Proof of the Lipschitz-Hankel integral formula

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We desire to prove the identity

$$\int_0^\infty e^{-y\lambda} J_\nu(r_1\lambda) J_\nu(r_2\lambda)\, d\lambda = \frac{1}{\pi}(r_1 r_2)^{-1/2}\, Q_{\nu-1/2}\Big(\frac{r_1^2+r_2^2+y^2}{2r_1r_2}\Big), \tag{1}$$

due to Lipschitz and Hankel, of great use for analysis on cones (cf. [CT]). We derive (1) from the identity

$$\int_0^\infty e^{-t\lambda^2} J_\nu(r_1\lambda) J_\nu(r_2\lambda)\lambda\, d\lambda = \frac{1}{2t} e^{-(r_1^2+r_2^2)/4t}\, I_\nu\Big(\frac{r_1r_2}{2t}\Big), \tag{2}$$

for which an elementary proof is given in (8.45) of [T2]. Here

$$I_\nu(y) = e^{-\pi i\nu/2}\, J_\nu(iy), \quad y > 0. \tag{3}$$

To work on (2), we use the subordination identity

$$\lambda^{-1} e^{-y\lambda} = \pi^{-1/2} \int_0^\infty e^{-y^2/4t} e^{-t\lambda^2} t^{-1/2}\, dt; \tag{4}$$

cf. [T1], (5.31) for a proof. Plugging this into the left side of (1), and using (2), we have

$$LHS(1) = \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-(r_1^2+r_2^2+y^2)/4t}\, I_\nu\Big(\frac{r_1r_2}{2t}\Big) t^{-3/2}\, dt. \tag{5}$$

The change of variable $s = r_1r_2/2t$ gives

$$LHS(1) = \sqrt{\frac{1}{2\pi}}\,(r_1r_2)^{-1/2} \int_0^\infty e^{-s(r_1^2+r_2^2+y^2)/2r_1r_2}\, I_\nu(s) s^{-1/2}\, ds. \tag{6}$$

Thus the asserted identity (1) follows from the identity

$$\int_0^\infty e^{-sz} I_\nu(s) s^{-1/2}\, ds = \sqrt{\frac{2}{\pi}} Q_{\nu-1/2}(z), \quad z > 0. \tag{7}$$

As for the validity of (7), we mention two identities. First, we have

$$\begin{aligned}\int_0^\infty e^{-sz} J_\nu(\lambda s) s^{\mu-1}\, ds = \Big(\frac{\lambda}{2}\Big)^\nu z^{-\mu-\nu} \frac{\Gamma(\mu+\nu)}{\Gamma(\nu+1)} \\ \cdot\, {}_2F_1\Big(\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}, \frac{\mu}{2}+\frac{\nu}{2}; \nu+1; -\frac{\lambda^2}{z^2}\Big).\end{aligned} \tag{8}$$

A simple proof of this is given in (8.42) of [T2]. Next, there is the classical representation of the Legendre function $Q_{\nu-1/2}(z)$ as a hypergeometric function:

$$Q_{\nu-1/2}(z) = \frac{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)}\,(2z)^{-\nu-1/2}\, {}_2F_1\Big(\frac{\nu}{2}+\frac{3}{4}, \frac{\nu}{2}+\frac{1}{4}; \nu+1; \frac{1}{z^2}\Big); \tag{9}$$

cf. [Leb], (7.3.7). If we apply (8) with $\lambda = i,\ \mu = 1/2$, then (7) follows.

Remark. Formulas (1) and (2) are proven in the opposite order in [W].

References

[CT] J. Cheeger and M. Taylor, Diffraction of waves by conical singularities, Comm. Pure Appl. Math. 35 (1982), 275–331, 487–529.
[Leb] N. Lebedev, Special Functions and Their Applications, Dover, New York, 1972.
[T1] M. Taylor, Partial Differential Equations, Vol. 1, Chapter 3.
[T2] M. Taylor, Partial Differential Equations, Vol. 2, Chapter 8.
[W] G. Watson, Bessel Functions, Cambridge Univ. Press, 1944.