Remarks on M. Pinsky's Derivation of the Modulus of Continuity for Brownian Paths

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Let $\mathfrak{P}$ be path space for Brownian motion on the line, with Wiener measure P. It is known that there is an estimate

$$|X_t(\omega) - X_s(\omega)| \le M_1(\omega) h(|t-s|), \tag{1}$$

valid for $s, t \in [0,1]$, with $M_1(\omega) < \infty$ for almost all $\omega \in \mathfrak{P}$. Here

$$h(\delta) = \Bigl(\delta \log \frac{1}{\delta}\Bigr)^{1/2} \tag{2}$$

for $0 < \delta \le 1/e$, and we set $h(\delta) = h(1/e) = 1/\sqrt{e}$ for $\delta \ge 1/e$. Recently M. Pinsky [P] produced a pleasant proof of this, using Ciesielski's representation of $X_t(\omega)$ for $0 \le t \le 1$ as a Haar series:

$$X_t(\omega) = \sum_{N=1}^{\infty} Z_N(\omega) \int_0^t \varphi_N(s)\, ds, \tag{3}$$

where $\{\varphi_N : N \ge 1\}$ is the Haar orthonormal basis of $L^2([0,1])$. Path space over $0 \le t \le 1$ is parametrized by

$$\Omega = \prod_{N \ge 1} F_N, \quad F_N = \bigl(\mathbb{R}, (2\pi)^{-1/2} e^{-x^2/2}\, dx\bigr), \tag{4}$$

and $Z_N : \Omega \to \mathbb{R}$ is projection onto the Nth factor, identified with $\mathbb{R}$.

The estimate (1) can be compared and contrasted with the Lévy estimate

$$\limsup_{0 \le s,t \le 1, |s-t| \searrow 0} \frac{|X_t(\omega) - X_s(\omega)|}{h(|t-s|)} = \sqrt{2}, \tag{5}$$

valid for almost all $\omega \in \mathfrak{P}$. For one, (5) implies (1), but with no effective bound on $M_1(\omega)$. In fact, $M_1(\omega)$ cannot be essentially bounded on $\mathfrak{P}$; if it were, one would have for some $K < \infty$ an estimate

$$|X_t(\omega) - X_s(\omega)| \le K h(|t-s|), \tag{6}$$

valid for almost every $\omega \in \mathfrak{P}$, for all $s, t \in [0,1]$. In fact, for *fixed* $s < t \in (0,1]$ the Gaussian statistics for $X_t(\omega) - X_s(\omega)$ guarantee that (6) is violated for a set of ωs of positive measure.

Maximal estimates on Gaussian processes, such as given in Theorem 1.3.3 of [F], imply that once one has $M_1(\omega) < \infty$ almost everywhere in (1), then there is a bound

$$P\big(M_1(\omega) > \lambda\big) \le Ce^{-a\lambda^2}, \tag{7}$$

for some $C < \infty,\ a > 0$. There is even a sharp result on the optimal value of a in (7).

In this note we show that the method of proof of (1) in [P] can be pushed a little further to establish (7) directly (though without a sharp estimate for a). On the other hand, the estimate we establish in Proposition 1 below is in some ways more precise than (7).

To get started, we recall the ingredients of the proof of (1) in [P]. One ingredient is the following set of estimates on the Haar functions:

$$\|\varphi_N\|_{L^\infty(I)} \le CN^{1/2}, \tag{8}$$

and

$$\|\varphi_N\|_{L^1(I)} \le CN^{-1/2}, \tag{9}$$

(where $I = [0,1]$) plus the fact that, over each range $2^{\nu-1} < N \le 2^\nu$, the functions

$$\psi_N(t) = \int_0^t \varphi_N(s)\, ds \tag{10}$$

have disjoint supports. Another ingredient is a study of the function

$$A(\omega) = \sup_{N\ge 2} \frac{|Z_N(\omega)|}{\sqrt{\log N}}. \tag{11}$$

It is shown in Lemma 1 of [P] that $A(\omega) < \infty$ for almost all $\omega \in \Omega$. Then the sum

$$X_t(\omega) - X_s(\omega) = \sum_{N=1}^{\infty} Z_N(\omega)[\psi_N(t) - \psi_N(s)] \tag{12}$$

is broken into two pieces and the estimate (1) is obtained, with $M_1(\omega) \le 1 + 2A(\omega)$. Hence (7) will follow from an associated estimate on $A(\omega)$.

We find it of interest to consider more generally

$$A_\mu(\omega) = \sup_{N\ge \mu} \frac{|Z_N(\omega)|}{\sqrt{\log N}}, \tag{13}$$

for $\mu \ge 2$. Now the nature of Z_N as a Gaussian random variable gives

$$P\big(|Z_N(\omega)| \ge x\big) \le e^{-x^2/2}, \tag{14}$$

hence

$$\mathcal{S}_{N,\lambda} = \{\omega \in \Omega : |Z_N(\omega)| \geq \lambda\sqrt{\log N}\} \Longrightarrow P(\mathcal{S}_{N,\lambda}) \leq N^{-\lambda^2/2}. \tag{15}$$

Now

$$\{\omega \in \Omega : A_\mu(\omega) \geq \lambda\} = \bigcup_{N\geq\mu} \mathcal{S}_{N,\lambda}, \tag{16}$$

so

$$P\big(A_\mu(\omega) \geq \lambda\big) \leq \sum_{N\geq\mu} N^{-\lambda^2/2}. \tag{17}$$

The convexity of the function $f(y) = y^{-s}$ implies that, for $s \geq 2$,

$$\sum_{N\geq\mu} N^{-s} \leq \mu^{-s} + \int_{\mu+1/2}^{\infty} y^{-s}\, dy \leq \Big(\mu + \frac{3}{2}\Big)\mu^{-s}, \tag{18}$$

so we have, for $\lambda \geq 2,\ \mu \geq 2$,

$$P\big(A_\mu(\omega) \geq \lambda\big) \leq \Big(\mu + \frac{3}{2}\Big)\mu^{-\lambda^2/2} \leq C_\mu e^{-K(\mu)\lambda^2}, \tag{19}$$

with $C_\mu = \mu + 3/2,\ K(\mu) = (1/2)\log\mu$.

The $\mu = 2$ case of this estimate is already enough to establish (7), in view of the estimate $M_1(\omega) \leq 1 + 2A_2(\omega)$ established in [P]. However, we will go further (in a parallel fashion). Suppose $\mu = 2^\alpha + 1$ and $\alpha \geq 1$. We will estimate

$$\sum_{2^\nu < N \leq 2^{\nu+1}} Z_N(\omega)[\psi_N(t) - \psi_N(s)] = D_\nu(t, s, \omega) \tag{20}$$

for $\nu \geq \alpha$. The observations in (8)–(10) imply the following two estimates:

$$\begin{aligned} |D_\nu(t, s, \omega)| &\leq C A_\mu(\omega)\sqrt{\nu}2^{\nu/2}|t - s|, \\ |D_\nu(t, s, \omega)| &\leq C A_\mu(\omega)\sqrt{\nu}2^{-\nu/2}. \end{aligned} \tag{21}$$

Hence

$$\begin{aligned} &\Big|\sum_{M\geq\mu} Z_N(\omega)[\psi_N(t) - \psi_N(s)]\Big| \\ &\quad\leq \sum_{\nu\geq\alpha} |D_\nu(t, s, \omega)| \\ &\quad\leq C A_\mu(\omega)\Big[\sum_{\alpha\leq\nu\leq\beta} 2^{\nu/2}\nu^{1/2}|t - s| + \sum_{\nu>\beta} 2^{-\nu/2}\nu^{1/2}\Big] \\ &\quad\leq C_2 A_\mu(\omega)\beta^{1/2}\big[2^{\beta/2}|t - s| + 2^{-\beta/2}\big]. \end{aligned} \tag{22}$$

We can optimize this by picking β such that $2^{-\beta/2} \approx |t-s|$, as long as $|t-s| \le 2^{-\alpha/2}$, say $|t-s| \le \mu^{-1/2}$. This gives

$$\Big|\sum_{N\ge\mu} Z_N(\omega)[\psi_N(t)-\psi_N(s)]\Big| \le C_3 A_\mu(\omega) h(|t-s|), \tag{23}$$

with $h(\delta)$ as in (2). As for the rest of (12), we crudely have

$$\Big|\sum_{N<\mu} Z_N(\omega)[\psi_N(t)-\psi_N(s)]\Big| \le C B_\mu(\omega)|t-s|, \quad B_\mu(\omega)=\mu^{1/2}\sum_{N<\mu}|Z_N(\omega)|. \tag{24}$$

This establishes the following (with slight change in notation):

Proposition 1. *Fix $K \in (0,\infty)$ and set $\delta = e^{-K}$. There exists $a = a(K) > 0$ and $C_j = C_j(K)$ such that, for $t, s \in [0,1]$, $|t-s| \le \delta$,*

$$|X_t(\omega) - X_s(\omega)| \le A_K(\omega) h(|t-s|) + B_K(\omega)|t-s|, \tag{25}$$

with

$$P\big(A_K(\omega)\ge\lambda\big) \le C_1 e^{-K\lambda^2}, \quad P\big(B_K(\omega)\ge\lambda\big)\le C_2 e^{-a\lambda^2}. \tag{26}$$

Returning to the context of the estimate (1), we make a concluding comment. It similarly follows that there is for each $k \in \mathbb{Z}^+$ an estimate

$$|X_t(\omega)-X_s(\omega)| \le M_k(\omega) h(|t-s|), \quad s,t\in[k-1,k]. \tag{27}$$

The functions M_k on $\mathfrak{P}$ can be taken to be independent random variables that are identically distributed.

References

[F] X. Fernique, Regularite des trajectoires des fonctions aleatoires Gaussiennes, pp. 1–96 in LNM #480, Springer-Verlag, New York, 1975.

[P] M. Pinsky, Brownian continuity modulus via series expansions, Jour. of Theoretical Probability, to appear.