The Green Function for $\Delta - V$ with a Rough Potential

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1. Introduction

Let M be a compact Riemannian manifold, with smooth metric tensor, and Laplace-Beltrami operator Δ. Then $(\Delta - 1)^{-1}$ is a pseudodifferential operator of order -2, of classical type, whose integral kernel $E_1(x,y)$ has a well known behavior. Now let V be a real valued function on M. Assume $V \in L^p(M)$ with $p > n/2$. Then the operator M_V of multiplication by V has the property

$$M_V : H^{1,2}(M) \longrightarrow H^{-1,2}(M),$$

and this map is compact, so $\Delta - V$ is Fredholm, of index zero. We assume the null space is zero, so

$$(\Delta - V)^{-1} : H^{-1,2}(M) \longrightarrow H^{1,2}(M). \tag{1.1}$$

We desire to compare the integral kernel $E_V(x,y)$ of $(\Delta - V)^{-1}$ with $E_1(x,y)$, and estimate the difference. Estimates of this nature were obtained in Appendix A of [DHR], assuming V is Hölder continuous, and we aim to produce much stronger estimates, namely the following:

Theorem 1.1. *If M is a compact manifold of dimension n with a smooth metric tensor, $V \in L^p(M)$ is real valued and $p \in (n/2, \infty]$, and if $\Delta - V$ has zero null space, then*

$$|E_V(x,y) - E_1(x,y)| \le C\Big(d(x,y)^{4-n-n/p} + 1\Big), \tag{1.2}$$

and

$$|\nabla_x E_V(x,y) - \nabla_x E_1(x,y)| \le C\Big(d(x,y)^{3-n-n/p} + 1\Big). \tag{1.3}$$

In both cases, one replaces the power of $d(x,y)$ by $\log C/d(x,y)$ if the exponent is zero.

Here $d(x,y)$ denotes the distance from x to y. Our proof starts with a sequence of simple formulas relating E_V and E_1. (As is common, we identify operators and their intgral kernels.) From

$$(\Delta - V)E_V = I, \quad (\Delta - V)E_1 = I + (1 - V)E_1, \tag{1.4}$$

we have $(\Delta - V)(E_V - E_1) = (V - 1)E_1$, and hence

$$E_V = E_1 + E_V(V - 1)E_1. \tag{1.5}$$

Iterating (1.5) produces

$$E_V = E_1 + E_1(V-1)E_1 + E_1(V-1)E_1(V-1)E_1 + \cdots \\ + E_V(V-1)E_1 \cdots (V-1)E_1, \tag{1.6}$$

where the last term has one factor of E_V, followed by k factors of $(V-1)E_1$. Alternatively (as one sees immediately by self-adjointness),

$$E_V = E_1 + E_1(V-1)E_1 + E_1(V-1)E_1(V-1)E_1 + \cdots \\ + E_1(V-1)E_1 \cdots (V-1)E_V, \tag{1.7}$$

where the last term has k factors of $E_1(V-1)$, followed by one factor of E_V.

Since $-E_1$ and $-E_V$ are positivity-preserving, it is clear that the first $k-1$ terms after E_1 on the right side of (1.6) have the same integral kernel bounds as pseudodifferential operators of order -4, -6, etc., if $V \in L^\infty(M)$. Once one shows that

$$E_V : L^p(M) \longrightarrow L^\infty(M), \quad p > \frac{n}{2}, \tag{1.8}$$

which is rather easy if $V \in L^\infty(M)$, it follows that the last term in (1.6) also has such an integral kernel bound. This proves (1.2) when $p = \infty$. Having this, one can apply ∇_x to both sides of (1.7) and apply similar reasoning to get (1.3) for $p = \infty$.

In §2 we will derive appropriate estimates on all the terms but the last on the right side of (1.6) in case $p \in (n/2, \infty)$, and in §3 we show that (a stronger result than) (1.8) holds when $V \in L^p(M)$ and $p > n/2$, to finish the proof of Theorem 1.1.

The integral kernel estimates of §2 work not only for $V \in L^p(M)$, but more generally for V in the Morrey space $M_1^p(M)$ (provided $p > n/2$). Morrey spaces are defined as follows. Assume $1 \le q \le p < \infty$. We say a function $f \in L^1(M)$ belongs to $M_q^p(M)$ if and only if for each ball $B_R \subset M$ of radius R,

$$\int_{B_R} |f(x)|^q \, dV(x) \le C R^{n(1-q/p)}. \tag{1.9}$$

A simple consequence of Hölder's inequality is that $M_1^p(M) \subset M_q^p(M) \subset L^p(M)$. In §3 we also obtain an analogue of (1.8) with $L^p(M)$ replaced by $M_q^p(M)$, provided that also $q > 1$. This produces the following extension of Theorem 1.1:

Theorem 1.2. *The results of Theorem 1.1 hold whenever $V \in M_q^p(M)$, with $p > n/2$, $q > 1$, provided $\Delta - V$ has zero null space.*

2. Composition estimates

Estimates on

$$E_1(V-1)E_1 \cdots E_1(V-1)E_1, \tag{2.1}$$

consisting of k factors of $E_1(V-1)$ followed by one factor of E_1, will be derived from the following general result. We work in a coordinate chart in $\mathbb{R}^n$. Assume

$$|p_\alpha(x,y)| \le |x-y|^{\alpha-n}, \quad \forall\, x,y \in \mathbb{R}^n, \tag{2.2}$$

and that $\operatorname{supp} p_\alpha \subset \{(x,y) : |x|, |y| \le 1\}$. Take

$$f \in L^p(\mathbb{R}^n), \quad p > \frac{n}{\alpha},\ p > \frac{n}{\beta}. \tag{2.3}$$

We aim to show that

$$q_{\alpha\beta}(x) = \int p_\alpha(x,y) f(y) p_\beta(y,0)\, dy \tag{2.4}$$

satisfies (for $|x| \le 1/2$) the estimate

$$\begin{aligned} |q_{\alpha\beta}(x)| \le C|x|^{\alpha+\beta-n-n/p} \quad & \text{if } \alpha+\beta-n-n/p < 0, \\ C \log \frac{1}{|x|} \quad & \text{if } \alpha+\beta-n-n/p = 0, \\ C \quad & \text{if } \alpha+\beta-n-n/p > 0. \end{aligned} \tag{2.5}$$

To prove this, take $|x| = \varepsilon = 2^{-\mu}$ and estimate

$$\int_X |p_\alpha(x,y) f(y) p_\beta(y,0)\, dy| \tag{2.6}$$

over sets of the form

$$X = C_k,\ E_\mu,\ A_k,\ B_k, \tag{2.7}$$

where

$$\begin{aligned} &C_k \text{ is a shell } 2^{k-1} \le |y| \le 2^{-k},\ -1 \le k \le \mu-1, \\ &E_\mu \text{ is the ball } |y| \le 2\varepsilon = 2^{-\mu+1}, \\ &\qquad\qquad \text{with some neighborhoods of 0 and } |x| \text{ excised.} \\ &A_k \text{ is a shell around 0 where } |y| \approx 2^{-k},\ k \ge \mu+1, \\ &B_k \text{ is a shell around } x \text{ where } |x-y| \approx 2^{-k},\ k \ge \mu+1. \end{aligned} \tag{2.8}$$

We have

$$\int_{C_k} |p_\alpha(x,y) f(y) p_\beta(y,0)\, dy| \le C 2^{-k(\alpha+\beta-2n)} \int_{C_k} |f(y)|\, dy.$$

Now

$$\int_{C_k} |f(y)|\, dy \le C 2^{-kn/p'} \|f\|_{M_1^p},$$

where $M_1^p(\mathbb{R}^n)$ is the Morrey space, defined by the property that (1.9) holds for all balls of radius $R \le 1$. A well known consequence of Hölder's inequality is that $L^p(\mathbb{R}^n) \subset M_1^p(\mathbb{R}^n)$. The two estimates above yield

$$\int_{C_k} |p_\alpha(x,y) f(y) p_\beta(y,0)|\, dy \le C 2^{-k(\alpha+\beta-n-n/p)} \|f\|_{M_1^p}. \tag{2.9}$$

It follows that (for $\varepsilon \le 1/2$)

$$\begin{aligned}\sum_{k=-1}^{\mu-1} \int_{C_k} &|p_\alpha(x,y) f(y) p_\beta(y,0)|\, dy \\ &\le C\varepsilon^{\alpha+\beta-n-n/p} \|f\|_{M_1^p}, \quad \text{if } \alpha+\beta-n-n/p<0, \\ &\quad C \log \frac{1}{\varepsilon}\, \|f\|_{M_1^p}, \qquad \text{if } \alpha+\beta-n-n/p=0, \\ &\quad C, \qquad\qquad\qquad\quad \text{if } \alpha+\beta-n-n/p>0.\end{aligned} \tag{2.10}$$

Next we have

$$\begin{aligned}\int_{E_\mu} |p_\alpha(x,y) f(y) p_\beta(y,0)|\, dy &\le C\varepsilon^{\alpha+\beta-2n} \int_{E_\mu} |f(y)|\, dy \\ &\le C\varepsilon^{\alpha+\beta-2n} \varepsilon^{n/p'} \|f\|_{M_1^p} \\ &= C\varepsilon^{\alpha+\beta-n-n/p} \|f\|_{M_1^p}.\end{aligned} \tag{2.11}$$

Next,

$$\begin{aligned}\int_{A_k} |p_\alpha(x,y) f(y) p_\beta(y,0)|\, dy &\le C\varepsilon^{\alpha-n} \int_{A_k} |f(y) p_\beta(y,0)|\, dy \\ &\le C\varepsilon^{\alpha-n} 2^{-k(\beta-n)} 2^{-kn/p'} \|f\|_{M_1^p},\end{aligned} \tag{2.12}$$

and hence (given $\beta > n/p$)

$$\begin{aligned}\sum_{k=\mu+1}^{\infty} \int_{A_k} |p_\alpha(x,y) f(y) p_\beta(y,0)|\, dy &\le C\varepsilon^{\alpha-n} \sum_{k=\mu+1}^{\infty} 2^{-k(\beta-n/p)} \|f\|_{M_1^p} \\ &\le C\varepsilon^{\alpha+\beta-n-n/p} \|f\|_{M_1^p}.\end{aligned} \tag{2.13}$$

Similarly

$$\sum_{k=\mu+1}^{\infty} \int_{B_k} |p_\alpha(x,y) f(y) p_\beta(y,0)|\, dy \le C\varepsilon^{\alpha+\beta-n-n/p} \|f\|_{M_1^p}. \tag{2.14}$$

Collecting these estimates, we have (2.5), under the hypotheses (2.2)–(2.3), with (2.3) generalized to

$$f \in M_1^p(\mathbb{R}^n), \quad p > \frac{n}{\alpha},\ p > \frac{n}{\beta}. \tag{2.15}$$

The general estimate (2.5) applies to

$$p_1 = E_1(V-1)E_1, \cdots, p_{k+1}(x,y) = E_1(V-1)p_k. \tag{2.16}$$

Again we identify integral kernels and operators, and use operator products. We have, given $V \in M_1^p,\ p > n/2$,

$$|p_1(x,y)| \le C\Big(|x-y|^{4-n-n/p} + 1\Big), \tag{2.17}$$

with a $\log 1/|x-y|$ included if $4-n-n/p = 0$. Iterating gives

$$|p_k(x,y)| \le C\Big(|x-y|^{2+k(2-n/p)-n} + 1\Big), \tag{2.18}$$

with a $\log 1/|x-y|$ included if $2+k(2-n/p)-n = 0$.

3. Mapping property of $(\Delta - V)^{-1}$

Our first goal in this section is to prove the following.

Proposition 3.1. *Let M be a compact Riemannian manifold of dimension n, with smooth metric tensor. Assume $V \in L^p(M),\ p > n/2$. Then*

$$\Delta - V : H^{2,p}(M) \longrightarrow L^p(M) \quad \text{is Fredholm, of index 0.} \tag{3.1}$$

If the null space of $\Delta - V$ is zero, then

$$(\Delta - V)^{-1} : L^p(M) \to H^{2,p}(M) \subset C(M). \tag{3.2}$$

Proof. First, since $p > n/2 \Rightarrow H^{2,p}(M) \hookrightarrow C(M)$, compactly embedded, we clearly have

$$M_V : H^{2,p}(M) \longrightarrow L^p(M), \quad \text{compact.} \tag{3.3}$$

Thus $\Delta - V$ is a compact perturbation of $\Delta - 1 : H^{2,p}(M) \to L^p(M)$, which is an isomorphism, so we have (3.1). If the null space of $\Delta - V$ is zero, then $\Delta - V$ is an isomorphism in (3.1), and we have (3.2).

Remark. Clearly the restriction of $(\Delta - V)^{-1}$ in (1.1) to $L^p(M)$ agrees with $(\Delta - V)^{-1}$ in (3.2).

As noted in §1, Proposition 3.1 completes the proof of Theorem 1.1. We now prove Theorem 1.2. By the estimates of §2, it remains to prove that

$$V \in M_q^p(M),\ p > \frac{n}{2},\ q > 1 \Longrightarrow (\Delta - V)^{-1} : M_q^p(M) \to L^\infty(M). \tag{3.4}$$

In order to do this, via a result parallel to Proposition 3.1, we bring in Morrey scales. We define $M_q^{p,k}(M)$ for $k \in \mathbb{Z}^+$ by

$$u \in M_q^{p,k}(M) \Longleftrightarrow Lu \in M_q^p(M),\ \forall\, L \in \mathrm{Diff}^k(M), \tag{3.5}$$

where $\mathrm{Diff}^k(M)$ denotes the space of differential operators on M, with smooth coefficients, of order $\le k$. Many analytical results on such spaces are discussed in [T]. We mention a few here. First, Morrey's embedding theorem, which is sharper than Sobolev's embedding theorem, implies

$$qk > n \Longrightarrow M^{q,k}_r(M) \hookrightarrow C(M), \text{ compactly.} \tag{3.6}$$

Next,

$$1 < q \le p < \infty \Longrightarrow \Delta - 1 : M^{p,2}_q(M) \rightarrow M^p_q(M), \text{ isomorphically.} \tag{3.7}$$

We can now establish:

Proposition 3.2. *Let M be as in Proposition 3.1. Assume $V \in M^p_q(M)$, with $q > 1,\ p > n/2$. Then*

$$\Delta - V : M^{p,2}_q(M) \longrightarrow M^p_q(M) \ \text{ is Fredholm, of index 0.} \tag{3.8}$$

If the null space of $\Delta - V$ is zero, then

$$(\Delta - V)^{-1} : M^p_q(M) \longrightarrow M^{p,2}_q(M) \subset C(M). \tag{3.9}$$

Proof. Parallel to (3.3), we have

$$M_V : M^{p,2}_q(M) \longrightarrow M^p_q(M), \quad \text{compact,} \tag{3.10}$$

via (3.6). This and (3.7) yield (3.8), and (3.9) follows.

We now have (3.4), and Theorem 1.2 is proven.

References

[DHR] O. Druet, E. Hebey, and F. Robert, Blow-up Theory for Elliptic PDEs in Riemannian Geometry, Princeton Univ. Press, Princeton, NJ, 2004.

[T] M. Taylor, Microlocal analysis on Morrey spaces, pp. 97–135 in *Singularities and Oscillations* (J. Rauch and M. Taylor, eds.) IMA Vol. 91, Springer-Verlag, New York, 1997.