The Green Function on a Compact 2D Manifold

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Let M be a compact 2D manifold, with a Riemannian metric tensor g. Say M has a coordinate system for which the components of this tensor are Hölder continuous, in C^r, for some $r > 0$. Let Δ denote the Laplace-Beltrami operator, and $\mathcal{G}$ the Green operator,

$$
\begin{aligned}
\mathcal{G} = \Delta^{-1} &\quad \text{on} \quad V = \{u \in L^2(M) : \int u(x)\, dV(x) = 0\}, \\
0 &\quad \text{on} \quad V^{\perp}.
\end{aligned} \tag{1}
$$

Let $G(x, y)$ denote the integral kernel of $\mathcal{G}$, so

$$
\mathcal{G}u(x) = \int_M G(x, y) u(y)\, dV(y). \tag{2}
$$

Here and above, dV is the volume element on M associated to its metric tensor. One might think that if g were rough then G might be hard to analyze with any precision. Here we show that one can say a good bit about it, thanks to special structure in 2D. For simplicity we assume M is oriented, though we could remove that hypothesis.

First, note that g provides for M a canonical smooth structure. Indeed, under the hypotheses above, M has local isothermal coordinates (cf. [T], §III.9). These coordinates give M the structure of a Riemann surface, hence of a C^∞ manifold. In these coordinates, the components of g have the same regularity as hypothesized above. Furthermore, there exist *smooth* metric tensors on M in the same conformal class as g (including "canonical" metrics, with constant curvature). Pick one and denote it g_2.

Call the original matric tensor g_1 and its Laplace operator, Green operator, and Green function $\Delta_1, \mathcal{G}_1$, and $G_1(x, y)$, and denote the parallel objects associated to g_2 by $\Delta_2, \mathcal{G}_2$, and $G_2(x, y)$, so

$$
\mathcal{G}_2 u(x) = \int_M G_2(x, y) u(y)\, dV_2(y), \tag{3}
$$

where dV_2 is the volume element on M associated to g_2. Now $G_2(x, y)$ is the integral kernel of a classical pseudodifferenetial operator of order -2; it is an object with transparent structure. It remains to compare $G_1(x, y)$ and $G_2(x, y)$.

Note that

$$
\Delta_j = *_j\, d * d, \tag{4}
$$

where $*$ is the Hodge star on 1-forms and $*_j$ takes 2-forms to 0-forms. Thus

$$\Delta_1 = A\,\Delta_2, \quad A = \frac{dV_2}{dV_1}. \tag{5}$$

If $\int_M u(y)\,dV_1(y) = 0$, then $\int_M u(y)A(y)^{-1}\,dV_2(y) = 0$, and

$$\mathcal{G}_1 u(x) = \mathcal{G}_2\,A^{-1}u(x) \quad \text{mod const.} \tag{6}$$

Now

$$\begin{aligned} \mathcal{G}_2\,A^{-1}u(x) &= \int\limits_M G_2(x,y)A(y)^{-1}u(y)\,dV_2(y) \\ &= \int\limits_M G_2(x,y)u(y)\,dV_1(y), \end{aligned} \tag{7}$$

so

$$\mathcal{G}_1 u(x) = \int\limits_M G_2(x,y)u(y)\,dV_1(y) \quad \text{mod const., if} \quad \int\limits_M u(y)\,dV_1(y) = 0. \tag{8}$$

Of course

$$\mathcal{G}_1 u(x) = \int\limits_M G_1(x,y)u(y)\,dV_1(y). \tag{9}$$

From here a brief calculation yields

$$G_1(x,y) = G_2(x,y) - \frac{\eta(x)+\eta(y)}{V_1(M)} + \frac{B}{V_1(M)}, \tag{10}$$

where $V_1(M) = \int_M 1\,dV_1(M)$ and

$$\eta(x) = \int\limits_M G_2(x,y)\,dV_1(y) = \mathcal{G}_2\Big(\frac{dV_1}{dV_2}\Big)(x), \tag{11}$$

and

$$B = \frac{1}{V_1(M)}\int\limits_M \eta(y)\,dV_1(y). \tag{12}$$

Recall we are trying to understand the behavior of $G_1(x,y)$, and as we have said the behavior of $G_2(x,y)$ is well understood. Now if the components of g_1 are of class C^r in local isothermal coordinates, we have

$$\frac{dV_1}{dV_2} \in C^r(M), \tag{13}$$

and hence, since the coefficients of Δ_2 are smooth,

$$\eta \in C^{r+2}(M). \tag{14}$$

Now let's say we are in the following situation. In some (possibly poorly chosen) coordinate system, g_1 has components that are Hölder continuous and have one derivative in L^2. Then the Gauss curvature K is a well defined distribution. Assume

$$K \in L^4(M). \tag{15}$$

Now move to isothermal coordinates. Then the components of g_1 belong to an L^4-Sobolev space:

$$H^{2,4} \subset C^{3/2}. \tag{16}$$

Hence, in isothermal coordinates, we have

$$\frac{dV_1}{dV_2} \in H^{2,4} \tag{17}$$

and hence (10) holds, with

$$\eta \in H^{4,4}(M) \subset C^{7/2}(M). \tag{18}$$

This is the degree of regularity one has in isothermal coordinates. In bad coordinates the regularity might be less.

Example. Suppose $M \subset \mathbb{R}^3$ is a surface smooth of class $C^{1+r} \cap H^{2,p},\ 0 < r < 1,\ 2 < p < \infty$, i.e., M is locally the graph of functions with that regularity. One can use such "graph coordinates," in which the metric tensor induced from the Euclidean metric on $\mathbb{R}^3$ has components in $C^r \cap H^{1,p}$. Hence local isothermal coordinates exist. Also the Gauss map is smooth of class $C^r \cap H^{1,p}$, so

$$K \in L^q, \quad q = \frac{p}{2}. \tag{19}$$

Thus, in local isothermal coordinates the metric tensor has components in $H^{2,q}$, and we have (10) with $\eta \in H^{4,q}$, in isothermal coordinates. If $p = 8$, then (18) holds. Isothermal coordinates and graph coordinates are related by diffeomorphisms smooth of class $C^{1+r} \cap H^{2,p}$. Hence, if $p = 8$, one has $\eta \in C^{7/2}(M)$ in isothermal coordinates, but only $\eta \in C^{1+r}(M)$ in graph coordinates. Note that we have the same function η in both cases, just a finer differential structure on M in one case.

Reference

[T] M. Taylor, Tools for PDE, Math. Surveys and Monogr. #81, AMS, Providence, RI, 2000.