frac{c_0}{F(z)^2}\cot\Big(\chi(z) + \frac{1}{6}\pi\Big). \end{aligned} \tag{A.3.2}$$

Using the Wronskian relation

$$A'_\pm(z)Ai(z) - Ai'(z)A_\pm(z) = c_\pm, \tag{A.3.3}$$

one obtains

$$\Phi_\pm(z) - \Phi i(z) = c_\pm[A_\pm(z)Ai(z)]^{-1}. \tag{A.3.4}$$

From the formulæ above

$$\begin{aligned} A_\pm(z)Ai(z) &= \omega^{\mp 1}F(z)^2\big[e^{\pm 2i\chi(z)} + \omega^{\pm 2}\big] \\ &= \omega^{\pm 1}F(z)^2\big[e^{\pm 2i(\chi(z)-\pi/3)} + 1\big]. \end{aligned} \tag{A.3.5}$$

Directly from Airy's equation the Airy quotients satisfy a nonlinear differential equation of first order:

$$\Phi'(z) = z - \Phi(z)^2, \tag{A.3.6}$$

for $\Phi(z) = \Phi i(z)$ or $\Phi_\pm(z)$. Note that

$$\Phi_\pm(z) = \omega^{\mp 2}\Phi i(\omega^{\mp 2}z). \tag{A.3.7}$$

The poles of $\Phi_+(z)$ lie on the ray $e^{-i\pi/3}[-s_0,\infty)$ which is contained in the fourth quadrant. The poles of $\Phi_-(z)$ lie on the ray $e^{i\pi/3}[-s_0,\infty)$ in the first quadrant. Outside any conic neighborhood of the respective rays there are asymptotic expansions:

$$\Phi_\pm(z) \sim z^{1/2}\sum_{j=0}^{\infty} b_j^\pm z^{-3j/2}, \quad |z| \to \infty. \tag{A.3.8}$$

In particular, (A.3.8) holds for $\Phi_+(z)$ for z in the upper half plane $\{\operatorname{Im} z \geq 0\}$, and a similar expansion holds for $\Phi_-(z)$ in the lower half plane since

$$\Phi_+(z) = \overline{\Phi_-(\overline{z})}. \tag{A.3.9}$$

The first constant is:

$$b_0^\pm = 1. \tag{A.3.10}$$

We wish to consider the manner in which $\Phi_+(z)$ maps the upper half plane into itself. The asymptotic expansion (A.3.8) shows that for $|z|$ large, and $\operatorname{Im} z \geq 0$, $\Phi_+(z)$ lies in an arbitrarily small conic neighborhood of the first quadrant,

$\operatorname{Re}\Phi_+ \geq 0$, $\operatorname{Im}\Phi_+ \geq 0$. In fact examination of (A.3.1) shows that for $|z|$ large, $\operatorname{Im} z \geq 0$, $\operatorname{Re}\Phi_+, \operatorname{Im}\Phi_+ > 0$. Indeed as $|z| \to \infty$ in a conic neighborhood of $\mathbb{R}^-$,

$$\frac{F'(z)}{F(z)} \sim z^{-1} \sum_{j=0}^{\infty} \alpha_j z^{-3j/2}, \quad F(z)^{-2} \sim (-z)^{1/2} \sum_{j=0}^{\infty} g_j z^{-3j/2}, \tag{A.3.11}$$

and as $|z| \to \infty$ in a conic neighborhood of $\mathbb{R}^+$,
(A.3.12)

$$\frac{F'(z)}{F(z)} \sim z^{1/2} \sum_{j=0}^{\infty} \tilde{\alpha}_j z^{-3j/2}, \quad F(z)^{-2} \sim z^{-1/2} \exp\Big(-\frac{4}{3} z^{3/2}\Big) \sum_{j=0}^{\infty} \tilde{g}_j z^{-3j/2},$$

where all the coefficients are real.

Next it will be shown that the closed upper half plane

$$\mathbb{C}^+ = \{z \in \mathbb{C}; \operatorname{Im}(z) \geq 0\}$$

is mapped by Φ_+ into the open first quadrant

$$\mathcal{Q}_1 = \{0 < \arg(\Phi) < \frac{1}{2}\pi; |\Phi| > 0\}.$$

Since $F(s)$ is real for real s, (A.3.1) implies that $\operatorname{Im}\Phi_+(s) > 0$ for $s \in \mathbb{R}$. Thus, $\operatorname{Im}(\Phi_+(z))$ is positive for $z \in \mathbb{R}$ and near infinity in $\mathbb{C}+$. Hence it must be strictly positive for $z \in \mathbb{C}^+$ by the maximum principle, i.e.,

$$\operatorname{Im}\Phi_+(z) > 0, \quad z \in \mathbb{C}^+. \tag{A.3.13}$$

Next consider the real part of $\Phi_+(z)$. Certainly $\operatorname{Re}\Phi_+(z) > 0$ outside a compact subset $K \subset \mathbb{C}^+$. Let $z_0 = x_0 + iy_0$ be a point with maximal imaginary part at which $\operatorname{Re}\Phi_+(z)$ vanishes. From the differential equation (A.3.6), $\operatorname{Im}\Phi_+'(z_0) = y_0$, so if $y_0 > 0$,

$$\operatorname{Re}\Phi_+(z_0 + it) = -y_0 t + O(t^2) < 0,$$

if $t > 0$ is small. This contradicts the maximality of y_0, so the only possibility left is $y_0 = 0$. At such a point, $\Phi_+'(z_0)$ would be real, by (A.3.6) but since were have already shown $\operatorname{Re}\Phi_+(z) \geq 0$ this implies that $\Phi_+'(z_0) = 0$. Near such a zero of order two or higher the image of a half disc in $\mathbb{C}^+$ cannot satisfy $\operatorname{Re}\Phi_+(z) \geq 0$, so this possibility is eliminated; we have proved that:

$$\operatorname{Re}\Phi_+(z) > 0, \quad z \in \mathbb{C}^+. \tag{A.3.14}$$

One consequence of (A.3.14) and (A.3.1) is:

$$F'(s) > 0, \quad s \in \mathbb{R}, \tag{A.3.15}$$

which is equivalent to

$$A_+(s)A_-(s) = |A(s)|^2 \text{ is monotone increasing for } s \in \mathbb{R}. \tag{A.3.16}$$

From (A.2.26) it is clear that $\chi(s)$ is monotone for $s \in \mathbb{R}$. Again by (A.3.1)

$$\frac{d \operatorname{Im} \Phi_+(s)}{ds} < 0, \quad s \in \mathbb{R}, \tag{A.3.17}$$

so

$$\operatorname{Im} \Phi_+(s) \text{ is monotone decreasing for } s \in \mathbb{R}. \tag{A.3.18}$$

This shows that the curve $\mathbb{R} \ni s \longrightarrow \Phi_+(s)$, has no self-intersections and that its image in the Riemann sphere has winding number one about an interior point of $\Phi_+(\mathbb{C}^+)$. This completes the proof of:

Theorem A.3.19. $\Phi_+ : \mathbb{C}^+ \longrightarrow \mathcal{Q}_1$ *is a biholomorphism onto its image, which is contained in the open first quadrant.*

Assertions (A.3.13) and (A.3.14) were proved in [MeS2]. The fact that $\operatorname{Re} \Phi_+(s) > 0$, $\operatorname{Im} \Phi_+(s) > 0$ for $s \in \mathbb{R}$ was used by Imai and Shirota [ImSh], who show that this is equivalent to the monotonicity (A.3.16) of $|A_+(s)|^2$ and refer to Miller [Mil] for this result. Since $|A_+(s)|^2 = Ai(s)^2 + Bi(s)^2$, the graph on [Mi1], page B16 is consistent with (A.3.16) but an explicit proof does not seem to be given there. We present here a graph of the curve $\Phi_+(s)$ in $\mathbb{C}$, as s runs over $\mathbb{R}$. See Fig. A.4. This graph was produced by numerically integrating the ODE (A.3.6) for Φ_+, with initial data

$$\Phi_+(0) = -e^{-2\pi i/3}\, 3^{1/3}\, \frac{\Gamma(2/3)}{\Gamma(1/3)} = -e^{-2\pi i/3}\, \frac{\sqrt{\pi} 2^{2/3} 3^{1/3}}{\Gamma(1/6)}.$$

Note how rapidly the curve approaches the x-axis, which is to be expected, given (A.3.1) and the behavior (A.3.12) of $F(s)^{-2} = |A_+(s)|^{-2}$ as $s \to +\infty$. Of course, these formulas make it clear that $\Phi_+(s)$ has positive imaginary part for $s \in \mathbb{R}$; this is the simplest part of Theorem A.3.19.

We next consider how close $\Phi_+(z)$ is to $z^{1/2}$ by examining the difference between $\Phi_+(z)^2$ and z. From (A.3.6)

$$\Phi_+(z)^2 = z - \Phi_+'(z), \tag{A.3.20}$$

so

$$\Phi_+(z)^2 \sim z + \sum_{j=0}^{\infty} \gamma_j z^{-1/2-3j/2}, \quad \text{as } |z| \to \infty. \tag{A.3.21}$$

Combining (A.3.13), (A.3.14) with this and Theorem A.3.19 we have:

FIGURE A.4

Corollary A.3.22. Φ_+^2 *is biholomorphic from* $\mathbb{C}^+$ *to its image, which is contained in the interior of* $\mathbb{C}^+$.

Note from (A.3.12) that for some positive constant C,

$$\operatorname{Im}\Phi_+(s)^2 \geq \begin{cases} C(1+|s|)^{-3/2}, & s \leq 0, \\ C\exp(-(4/3)s^{3/2}), s \geq 0. \end{cases}$$

Together with Corollary (A.3.22) this implies:

$$\operatorname{Im}\Phi i(x+iy)^2 \geq \begin{cases} C(1+|x|)^{-3/2} + Cy, & y \geq 0, x \leq 0, \\ C\exp(-(4/3)x^{3/2}) + Cy, & y \geq 0, x \geq 0. \end{cases} \tag{A.3.23}$$

Since $\operatorname{Re}\Phi i(x+iy)^2 = x + O((1+|x|^2+|y|^2)^{-1/4})$ we therefore have:

$$\operatorname{Re}\Phi_+(x+iy) \geq C(1+|x|)^{-1/2}\Big(y + (1+|x|)^{-3/2}\Big), \text{ if } y \geq 0, x \leq 0, \tag{A.3.24}$$

and

$$\operatorname{Im}\Phi_+(x+iy) \geq C(1+|x|)^{-1/2}\Big(y + \exp(-(4/3)x^{3/2})\Big) \text{ if } y \geq 0, x \geq 0. \tag{A.3.25}$$

We next turn to the examination of $\Phi i(z) = Ai'(z)/Ai(z)$. Note that $\Phi i(s)$ is real for real s. In fact, $\Phi i(s) > 0$ for $s > \sigma_0$, where

$$\{\sigma_j; j = 0, 1, 2, \dots\} = \{\sigma; Ai'(\sigma) = 0\}. \tag{A.3.26}$$

Thus, $\Phi i(\sigma_j) = 0$ and $\Phi i(z)$ has a simple pole at each of the zeroes, $z = s_j$, of $Ai(z)$. Note that

$$0 > \sigma_0 > s_0 > \sigma_1 > s_1 > \cdots . \tag{A.3.27}$$

For any fixed $\delta > 0$, the behaviour of $\Phi i(z)$ on the set

$$\mathfrak{A}_\delta = \{z \in \mathbb{C}; |\arg(z)| \leq \pi - \delta\} \tag{A.3.28}$$

is rather obvious. From the expansion (A.1.3), (A.1.4)

$$\Phi i(z) \sim z^{1/2} \sum_{j=0}^{\infty} \gamma_j z^{-3j/2}, \quad |z| \to \infty \text{ in } \mathfrak{A}_\delta. \tag{A.3.29}$$

Since $\Phi i(s)$ is real and positive for $s \in \mathbb{R}^+$, all the γ_j in (A.3.29) are real with $\gamma_0 > 0$. From (A.3.7) and Theorem A.3.19 we obtain:

Proposition A.3.30. *Φi maps $\mathfrak{A}_{\pi/3}$ biholomorphically onto a domain in $\{|\arg(z)| < \pi/3\}$.*

§A.4: Behaviour of Φi near $(-\infty, 0]$

It remains to examine $\Phi i(z)$ in detail in a conic neighborhood of the negative real axis. To do so it is useful to obtain formulae parallel to (A.2.21) and (A.2.27), using the functions:

$$G(z) = [A'_+(z)A'_-(z)]^{1/2}, \quad \psi(z) = \frac{1}{2i} \log\left[\frac{A'_+(z)}{A'_-(z)}\right], \tag{A.4.1}$$

for z in the complex plane slit along two rays connecting, respectively, the zeroes of $A'_+(z)$ and those of $A'_-(z)$; cf. Figure A.3. Then

$$A'_\pm(z) = G(z)e^{\pm i\psi(z)}, \tag{A.4.2}$$

and

$$Ai'(z) = 2G(z)\sin\Big(\psi(z) + \frac{1}{6}\pi\Big). \tag{A.4.3}$$

Since $A'_+(z) = \overline{A'_-(\overline{z})}$,

$$G,\ \psi : \mathbb{R} \longrightarrow \mathbb{R}. \tag{A.4.4}$$

Differentiating the asymptotic expansion (A.1.3), (A.1.4), rotated to apply to $A'_\pm(z)$ we deduce that:

$$G(z) \sim (-z)^{1/4} \sum_{j=0}^{\infty} g_j(-z)^{-3j/2} \tag{A.4.5}$$

and

$$\psi(z) \sim \frac{2}{3}(-z)^{3/2} \sum_{j=0}^{\infty} e_j (-z)^{-3j/2} \tag{A.4.6}$$

as $|z| \to \infty$ in $\operatorname{Re} z \le 0$; cf. (A.2.22), (A.2.23).

In place of (A.2.26) we obtain

$$2\psi'(z) = -c_0 \frac{z}{G(z)^2}. \tag{A.4.7}$$

Unlike $\chi(s)$, which is monotonic on the real line, $\psi(s)$ is monotonic increasing for $s < 0$ and monotonic decreasing for $s > 0$. In fact in $s < 0, \psi(s)$ is closely related to $\chi(s)$. From (A.2.27) and (A.4.3) and noting that the zeroes of $Ai(s)$ and $Ai'(s)$ are interlaced, it follows that $\chi(s)+\pi/6$ and $\psi(s)+\pi/6$ alternately assume values which are integer multiples of π, so the difference must be bounded. In fact, (A.2.23), (A.4.6) together give:

$$\chi(z) - \psi(z) \sim \frac{1}{2}\pi - \sum_{j=1}^{\infty} \sigma_j z^{-3j/2}, \tag{A.4.8}$$

as $|z| \to \infty$ in $\{\operatorname{Re} z \le 0\}$.

Differentiating (A.4.2) and proceeding as in the derivation of (A.3.1) yields

$$\Phi_{\pm}(z)^{-1} = \frac{1}{z}\frac{G'(z)}{G(z)} \mp \frac{c_0 i}{2} G(z)^2. \tag{A.4.9}$$

Then, (A.3.14) and (A.3.15) imply that $\Phi_+^{-1}(s)$ lies in the first quadrant, so:

$$G'(s) \text{ has the same sign as } s, \ \ s \in \mathbb{R}. \tag{A.4.10}$$

Comparison of (A.3.1) and (A.4.9) also gives

$$G^2 = \left(\tfrac{1}{2}c_0\right)^2 F^{-2} + (F')^2. \tag{A.4.11}$$

To resume the discussion of the behaviour of $\Phi i(z)$ for z in a conic neighborhood of $\mathbb{R}^-$, consider (A.2.27) and (A.4.3), which show:

$$\Phi i(z) = \frac{G \sin(\psi + \pi/6)}{F \sin(\chi + \pi/6)}. \tag{A.4.12}$$

From the definitions of F and G,

$$\frac{G}{F}(z) = [\Phi_+(z)\Phi_-(z)]^{1/2}. \tag{A.4.13}$$

FIGURE A.5

The formula (A.4.12) can be used to describe $\Phi i(z)$ in the set

$$\mathcal{D} = \left\{ z \in \mathbb{C}; \operatorname{Re}(z) \leq -C, 0 \leq \operatorname{Im}(z) \leq C(1+|z|)^{-1/2} \right\}. \tag{A.4.14}$$

Divide $\mathcal{D}$ as follows. Pick the half-way points between the zeroes and the poles of $\Phi i(z)$,

$$\alpha_j = \frac{1}{2}(\sigma_j + s_j), \quad \beta_j = \frac{1}{2}(s_j + \sigma_{j+1}), \quad j \geq 0.$$

Then consider the parts:

$$\begin{aligned} \mathcal{E}_j &= \{ z \in \mathcal{D}; \beta_j \leq \operatorname{Re} z \leq \alpha_j \}, \quad j \geq 0, \\ \mathcal{F}_j &= \{ z \in \mathcal{D}; \alpha_j \leq \operatorname{Re} z \leq \alpha_{j-1} \}, \quad j \geq 1, \end{aligned} \tag{A.4.15}$$

as illustrated in Figure A.5.

The lower boundary of $\mathcal{E}_j$ is roughly centered at s_j, that of $\mathcal{F}_j$ at σ_j. Note that

$$s_j - s_{j+1} \sim \sigma_j - \sigma_{j+1} \sim c(-s_j)^{-1/2}.$$

By (A.2.27) and (A.4.3), $\chi + \pi/6$ maps $[s_{j+1}, s_j]$ to $[-(j+1)\pi, -j\pi]$. Thus the map:

$$\chi_j = \chi + \frac{1}{6}\pi + j\pi$$

maps s_j to the origin. From the asymptotic expansion for χ, it follows that

$$\chi_j(\mathcal{E}_j) \subset \mathcal{R},$$

where $\mathcal{R}$ is a rectangle in the upper half plane with base on the real axis centered at the origin. In fact for large j each χ_j has inverse, κ_j, holomorphic in a neighborhood of $\mathcal{R}$ with range containing $\mathcal{E}_j$. Set

$$v_j(z) = j^{-1/3}\Phi i\big(\kappa_j(z)\big). \tag{A.4.16}$$

From (A.4.12), (A.4.13), the asymptotic expansions (A.4.6) and (A.4.8), it follows that as $j \to \infty$, for some constant v,

$$v_j(z) \to v\tan(z) \tag{A.4.17}$$

uniformly on $\mathcal{R}$. Similar arguments apply to the function ψ defined on $\mathcal{F}_j$, their normalizations $\psi + (1/6 + j)\pi$ with inverses λ_j so that the functions:

$$w_j(z) = \frac{j^{1/3}}{\Phi i\big(\lambda_j(z)\big)} \to w\tan(z), \tag{A.4.18}$$

uniformly on $\mathcal{R}$ for some constant w.

From (A.4.16) it follows that, for large j,

$$|\Phi i(z)| \le cj^{1/3} \le C(1+|z|)^{1/2}, \quad z \in \mathcal{F}_j, \tag{A.4.19}$$

and

$$\operatorname{Im}\Phi i(z) \ge cj^{1/3}\operatorname{Im}(j^{1/3}z) \ge C(1+|z|)\operatorname{Im} z, \quad z \in \mathcal{F}_j, \tag{A.4.20}$$

with the constants positive and independent of j. Simlarly from (A.4.17),

$$|\Phi i(z)|^{-1} \le cj^{-1/3} \le C(1+|z|)^{-1/2}, \quad z \in \mathcal{E}_j \tag{A.4.21}$$

and

$$\operatorname{Im}\Phi i(z)^{-1} \ge cj^{-1/3}\operatorname{Im}(j^{1/3}z) = C\operatorname{Im} z, \quad z \in \mathcal{E}_j. \tag{A.4.22}$$

These last inequalities give in particular:

$$|\Phi i(z)| \le C|\operatorname{Im}(z)|^{-1}, \quad z \in \mathcal{E}_j, \tag{A.4.23}$$

and

$$\operatorname{Im}\Phi i(z) \ge Cj^{1/3} \ge C(1+|z|)^{1/2}, \quad z \in \mathcal{E}_j. \tag{A.4.24}$$

These inequalities have been proved uniformly for large j, but of course are simple to demonstrate for any finite value of j so hold uniformly, with different constants, for all j.

Combining (A.4.19) and (A.4.23) gives

$$|\Phi i(z)| \le C|\operatorname{Im}(z)|^{-1}, \quad z \in \mathcal{D}, \tag{A.4.25}$$

and combining (A.4.20) and (A.4.24) gives:

$$\operatorname{Im}\Phi i(z) \ge C\big(1+|z|\big)\ |\operatorname{Im}(z)|, \quad z \in \mathcal{D}. \tag{A.4.26}$$

Note also that

$$\operatorname{Im}\{\Phi i(z)^{-1}\} \ge C \operatorname{Im} z, \quad z \in \mathcal{D}. \tag{A.4.27}$$

It is useful to get similar bounds for the Airy function $Ai(z)$ and its derivative $Ai'(z)$, for $z \in \mathcal{D}$. Indeed, starting from (A.2.27) and using reasoning similar to that in the derivation of (A.4.25) and (A.14.26) one finds that:

$$\operatorname{Im} Ai(z) \ge C(1+|z|)^{1/4} \operatorname{Im} z, \quad z \in \mathcal{D}, \tag{A.4.28}$$

and

$$|Ai(z)^{-1}| \le C(1+|z|)^{-1/4}|\operatorname{Im}(z)|^{-1}, \quad z \in \mathcal{D}. \tag{A.4.29}$$

Further estimation of the same type leads to

$$\operatorname{Im} Ai'(z) \ge C(1+|z|)^{3/4} \operatorname{Im} z, \quad z \in \mathcal{D}, \tag{A.4.30}$$

and

$$|Ai'(z)^{-1}| \le C(1+|z|)^{-3/4}|\operatorname{Im}(z)|^{-1}, \quad z \in \mathcal{D}. \tag{A.4.31}$$

The region $\mathcal{D}$ used above is particularly convenient for such estimates but there is in fact no difficulty in extending the same type of argument to a larger region such as:

$$\mathcal{D}^{\#} = \{z \in \mathbb{C}; \operatorname{Re} z \le 0,\ 0 \le \operatorname{Im} z \le C\}. \tag{A.4.32}$$

We leave to the reader the details, and only note that the estimate $\operatorname{Im} z \le C(1+|z|)^{-1/2}$ valid in $\mathcal{D}$ can no longer be used, so one arrives at estimates such as:

$$|\Phi i(z)| \le C\big(|\operatorname{Im}(z)|^{-1} + |z|^{1/2}\big), \quad z \in \mathcal{D}^{\#}. \tag{A.4.33}$$

Finally, we mention estimates of $\Phi i(z)$ and $\Phi i(z)^{-1}$ on

$$\mathfrak{U}^{\#} = \{z \in \mathbb{C} : \operatorname{Im} z \ge B\}, \tag{A.4.34}$$

given $B > 0$, which follow from (A.3.29) for $z \in \mathfrak{U}^{\#} \cap \mathfrak{A}_\delta$ and from (A.4.12) and the analysis of its ingredients, via (A.4.13) and (A.4.6)–(A.4.8), for $z \in \mathfrak{U}^{\#} \setminus \mathfrak{A}_\delta$. We have

$$|\Phi i(z)| \le C|z|^{1/2}, \quad |\Phi i(z)^{-1}| \le C|z|^{-1/2}, \quad z \in \mathfrak{U}^{\#}. \tag{A.4.35}$$

Appendix B: Scattering of waves by a sphere, and harmonic analysis on spheres

In this appendix we will examine a very classical boundary problem for the wave equation for $u(t, x)$,

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u = 0, \tag{B.0.1}$$

namely that (B.0.1) hold for $t \in \mathbb{R}$, $x \in \mathbb{R}^n \backslash \mathbf{B}_1$, $\mathbf{B}_1$ being the unit ball in $\mathbb{R}^n$ centered at the origin, and that the Dirichlet boundary condition

$$u\big|_{\mathbb{R}\times\mathbf{S}^{n-1}} = f \in \mathcal{E}'(\mathbb{R} \times \mathbf{S}^{n-1}). \tag{B.0.2}$$

hold at the boundary. Thus we are considering the problem of scattering of waves by the unit sphere $\mathbf{S}^{n-1}$. We impose the usual outgoing condition

$$u = 0 \quad \text{for} \quad t << 0. \tag{B.0.3}$$

There is a large literature on this problem, including the paper of Watson [Wat2], and much more recent papers, such as Nussensweig [Nus], and the book of Bowman, Senior, and Uslenghi [BoSU], where the reader can find many more references. We will restrict our attention to odd n, the main physical example being scattering of waves in $\mathbb{R}^3$ by the two dimensional sphere.

Here as in these other places, we use separation of variables, writing the Laplacian Δ on $\mathbb{R}^n$ in polar coordinates

$$\Delta v = r^{-2}\left[r^2\frac{\partial^2}{\partial r^2} + (n-1)r\frac{\partial}{\partial r} + \Delta_{\mathbf{S}}\right] v, \tag{B.0.4}$$

where $\Delta_{\mathbf{S}}$ denotes the Laplace operator on the unit sphere $\mathbf{S}^{n-1}$.

We will reduce the boundary problem (B.0.1)–(B.0.3) to a problem in harmonic analysis on the boundary $\mathbb{R} \times \mathbf{S}^{n-1}$, by the traditional use of Bessel functions. We combine this use of classical analysis with a more contemporary approach to such harmonic analysis, largely avoiding the use of special function theory in examining the properties of eigenfunction expansions in terms of spherical harmonics. We will use some basic properties of solutions to hyperbolic equations as a tool in this harmonic analysis. Such an approach to scattering by a sphere has been discussed by Taylor [Tay5], Cheeger and Taylor [CT], and Melrose and Taylor [MeT1].

§B.1: Analysis via Bessel's equation

If we take the partial Fourier tansform with respect to t,

$$v(x, \lambda) = \int_{-\infty}^{\infty} u(t, x)e^{i\lambda t}\, dt, \quad g(x, \lambda) = \int_{-\infty}^{\infty} f(t, x)e^{i\lambda t}\, dt, \tag{B.1.1}$$

the system (B.0.1)–(B.0.2) becomes the reduced wave equation

(B.1.2) $$(\Delta + \lambda^2)v = 0 \text{ for } |x| > 1,$$

(B.1.3) $$v\big|_{\mathbf{S}^{n-1}} = g(x, \lambda).$$

The outgoing condition (B.0.3) could be translated to a condition on $g(\lambda)$ for $\mathrm{Im}\lambda \geq 0$, via the Paley-Wiener theorem. Also it could be characterized by the Sommerfeld radiation condition:

(B.1.4) $$r^{(n-1)/2}\left(\frac{\partial v}{\partial r} - i\lambda v\right) \longrightarrow 0 \text{ as } r \to \infty,$$

Now, by (B.0.4), the reduced wave equation (B.1.2) in polar coordinates is

(B.1.5) $$r^2\frac{\partial^2 v}{\partial r^2} + (n-1)r\frac{\partial v}{\partial r} + (\lambda^2 r^2 + \Delta_{\mathbf{S}})v = 0.$$

This can be transformed into Bessel's equation

(B.1.6) $$y''(t) + \frac{1}{t}y'(t) + \left(1 - \frac{\nu^2}{t^2}\right)y(t) = 0$$

by the change of variable

$$v(r) = r^{\alpha}y(\lambda r), \quad \alpha = -\frac{n-2}{2}.$$

The outgoing condition requires that we use the Hankel function $H_\nu^{(1)}(z)$. We get

(B.1.7) $$v(x, \lambda) = r^{-(n-2)/2}\ \frac{H_\nu^{(1)}(\lambda r)}{H_\nu^{(1)}(\lambda)}g(x, \lambda).$$

Here we have set

(B.1.8) $$\nu = \left(-\Delta_{\mathbf{S}} + \frac{(n-2)^2}{4}\right)^{1/2}.$$

Thus ν is a self adjoint operator on functions on the sphere $\mathbf{S}^{n-1}$; it is an elliptic operator in OPS^1. Thus in (B.1.7) on the right side we have a family of operators, operating on a family of distributions on $\mathbf{S}^{n-1}$, parametrized by λ. If we take the inverse Fourier transform in λ, still using polar coordinates $x = r\omega,\ \omega \in \mathbf{S}^{n-1}$, we have for the solution u of (B.0.1)–(B.0.3),

(B.1.9) $$u(t, r\omega) = r^{-(n-2)/2}\ \frac{H_\nu^{(1)}(rD_t)}{H_\nu^{(1)}(D_t)}f(t, \omega).$$

§B.2: The Neumann operator and Hankel quotients

Before we attack the formula (B.1.9) directly, we will use the representation for the solution to (B.0.1)–(B.0.3) given by Kirchhoff's formula:

$$u(t,x)=\int_{\mathbb{R}\times\mathbf{S}^{n-1}}\Big[u(s,y)\frac{\partial G}{\partial\nu}(t-s,x-y)-\frac{\partial u}{\partial\nu}(s,y)G(t-s,x-y)\Big]ds\,d\mathbf{S}(y), \tag{B.2.1}$$

where $G(t,x)$ is the free space fundamental solution to the wave equation. Recall that, in case $n=3$, we have

$$G(t,x)=(4\pi t)^{-1}\delta(|x|-t). \tag{B.2.2}$$

As emphasized in Chapter 8 for the wave equation with general convex or concave boundary, this formula induces us to study the Neumann operator N, defined on $f\in\mathcal{E}'(\mathbb{R}\times\mathbf{S}^{n-1})$ by

$$Nf=\frac{\partial u}{\partial\nu}\Big|_{\mathbb{R}\times\mathbf{S}^{n-1}}, \tag{B.2.3}$$

where u satisfies (B.0.1)–(B.0.3). Given a good analysis of N, the formula (B.2.1) is an effective tool in providing an analysis of the solution u. Now, in our case here of a spherical boundary, the formula (B.1.9) gives

$$N=D_t\frac{H_\nu^{(1)\prime}(D_t)}{H_\nu^{(1)}(D_t)}-\frac{n-2}{2}. \tag{B.2.4}$$

Thus we are required to study the operator $F(\nu,D_t)$, where

$$F(\mu,\lambda)=\lambda\frac{H_\mu^{(1)\prime}(\lambda)}{H_\mu^{(1)}(\lambda)}-\frac{n-2}{2}. \tag{B.2.5}$$

In particular we want to understand the Hankel quotient

$$Q(\mu,\lambda)=\frac{H_\mu^{(1)\prime}(\lambda)}{H_\mu^{(1)}(\lambda)}. \tag{B.2.6}$$

We begin by listing a few elementary properties of Hankel functions, which can be found in the book of Watson [Wat2]; see also Lebedev [Leb], or Olver [Ol1]. The Hankel function $H_\mu^{(1)}(\lambda)$ is a solution to Bessel's equation

$$y''(\lambda)+\frac{1}{\lambda}y'(\lambda)+\Big(1-\frac{\mu^2}{\lambda^2}\Big)y(\lambda)=0 \tag{B.2.7}$$

and is given λ in the upper half plane, $0 < \arg \lambda < \pi$, by the integral formula

$$H_\mu^{(1)}(\lambda) = \frac{2e^{-\pi i\mu}}{i\sqrt{\pi}\Gamma(\mu+1/2)} \Bigl(\frac{\lambda}{2}\Bigr)^\mu \int_1^\infty e^{i\lambda t}(t^2-1)^{\mu-1/2}\, dt, \tag{B.2.8}$$

if $\operatorname{Re} \mu > -1/2$. In our case μ takes only positive values, but one can analytically continue to arbitrary μ by the identity

$$H_\mu^{(1)}(\lambda) = e^{-\pi i\mu} H_{-\mu}^{(1)}(\lambda). \tag{B.2.9}$$

Since $H_\mu^{(1)}$ solves (B.2.7), it is holomorphic in λ on the logarithmic covering surface $\mathfrak{A}$ of $\mathbb{C} \setminus 0$. For the principal value, it is traditional to cut the complex plane along the negative imaginary axis, so $-\pi/2 < \arg \lambda < 3\pi/2$. An expansion about $\lambda = 0$ is given as follows. Another solution to (B.2.7) is the Bessel function $J_\mu(\lambda)$, with expansion

$$J_\mu(\lambda) = \sum_{k=0}^\infty \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\mu+1)} \Bigl(\frac{\lambda}{2}\Bigr)^{\mu+2k}. \tag{B.2.10}$$

When μ is not an integer, we have the identity

$$H_\mu^{(1)}(\lambda) = \frac{J_{-\mu}(\lambda) - e^{-\pi i\mu} J_\mu(\lambda)}{i\ \sin \pi\mu}. \tag{B.2.11}$$

Thus, for $\mu \notin \mathbb{Z}, H_\mu^{(1)}(\lambda)$ has an expansion about $\lambda = 0$ of the form

$$H_\mu^{(1)}(\lambda) = \lambda^{-\mu} \sum_{k=0}^\infty a_k(\mu)\lambda^{2k} + \lambda^\mu \sum_{k=0}^\infty b_k(\lambda)\lambda^{2k}. \tag{B.2.12}$$

For $m \in \mathbb{Z}$, one can let $\mu \to m$ in (B.2.11) and obtain

$$H_m^{(1)}(\lambda) = \lambda^{-m} \sum_{k=0}^\infty \alpha_k(m)\lambda^{2k} + (\log\,\lambda)\lambda^m \sum_{k=0}^\infty \beta_k(m)\lambda^{2k} + \lambda^m \sum_{k=0}^\infty \gamma_k(m)\lambda^{2k}. \tag{B.2.13}$$

Note that, precisely when $\mu = m + 1/2,\ m \in \mathbb{Z}$, all the exponents in (B.2.12) differ by integers, so, only in that case, we have

$$H_{m+1/2}^{(1)}(\lambda) = \lambda^{-m-1/2} \sum_{k=0}^\infty c_k(m)\lambda^k. \tag{B.2.14}$$

In fact, these Hankel functions have a simpler structure, and with the notation

$$h_m(\lambda) = \Bigl(\frac{\pi\lambda}{2}\Bigr)^{1/2} H_{m+1/2}^{(1)}(\lambda), \tag{B.2.15}$$

one has

$$
\text{(B.2.16)} \qquad \begin{aligned} h_m(\lambda) &= -i(-1)^m \Big(\frac{d}{\lambda d\lambda}\Big)^m \Big(\frac{e^{i\lambda}}{\lambda}\Big) \\ &= (i\lambda)^{-m-1}\ e^{i\lambda} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!(m-k)!} \Big(\frac{i}{2}\Big)^k \lambda^{m-k} \\ &= \lambda^{-m-1} e^{i\lambda}\ p_m(\lambda), \end{aligned}
$$

where $p_m(\lambda)$ is a polynomial of order m in λ. Since all the eigenvalues of the operator $\nu = \big(-\Delta_{\mathbf{S}} + \frac{1}{4}(n-2)^2\big)^{1/2}$ on an even dimensional sphere are half integers, special properties of $H^{(1)}_{m+1/2}(\lambda)$ have special implications for scattering of waves in $\mathbb{R}^n$ by a sphere, when n is odd, as we will see.

To understand the Hankel quotient (B.2.6), we need to understand the zeros of $H^{(1)}_\mu(\lambda)$. For $\mu \in \mathbb{R}, H^{(1)}_\mu(\lambda)$ has no zeros in the upper half plane $0 \le \arg \lambda \le \pi$. In fact, by virtue of the identity

$$
\text{(B.2.17)} \qquad H^{(1)}_\mu(\lambda) = \frac{2}{\pi i} e^{-\pi i \mu/2} K_\mu(-i\lambda),
$$

this fact follows from a result we proved in Appendix A, Proposition A.2.4. It is also known (see Watson) that, on the region

$$
\text{(B.2.18)} \qquad \mathfrak{A}_o = \Big\{\lambda \in \mathbb{C} \setminus 0 : -\frac{\pi}{2} \le \arg \lambda < \frac{3\pi}{2}\Big\},
$$

the plane slit along the negative imaginary axis, for $\mu > 0$, the number of zeros of $H^{(1)}_\mu(\lambda)$ is the even integer closest to $\mu - 1/2$, unless $\mu = m + 1/2$, when the number of zeros (by formulas (B.2.15) and (B.2.16)) is m. To reiterate, all these zeros in $\mathfrak{A}_o$ of $H^{(1)}_\mu(\lambda)$ lie in the lower half plane.

Now the Hankel quotient (B.2.6) is seen to be meromorphic on $(\mu, \lambda) \in \mathbb{C} \times \mathfrak{A}$, and its restriction to $\mathbb{R} \times \mathfrak{A}_o$ is holomorphic for λ in the upper half plane, $0 \le \arg \lambda \le \pi$. $Q(\mu, \lambda)$ has poles where $H^{(1)}_\mu(\lambda) = 0$, all simple since $(\partial/\partial\lambda) H^{(1)}_\mu(\lambda) \ne 0$ at a zero. For $\mu = m + 1/2$, the restriction of $Q(m + 1/2, \lambda)$ to $\lambda \in \mathfrak{A}_o$ actually extends to a meromorphic function on $\lambda \in \mathbb{C}$, and (B.2.15), (B.2.16) give

$$
\text{(B.2.19)} \qquad Q\Big(m + \frac{1}{2}, \lambda\Big) = -\Big(m + \frac{1}{2}\Big)\lambda^{-1} + i + \frac{p_m'(\lambda)}{p_m(\lambda)},
$$

where $p_m(\lambda)$ is the polynomial of degree m defined by (B.2.16). Thus

$$
\text{(B.2.20)} \qquad \lambda Q\Big(m + \frac{1}{2}, \lambda\Big) = i\lambda - \Big(m + \frac{1}{2}\Big) + \lambda\, \frac{p_m'(\lambda)}{p_m(\lambda)}
$$

is holomorphic in the upper half plane $\mathrm{Im}\lambda \ge 0$ and also in a horizontal strip about the real axis. As we see from (B.2.12) and (B.2.13), if $\mu \ne m + 1/2$ for any integer

$m, Q(\mu, \lambda)$ cannot be continued as a single valued meromorphic function across the branch cut $\{-ir : r \geq 0\}$, not even in any neighborhood of $\lambda = 0$.

If $\zeta_{ml}, \dots \zeta_{mm}$ denote the zeros of $H^{(1)}_{m+1/2}(\lambda)$ in $\mathfrak{A}_o$, which are hence the poles of $\lambda Q(m + 1/2, \lambda)$, we have

$$\lambda Q\Big(m + \frac{1}{2}, \lambda\Big) = i\lambda - \Big(m + \frac{1}{2}\Big) + \lambda \sum_{j=1}^{m} (\lambda - \zeta_{mj})^{-1}. \tag{B.2.21}$$

Now it is known that, not only is $\operatorname{Im} \zeta_{mj}$ always negative, but there is a constant $C > 0$, independent of m, such that

$$\operatorname{Im} \zeta_{mj} \leq -2C(m + 1)^{1/3}, \tag{B.2.22}$$

while one also has

$$C_1 m \leq |\zeta_{mj}| \leq C_2 m, \quad m \geq 1. \tag{B.2.23}$$

Indeed, the asymptotic behavior of ζ_{mj} for large m has been evaluated. See Abramowitz and Stegun [AbSt], p. 441, and references given there.

The fact that $\lambda Q(m + 1/2, \lambda)$, given by (B.2.21), is holomorphic in the strip $|\mathrm{Im}\lambda| \leq C$ implies the familiar and important principle that there is local exponential decay of solutions to (B.0.1)–(B.0.3) for odd n. In fact, we claim that, given a delta function δ_{o,ω_o} on $\mathbb{R} \times \mathbf{S}^{n-1}, n - 1$ even,

$$N\delta_{o,\omega_o}(t, x) = D_t Q(\nu, D_t)\delta_{o,\omega_o} - \frac{n-2}{2}\delta_{o,\omega_o} \tag{B.2.24}$$

is exponentially decreasing as $t \longrightarrow \infty$, together with all derivatives. The partial Fourier transform with respect to t is given by

$$\int_{-\infty}^{\infty} e^{i\lambda t} N\delta_{\omega_o}(t, x)dt = \lambda Q(\nu, \lambda)\delta_{\omega_o} - \frac{n-2}{2}\delta_{\omega_o}.$$

On $\mathbf{S}^{n-1}$ for $n-1$ even, all the eigenvalues of ν are half integers (we will include a demonstration of this fact near the end of the appendix), so the results (B.2.19), (B.2.20) on $\lambda Q(m + 1/2)$ apply. To get such asserted decay of (B.2.24) , we need to estimate the derivatives $D^j_\lambda \lambda Q(m + 1/2, \lambda)$ for $|\mathrm{Im}\lambda| \leq B$. By (B.2.21)–(B.2.23), we have, for $j \geq 2$,

$$\begin{aligned} \Big|D^j_\lambda\ \lambda Q\Big(m + \frac{1}{2}, \lambda\Big)\Big| &\leq C_j\big(1 + |\lambda|\big) \sum_{j=1}^{m} \Big(|\lambda - \operatorname{Re} \zeta_{mj}| + (m + 1)^{1/3}\Big)^{-j} \\ &\leq C'_j m\big(1 + |\lambda|\big)\big(|\lambda| + m + 1\big)^{-j/3}, \quad (???) \end{aligned} \tag{B.2.25}$$

for $|\mathrm{Im}\lambda| \leq B$. From this inequality, the asserted exponential decay of (B.2.24) and all its derivatives is a simple consequence. Local exponential decay of solutions to (B.0.1)–(B.0.3) for n odd follows from this fact and the Kirchhoff formula (B.2.1). It also follows from (B.2.25) that $N\delta_{o,\omega_o}(t, x)$ is C^∞ for $t \neq 0$. Later we will see that it is C^∞ except at $t = 0, x = \omega_o$, and in fact N will be seen to be a pseudodifferential operator. Of course, this property of N is a special case of the general results proved in Section 8.

§B.3: Microlocal study of the Neumann operator

We can use the exponential decay as $t \to \infty$, established above, to transfer our analysis from the operator N on $\mathbb{R} \times \mathbf{S}^{n-1}$ to such an operator on the compact manifold $\mathbf{S}^1 \times \mathbf{S}^{n-1}$, as follows. Suppose, in (B.0.1)–(B.0.3), f has support on $|t| < T$. Then, by local exponential decay, the sum

$$\sum_{j=-\infty}^{\infty} u(t + 2jT, x) \tag{B.3.1}$$

is convergent, and its normal derivative, on $\mathbb{R} \times \mathbf{S}^{n-1}$, is periodic of period $2T$, so it can be considered as a distribution on $\mathbf{S}^1 \times \mathbf{S}^{n-1}$, with $\mathbf{S}^1 = \mathbb{R}/2T\mathbb{Z}$. Thus we have defined the operator

$$N : \mathcal{D}'(\mathbf{S}^1 \times \mathbf{S}^{n-1}) \longrightarrow \mathcal{D}'(\mathbf{S}^1 \times \mathbf{S}^{n-1}) \tag{B.3.2}$$

with the formula (B.2.4), (B.2.5), where now $D_t : \mathcal{D}'(\mathbf{S}^1) \to \mathcal{D}'(\mathbf{S}^1)$.

We next intend to reduce the analysis of the Neumann operator N to the study of the Hankel quotient $Q(\mu, \lambda)$ on some conic neighborhood of $|\lambda| = |\mu|$. In order to accomplish this, we will use an elementary property of the operator N, applied to $f \in \mathcal{E}'(\mathbb{R} \times \mathbf{S}^{n-1})$. Namely, if $WF(f)$ is disjoint from the subset of $T^*(\mathbb{R} \times \mathbf{S}^{n-1})$ where $|\tau| = |\xi|$, the set over which grazing rays for (B.0.1) pass, then the solution u to (B.0.1)–(B.0.3) is given by the ordinary constructions of geometrical optics for that part of f with wave front set in $|\tau| > |\xi|$ and by a Poisson integral for that part of f with wave front set in $|\tau| < |\xi|$. Thus Nf is easily analyzed as a classical pseudodifferential operator in OPS^1 applied to f, for $WF(f)$ disjoint from $|\tau| = |\xi|$.

Consequently it remains only to analyze NF when $WF(f)$ is contained in a small conic neighborhood of $|\tau| = |\xi|$. We claim that, for $f \in \mathcal{D}'(\mathbf{S}^1 \times \mathbf{S}^{n-1})$ with $WF(f)$ contained in a small conic neighborhood of $|\tau| = |\xi|$, the spectrum of f with respect to D_t and $\nu = (-\Delta_{\mathbf{S}} + \frac{1}{4}(n-2)^2)^{1/2}$ is concentrated near $|\mu| = |\lambda|$, in the sense that, if $V_{\mu,\lambda}$ denotes the joint eigenspace for D_t, ν with eigenvalues λ, μ, respectively, and P is the orthogonal projection onto any direct sum of $V_{\mu,\lambda}$ with (μ, λ) outside a conic neighborhood of $|\mu| = |\lambda|$, then $Pf \in C^\infty(\mathbf{S}^1 \times \mathbf{S}^{n-1})$. Actually, we will take a function $p(\mu, \lambda)$ which belongs to $S^0(\mathbb{R}^2)$, is even in μ, and vanishes on some conic neighborhood of $|\mu| = |\lambda|$, and let

$$P_o = p(\nu, D_t). \tag{B.3.3}$$

We claim that P_o is a pseudodifferential operator on $\mathbf{S}^1 \times \mathbf{S}^{n-1}$ with symbol of order $-\infty$ on a conic neighborhood of $|\tau| = |\xi|$. It follows from this claim that

$$P_o f \in C^\infty(\mathbf{S}^1 \times \mathbf{S}^{n-1}),$$

if $WF(f)$ is contained in a sufficiently small conic neighborhood of $|\tau| = |x|$. This of course implies the spectrum of f is concentrated near $|\mu| = |\lambda|$ in the sense given above.

We thus are motivated to make a brief study of operators of the form (B.3.3) under the hypotheses

$$p(\mu, \lambda) = p(-\mu, \lambda), \tag{B.3.4}$$

$$p(\mu, \lambda) \in S^m_{\rho,0}(\mathbb{R}^2), \quad \rho > 0. \tag{B.3.5}$$

The analysis we give here extends to a larger class of elliptic self adjoint operators on compact manifolds. See Chapter 12 of Taylor [Tay7] for more on this, and implications for the study of harmonic analysis on compact manifolds. Note that, for $f \in \mathcal{D}'(\mathbf{S}^1 \times \mathbf{S}^{n-1})$,

$$p(\nu, D_t)f = \int \hat{p}(\sigma, \tau) e^{i\sigma\nu + i\tau D_t} f \, d\sigma \, d\tau, \tag{B.3.6}$$

by the Fourier inversion formula applied to the spectral representation of the commuting self adjoint operators ν and D_t. By (B.3.4), $\hat{p}(\sigma, \tau) = \hat{p}(-\sigma, \tau)$, so, using $e^{i\sigma\nu + i\tau D_t} = e^{i\sigma\nu} e^{i\tau D_t}$, we can rewrite (B.3.6) as

$$p(\nu, D_t)f = \int \hat{p}(\sigma, \tau) \cos \sigma\nu \; e^{i\tau D_t} f \; d\sigma \; d\tau. \tag{B.3.7}$$

As is well known, if $p(\nu, \lambda)$ satisfies (B.3.5), then $\hat{p}(\sigma, \tau)$ is C^∞ away from $\sigma = \tau = 0$, and all its derivatives are rapidly decreasing as $|\sigma| + |\tau|$ goes to ∞. Thus, for any $\varepsilon > 0$, we can write

$$\hat{p}(\sigma, \tau) = \hat{p}_1(\sigma, \tau) + \hat{p}_2(\sigma, \tau) \tag{B.3.8}$$

with

$$\operatorname{supp} \hat{p}_1 \subset \{|\sigma|^2 + |\tau|^2 < \varepsilon^2\}, \quad \hat{p}_2 \in \mathcal{S}(\mathbb{R}^2). \tag{B.3.9}$$

Now

(B.3.10)
$$p_2(\nu, D_t) = (-1)^k \big(1 + \nu^2 + D_t^2\big)^{-k} \int \big(1 + D_\sigma^2 + D_\tau^2\big)^k \hat{p}_2(\sigma, \tau) e^{i\sigma\nu + i\tau D_t} \, d\sigma \, d\tau,$$

so $p_2(\nu, D_t)$ is a smoothing operator:

$$p_2(\nu, D_t) : \mathcal{D}'(\mathbf{S}^1 \times \mathbf{S}^{n-1}) \longrightarrow C^\infty(\mathbf{S}^1 \times \mathbf{S}^{n-1}). \tag{B.3.11}$$

Meanwhile, since $\cos\sigma\nu$ and $e^{i\tau D_t}$ both enjoy finite propagation speed, we have

$$\text{(B.3.12)} \quad \text{supp}\,(\cos\sigma\nu\ e^{i\tau D_t} f) \subset \{p \in \mathbf{S}^1 \times \mathbf{S}^{n-1} : \text{dist}\,(p, \text{supp}\ f) \leq |\sigma| + |\tau|\}.$$

It follows from (B.3.11) and (B.3.12) that

$$\text{(B.3.13)} \qquad \text{sing supp}\ p(\nu, D_t)f \subset \text{ sing supp}\ f.$$

The assumption (B.3.4) that $p(\mu, \lambda)$ is even in μ is necessary to derive (B.3.13). Indeed, the operator ν itself does not satisfy this property; $\nu\delta_{o,\omega_o}(t, x)$ is singular on the entire surface $t = 0$. Note that ν is not a pseudodifferential operator on the product manifold $\mathbf{S}^1 \times \mathbf{S}^{n-1}$; its symbol is singular on $|\xi| = 0$. Of course, $\nu^2 = -\Delta_{\mathbf{S}} + (n-2)^2/4$ is a differential operator, with regular symbol.

We now show that, if $p(\mu, \lambda)$ satisfies (B.3.4), (B.3.5) with $1/2 < \rho \leq 1$, then we have $p(\nu, D_t) \in OPS^m_{\rho,1-\rho}$. We employ the observation that

$$\text{(B.3.14)} \qquad p(\mu, \lambda) = p^\#\big(\sqrt{\mu^2 + \lambda^2}, \lambda\big),$$

with

$$\text{(B.3.15)} \qquad p^\#(\eta, \lambda) \in S^m_{\rho,0}(\mathbb{R}^2 \setminus 0).$$

Thus,

$$\text{(B.3.16)} \qquad p(\nu, D_t) = p^\#(A, D_t)$$

with

$$\text{(B.3.17)} \qquad A = \big(\nu^2 + D_t^2\big)^{1/2}.$$

The advantage of (B.3.17) is that A, being the square root of an elliptic differential operator, is an elliptic pseudodifferential operator on $\mathbf{S}^1 \times \mathbf{S}^{n-1}$, belonging to OPS^1. Now we have

$$\text{(B.3.18)} \qquad p^\#(A, D_t)f = \int \hat{p}^\#(\sigma, \tau) e^{i\sigma A + i\tau D_t} f\, d\sigma\, d\tau.$$

If we write $\hat{p}^\# = \hat{p}^\#_1 + \hat{p}^\#_2$ as before, we get $p^\#(A, D_t)$ as a sum of a smoothing operator and an operator which moves around the wave front set of f an arbitrarily small amount. Hence

$$\text{(B.3.19)} \qquad WF\ p^\#(A, D_t)f \subset WF\ f,$$

which refines (B.3.13). If $\rho > 1/2$, we can analyze (B.3.18) as a pseudodifferential operator, as follows. For $|\sigma|$ and $|\tau|$ small enough, in local coordinates on $\mathbf{S}^1 \times \mathbf{S}^{n-1}$, we can write

$$\text{(B.3.20)} \qquad e^{i\sigma A + i\tau D_t} f(t, x) = \int a(\sigma, \tau, t, x, \lambda, \xi) e^{i\phi} \hat{f}(\lambda, \xi)\, d\lambda\, d\xi$$

where a is a symbol in $S^m_{\rho,1-\rho}$, with phase variables λ, ξ, and $\phi(\sigma,\tau,t,x,\lambda,\xi)$ is homogeneous of degree 1 in (λ,ξ), determined by the usual methods of geometrical optics. Applying (B.3.18) gives

$$\begin{aligned} p^{\#}(A,D_t)f &= \int p^{\#}(D_\sigma,D_\tau)(ae^{i\phi})|_{\sigma=\tau=0}\hat{f}(\lambda,\xi)\,d\lambda\,d\xi \\ &= \int b(t,x,\lambda,\xi)e^{i(t\lambda+x\cdot\xi)}\hat{f}(\lambda,\xi)\,d\lambda\,d\xi, \end{aligned} \tag{B.3.21}$$

with $b(t,x,\lambda,\xi) \in S^m_{\rho,1-\rho}$, determined by the fundamental asymptotic expansion lemma for pseudodifferential operators. The principal symbol of b is $p^{\#}\big(a_1(t,x,\lambda,\xi),\lambda\big)$, where a_1 is the principal symbol of A. Note also that, if $p^{\#}(\eta,\lambda)$ has order $-\infty$ on a conic neighborhood of $\eta = \alpha\lambda$ (α a constant) then $p^{\#}(A,D_t)$ has order $-\infty$ on a conic neighborhood of the set of (t,x,λ,ξ) where $a_1(t,x,\lambda,\xi) = \alpha\lambda$. This is particular establishes our original claim about (B.3.3). In particular, if WFf is contained in a small conic neighborhood of the grazing set $|\tau| = |\xi|$, and if $R(\mu,\lambda) \in S^0_{1,0}$ is supported in a conic neighborhood of $|\mu| = |\lambda|$, but equal to 1 on an appropriate conic neighborhood of $|\mu| = |\lambda|$, then

$$Nf = \Big[D_t Q(\nu,D_t) - \frac{1}{2}(n-2)\Big]R(\nu,D_t)f \mod C^\infty. \tag{B.3.22}$$

This formula shows that we need only analyze the behavior of the Hankel quotient

$$Q(\mu,\lambda) = \frac{H^{(1)\prime}_\mu(\lambda)}{H^{(1)}_\mu(\lambda)},$$

for (μ,λ) in a conic neighborhood of $|\mu| = |\lambda|$. Note that

$$Q(-\mu,\lambda) = Q(\mu,\lambda), \tag{B.3.23}$$

and

$$Q(\mu,-\lambda) = \overline{Q(\mu,\bar{\lambda})}. \tag{B.3.24}$$

Thus it suffices to analyze $Q(\mu,\lambda)$ on a conic neighborhood in $\mathbb{R}^2$ of the ray $\mu = \lambda > 0$. This analysis proceeds from the following uniform asymptotic expansion of the Hankel function $H^{(1)}_\mu(\lambda)$ and its derivative, valid for $|\mu-\lambda| \le C_0|\lambda|$. In fact, for z in a neighborhood of 1, we have

$$\begin{aligned} H^{(1)}_\mu(\mu z) \sim 2e^{-\pi i/3}\Big(\frac{4\zeta}{1-z^2}\Big)^{1/4}\Big\{&A_+(\mu^{2/3}\zeta)\mu^{1/3}\sum_{k=0}^{\infty} a_k(\zeta)\mu^{-2k} \\ &+ A_+'(\mu^{2/3}\zeta)\mu^{5/3}\sum_{k=0}^{\infty} b_k(\zeta)\mu^{-2k}\Big\}, \end{aligned} \tag{B.3.25}$$

$$H_\mu^{(1)\prime}(\mu) \sim \frac{4}{z} e^{2\pi i/3} \Bigl(\frac{1-z^2}{4\zeta}\Bigr)^{1/4} \Bigl\{ A_+'(\mu^{2/3}\zeta)\mu^{-2/3} \sum_{k=0}^{\infty} d_k(\zeta)\mu^{-2k} + A_+(\mu^{2/3}\zeta)\mu^{4/3} \sum_{k=0}^{\infty} c_k(\zeta)\mu^{-2k} \Bigr\}. \tag{B.3.26}$$

Here $A_+(s)$ is the Airy function used in the main text. The function ζ is defined by

$$\frac{2}{3}\zeta^{3/2} = \int_z^1 \sqrt{1-t^2}\ \frac{dt}{t} = \log\big[(1+\sqrt{1-z^2})/z\big] - \sqrt{1-z^2}. \tag{B.3.27}$$

We remark that ζ is analytic in z, even at $z=1$, and $d\zeta/dz < 0$ there; also, at $z=1$, $\zeta = 0$ and $(1-z^2)^{-1}\zeta = 2^{-\frac{2}{3}}$. The uniform expansions (B.3.25), (B.3.26), and related expansions for $J_\mu(\mu z)$ and other Bessel functions, are among the deepest and most important results in the theory of Bessel functions. These results unfortunately do not appear in the treatise of Watson, having been established after Watson's second edition was published. See Olver [Ol2], and references given there, and also Abramowitz and Stegun [AbSt]. From (B.3.25) and (B.3.26) we deduce that, for z in a neighborhood of 1,

$$Q(\mu,\mu z) \sim \alpha(z)\mu^{1/3}\Phi_+(\mu^{2/3}\zeta)\ \frac{d(z,\mu) + \mu^{-2/3}\Phi_+(\mu^{2/3}\zeta)^{-1}c(z,\mu)}{a(z,\mu) + \mu^{-4/3}\Phi_+(\mu^{2/3}\zeta)b(z,\mu)}, \tag{B.3.28}$$

where, as in the text,

$$\Phi_+(s) = \frac{A_+'}{A_+}(s).$$

The amplitudes $a(z,\mu),\dots,d(z,\mu)$ belong to $S^0_{1,0}(\mathbb{R}\times\mathbb{R})$, with a and d elliptic. In particular, (B.3.28) shows that

$$\lambda Q(\mu,\lambda) \in S^1_{1/3,0} \text{ on a conic neighborhood of } \lambda = \mu. \tag{B.3.29}$$

In conjunction with (B.3.22) and the discussion of the classical behavior of the Neumann operator on distributions with wave front set disjoint from the grazing set, this implies that N is microlocal : $WF(Nf) \subset WF(f)$. We can now show directly that, microlocally near the grazing set $|\xi| = |\tau|, N$ is the sort of Airy operator discussed in Chapter 8. One way is to note that there is an elliptic Fourier integral operator J_1 which microlocally conjugates ν to $D_1 = D_{x_1}$ and D_t to $D_2 = D_{x_2}$, and then, microlocally,

$$N = J_1\Bigl[D_2 Q(D_1,D_2) - \frac{1}{2}(n-2)\Bigr]J_1^{-1}. \tag{B.3.30}$$

By (B.3.28), microlocally $D_2Q(D_1,D_2)\in OPS^1_{1/3,0}$ with asymptotic expansion in terms of the Airy quotient. We can make N look more like the Airy operators of Chapter 8 by choosing an elliptic Fourier integral operator J_2 which microlocally conjugates ν to D_1 and conjugates $\zeta(\nu^{-1}D_t)$ to $D_1^{-1}D_2$. Then, microlocally, by (B.3.28), we have

$$N\sim J_2\big[A\Phi_+(D_1^{-1/3}D_2)+\cdots\big]J_2^{-1}, \tag{B.3.31}$$

a form like that given in Chapter 8 for the Neumann operator in general, in the diffractive case.

§B.4: The grazing ray parametrix

We can also transform the solution operator (B.1.9) for the boundary problem (B.0.1)–(B.0.3) to the sort of Fourier-Airy integral operator which was constructed as a parametrix in the general diffractive case, as follows. Applying the uniform expansion (B.3.25) with $\mu=\nu,\mu z=rD_t$, setting $\sigma(z)=2e^{-\pi i/3}\big[\zeta(z)/(1-z^2)\big]^{1/4}$, and

$$a(z,\mu)\sim\sigma(z)\sum_{k\geq 0}a_k(\zeta)\mu^{-2k},\quad b(z,\mu)\sim\sigma(z)\sum_{k\geq 0}b_k(\zeta)\mu^{-2k},$$

we see that if $WF(f)$ is contained in a small conic neighborhood of the grazing set, then, on a neighborhood of the boundary we have, mod C^∞,

$$\begin{aligned}u=\Big[&a(r\nu^{-1}D_t,\nu)A_+\big(\nu^{2/3}\zeta(r\nu^{-1}D_t)\big)\\&+b(r\nu^{-1}D_t,\nu)A_+'\big(\nu^{2/3}\zeta(r\nu^{-1}D_t)\big)\Big]A_+\big(\nu^{2/3}\zeta(\nu^{-1}D_t)\big)^{-1}F_1,\end{aligned} \tag{B.4.1}$$

with

$$F_1=A_+\big(\nu^{2/3}\zeta(\nu^{-1}D_t)\big)H_\nu^{(1)}(D_t)^{-1}f=Lf.$$

Applying the expansion (B.3.25) again, with $r=1$, allows us to analyze L as an Airy operator. Indeed, we have, mod $OPS^{-\infty}$,

$$L^{-1}=a(\nu^{-1}D_t,\nu)+b(\nu^{-1}D_t,\nu)\Phi_+\big(\nu^{2/3}\zeta(\nu^{-1}D_t)\big), \tag{B.4.2}$$

and, microlocally near the grazing set,

$$A=a(\nu^{-1}D_t,\nu)\in OPS^{-1/3},\ \text{elliptic}\ ;B=b(\nu^{-1}D_t,\nu)\in OPS^{-5/3}. \tag{B.4.3}$$

Thus

$$L^{-1}=A+BJ_2\Phi_+(D_1^{-1/3}D_2)J_2^{-1}, \tag{B.4.4}$$

or

$$J_2^{-1}L^{-1}J_2=\tilde{A}+\tilde{B}\Phi_+,\ \text{elliptic in}\ OPS^{-1/3}_{1/3,0}. \tag{B.4.5}$$

Consequently L is an Airy operator such as studied in Chapter 9. Meanwhile the operator in (B.4.1) can be rewritten as follows. If J_2 conjugates ν to D_1 and $\zeta(\nu^{-1}D_t)$ to $D_1^{-1}D_2$, say it conjugates $\zeta(r\nu^{-1}D_t)$ to $\rho(r,D)$, and $\nu^{-1}D_t$ to $\beta(D)$. Then (B.4.1) becomes

$$\begin{aligned} u = J_2\Big[&a(r\beta(D),D_1)A_+\big(D_1^{2/3}\rho(r,D)\big) \\ &+ b(r\beta(D),D_1)A_+'\big(D_1^{2/3}\rho(r,D)\big)\Big]A_+(D_1^{-1/3}D_2)^{-1}J_2^{-1}F_1, \end{aligned} \tag{B.4.6}$$

and with

$$F = J_2^{-1}F_1 = J_2^{-1}Lf, \tag{B.4.7}$$

if

$$J_2 g(x) = \int c(x,\xi)e^{i\phi(x,\xi)}\hat{g}(\xi)\,d\xi, \quad x \in \mathbf{S}^{n-1}, \tag{B.4.8}$$

we have
(B.4.9)

$$\begin{aligned} u = \int e^{i\phi(x,\xi)}c(x,\xi)\Big[&a\big(r\beta(\xi),\xi_1\big)A_+\big(\xi_1^{2/3}\rho(r,\xi)\big) \\ &+ b\big(r\beta(\xi),\xi_1\big)A_+'\big(\xi_1^{2/3}\rho(r,\xi)\big)\Big]A_+(\xi_1^{-1/3}\xi_2)^{-1}\hat{F}(\xi)\,d\xi, \end{aligned}$$

which is the sort of Fourier-Airy integral operator analyzed in Chapter 5.

§B.5: The reduced Neumann operator

Going back to the partial Fourier transform with respect to t, the normal derivative at $\mathbf{S}^{n-1}$ of the outgoing solution to the reduced wave equation

$$(\Delta+\lambda^2)v = 0 \ \text{ for } |x|>1, \quad v\big|_{\mathbf{S}^{n-1}} = g(x),$$

defines what we might call the reduced Neumann operator:

$$N(\lambda) : \mathcal{D}'(\mathbf{S}^{n-1}) \longrightarrow \mathcal{D}'(\mathbf{S}^{n-1}), \quad N(\lambda)g = \frac{\partial u}{\partial \nu}, \tag{B.5.1}$$

and, by (B.1.7), we have

$$N(\lambda) = \lambda Q(\nu,\lambda) - \frac{n-2}{2} = \lambda \frac{H_\nu^{(1)\prime}(\lambda)}{H_\nu^{(1)}(\lambda)} - \frac{n-2}{2}. \tag{B.5.2}$$

To study families of functions of the operator ν on $\mathcal{D}'(\mathbf{S}^{n-1})$, we can make use of the following analogue of (B.3.6):

$$p_\lambda(\nu)f = \int_{-\infty}^{\infty} \hat{p}_\lambda(\sigma)e^{i\sigma\nu}f\,d\sigma. \tag{B.5.3}$$

For many classes of functions $p_\lambda(\mu)$, this can be analyzed qualitatively by replacing the solution operator $e^{i\sigma\nu}$ to the wave equation on $\mathbf{S}^{n-1}$ by its geometrical optics approximation, parallel to (B.3.20). However, it is also useful to know that a simple closed form of the solution operator $e^{i\sigma\nu}$ can be obtained. We turn to a construction of such a formula, which was used in §2 of Cheeger and Taylor [CT].

§B.6: Harmonic analysis on spheres

Using the representation (B.0.4) for the Laplace operator in polar coordinates, one can write the kernel $K(x_1, x_2)$ for the operator Δ^{-1} on $\mathbb{R}^n$ in polar coordinates as

$$(r_1 r_2)^{-(n-2)/2}(2\nu)^{-1}(r_1/r_2)^{\nu}, \quad \text{if } r_1 < r_2, \tag{B.6.1}$$

where we identify operators on $\mathcal{D}'(\mathbf{S}^{n-1})$ with distributions in $\mathcal{D}'(\mathbf{S}^{n-1} \times \mathbf{S}^{n-1})$. On the other hand, we also have the explicit formula

$$\begin{aligned} &[(n-2)V_{n-1}]^{-1}\|x_1 - x_2\|^{2-n} \\ &= [(n-2)V_{n-1}]^{-1}\big(r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta\big)^{-(n-2)/2} \end{aligned} \tag{B.6.2}$$

for this kernel, using the notation $x_j = r_j \omega_j$ and θ denoting the spherical geodesic distance between ω_1 and ω_2, so $\cos\theta = \omega_1 \cdot \omega_2$. V_{n-1} is the volume of $\mathbf{S}^{n-1}$. Thus these two expressions must be equal, for $r_1 < r_2$. Setting $r_1 = e^{-t} r_2$, we get the formula

$$\nu^{-1} e^{-t\nu} = 2[(n-2)V_{n-1}]^{-1}(2\cosh t - 2\cos\theta)^{-(n-2)/2}, \quad t > 0, \tag{B.6.3}$$

for the kernel $k_t(\omega_1, \omega_2)$ of $\nu^{-1} e^{-t\nu}$, $t > 0$.

Differentiating with respect to t gives the formula

$$e^{-t\nu} = V_{n-1}^{-1}(2\sinh t)(2\cosh t - 2\cos\theta)^{-n/2}, \tag{B.6.4}$$

for the Poisson kernel. Now we can analytically continue these identities to $Re\, t > 0$, and pass to the limit as $Re\ t \downarrow 0$, obtaining

(B.6.5)
$$\nu^{-1} e^{it\nu} = \lim_{\varepsilon \searrow 0}\ 2[(n-2)V_{n-1}]^{-1}\big(2\cosh\varepsilon\ \cos t + 2i\sinh\varepsilon\ \sin t - \cos\theta\big)^{-(n-2)/2},$$

and

$$e^{it\nu} = \lim_{\varepsilon \searrow 0}\ V_{n-1}^{-1} 2i\ \sin t (2\cosh\varepsilon\ \cos t + 2i\sinh\varepsilon\ \sin t - \cos\theta)^{-n/2}. \tag{B.6.6}$$

Taking real and imaginary parts yields formulas for $\nu^{-1}\cos t\nu$, $\nu^{-1}\sin t\nu$, and also for $\cos t\nu$, $\sin t\nu$. From these formulas we can read off basic qualitative properties of the wave kernel on $\mathbf{S}^{n-1}$, such as the fact that if $n-1$ is odd the strict Huygens principle holds: $\nu^{-1}\sin t\nu$ and $\cos t\nu$ are supported on $|\theta| = |t|$. Note that (B.6.5) and (B.6.6) are periodic in t, of period 2π if $n-1$ is odd and of period 4π if $n-1$ is even, reflecting the well known fact that all the eigenvalues of ν are integers if $n-1$ is odd and half integers if $n-1$ is even, a fact that can also be seen by noting that (B.6.1) must be smooth at $x_1 = 0$, if $r_2 > 0$.

We also note that the formula (B.6.4) is related to the solution operator for the Dirichlet problem on the unit ball

$$\Delta v = 0, \quad v = f \ \text{ for } |x| = 1. \tag{B.6.7}$$

Indeed, with $x = r\omega, \omega \in \mathbf{S}^{n-1}$, we have, for $n \geq 2$,

$$v(x) = r^{-(n-2)/2+\nu} \, f(\omega), \quad r \leq 1, \tag{B.6.8}$$

so if $n \geq 3$, (B.6.4) is equivalent to the Poisson kernel for the solution to (B.6.7).

We finally remark that one can use (B.6.6) to analyze

$$E_k = T^{-1} \int_0^T e^{-ikT} e^{it\nu} \, dt, \tag{B.6.9}$$

with $T = 2\pi$ or $4\pi, k$ an integer or half-integer, depending on whether $n-1$ is odd or even. One thus obtains formulas for the orthogonal projections onto the various eigenspaces of ν (hence of $\Delta_{\mathbf{S}}$). These formulas then lead one back to the classical theory of spherical harmonics. For example, on $\mathbf{S}^2$, formula (B.6.6) applied to (B.6.9) gives

$$E_{k+1/2} = \frac{k+1/2}{2\pi^2} \int_{-\infty}^{\infty} (2\cos t - 2\cos\theta)^{-1/2} e^{i(k+1/2)t} \, dt, \tag{B.6.10}$$

which is equivalent to Mehler's formula

$$P_k(\cos\theta) = \frac{1}{\pi} \int_{-\theta}^{\theta} (2\cos t - 2\cos\theta)^{-1/2} e^{i(k+1/2)t} \, dt \tag{B.6.11}$$

for the Legendre polynomials. In this case, the formula (B.6.3) is equivalent to the generating function identity

$$\big(1 + r^2 - 2r\cos\theta\big)^{-1/2} = \sum_{k=0}^{\infty} r^k P_k(\cos\theta), \quad r < 1. \tag{B.6.12}$$

For more on this approach to harmonic analysis on spheres, see Chapter 4 of the monograph [Tay11], or Chapter 8 of [Tay13].

C: Wave front sets on bounded regions

Here we present a rough and ready definition of the wave front set of a distribution u on a bounded region $\overline{\Omega}$, as a subset of

$$ {}^bT^*\overline{\Omega} \setminus 0 = (T^*\Omega \setminus 0) \cup (T^*\partial\Omega \setminus 0), \tag{C.0.1} $$

where Ω is the interior of $\overline{\Omega}$ and $\partial\Omega$ its boundary. The 'wave front set' of u will be denoted $WF_b\, u$:

$$ WF_b\, u \subset {}^bT^*\overline{\Omega} \setminus 0. \tag{C.0.2} $$

We define this object not for all $u \in \mathcal{C}^{-\infty}(\Omega)$, but for a subspace, consisting of

$$ u \in \mathcal{C}^\infty(I, \mathcal{C}^{-\infty}(\partial\Omega)), \tag{C.0.3} $$

where, locally near $\partial\Omega$, we write $\overline{\Omega}$ as a product $I \times \partial\Omega,\ I = [0, 1)$. First we define

$$ (WF_b\, u) \cap (T^*\Omega \setminus 0) = WF\, u, \tag{C.0.4} $$

the usual wave front set of u on Ω. It remains to define $(WF_b\, u) \cap (T^*\partial\Omega \setminus 0)$. We do the following. Pick $(y_0, \eta_0) \in T^*\partial\Omega \setminus 0$. Then we say

$$ (y_0, \eta_0) \notin WF_b\, u \tag{C.0.5} $$

provided there exists $A = A(y, D_y) \in OPS^0(\partial\Omega)$ that is elliptic at (y_0, η_0), such that

$$ A(y, D_y)u \in \mathcal{C}^\infty(\overline{\Omega}). \tag{C.0.5} $$

This definition appears to depend on the chioce of local splitting $I \times \partial\Omega$ near $\partial\Omega$. The following comments are in order. Suppose P is a differential operator with coefficients in $\mathcal{C}^\infty(\overline{\Omega})$ and assume $\partial\Omega$ is non-characteristic for P at each point. Then if $u \in \mathcal{C}^{-\infty}(\Omega)$ and $Pu \in \mathcal{C}^\infty(\overline{\Omega})$, it follows that u satisfies (C.0.3) and that the characterization (C.0.5)–(C.0.6) of $(WF_b\, u) \cap (T^*\partial\Omega \setminus 0)$ is independent of choices. For more details on this, and also more intrinsic characterizations, see [Me7] or [Me10].

D: Fourier integral operators with singular phase functions

In Chapters 5 and 6 we encountered Fourier integral operators with singular phase, of the form

$$Au(x) = \int a(x,\xi) e^{i\varphi(x,\xi)} \hat{u}(\xi)\, d\xi, \tag{D.0.1}$$

where φ is singular, in a fashion to be discussed below. We want to derive basic properties of these operators here.

§D.1: General set-up

We suppose that the amplitude $a(x,\xi)$ has support in an open cone Γ with smooth boundary; say Γ is given (locally) by $\gamma(x,\xi) > 0$, $\gamma(x,\xi)$ smooth and homogeneous of degree 0 in ξ, $\nabla_{x,\xi}\gamma(x,\xi) \neq 0$ on $\partial\Gamma$. As usual we suppose $\varphi(x,\xi)$ is real valued and homogeneous of degree 1 in ξ, bit instead of supposing $\varphi \in C^\infty(\overline{\Gamma} \setminus 0)$, we make the following hypothesis:

$$\varphi(x,\xi) = \theta(x,\xi) + \gamma(x,\xi)^{1+a}\beta(x,\xi) \quad \text{on} \quad \overline{\Gamma} \setminus 0, \tag{D.1.1}$$

where θ, β are smooth and homogeneous of degree 1 in ξ, and $a \in (0,1)$. Typically, $a = 1/2$. We also assume

$$|\nabla_x \varphi(x,\xi)| \geq C|\xi| \quad \text{on} \quad \overline{\Gamma} \setminus 0. \tag{D.1.2}$$

As for the amplitude in (D.1), we assume

$$a(x,\xi) \in S^m_{\rho,\delta} \quad \text{is supported on} \quad \gamma(x,\xi) \geq C_0|\xi|^{-b} \ (C_0 > 0). \tag{D.1.3}$$

We can rewrite (D.0.1) in the form

$$Au(x) = \int a(x,\xi) e^{i\psi(x,\xi)} \hat{u}(\xi)\, d\xi, \tag{D.1.4}$$

where $\psi(x,\xi) \in C^\infty(\overline{\Gamma} \setminus 0)$ is an inhomogeneous phase function, as follows. Let

$$\psi(x,\xi) = \theta(x,\xi) + f\big(|\xi|^b\gamma(x,\xi)\big)|\xi|^{-(1+a)b}\beta(x,\xi), \tag{D.1.5}$$

where $f \in C^\infty$ and

$$f(s) = s^{1+a} \quad \text{for} \quad s \geq \frac{C_0}{2}. \tag{D.1.6}$$

Thus $\psi = \varphi$ on the support of $a(x,\xi)$. We see that ψ satisfies the following conditions:

$$\psi(x,\xi),\ \nabla_x\psi(x,\xi) \in S^1_{1-b,b}(\overline{\Gamma}\setminus 0), \quad \text{real valued,} \tag{D.1.7}$$

$$\nabla_\xi\psi(x,\xi) \in S^0_{1-b,b}(\overline{\Gamma}\setminus 0), \tag{D.1.8}$$

$$|\nabla_x\psi(x,\xi)| \ge C|\xi| \ \text{ on } \ \overline{\Gamma}\setminus 0, \tag{D.1.9}$$

the last condition holding at least on a conic neighborhood of $\partial\Gamma$. Note that

$$\begin{aligned}\nabla_x\psi(x,\xi) = \nabla_x\theta(x,\xi) &+ f\big(|\xi|^b\gamma(x,\xi)\big)|\xi|^{-(1+a)b}\nabla_x\beta(x,\xi)\\ &+ f'\big(|\xi|^b\gamma(x,\xi)\big)|\xi|^{-ab}\beta(x,\xi)\nabla_x\gamma(x,\xi).\end{aligned} \tag{D.1.10}$$

The decomposition of Fourier-Airy operators naturally gives rise to operators of the form (D.1.4) if we use the asymptotic expansion

$$A_\pm(z) = F(z)\, e^{\pm i\chi(x)}, \quad \operatorname{Re} z \le 0, \tag{D.1.11}$$

instead of

$$A_\pm(z) = \Psi_\pm(z) e^{\mp 3i(-z)^{3/2}/2}, \quad \operatorname{Re} z \le 0. \tag{D.1.12}$$

In such a case the phase function in (D.1.4) is the sum of a smooth term and the real part of $\chi(\xi_1^{-1/3}(\xi_n + iT))$.

§D.2: Action on distributions

The study of operators of the form (D.1.4) with inhomogeneous phase functions satisfying hypotheses (D.1.7)–(D.1.9) is accomplished in almost exact parallel with the study of operators with smooth homogeneous phase functions, as we will now see.

We show that (D.1.4) is well defined for $u \in \mathcal{E}'(\mathbb{R}^n)$. Let

$$L = i^{-1}|\nabla_x\psi|^{-2}\nabla_x\psi\cdot\nabla_x. \tag{D.2.1}$$

Then L is a first order differential operator in x with coefficients that belong to the symbol class $S^{-1}_{1-b,b}$, and

$$Le^{i\psi} = e^{i\psi}. \tag{D.2.2}$$

Thus, for $u \in \mathcal{E}',\ v \in C_0^\infty$, we have, formally,

$$\langle Au, v\rangle = \iint (L^t)^k\big(v(x)a(x,\xi)\big)e^{i\psi}\,\hat{u}(\xi)\,d\xi\,dx. \tag{D.2.3}$$

Note that

$$
\text{(D.2.4)} \qquad \begin{aligned} L^t &= -i^{-1}|\nabla_x \psi|^{-2}\nabla_x\psi \cdot \nabla_x + i^{-1}\nabla_x \cdot (|\nabla_x\psi|^{-2}\nabla_x\psi) \\ &= \sum_{|\alpha|=1} a_\alpha(x,\xi) D_x^\alpha + b(x,\xi), \end{aligned}
$$

with

$$
\text{(D.2.5)} \qquad a_\alpha(x,\xi) \in S^{-1}_{1-b,b}, \quad b(x,\xi) \in S^{-(1-b)}_{1-b,b}.
$$

It follows that

$$
\text{(D.2.6)} \qquad (L^t)^k = \sum_{|\sigma| \le k} A_{k\sigma}(x,\xi) D_x^\sigma,
$$

with

$$
\text{(D.2.7)} \qquad A_{k\sigma}(x,\xi) \in S^{-k(1-b)-|\sigma|b}_{1-b,b}.
$$

This makes it clear that (D.2.3) is an absolutely convergent integral if k is large enough, provided $\delta < 1$ in (D.1.3) and, with $\alpha(\xi) \in C_0^\infty(\mathbb{R}^n)$,

$$
\text{(D.2.8)} \qquad Au = \lim_{\varepsilon \to 0} \int \alpha(\varepsilon\xi) a(x,\xi) e^{i\psi(x,\xi)}\, d\xi,
$$

so A is independent of the choice of k in (D.2.3).

This argument shows that differential operators can be brought under the integral sign:

$$
\text{(D.2.9)} \qquad \frac{\partial}{\partial x_j} Au = \int \Big[i \frac{\partial \psi}{\partial x_j}\, a + \frac{\partial a}{\partial x_j} \Big] e^{i\psi} \hat{u}(\xi)\, d\xi,
$$

etc., as is the case with ordinary Fourier integral operators.

§D.3: Wave front relation

We can also analyze the relation between $WF(Au)$ and $WF(u)$ along the same lines as for ordinary Fourier integral operators. Recall that $(x_0,\xi_0) \notin WF(Au)$ if and only if for some $\chi \in C_0^\infty$, $\chi(x) = 1$ near x_0,

$$
\text{(D.3.1)} \qquad |\langle \chi(x) e^{-ix\cdot\theta}, Au\rangle| \le C_N |\theta|^{-N}, \quad \theta \in \Sigma,
$$

where Σ is some conic neighborhood of ξ_0. Now we have

$$
\text{(D.3.2)} \qquad \begin{aligned} &\langle \chi(x) e^{-ix\cdot\theta}, Au\rangle \\ &= \iiint u(y)\chi(x) a(x,\xi) e^{i\psi(x,\xi) - iy\cdot\xi - ix\cdot\theta}\, dy\, dx\, d\xi. \end{aligned}
$$

Before we proceed with an analysis of (D.3.2), let us make some preliminary observations that will simplify the analysis. Suppose $WF(u)$ is contained in some small conic neighborhood $U \times \Omega$ of (y_0, η_0). We can assume $u(y)$ is supported near y_0. Also, since Au defined by (D.1.4) would only be altered by a smooth function, we may as well suppose that $a(x,\xi)$ is supported for ξ is a small conic neighborhood of η_0, and that $a(x,\xi) = 0$ for $|\xi| < 1$. Finally, without loss of generality we can suppose $u(y)$ is continuous. Indeed, any $u \in \mathcal{E}'$ can be smoothed out by a finite degree by applying some negative power of the Laplace operator, which would be compensated in (D.1.4) if $a(x,\xi)$ had its order increased.

With these hypotheses, we are going to show that (D.3.1) holds on any cone Σ with the property that, for $\theta \in \Sigma,\ \xi \in \Omega$, the phase function

$$\Phi(x,y,\xi,\theta) = \psi(x,\xi) - y \cdot \xi = x \cdot \theta \tag{D.3.3}$$

satisfies

$$|\nabla_x \Phi| + (|\xi| + |\theta|)|\nabla_\xi \Phi| \geq C(|\xi| + |\theta|), \tag{D.3.4}$$

that is,

$$|\nabla_x \psi - \theta| + (|\xi| + |\theta|)|\nabla_\xi \psi - y| \geq C(|\xi| + |\theta|). \tag{D.3.5}$$

Let

$$M = \big[|\nabla_x \Phi|^2 + (|\xi|^2 + |\theta|^2)|\nabla_\xi \Phi|^2\big]^{-1}\big[\nabla_x \Phi \cdot \nabla_x + (|\xi|^2 + |\theta|^2)\nabla_\xi \Phi \cdot \nabla_\xi\big]. \tag{D.3.6}$$

It follows that

$$Me^{i\Phi} = ie^{i\Phi}. \tag{D.3.7}$$

We see that the coefficients of ∇_x in M belong to $S^{-1}_{1-b,b;1,0}(\overline{\Gamma} \times \Sigma)$, and those of ∇_ξ in M belong to $S^0_{1-b,b;1,0}(\overline{\Gamma} \times \Sigma)$, where we say

$$a(x,y,\xi,\theta) \in S^m_{\rho,\delta,\rho',\delta'}(\overline{\Gamma} \times \Sigma) \tag{D.3.8}$$

if and only if

$$|D_x^\beta D_\xi^\alpha D_y^\gamma D_\theta^\sigma a(x,y,\xi,\theta)| \leq C(1 + |\xi| + |\theta|)^{m - \rho|\alpha| - \rho'|\sigma| + \delta|\beta| + \delta'|\gamma|}, \tag{D.3.9}$$

for $(x,\xi) \in \overline{\Gamma},\ \theta \in \Sigma$. Integration by parts gives

$$\langle \chi(x) e^{-ix\cdot\theta}, Au\rangle = \iiint u(y)[(M^t)^k(\chi a)(x,\xi)]e^{i\Phi}\,dy\,dx\,d\xi. \tag{D.3.10}$$

For notational convenience we replace χa by a below. Note that if we write

$$M = A \cdot \nabla_x + B \cdot \nabla_\xi, \tag{D.3.11}$$

we have

$$\begin{aligned} M^t &= -A \cdot \nabla_x - B \cdot \nabla_\xi + (\nabla_x \cdot A) + (\nabla_\xi \cdot B) \\ &= -A \cdot \nabla_x - B \cdot \nabla_\xi + C. \end{aligned} \tag{D.3.12}$$

Recall that

$$A \in S^{-1}_{1-b,b;1,0}, \quad B \in S^0_{1-b,b;1,0}. \tag{D.3.13}$$

Thus

$$C \in S^{-(1-b)}_{1-b,b;1,0}. \tag{D.3.14}$$

It follows that

$$(M^t)^k = \sum_{|\alpha|+|\beta|\le k} (|\theta| + |\xi|)^{|\alpha|} b_{\alpha\beta k}(x,y,\xi,\theta) D_x^\beta D_\xi^\alpha, \tag{D.3.15}$$

with

$$b_{\alpha\beta k}(x,y,\xi,\theta) \in S^{-k(1-b)-(|\alpha|+|\beta|)b}_{1-b,b;1,0}, \tag{D.3.16}$$

or, with $e_{\alpha\beta k}(x,y,\xi,\theta) = (|\theta| + |\xi|)^{|\alpha|} b_{\alpha\beta k}(x,y,\xi,\theta)$,

$$e_{\alpha\beta k}(x,y,\xi,\theta) \in S^{-k(1-b)+(1-b)|\alpha|-b|\beta|}_{1-b,b;1,0}. \tag{D.3.17}$$

Thus

$$(M^t)^k a(x,\xi) = \sum_{|\alpha|+\beta|\le k} e_{\alpha\beta k}(x,y,\xi,\theta) D_x^\beta D_\xi^\alpha a(x,\xi), \tag{D.3.18}$$

with

$$e_{\alpha\beta k}(x,y,\xi,\theta) D_x^\beta D_\xi^\alpha a(x,\xi) \in S^{m-k(1-b)-\rho|\alpha|+(1-b)|\alpha|+\delta|\beta|-b|\beta|}_{\rho_0,\delta_0;1,0}, \tag{D.3.19}$$

where

$$\rho_0 = \min(\rho, 1-b), \quad \delta_0 = \max(\delta, b).$$

Since $|\alpha| + |\beta| \le k$ in (D.3.18), we see that (D.3.19) is contained in

$$S^{m-(k-|\alpha|-|\beta|)(1-b)-\rho|\alpha|-(1-\delta)|\beta|}_{\rho_0,\delta_0;1,0},$$

and hence

$$A_k(x,y,\xi,\theta) = (M^t)^k a(x,\xi) \in S^{m-\rho_1 k}_{\rho_0,\delta_0;1,0}, \tag{D.3.20}$$

where

$$\rho_1 = \min(\rho, 1-\delta, 1-b). \tag{D.3.21}$$

We apply (D.3.20) to the analysis of (D.3.10). We have

$$\langle \chi(x) e^{-ix\cdot\theta}, Au \rangle = \iiint u(y) A_k(x,y,\xi,\theta) e^{i\Phi}\, dy\, dx\, d\xi. \tag{D.3.22}$$

Thus, if we pick k so large that $\rho_1 k > m + n$, we get

$$|\langle \chi(x) e^{-ix\cdot\theta}, Au \rangle| \le C_k (1 + |\theta|)^{m+n-\rho_1 k}. \tag{D.3.23}$$

Since any $u \in \mathcal{E}'$ may be decomposed into a finite sum $\sum u_j$, each term having small wave front set, the argument above establishes the following.

Theorem D.3.24. *If A is given by (D.1.4), under the hypotheses above, $WF(Au)$ is contained in the set of points not bounded away from the cones generated by*

$$\{(x,\xi) : (\nabla_\xi \psi, \xi) \in WF(u) \text{ for some } (x,\xi) \in \operatorname{supp} a(x,\xi) \text{ with } \\ \nabla_x \psi(x,\xi) = \theta,\ |\xi| \geq R\},$$

as $R \to \infty$.

In the case where ψ arises from (D.1.1), we have:

Corollary D.3.25. *If A is given by (D.0.1), we have*

$$WF(Au) \subset \{(x,\theta) : (\nabla_\xi \varphi, \xi) \in WF(u) \quad \text{for some} \\ (x,\xi) \in \text{ conic supp } a(x,\xi), \nabla_x \varphi(x,\xi) = \theta\}.$$

Thus in this case the wave front relation is

$$(\nabla_\xi \varphi, \xi) \mapsto (x, \nabla_x \varphi), \tag{D.3.26}$$

just as it is for smooth phase functions.

We note the following generalization of the development of operators of the form (D.1.4). Namely, granted condition (D.1.3) on the amplitude $a(x,\xi)$, the hypotheses (D.1.7)–(D.1.9) on the phase function $\psi(x,\xi)$, or more precisely the estimates to which these hypotheses are equivalent, need only hold on the support of $a(x,\xi)$, not on the entire cone $\overline{\Gamma} \setminus 0$. In particular, such a weaker hypothesis immediately applies to operators of the form (D.0.1) where $\varphi(x,\xi)$ is homogeneous of degree 1 in ξ and where hypothesis (D.1.1) is weakened to

$$\varphi(x,\xi) \in C^1(\overline{\Gamma} \setminus 0), \tag{D.3.27}$$

$$|D_x^\beta D_\xi^\alpha \nabla_x \varphi(x,\xi)| \leq C|\xi|^{1-|\alpha|} \gamma(x,\xi)^{a-|\alpha|-|\beta|}, \text{ if } |\alpha| + |\beta| \geq 1, \tag{D.3.28}$$

$$|D_x^\beta D_\xi^\alpha \nabla_\xi \varphi(x,\xi)| \leq C|\xi|^{-|\alpha|} \gamma(x,\xi)^{a-|\alpha|-|\beta|}, \text{ if } |\alpha| + |\beta| \geq 1. \tag{D.3.29}$$

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