Multiple Eigenvalues of Operators with Noncommutative Symmetry Groups

Michael E. Taylor

Abstract. The presence of a noncommutative finite group G of symmetries of an elliptic, self-adjoint differential operator L, with discrete spectrum, can force the existence of an infinite number of multiple eigenvalues of L. In §1 we show that such a phenomenon occurs rather generally, due to the fact that some irreducible representation ρ of G of degree > 1 must be contained in infinitely many eigenspaces of L. In §2 we show that, under mild assumptions, the relative frequency of occurrence of each irreducible representation ρ of G in the sum of all the eigenspaces with eigenvalues $\le R$ tends as $R \to \infty$ to a limit equal to the relative frequency that ρ occurs in the regular representation of G. In §3 we study an example where the multiplicities can get arbitrarily large.

1. Inevitability of multiple eigenvalues

Let G be a finite group of measure-preserving transformations of a non-atomic, σ-finite measure space $(X, \mathcal{B}, \mu)$. Thus G has a unitary representation on $H = L^2(X, \mu)$, given by $U(g)f(x) = f(g^{-1}x)$. We make the hypothesis that

$$U : G \longrightarrow \mathcal{U}(H) \quad \text{is injective.} \tag{1.1}$$

Our first result is the following.

Proposition 1.1. *Let K be a compact self-adjoint operator on $H = L^2(X, \mu)$. Assume also that K is injective. Assume G is noncommutative, and that (1.1) holds. If K commutes with $U(g)$ for all $g \in G$, then K has infinitely many multiple eigenvalues.*

Proof. As a preliminary comment, we note that, if H_0 is the closed linear span of all the 1-dimensional eigenspaces of K, then G acts on H_0, and the restriction of U, given by $V(g) = U(g)\big|_{H_0}$, has the property that $G/\ker V$ is commutative. Thus H_0 cannot equal H, since then (1.1) would imply G is commutative. The content of the proposition is that the orthogonal complement H_1 of H has infinite dimension.

To see this, we first note that there is an irreducible representation ρ of G, on a space V_ρ of dimension greater than 1, such that ρ is contained in U. This follows by the same sort of argument as above; if every irreducible representation of G contained in U were one-dimensional, then G would act as a commutative group of transformations of H, and this contradicts our hypotheses.

Now, given such ρ, consider the orthogonal projection P_ρ of H onto the subspace on which G acts as a sum of copies of ρ; it is given by

$$P_\rho = \frac{d(\rho)}{o(G)} \sum_{g \in G} \overline{\chi_\rho(g)}\, U(g), \tag{1.2}$$

where $\chi_\rho(g) = \mathrm{Tr}\ \rho(g)$ and $d(\rho) = \dim V_\rho$. In other words,

$$P_\rho f(x) = \frac{d(\rho)}{o(G)} \sum_{g \in G} \overline{\chi_\rho(g)}\, f(g^{-1}x). \tag{1.3}$$

We know that $P_\rho \neq 0$. The proof will be complete when we show that the rank of P_ρ is infinite, since no eigenspace of K can contain infinitely many copies of ρ.

Now, if P_ρ has finite rank, it would be a Hilbert-Schmidt operator, so there would exist $\varphi \in L^2(X \times X)$ such that

$$P_\rho f(x) = \int_X \varphi(x, y) f(y)\ d\mu(y). \tag{1.4}$$

However, as long as $P_\rho \neq 0$, (1.3) and (1.4) are incompatible if X has no atoms. In fact, we see that, for any $A \in \mathcal{B}$, the set

$$\widetilde{A} = \{x \in X : \text{not } \varphi(x, y) = 0 \text{ a.e. } y \in A\}$$

satisfies

$$\mu(\widetilde{A}) \leq o(G)\, \mu(A). \tag{1.5}$$

If X has no atoms, this implies $\varphi = 0$ a.e.

Example 1. Let X be the circle S^1 with its standard arc-length measure. Let $V \in C(S^1)$ be real valued. Then the differential operator

$$L = \frac{d^2}{d\theta^2} - V(\theta) \tag{1.6}$$

has compact resolvent. Take $K = (L - \lambda)^{-1}$ for some sufficiently large $\lambda \in (0, \infty)$. We deduce that:

Corollary 1.2. *If $V(\theta)$ is invariant under a rotation through $2\pi/\ell$, for some $\ell \geq 3$, and also invariant under a reflection, then L has infinitely many double eigenvalues.*

Of course, each eigenspace of L has dimension 1 or 2. The argument just recounted arose in a conversation of the author and E. Trubowitz, in 1975.

Example 2. Let X be an equilateral triangle in the plane. Consider the Laplace operator Δ on X, with the Dirichlet (or Neumann) boundary condition. Then Δ is invariant under the group of isometries of X, a group isomorphic to S_3. It follows that Δ has infinitely many multiple eigenvalues.

M. Pinsky [P] has shown that, in this case, Δ has eigenspaces of arbitrarily large dimension. We have a similar situation when X is a square. The number theoretic explanation that Δ has eigenspaces of arbitrarily high dimension is well known in that case.

Note that Example 2 can be extended to every regular polygon in $\mathbb{R}^2$, and also to every regular polyhedron in $\mathbb{R}^3$. I do not know if Δ has eigenspaces of arbitrarily high dimension in all these cases.

There are other variations of Example 2, to which Proposition 1.1 applies. For example, X cound be a wriggly perturbation of an equilateral triangle, still having S_3 as a symmetry group.

To take a variation of Example 1, consider the action of S_5 on $\mathbb{R}^3$, as the group of isometries of the regular icosahedron. This also provides a group of isometries of the unit sphere S^2. One can then consider

$$L = \Delta - V, \tag{1.7}$$

where $V \in C(S^2)$ is real valued and invariant under this action of S_5.

2. Asymptotic density

Here we will specialize, as follows. We take X to be Ω, an open subset of some smooth Riemannian manifold M, such that $\overline{\Omega}$ is compact. We make the following geometrical hypothesis on the action of G on Ω, which implies (1.1):

$$g \neq e \Longrightarrow \text{vol}\,\{x \in \Omega : gx = x\} = 0. \tag{2.1}$$

We suppose L is a strongly elliptic, second order, negative semidefinite, differential operator on Ω, and that the action of U on $L^2(\Omega)$ commutes with the semigroup e^{tL}. Assume either that L has the Dirichlet boundary condition, or that it has some other coercive boundary condition, such as the Neumann boundary condition, and $\partial\Omega$ is sufficiently regular, so that the standard asymptotic analysis of the integral kernel $p(t,x,y)$ of e^{tL} is valid.

In such a case, e^{tL} is trace class for each $t > 0$. For each irreducible representation ρ of G, the operator P_ρ given by (1.2) commutes with e^{tL}, and we have the following two identities. On the one hand,

$$\text{Tr}\ P_\rho e^{tL} = \sum (\dim\ E_{\rho,\lambda}) e^{-t\lambda}, \tag{2.2}$$

where the sum is over $\lambda \in \operatorname{spec}(-L)$ and $E_{\rho,\lambda}$ is the subspace of the λ-eigenspace E_λ of $-L$ on which U acts as a sum of copies of ρ. On the other hand, by (1.3),

$$\operatorname{Tr} P_\rho e^{tL} = \frac{d(\rho)}{o(G)} \sum_{g\in G} \overline{\chi_\rho(g)} \int_\Omega p(t, g^{-1}x, x)\, dV(x). \tag{2.3}$$

The asymptotic analysis of $p(t,x,y)$ alluded to above implies

$$\int_\Omega p(t,x,x)\, dV(x) = (\text{vol } \Omega)(4\pi t)^{-n/2} + o(t^{-n/2}), \tag{2.4}$$

as $t \searrow 0$. On the other hand, the behavior of $p(t,x,y)$ off the diagonal yields the following, in cases where (2.1) holds:

$$g \neq e \Longrightarrow \int_\Omega p(t, g^{-1}x, x)\, dV(x) = o(t^{-n/2}). \tag{2.5}$$

Hence, under these hypotheses, we have

$$\operatorname{Tr} P_\rho e^{tL} = \frac{d(\rho)^2}{o(G)} (\text{vol } \Omega)(4\pi t)^{-n/2} + o(t^{-n/2}), \quad t \searrow 0. \tag{2.6}$$

We are ready to prove the following.

Proposition 2.1. *For $R \in (0,\infty)$, set*

$$F_R = \bigoplus_{\lambda \le R} E_\lambda, \quad F_{\rho,R} = \bigoplus_{\lambda\le R} E_{\rho,\lambda}. \tag{2.7}$$

Then, for each irreducible representation ρ of G,

$$\lim_{R\to\infty} \frac{\dim F_{\rho,R}}{\dim F_R} = \frac{d(\rho)^2}{o(G)}. \tag{2.8}$$

Proof. The asymptotic behavior

$$\dim F_R = \frac{\text{vol } \Omega}{\Gamma(\frac{n}{2}+1)(4\pi)^{n/2}} R^{n/2} + o(R^{n/2}), \quad R \to \infty, \tag{2.9}$$

follows from (2.4), via a well known Tauberian argument. The same argument applied to (2.6) yields

$$\dim F_{\rho,R} = \frac{d(\rho)^2}{o(G)} \frac{\text{vol } \Omega}{\Gamma(\frac{n}{2}+1)(4\pi)^{n/2}} R^{n/2} + o(R^{n/2}), \quad R \to \infty, \tag{2.10}$$

and then (2.8) follows.

Note that Proposition 2.1 applies to all the examples mentioned in §1.

Regarding the right side of (2.8), we note that the subspace of $\ell^2(G)$ on which the regular reprresentation of G acts like copies of ρ is a space of dimension $d(\rho)^2$.

3. D_4 acting on $\mathbb{T}^2$: high multiplicities

The dihedral group D_4 acts as a group of isometries of $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z}^2)$, hence as a unitary group on $L^2(\mathbb{T}^2)$, leaving invariant each eigenspace of the Laplace operator Δ.

Proposition 3.1. *Each irreducible representation ρ of D_4 has the property that there are eigenspaces of Δ containing arbitrarily many copies of ρ.*

To see this, first recall one way of showing that there are eigenspaces of Δ of arbitrarily high dimension. Namely,

$$\mathrm{Spec}(-\Delta) = \{j^2 + k^2 : j, k \in \mathbb{Z}\}, \tag{3.1}$$

and if $\nu = j^2 + k^2$, the dimension of the ν-eigenspace of $-\Delta$ is equal to the number of pairs $(j, k) \in \mathbb{Z} \times \mathbb{Z}$ such that $j^2 + k^2 = \nu$. Now, number theoretical constraints imply that the set of sums of two squares has mean density zero in $\mathbb{Z}^+$. On the other hand, the sum of the dimensions of the ν-eigenspaces of $-\Delta$, for $\nu \le R$, which is the number of integer lattice points within a disk of radius $\sqrt{R}$, behaves like πR as $R \to \infty$. It follows that some eigenspaces must have arbitrarily large dimension.

Now, by Proposition 2.1, the same argument extends to the parts of the eigenspaces of Δ on which D_4 acts like copies of ρ, so the proposition follows.