Remarks on a Class of Greenian Domains

Michael Taylor

1. Introduction

Let $\overline{M}$ be a connected, noncompact, complete, n-dimensional Riemannian manifold, with nonempty, smooth, compact boundary ∂M. We seek conditions guaranteeing unique solvability of the Dirichlet problem

$$u\big|_{\partial M} = f, \quad \Delta u = 0 \ \text{ on } \ M, \quad \lim_{x\to\infty} u(x) = 0, \tag{1.1}$$

given f on ∂M, in various spaces, including $f \in C(\partial M)$, $L^p(\partial M)$, and $\mathcal{D}'(\partial M)$. We will show that such a Dirichlet problem can be solved under the following assumption, which we call "Hypothesis H."

Hypothesis H. *The Riemannian manifold $\overline{M}$ is as above and there is a compact $X \subset \overline{M}$, with complement $U = \overline{M} \setminus X$, and $H \in C^\infty(U)$ satisfying*

$$H > 0, \quad \Delta H \le 0, \quad \lim_{x\to\infty} H(x) = 0. \tag{1.2}$$

We will demonstrate the following.

Theorem 1.1. *If $\overline{M}$ satisfies Hypothesis H, then*
(A) for each $f \in C(\partial M)$, (1.1) has a unique solution.
More generally,
(B) for each $f \in \mathcal{D}'(\partial M)$, (1.1) has a unique solution.

Regarding the behavior of such a solution near ∂M, standard local regularity results apply. In particular, take $\overline{N}$ to be a smoothly bounded, compact, n-dimensional submanifold of $\overline{M}$, containing a neighborhood of ∂M, set $\partial N = \partial M \cup \Sigma$, and consider the unique solution v to

$$\Delta v = 0 \ \text{ on } \ N, \quad v\big|_{\partial M} = f, \quad v\big|_{\Sigma} = 0. \tag{1.3}$$

Then

$$u\big|_{\overline{N}} - v \in C^\infty(\overline{N}). \tag{1.4}$$

We proceed as follows. In §2 we prove part (A) of Theorem 1.1. Our strategy for establishing part (B) will be to take v satisfying (1.3), take $\chi \in C_0^\infty(\overline{M})$ supported in $\overline{N}$, equal to 1 on a neighborhood of ∂M and to 0 on a neighborhood of Σ, and find u in the form

$$u = \chi v + w, \tag{1.5}$$

where w satisfies

$$\Delta w = g \ \text{ on } \ M, \quad w\big|_{\partial M} = 0, \quad \lim_{x\to\infty} w(x) = 0. \tag{1.6}$$

For this treatment, $g = -\Delta(\chi v) \in C_0^\infty(M)$. (Here $M = \overline{M} \setminus \partial M$ is the interior of $\overline{M}$.) In §3 we examine (1.6), making use of part (A) of Theorem 1.1. Then we establish part (B) of Theorem 1.1 in §4, and make some comments on the behavior of the solution when $f \in L^p(\partial M)$, or more generally $f \in H^{s,p}(\partial M)$. In §5 we show that whenever $\overline{M}$ is asymptotically Euclidean and has dimension $n \geq 3$, Hypothesis H holds and hence Theorem 1.1 applies.

2. Continuous boundary values

Here we prove part (A) of Theorem 1.1. Let us fix a sequence $\overline{M}_k$ of compact, connected manifolds with smooth boundary, such that

$$\partial M \subset \overline{M}_1 \subset \cdots \subset \overline{M}_k \nearrow \overline{M}, \tag{2.1}$$

the convergence to $\overline{M}$ meaning that for each compact $K \subset \overline{M}$, we have $K \subset \overline{M}_k$ for k sufficiently large. Let $X \subset \overline{M}$ be as in Hypothesis H. We can assume $X \subset \overline{M}_1 \backslash \Sigma_1$, where $\partial \overline{M}_k = \partial M \cup \Sigma_k$.

It suffices to prove part (A) when

$$f \in C(\partial M), \quad 0 \leq f \leq 1, \tag{2.2}$$

and this is what we assume below. Define $u_k \in C(\overline{M}_k) \cap C^\infty(M_k)$ by

$$u_k\big|_{\partial M} = f, \quad u_k\big|_{\Sigma_k} = 0, \quad \Delta u_k = 0 \ \text{ on } \ M_k. \tag{2.3}$$

Hence $M_k = \overline{M}_k \setminus \partial M_k$ and $\partial M_k = \partial M \cup \Sigma_k$. If (2.2) holds, we have $0 \leq u_k \leq 1$ on $\overline{M}_k$. We extend u_k to be 0 on $\overline{M} \setminus \overline{M}_k$. The maximum principle gives $u_k \nearrow$, so there is a unique limit:

$$u_k \nearrow u \ \text{ on } \ \overline{M}. \tag{2.4}$$

Clearly $0 \leq u \leq 1$, $u\big|_{\partial M} = f$, and $\Delta u = 0$ on M. It remains to verify that

$$u(x) \to 0 \ \text{ as } \ x \to \infty. \tag{2.5}$$

It is here that we bring in H. Take

$$A = \inf_{x \in \Sigma_1} H(x), \quad A > 0. \tag{2.6}$$

Another application of the maximum principle gives

$$u_k(x) \leq \frac{1}{A} H(x), \ \text{ on } \ \overline{M} \setminus \overline{M}_1, \quad \forall\, k \geq 2, \tag{2.7}$$

which yields (2.5).

3. The equation $\Delta w = g$

As indicated in §1, we can go from part (A) of Theorem 1.1 to part (B) if we establish the following.

Proposition 3.1. *If $\overline{M}$ satisfies Hypothesis H and $g \in C_0^\infty(M)$, there exists a unique $w \in C^\infty(\overline{M})$ satisfying*

$$\Delta w = g, \quad w\big|_{\partial M} = 0, \quad \lim_{x\to\infty} w(x) = 0. \tag{3.1}$$

In order to prove this, it is convenient to establish the following.

Lemma 3.2. *Assume $\overline{M}$ satisfies Hypothesis H and $K \subset M$ is compact. Then there exists $U \in C(\overline{M})$ satisfying*

$$U > 0 \;\; on \;\; M, \quad U\big|_{\partial M} = 0, \quad \lim_{x\to\infty} U(x) = 0. \tag{3.2}$$

and also

$$\Delta U \le 0 \;\; on \;\; M, \quad \Delta U \le -1 \;\; on \;\; K. \tag{3.3}$$

(Here $\Delta U \le 0$ means $-\Delta U$ is a positive measure.)

Proof. Pick $p \in K$. Part (A) of Theorem 1.1 applies to $\overline{M} \setminus B_r(p)$, for small $r > 0$, to produce U_p, harmonic on $M \setminus B_r(p)$, satisfying

$$U_p \ge 0, \quad U_p\big|_{\partial B_r(p)} = 1, \quad U_p\big|_{\partial M} = 0, \quad \lim_{x\to\infty} U_p(x) = 0. \tag{3.4}$$

If we extend U_p as 1 on $B_r(p)$, the extended function is subharmonic. That is, $\Delta U_p \le 0$; in fact it is a negative measure supported on $\partial B_r(p)$. Now we take $a > 0$ and set

$$\begin{aligned} V_p(x) = U_p(x), & \qquad x \in \overline{M} \setminus B_r(p), \\ 1 + a - \frac{a}{r^2}\rho_p(x)^2, & \qquad x \in B_r(p), \end{aligned} \tag{3.5}$$

where $\rho_p(x) = \mathrm{dist}(x, p)$.

Note that $\Delta \rho_p(x)^2 = 2n$ at $x = p$. Hence, given $p \in K$, there exists $r = r_p > 0$ such that $\Delta \rho_p^2 \ge 1$ on $B_r(p)$. Having picked such r, we then find $a = a_p > 0$ such that V_p, defined by (3.5), satisfies

$$\Delta V_p \le -\frac{a}{r^2}\chi_{B_r(p)} \;\; \text{on} \;\; M. \tag{3.6}$$

Since $K \subset M$ is compact, one can cover it by a finite number of such balls $B_r(p)$, and let U be an appropriate positive linear combination of such V_p, to satisfy (3.2)–(3.3). This proves Lemma 3.2.

We now take up the proof of Proposition 3.1. It suffices to treat the case where $g \in C_0^\infty(M)$ is ≤ 0. Say supp $g \subset K$. We take $\overline{M}_k \nearrow \overline{M}$ as in §2, and we arrange that $K \subset \overline{M}_1 \setminus \Sigma_1$. Define w_k on $\overline{M}_k$ by

$$\Delta w_k = g \;\; \text{on} \;\; M_k, \quad w_k\big|_{\partial \overline{M}_k} = 0. \tag{3.7}$$

Since $g \le 0$, we have $w_k \ge 0$ on M_k. We also have $w_{k+1} \ge 0$ on $\partial \overline{M}_k$, and hence $w_k \nearrow$. On the other hand, if we take U as in Lemma 3.2, the maximum principle yields

$$w_k \le \|g\|_{L^\infty} U \;\; \text{on} \;\; M_k, \quad \forall\, k. \tag{3.8}$$

Hence $w_k \nearrow w$, a limit satisfying (3.1).

We can use a similar argument to construct a Green function G_p, with pole at $p \in M$, satisfying $G_p \in \mathcal{D}'(M) \cap C^\infty(\overline{M} \setminus p)$, $G_p > 0$ on $M \setminus p$, and

$$\Delta G_p = -\delta_p, \quad G_p\big|_{\partial M} = 0, \quad \lim_{x \to \infty} G_p(x) = 0. \tag{3.9}$$

4. Other spaces of boundary data

As already noted, the proof of part (B) of Theorem 1.1 is an immediate consequence of Proposition 3.1 and the construction involving (1.5)–(1.6). Here we record some further properties of the solution u to (1.1), when f belongs to various spaces of functions (or distributions). These results follow from the nature of a parametrix for the solution to (1.3) (hence for (1.1)) constructed, e.g., as in pages 199–200 of [T]. Taking a diffeomorphism mapping a collar neighborhood $\overline{N}$ of ∂M onto $[0,1] \times \partial M$, we then take $y \in [0,1]$, $x \in \partial M$, and write

$$u(y,x) = P(y,x,D_x)f(x), \quad \text{mod } C^\infty(\overline{N}), \tag{4.1}$$

where $P(y,x,D_x)$ is a bounded family of pseudodifferential operators on ∂M, with symbols given as in (2.11)–(2.15) of [T]. Here is one consequence.

Proposition 4.1. *Given $s \in \mathbb{R}$, $p \in (1,\infty)$, and $f \in H^{s,p}(\partial M)$ (the L^p-Sobolev space), the solution u to (1.1) satisfies*

$$u \in C([0,1], H^{s,p}(\partial M)) \tag{4.2}$$

near ∂M.

There are many other well known local regularity results, which can be found in the large literature on the subject.

5. Asymptotically Euclidean spaces

If $\overline{M}$ is asymptotically Euclidean, then there is a compact $X \subset \overline{M}$ whose complement $U = \overline{M} \setminus X$ is diffeomorphic to $(R,\infty) \times S$, for some $(n-1)$-dimensional S, and the metric tensor takes the form

$$ds^2 = dr^2 + r^2 g_S(r). \tag{5.1}$$

Here $g_S(r)$ is a family of metric tensors on S, depending smoothly on $r^{-1} \in [0,1/R)$, hence having an asymptotic expansion

$$g_S(r) \sim g_0 + r^{-1} g_1 + \cdots, \tag{5.2}$$

where g_0 is a metric tensor on S and, for $k \ge 1$, g_k are smooth symmetric second order tensors on S. The Laplace operator on U takes the form

$$\Delta = \partial_r^2 + M(r)\partial_r + r^{-2}\Delta_{S(r)}, \tag{5.3}$$

where $\Delta_{S(r)}$ is the Laplace operator on $S(r) = \{(r,x) : x \in S\}$, with metric tensor $g_S(r)$, and $M(r)$ is $(n-1$ times) the mean curvature of $S(r) \subset M$. One has

$$M(r) \sim \frac{n-1}{r} + \frac{\alpha}{r^2} + \cdots . \tag{5.4}$$

Consequently, if $n \geq 3$ and $\delta \in (0,1)$,

$$\begin{aligned} \Delta r^{-(n-2-\delta)} &= (n-2-\delta)(n-1-\delta) r^{-n+\delta} - (n-2-\delta) M(r) r^{-n+1+\delta} \\ &= -\delta r^{-n+\delta} + O(r^{-n-1+\delta}). \end{aligned} \tag{5.5}$$

Hence, if R_0 is large enough, $H = r^{-(n-2-\delta)}$ satisfies the condition (1.2) on $[R_0,\infty) \times S$. This proves:

Proposition 5.1. *If $\overline{M}$ is asymptotically Euclidean, of dimension ≥ 3, then Hypothesis H holds. Hence Theorem 1.1 applies.*

Reference

[T] M. Taylor, Pseudodifferential Operators, Princeton Univ. Press, Princeton NJ, 1981.