PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

COMMUTATOR ESTIMATES FOR HÖLDER CONTINUOUS AND BMO-SOBOLEV MULTIPLIERS

MICHAEL E. TAYLOR

(Communicated by A. Iosevich)

Abstract. We discuss conditions on a function f under which the commutator $[P, f]$ of a pseudodifferential operator P of order m with the operation of multiplication by f is an operator of order $m - r$ on various function spaces, namely Hölder-Zygmund spaces and L^p-Sobolev spaces, given $0 < r < 1$. We also establish an endpoint case involving $r = 1$, and we extend the scope to all $r > 0$ for a particularly significant case in 1 dimension.

1. Introduction

The fact that the commutator of a pseudodifferential operator P of order m with the operation of multiplication by a smooth function f is an operator of order $m-1$ is a central result, which has had important refinements. In particular, such a result holds for $f \in \mathrm{Lip}^1$, for a certain range of m; cf. [4], [6], [10], [1], [17]. Here we examine when a Hölder hypothesis on f, or some variant, implies $[P, f]$ has order $m - r$, for some $r \in (0, 1)$. Here is one sample result, when $m = 0,\ 0 < r < 1$:

$$\|[P, f]u\|_{C^r} \le C\|f\|_{C^r}(\|u\|_{L^\infty} + \|Pu\|_{L^\infty}). \tag{1.1}$$

Such a result is useful in the regularity theory of vortex patches; cf. [2], [5]. A proof when P is a classical singular integral operator of convolution type is given in [11], pp. 355–356. The estimate (1.1) is valid more generally for $P \in OPS^0_{1,\delta},\ \delta < 1$, and even more generally for $P \in OP\mathcal{B}S^0_{1,1}$, as we will see below. We recall that $P = p(x, D)$ belongs to $OPS^m_{1,\delta}$ if and only if its symbol $p(x, \xi)$ satisfies

$$|D^\beta_x D^\alpha_\xi p(x, \xi)| \le C_{\alpha\beta}(1 + |\xi|)^{m-|\alpha|+\delta|\beta|}. \tag{1.2}$$

We say $p(x, \xi) \in \mathcal{B}S^m_{1,1}$ provided $p(x, \xi) \in S^m_{1,1}$ and the partial Fourier transform $\hat{p}(\eta, \xi)$ has the property

$$\operatorname{supp} \hat{p} \subset \{(\eta, \xi) : |\eta| \le \rho|\xi|\}, \tag{1.3}$$

for some $\rho < 1$. This class was introduced in [13]. We remark that $OP\mathcal{B}S^m_{1,1}$ contains $OPS^m_{1,\delta}$ (modulo smoothing operators) for each $\delta < 1$.

2010 *Mathematics Subject Classification.* Primary 42B20, 42B25, 42B35, 35S05, 35S50.
Key words and phrases. Commutator, pseudodifferential operator, paraproducts, bounded mean oscillation.
Work supported by NSF grant DMS-1161620.

The analogue of (1.1) for $[P,f]$ acting on L^p-Sobolev spaces $H^{r,p}$ requires an hypothesis on f slightly stronger than $f\in C^r$, namely $f\in\mathfrak{h}^{r,\infty}$, the bmo-Sobolev space,

$$\mathfrak{h}^{r,\infty}=\Lambda^{-r}\,\mathrm{bmo},\quad \Lambda=(1-\Delta)^{1/2}, \tag{1.4}$$

where bmo is the inhomogeneous variant of the John-Nirenberg space BMO (cf. [9]). See Appendix A for a precise definition of the spaces $\mathfrak{h}^{r,\infty}$, and some of their basic properties. Parallel to (1.1), we will show that

$$\|[P,f]u\|_{H^{r,p}}\le C\|f\|_{\mathfrak{h}^{r,\infty}}\|u\|_{L^p}, \tag{1.5}$$

given $P\in OP\mathcal{B}S^0_{1,1},\ 0<r<1,\ 1<p<\infty$. We will establish the following further estimates, complementing (1.1) and (1.5). See also Appendix A for a definition of the spaces C^s_* arising in (1.7), and a description of some of their basic properties.

Proposition 1.1. *Let $P\in OP\mathcal{B}S^m_{1,1}$. Assume $0<r<1$ and*

$$-r<s<0,\quad s<m<r+s. \tag{1.6}$$

Then

$$f\in C^r\Longrightarrow [P,f]:C^s_*\to C^{r+s-m}_*. \tag{1.7}$$

Proposition 1.2. *Let $P\in OP\mathcal{B}S^m_{1,1}$. Assume $0<r<1,\ 1<p<\infty$, and*

$$-r<s\le 0,\quad s\le m\le r+s. \tag{1.8}$$

Then

$$f\in\mathfrak{h}^{r,\infty}\Longrightarrow [P,f]:H^{s,p}\to H^{r+s-m,p}. \tag{1.9}$$

Note that Proposition 1.2 provides a strict extension of (1.5) (where $m=0$), while (with $m=0$) the $s=0$ limit of Proposition 1.1, given in (1.1), takes $u,Pu\in L^\infty$ rather than in C^0_*.

In §2 we start the proofs of the results stated above, using paraproducts. Crucial paraproduct estimates are established in §3, to complete the proofs of these results. In §4 we supplement (1.9) with estimates on $\|[P,f]u\|_{\mathfrak{h}^{r,\infty}}$. We also get such an estimate for $f\in\mathrm{Lip}^1$, supplementing estimates of Calderon-Coifman-Meyer type. In §5 we extend the scope of such estimates to all $r>0$ when P is a particularly important singular integral integral operator on the circle (essentially the Hilbert transform). We end with Appendix A, advertised above, which provides some material on the spaces $\mathfrak{h}^{r,\infty}$ and C^s_*, appearing in Propositions 1.1 and 1.2.

Remark. The case $r=0$ of (1.5) is also valid. This was established in [7] for a classical pseudodifferential operator of order zero, and in [1] for $P\in OP\mathcal{B}S^0_{1,1}$. The case $m=r+s, s=0$ of Proposition 1.2 was established in [14], for classical pseudodifferential operators of convolution type, in dimension 1.

2. Paraproduct decompositions and preliminary estimates

An essential ingredient in our analysis is the paraproduct operation of J.-M. Bony:

$$T_f u=\sum_{k\ge 0} f_k\,\psi_k(D)u, \tag{2.1}$$

where $\{\psi_k\}$ is a Littlewood-Paley partition of unity and $f_k = \sum_{j\le k-3}\psi_j(D)f$. Along the lines of arguments used in [15] and [1], we write

$$P(fu) = PT_f u + PT_u f + PR(f,u),\qquad f\,Pu = T_f Pu + T_{Pu} f + R(f,Pu). \tag{2.2}$$

Here

$$R(f,u) = \sum_{|j-k|\le 2} \psi_j(D)f\cdot\psi_k(D)u. \tag{2.3}$$

To begin, we have, for $0<r<1$, $P\in OP\mathcal{B}S^m_{1,1}$, with ρ in (1.3) sufficiently small,

$$f\in C^r \Longrightarrow [P,T_f]\in OP\mathcal{B}S^{m-r}_{1,1}. \tag{2.4}$$

Such a result follows from the paradifferential operator calculus initiated in [3] and [13]; cf. also Proposition 7.3 in Chapter I of [16]. From (2.4) we have

$$[P,T_f]: C^s_* \longrightarrow C^{r+s-m}_*,\quad [P,T_f]: H^{s,p}\longrightarrow H^{r+s-m,p}, \tag{2.5}$$

for all $s,m\in\mathbb{R},\ p\in(1,\infty)$, given $f\in C^r,\ 0<r<1$.

To proceed, we have the following information on the operator $R_f u = R(f,u)$:

$$f\in C^r_*\Longrightarrow R_f\in OPS^{-r}_{1,1}; \tag{2.6}$$

this holds for all $r\in\mathbb{R}$; cf. [15], (3.5.11). Hence, given $P\in OP\mathcal{B}S^m_{1,1}$,

$$\|PR(f,u)\|_{C^{r+s-m}_*}\le C\|f\|_{C^r_*}\|u\|_{C^s_*}, \tag{2.7}$$

provided $r+s>0$, and

$$\|R(f,Pu)\|_{C^{r+s-m}_*}\le C\|f\|_{C^r_*}\|u\|_{C^s_*}, \tag{2.8}$$

provided $r+s>m$. Regarding Sobolev estimates, if $1<p<\infty$, we have

$$\|PR(f,u)\|_{H^{r+s-m,p}}\le C\|f\|_{C^r_*}\|u\|_{H^{s,p}}, \tag{2.9}$$

provided $r+s>0$, and

$$\|R(f,Pu)\|_{H^{r+s-m,p}}\le C\|f\|_{C^r_*}\|u\|_{H^{s,p}}, \tag{2.10}$$

provided $r+s>m$.

To complete the proofs of the results stated in §1, it remains to estimate PT_uf and $T_{Pu}f$, and to supplement (2.10) by

$$\|R(f,Pu)\|_{L^p}\le C\|f\|_{\mathfrak{h}^{r,\infty}}\|u\|_{H^{s,p}},\quad m=r+s,\ r\ge 0. \tag{2.11}$$

We undertake these estimates in the next section.

3. Complementary paraproduct estimates

Here we complete the proof of the results stated in §1, via estimates on T_vf. One basic estimate comes from

$$v\in L^\infty\Longrightarrow T_v\in OP\mathcal{B}S^0_{1,1}. \tag{3.1}$$

In particular, given $P\in OP\mathcal{B}S^0_{1,1},\ r\in\mathbb{R}$,

$$\|PT_uf\|_{C^r}\le C\|u\|_{L^\infty}\|f\|_{C^r},\quad \|T_{Pu}f\|_{C^r}\le C\|Pu\|_{L^\infty}\|f\|_{C^r}; \tag{3.2}$$

cf. [15], (3.5.5). This, together with estimates of §2, is enough to establish (1.1).

Another basic estimate comes from

$$v\in C^{-s}_*,\quad s>0\Longrightarrow T_v\in OP\mathcal{B}S^s_{1,1}; \tag{3.3}$$

cf. [15], (3.5.7). Thus, given $P \in OP\mathcal{B}S^m_{1,1},\ 0 < r < 1$,

$$s < 0, \quad u \in C^s_*, \quad f \in C^r \Longrightarrow PT_u f \in C_*^{-m+s+r}, \tag{3.4}$$

and

$$s - m < 0, \quad u \in C^s_*, \quad f \in C^r \Longrightarrow T_{Pu} f \in C_*^{-m+s+r}. \tag{3.5}$$

These results complete the proof of Proposition 1.1.

To obtain paraproduct estimates to establish (1.5) and Proposition 1.2, we start with the Coifman-Meyer estimate (cf. [6])

$$\|\tau(f,v)\|_{L^p} \le C\|f\|_{\mathrm{BMO}}\|v\|_{L^p}, \tag{3.6}$$

valid for $p \in (1,\infty)$, for a number of paraproduct operators, including

$$\tau(f,v) = T_v f, \quad \tau(f,v) = R(f,v). \tag{3.7}$$

The following consequence of (3.6) was demonstrated in Proposition 3.5.F of [15].

Lemma 3.1. *For $\tau(f,v)$ as in (3.7), we have, for each $p \in (1,\infty),\ r \in \mathbb{Z}^+$,*

$$\|\tau(f,v)\|_{H^{s,p}} \le C\|f\|_{\mathfrak{h}^{r,\infty}}\|v\|_{H^{s-r,p}}, \quad 0 \le s \le r. \tag{3.8}$$

We produce further extensions of this result. To rephrase (3.8) in case $s = r = k$, if we set $\tau_v f = \tau(f,v)$, then, for $p \in (1,\infty)$,

$$v \in L^p \Longrightarrow \tau_v : \mathfrak{h}^{k,\infty} \to H^{k,p}, \quad k \in \mathbb{Z}^+. \tag{3.9}$$

Interpolation (cf. [20], Proposition 5.3) gives

$$v \in L^p \Longrightarrow \tau_v : \mathfrak{h}^{r,\infty} \to H^{r,p}, \quad r \in [0,\infty). \tag{3.10}$$

In case $\tau_v f = T_v f$, we deduce that for $P \in OP\mathcal{B}S^0_{1,1},\ r > 0,\ p \in (1,\infty)$,

$$\|PT_u f\|_{H^{r,p}} \le C\|f\|_{\mathfrak{h}^{r,\infty}}\|u\|_{L^p}, \quad \|T_{Pu} f\|_{H^{r,p}} \le C\|f\|_{\mathfrak{h}^{r,\infty}}\|u\|_{L^p}. \tag{3.11}$$

This, together with estimates of §2, establishes (1.5).

To complete the proof of Proposition 1.2, we will establish the following extension of Lemma 3.1.

Proposition 3.2. *Given τ as in Lemma 3.1 and $p \in (1,\infty)$, the conclusion of Lemma 3.1 holds for all $r \in [0,\infty)$.*

Proof. We find it convenient to change notation slightly, and show that

$$s,\sigma \ge 0,\ f \in \mathfrak{h}^{s+\sigma,\infty},\ v \in H^{-\sigma,p} \Longrightarrow \tau(f,v) \in H^{s,p}. \tag{3.12}$$

For $\sigma = 0$, this follows from (3.10). We next claim it holds for each $s \in [0,\infty)$ and $\sigma = k \in \mathbb{Z}^+$. The proof is inductive. If (3.12) is valid for $\sigma = k \le \ell$ and if $v \in H^{-\ell-1,p}$, with $v = \partial_j v_j,\ v_j \in H^{-\ell,p}$, while $f \in \mathfrak{h}^{s+\ell+1,\infty}$, use

$$\tau(f,v) = \partial_j \tau(f,v_j) - \tau(\partial_j f, v_j) \tag{3.13}$$

to get (3.12) for $\sigma = \ell + 1$.

To finish the proof of Proposition 3.2, let us rephrase the result (3.12) as

$$\|\tau(\Lambda^{-\sigma} g, \Lambda^\sigma u)\|_{H^{s,p}} \le C\|g\|_{\mathfrak{h}^{s,\infty}}\|u\|_{L^p}, \quad s,\sigma \ge 0. \tag{3.14}$$

So far we have this for $\sigma = k \in \mathbb{Z}^+$. Let us set

$$\Phi(z) = \tau(\Lambda^{-z} g, \Lambda^z u), \quad \operatorname{Re} z \ge 0. \tag{3.15}$$

Then, for $k \in \mathbb{Z}^+, y \in \mathbb{R}$,

$$\Phi(k+iy) = \tau(\Lambda^{-k-iy} g, \Lambda^{k+iy} u) \in H^{s,p}, \tag{3.16}$$

with mild bounds as $|y| \to \infty$. Hence a maximum principle argument for vector-valued holomorphic functions yields (3.14) and completes the proof of Proposition 3.2. $\square$

Applying (3.8) with $\tau(f,v) = T_v f$ to $PT_u f$ and $T_{Pu} f$, we have the following. Assume $P \in OP\mathcal{B}S^m_{1,1}$. Change notation in (3.8), replacing s by $-s$. then we have

$$r > 0,\ s \le 0,\ s + r \ge 0 \Longrightarrow \|PT_u f\|_{H^{s+r-m,p}} \le C\|f\|_{\mathfrak{h}^{r,\infty}} \|u\|_{H^{s,p}}, \tag{3.17}$$

and

$$r > 0,\ s \le m,\ s + r \ge m \Longrightarrow \|T_{Pu} f\|_{H^{s+r-m,p}} \le C\|f\|_{\mathfrak{h}^{r,\infty}} \|u\|_{H^{s,p}}. \tag{3.18}$$

On the other hand, using $\tau(f,v) = R(f,v)$ and taking $s = 0$, we have

$$\|R(f,v)\|_{L^p} \le C\|f\|_{\mathfrak{h}^{r,\infty}} \|v\|_{H^{-r,p}}, \tag{3.19}$$

whenever $r \ge 0,\ p \in (1,\infty)$, which implies (2.11). This completes the proof of Proposition 1.2.

Remark. The $s = r$ case of (3.8) also gives an endpoint case of (3.11) in [17].

4. bmo-Sobolev space estimates

Here we estimate $[P,f]u$ in the $\mathfrak{h}^{r,\infty}$-norm, providing a $p = \infty$ endpoint case of (1.9). For simplicity we take the order m of P to be zero. We establish the following.

Proposition 4.1. *Let $P \in OP\mathcal{B}S^0_{1,1}$. Then*

$$\|[P,f]u\|_{\mathfrak{h}^{r,\infty}} \le C_r \|f\|_{\mathfrak{h}^{r,\infty}} \bigl(\|u\|_{L^\infty} + \|Pu\|_{L^\infty}\bigr), \quad \textit{for } 0 < r < 1, \tag{4.1}$$

$$\|[P,f]u\|_{\mathfrak{h}^{1,\infty}} \le C\|f\|_{\mathrm{Lip}^1} \bigl(\|u\|_{L^\infty} + \|Pu\|_{L^\infty}\bigr). \tag{4.2}$$

The estimate (4.2) is an endpoint case of an estimate of Calderon-Coifman-Meyer type:

$$\|[P,f]u\|_{H^{1,p}} \le C\|f\|_{\mathrm{Lip}^1} \|u\|_{L^p}, \quad 1 < p < \infty. \tag{4.3}$$

To perform these estimates, we again use (2.2). We also make use of the fact that

$$P \in OPS^m_{1,1},\ s - m > 0 \Longrightarrow P : \mathfrak{h}^{s,\infty} \to \mathfrak{h}^{s-m,\infty}, \tag{4.4}$$

which is the endpoint case of the well known behavior on L^p-Sobolev spaces. This result is the case $p = \infty,\ q = 2$ of Theorem I of [21]. In light of this, (2.4) yields

$$\|[P,T_f]u\|_{\mathfrak{h}^{r,\infty}} \le C\|f\|_{C^r} \|u\|_{\mathrm{bmo}}, \quad 0 < r < 1. \tag{4.5}$$

Furthermore, complementary to (2.4), we have

$$f \in \mathrm{Lip}^1 \Longrightarrow [T_f, P] \in OP\mathcal{B}S^{-1}_{1,1}; \tag{4.6}$$

cf. [1], Proposition 4.2, or [16], Proposition 7.4 of Chapter I. Hence

$$\|[P,T_f]u\|_{\mathfrak{h}^{1,\infty}} \le C\|f\|_{\mathrm{Lip}^1} \|u\|_{\mathrm{bmo}}. \tag{4.7}$$

Also we can use (2.6) to obtain

$$f \in C^r_* \Longrightarrow R_f : \mathrm{bmo} \to \mathfrak{h}^{r,\infty}, \quad r > 0. \tag{4.8}$$

Hence

$$\|PR(f,u)\|_{\mathfrak{h}^{r,\infty}},\ \|R(f,Pu)\|_{\mathfrak{h}^{r,\infty}} \le C\|f\|_{C^r_*} \|u\|_{\mathrm{bmo}}, \quad r > 0. \tag{4.9}$$

Finally an application of (3.1) gives

$$\|PT_u f\|_{\mathfrak{h}^{r,\infty}} \le C\|f\|_{\mathfrak{h}^{r,\infty}}\|u\|_{L^\infty}, \quad r \in \mathbb{R}, \tag{4.10}$$

and

$$\|T_{Pu} f\|_{\mathfrak{h}^{r,\infty}} \le C\|f\|_{\mathfrak{h}^{r,\infty}}\|Pu\|_{L^\infty}, \quad r \in \mathbb{R}. \tag{4.11}$$

These estimates together establish (4.1)–(4.2).

5. Special estimates in one dimension

Estimates given in Propositions 1.1–1.2 and in Proposition 4.1 hold for a wider range of r when P is a particularly important singular integral operator on the circle S^1, namely

$$P_+ \sum_{\nu\in\mathbb{Z}} a_\nu e^{i\nu\theta} = \sum_{\nu\ge 0} a_\nu e^{i\nu\theta}. \tag{5.1}$$

What is special about this operator is that its symbol is constant on each connected component of $T^*S^1 \setminus 0$. Thus, standard symbol asymptotics give, for any $\delta < 1$,

$$A \in OPS^m_{1,\delta}(S^1) \Longrightarrow [A, P_+] \in OPS^{-\infty}. \tag{5.2}$$

The following related result was established in (15.14), Chapter I, of [16]:

$$f \in L^\infty(S^1) \Longrightarrow [T_f, P_+] \in OPS^{-\infty}. \tag{5.3}$$

For use in Proposition 5.2, we mention that this argument in [16] extends, and we can take $f \in \mathcal{D}'(S^1)$ in (5.3). Using (5.3), we will prove the following.

Proposition 5.1. *With P_+ given by* (5.1)*, the following commutator estimates hold:*

$$\|[P_+, f]u\|_{C^r_*} \le C\|f\|_{C^r_*}\big(\|u\|_{L^\infty} + \|P_+u\|_{L^\infty}\big), \quad r > 0, \tag{5.4}$$

$$\|[P_+, f]u\|_{H^{r,p}} \le C\|f\|_{\mathfrak{h}^{r,\infty}}\|u\|_{L^p}, \quad r > 0,\ 1 < p < \infty, \tag{5.5}$$

$$\|[P_+, f]u\|_{\mathfrak{h}^{r,\infty}} \le C\|f\|_{\mathfrak{h}^{r,\infty}}\big(\|u\|_{L^\infty} + \|P_+u\|_{L^\infty}\big), \quad r > 0. \tag{5.6}$$

Proof. Going back to (2.2), we see that adequate estimates on $P_+R(f,u)$ and $R(f, P_+u)$ already follow from (2.7)–(2.10) and (4.8), while adequate estimates on P_+T_uf and $T_{P_+u}f$ follow from (3.2), (3.11), and (4.10)–(4.11). This just leaves estimates on $[T_f, P_+]$, which follow immediately from (5.3). □

Proposition 5.1 has an application to loop group factorization, given in [18]. The following commutator estimate also has an application there.

Proposition 5.2. *For $1 < p < \infty$,*

$$\|[P_+, f]u\|_{H^{r,p}} \le C\|f\|_{H^{r,p}}\big(\|u\|_{L^\infty} + \|P_+u\|_{L^\infty}\big), \quad r > 0. \tag{5.7}$$

Proof. This follows from estimates of P_+T_uf and $T_{P_+u}f$ in $H^{r,p}$, estimates of $P_+R(f,u)$ and $R(f,P_+u)$ in $H^{r,p}$, and of $[T_f, P_+]u$ (using (5.3), strengthened to allow $f \in \mathcal{D}'(S^1)$), in a similar fashion to the arguments given above. We merely replace information on R_f in (2.6) by the implication $u \in L^\infty \Rightarrow R_u \in OPS^0_{1,1}$. □

We mention a version of Proposition 5.1 that holds when S^1 is replaced by $\mathbb{R}$. Namely, take

$$q \in C^\infty(\mathbb{R}), \quad q(\xi) = 0 \ \text{ for } \ \xi \le -1, \ \ q(\xi) = 0 \ \text{ for } \ \xi \ge 1, \tag{5.8}$$

and set

$$Q_+ = q(D), \quad Q_+ \in OPS^0(\mathbb{R}). \tag{5.9}$$

Parallel to (5.2), we have, for $\delta < 1$,

$$A \in OPS^m_{1,\delta}(\mathbb{R}) \Longrightarrow [A, Q_+] \in OPS^{-\infty}(\mathbb{R}). \tag{5.10}$$

Also, parallel to (5.3),

$$f \in L^\infty(\mathbb{R}) \Longrightarrow [T_f, Q_+] \in OPS^{-\infty}(\mathbb{R}). \tag{5.11}$$

In fact, the proof of (15.14) in Chapter 1 of [16] just relies on Proposition 6.1 in Chapter 1 of [16], which works in the setting of $\mathbb{R}^n$. Consequently, analogues of the estimates (5.4)–(5.6) hold for $[Q_+, f]u$, with f and u defined on $\mathbb{R}$.

Appendix A. The spaces C^r_* and $\mathfrak{h}^{r,\infty}$

The spaces $C^r_*(\mathbb{R}^n)$, sometimes called Zygmund spaces, extend to all $r \in \mathbb{R}$ the family of spaces $C^r(\mathbb{R}^n)$, defined for $r \in (0,\infty) \setminus \mathbb{Z}^+$ as follows. If $0 < r < 1$, $C^r(\mathbb{R}^n)$ consists of Hölder continuous functions with exponent r. If $r = k + s$, $k \in \mathbb{Z}^+$, $0 < s < 1$, then $u \in C^r(\mathbb{R}^n)$ if and only if $\partial^\alpha u \in C^s(\mathbb{R}^n)$ whenever $|\alpha| \le k$. The spaces $C^r_*(\mathbb{R}^n)$ are conveniently defined using a Littlewood-Paley partition of unity, $\{\psi_k : k \ge 0\}$. Take $\psi_0 \in C_0^\infty(\mathbb{R}^n)$, $\psi_0(\xi) = 1$ for $|\xi| \le 1$, 0 for $|\xi| \ge 2$, set $\varphi_k(\xi) = \psi_0(2^{-k}\xi)$, and set $\psi_k(\xi) = \varphi_k(\xi) - \varphi_{k-1}(\xi)$ for $k \ge 1$. Then, given a tempered distribution u on $\mathbb{R}^n$,

$$u \in C^r_*(\mathbb{R}^n) \Longleftrightarrow \sup_{k\ge 0} \|\psi_k(D)u\|_{L^\infty} < \infty. \tag{A.1}$$

One has (cf. [15], pp. 183–184) that $C^r_*(\mathbb{R}^n) = C^r(\mathbb{R}^n)$ whenever $r \in (0,\infty) \setminus \mathbb{Z}^+$. One also has

$$P \in OPS^m_{1,1}(\mathbb{R}^n) \Longrightarrow P : C^r_*(\mathbb{R}^n) \to C^{r-m}_*(\mathbb{R}^n), \quad \text{if } \ r - m > 0, \tag{A.2}$$

and

$$O \in OP\mathcal{B}S^m_{1,1}(\mathbb{R}^n) \Longrightarrow P : C^r_*(\mathbb{R}^n) \to C^{r-m}_*(\mathbb{R}^n), \quad \forall\, m, r \in \mathbb{R}. \tag{A.3}$$

In particular, if $0 \le \delta < 1$,

$$P \in OPS^m_{1,\delta}(\mathbb{R}^n) \Longrightarrow P : C^r_*(\mathbb{R}^n) \to C^{r-m}_*(\mathbb{R}^n), \quad \forall\, m, r \in \mathbb{R}. \tag{A.4}$$

It follows that, for $\Lambda = (1-\Delta)^{1/2}$, i.e., $\widehat{\Lambda u}(\xi) = (1+|\xi|^2)^{1/2}\hat{u}(\xi)$,

$$\Lambda^m : C^r_*(\mathbb{R}^n) \xrightarrow{\approx} C^{r-m}_*(\mathbb{R}^n), \quad \forall\, r, m \in \mathbb{R}. \tag{A.5}$$

See [15], Chapter 2, for more operator results. The characterization (A.1) also presents $C^r_*(\mathbb{R}^n)$ as a Besov space:

$$C^r_*(\mathbb{R}^n) = B^r_{\infty,\infty}(\mathbb{R}^n), \quad \forall\, r \in \mathbb{R}. \tag{A.6}$$

For more on this perspective, see [22], pp. 89–91.

We turn to the spaces $\mathfrak{h}^{r,\infty}(\mathbb{R}^n)$, defined in terms of $\mathrm{bmo}(\mathbb{R}^n)$. To start, we recall the John-Nirenberg space

$$\mathrm{BMO}(\mathbb{R}^n) = \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : u^\# \in L^\infty(\mathbb{R}^n)\} \tag{A.7}$$

where

$$u^\#(x) = \sup_{B \in \mathcal{B}(x)} \frac{1}{V(B)} \int_B |u(y) - u_B|\, dy, \tag{A.8}$$

with $\mathcal{B}(x) = \{B_r(x) : 0 < r < \infty\}$, $B_r(x)$ being the ball centered at x of radius r, and u_B the mean value of u on B. There are variants, giving the same space. For example, one could use cubes containing x instead of balls centered at x, and one could replace u_B in (A.8) by c_B, chosen to minimize the integral. We set $\|u\|_{\mathrm{BMO}} = \|u^\#\|_{L^\infty}$. This is not a norm, since $\|c\|_{\mathrm{BMO}} = 0$ if c is a constant; it is a seminorm. The space $\mathrm{bmo}(\mathbb{R}^n)$, introduced in [9], is defined by

$$\mathrm{bmo}(\mathbb{R}^n) = \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : {}^\# u \in L^\infty(\mathbb{R}^n)\}, \tag{A.9}$$

where

$$ {}^\# u(x) = \sup_{B \in \mathcal{B}_1(x)} \frac{1}{V(B)} \int_B |u(y) - u_B|\, dy + \frac{1}{V(B_1(x))} \int_{B_1(x)} |u(y)|\, dy, \tag{A.10}$$

with $\mathcal{B}_1(x) = \{B_r(x) : 0 < r \le 1\}$. We set $\|u\|_{\mathrm{bmo}} = \|{}^\# u\|_{L^\infty}$. This is a norm, and it has good localization properties. For example,

$$f \in C^r(\mathbb{R}^n),\ u \in \mathrm{bmo}(\mathbb{R}^n),\ r > 0 \Longrightarrow fu \in \mathrm{bmo}(\mathbb{R}^n). \tag{A.11}$$

Also (cf. [16], p. 30),

$$P \in OP\mathcal{B}S^0_{1,1}(\mathbb{R}^n) \Longrightarrow P : \mathrm{bmo}(\mathbb{R}^n) \to \mathrm{bmo}(\mathbb{R}^n), \tag{A.12}$$

so in particular, if $0 \le \delta < 1$,

$$P \in OPS^0_{1,\delta}(\mathbb{R}^n) \Longrightarrow P : \mathrm{bmo}(\mathbb{R}^n) \to \mathrm{bmo}(\mathbb{R}^n). \tag{A.13}$$

Now, given $r \in \mathbb{R}$, we define

$$\mathfrak{h}^{r,\infty}(\mathbb{R}^n) = \{\Lambda^{-r} u : u \in \mathrm{bmo}(\mathbb{R}^n)\}, \tag{A.14}$$

with Λ as in (A.5). Thus $\mathfrak{h}^{0,\infty}(\mathbb{R}^n) = \mathrm{bmo}(\mathbb{R}^n)$. It follows from (A.12)–(A.13) and pseudodifferential operator calculus that, given $r, m \in \mathbb{R},\ 0 \le \delta < 1$,

$$\begin{aligned} P \in OP\mathcal{B}S^m_{1,1}(\mathbb{R}^n) &\Longrightarrow P : \mathfrak{h}^{r,\infty}(\mathbb{R}^n) \to \mathfrak{h}^{r-m,\infty}(\mathbb{R}^n), \\ P \in OPS^m_{1,\delta}(\mathbb{R}^n) &\Longrightarrow P : \mathfrak{h}^{r,\infty}(\mathbb{R}^n) \to \mathfrak{h}^{r-m,\infty}(\mathbb{R}^n). \end{aligned} \tag{A.15}$$

We briefly indicate how to define these spaces on a compact Riemannian manifold M. The spaces $C^r_*(M)$ can be defined via a partition of unity and local coordinate charts, leading to elements of $C^r_*(\mathbb{R}^n)$. In case $r \in (0,\infty) \setminus \mathbb{Z}^+$, one clearly has $C^r_*(M) = C^r(M)$, classically defined. Also, one can deduce from (A.4) that

$$P \in OPS^m_{1,0}(M) \Longrightarrow P : C^r_*(M) \to C^{r-m}_*(M), \quad \forall\, r, m \in \mathbb{R}. \tag{A.16}$$

The spaces $\mathfrak{h}^{r,\infty}(M)$ can also be defined via a partition of unity and local coordinate charts. We refer to [19] for details, worked out there for the more general class of complete Riemannian manifolds with bounded geometry. Parallel to (A.16), we have

$$P \in OPS^m_{1,0}(M) \Longrightarrow P : \mathfrak{h}^{r,\infty}(M) \to \mathfrak{h}^{r-m,\infty}(M), \quad \forall\, r, m \in \mathbb{R}. \tag{A.17}$$

References

[1] P. Auscher and M. Taylor, Paradifferential operators and commutator estimates, Comm. PDE 20 (1995), 1743–1775.

[2] A. Bertozzi and P. Constantin, Global regularity for vortex patches, Commun. Math. Phys. 152 (1993), 19–28.

[3] J.-M. Bony, Calcul symbolique et propagation des singularitiés pour les équations aux dérivées nonlinéaires, Ann. Sci. Ecole Norm. Sup. 14 (1981), 209–246.

[4] A. Calderon, Commutators of singular integral operators, Proc. NAS, USA 53 (1965), 1092–1099.

[5] J.-Y. Chemin, Persistance de structures geometriques dans les fluides incompressibles bidimensionnels, Ann. Ecole Norm. Sup. 26 (1993), 1–16.
[6] R. Coifman and Y. Meyer, Au dela des opérateurs pseudodifferentiels, Asterisque #57, Soc. Math. de France, 1978.
[7] R. Coifman, R. Rochberg, and G. Weiss, Factorization theorems for Hardy spaces in several variables, Ann. of Math. 103 (1976), 611–635.
[8] C. Fefferman and E. Stein, H^p functions of several variables, Acta Math. 129 (1972), 137–193.
[9] D. Goldberg, A local version of real Hardy spaces, Duke Math. J. 46 (1979), 27–42.
[10] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, CPAM 41 (1988), 891–907.
[11] A. Majda and A. Bertozzi, Vorticity and Incompressible Flow, Cambridge Univ. Press, 2002.
[12] J. Marschall, Pseudodifferential operators with coefficients in Sobolev spaces, Trans. AMS 307 (1988), 335–361.
[13] Y. Meyer, Remarques sur un théorème de J.M. Bony, Rend. del Circolo mat. di Palermo (suppl. II:1) (1981), 1–20.
[14] M. Murray, Commutators with fractional differentiation and BMO Sobolev spaces, Indiana Univ. Math. J. 34 (1985), 205–215.
[15] M. Taylor, Pseudodifferential Operators and Nonlinear PDE, Birkhauser, Boston, 1991.
[16] M. Taylor, Tools for PDE, Math. Surveys and Monogr., #81, AMS, Providence, R.I., 2000.
[17] M. Taylor, Commutator estimates, Proc. AMS 131 (2003), 1501–1507.
[18] M. Taylor, Regularity of loop group factorization, Proc. AMS 133 (2005), 627–631.
[19] M. Taylor, Hardy spaces and bmo on manifolds with bounded geometry, J. Geometric Anal. 19 (2009), 137–190.
[20] M. Taylor, Variations on complex interpolation, Preprint, available at http://www.unc.edu/math/Faculty/met/cpxint.pdf.
[21] R. Torres, Continuity properties of pseudodifferential operators of type 1,1, Comm. PDE 15 (1990), 1313–1328.
[22] H. Triebel, Theory of Function Spaces, Birkhauser, Basel, 1983.

University of North Carolina, Chapel Hill NC 27599, USA