The Gauss-Green Formula
(And Elliptic Boundary Problems On Rough Domains)

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Dirichlet Problem on Ω (open in a compact Riemannian manifold M), dimension n:

$$Lu = 0 \text{ on } \Omega, \quad u = f \text{ on } \partial\Omega. \tag{1}$$

$L = \Delta - V,\ V \in L^\infty(M),\ V \geq 0,\ V > 0$ on a set of positive measure, on each connected component of $M \setminus \overline{\Omega}$. Assume $\partial\Omega$ is "rough".
Examples: C^1, Lipschitz, bmo_1, vmo_1, vanishing chord-arc ("Semmes-Kenig-Toro domains").
Layer potential method: Look for solution as

$$u = \mathcal{D}g, \tag{2}$$

where $\mathcal{D}$ is double layer potential:

$$\mathcal{D}g(x) = \int_{\partial\Omega} \partial_{\nu_y} E(x,y) g(y)\, d\sigma(y), \tag{3}$$

$E(x,y)$ integral kernel for $L^{-1} : H^{-1}(M) \to H^1(M)$.

$$g(x)^{1/2} E(x,y) = e_0(x, x-y) + e_1(x,y), \tag{4}$$

in local coordinates, where

$$e_0(x,z) = C_n \Big(\sum g_{jk}(x) z_j z_k \Big)^{-(n-2)/2}, \tag{5}$$

$e_1(x,y)$ somewhat tamer.

Layer Potential Estimates for various classes of Ω.

$$\|K^* g\|_{L^p(\partial\Omega)} \leq C \|g\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty. \tag{6}$$

$$
\begin{aligned}
K^* g(x) &= \sup_{0<\varepsilon<1} |K_\varepsilon g(x)|, \\
K_\varepsilon g(x) &= \int\limits_{y\in\partial\Omega, |x-y|<\varepsilon} \partial_{\nu_y} E(x,y) g(y)\, d\sigma(y).
\end{aligned}
$$

Nontangential maximal estimate:

$$
\|\mathcal{N}\mathcal{D}g\|_{L^p(\partial\Omega)} \le C\|g\|_{L^p(\partial\Omega)}, \quad 1<p<\infty. \tag{7}
$$

Boundary behavior,

$$
\lim_{y\to x} \mathcal{D}g(y) = \Big(\frac{1}{2}I + K\Big) g(x), \quad \text{a.e. on } \partial\Omega, \tag{8}
$$

as $y \to x$ nontangentially, from within Ω, where

$$
Kg(x) = \lim_{\varepsilon\to 0} K_\varepsilon g(x). \tag{9}
$$

Validity of these estimates and limits
"Elementary" if $\partial\Omega$ is smoother than C^1.
C^1 case: Fabes-Jodeit-Riviere (following Calderon)
Lipschitz case: (6)–(7) Coifman-McIntosh-Meyer, (8) Verchota ([MT] for variable coefficients)
More general case: Uniformly rectifiable boundary **(UR domain)**.
One assumes $\partial\Omega$ is **Ahlfors regular**, i.e., $\forall\, p \in \partial\Omega$,

$$
C_1 r^{n-1} \le \mathcal{H}^{n-1}(B_r(p)\cap\partial\Omega) \le C_2 r^{n-1},
$$

and one assumes $\partial\Omega$ contains "big pieces of Lipschitz surfaces" (uniformly, on all scales).
For UR domains, (6) is due to G. David, (7)–(8) to [HMT].

Solving Dirichlet Problem done by inverting $\frac{1}{2}I + K$. To show:

$$
\text{Fredholm of index zero on } \ L^p(\partial\Omega), \tag{10}
$$

$$
\text{Adjoint injective on } \ L^p(\partial\Omega). \tag{11}
$$

Done by Fabes-Jodeit-Riviere for $1<p<\infty$, for C^1 domains.
Done by Verchota for $2-\varepsilon<p<\infty$, for Lipschitz domains. ([MT] for variable coefficients)
Done by [HMT] for $1<p<\infty$, for regular Semmes-Kenig-Toro domains.

Definition. Ω is a regular SKT domain (vanishing chord-arc domain) provided it is Ahlfors regular, δ-Reifenberg flat for small δ, and

$$\begin{aligned}\lim_{r\to 0}\inf_{p\in\partial\Omega}\frac{\mathcal{H}^{n-1}(B_r(p)\cap\partial\Omega)}{\omega_{n-1}r^{n-1}} &= \\ \lim_{r\to 0}\sup_{p\in\partial\Omega}\frac{\mathcal{H}^{n-1}(B_r(p)\cap\partial\Omega)}{\omega_{n-1}r^{n-1}} &= 1.\end{aligned} \tag{12}$$

Example. Any VMO_1 domain.

Two ingredients to get (10)–(11).

Theorem 1 ([HMT]). If Ω is a regular SKT domain,

$$K : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \text{ is compact, } \forall\, p\in(1,\infty). \tag{13}$$

Theorem 2 ([HMT]). If Ω is a UR domain,

$$\frac{1}{2}I + K^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \text{ is injective, } \forall\, p\geq 2. \tag{14}$$

Corollary. If Ω is a regular SKT domain,

$$\frac{1}{2}I + K : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \text{ is bijective, } \forall\, p\in(1,\infty). \tag{15}$$

Proof. We have

$$\begin{aligned}(13) \text{ and } (14) &\Rightarrow (15) \text{ for } p\in(1,2] \\ &\Rightarrow \tfrac{1}{2}I + K \ \text{ injective on } L^p(\partial\Omega) \ \text{ for } p\in(1,\infty) \\ &\Rightarrow (15) \quad (\text{again invoking } (13)).\end{aligned}$$

Proof of Theorem 2. Assume $f\in L^2(\partial\Omega)$ in Ker $\frac{1}{2}I + K^*$, and set

$$u = \mathcal{S}f(x) = \int_{\partial\Omega} E(x,y)f(y)\,d\sigma(y), \tag{16}$$

single layer potential. Counterparts to (7)–(8):

$$\|\mathcal{N}\nabla\mathcal{S}f\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)}, \tag{17}$$

$$\partial_\nu \mathcal{S}f\big|_{\partial\Omega^\pm} = \Big(\mp\frac{1}{2}I + K^*\Big)f, \quad \Omega^+ = \Omega,\ \Omega^- = M\setminus\overline{\Omega}. \tag{18}$$

Take $v = u\nabla u$, so

$$\operatorname{div} v = |\nabla u|^2 + u\Delta u = |\nabla u|^2 + Vu^2 \ \text{ on } \ \Omega. \tag{19}$$

Apply Gauss-Green formula

$$\int\limits_{\Omega} \operatorname{div} v\, dV = \int\limits_{\partial\Omega} \nu\cdot v\, d\sigma, \tag{20}$$

with Ω replaced by Ω^-, to get

$$\int\limits_{\Omega^-} \big(|\nabla u|^2 + Vu^2\big)\, dV = \int\limits_{\partial\Omega} u\Big(\frac{1}{2}I + K^*\Big)f\, d\sigma = 0, \tag{21}$$

hence $|\nabla u|\equiv 0$ on Ω^-, hence $u\equiv 0$ on Ω^-. Hence $u = 0$ on $\partial\Omega$, so by a second application of (20) (on Ω),

$$\int\limits_{\Omega} \big(|\nabla u|^2 + Vu^2\big)\, dV = \int\limits_{\partial\Omega} u\Big(-\frac{1}{2}I + K^*\Big)f\, d\sigma = 0. \tag{22}$$

So $u = \text{const}$ on Ω. Also u, given by (16), does not jump across $\partial\Omega$, so $u\equiv 0$ on M. Then (18) gives, for both choices of sign,

$$\Big(\pm\frac{1}{2}I + K^*\Big)f = 0, \quad \text{hence} \ \ f = 0. \tag{23}$$

The proof is done, modulo:

Need to justify applying the Gauss-Green formula (20).
What is known:

$$\begin{aligned} f\in L^2(\partial\Omega) &\Rightarrow \mathcal{N}\nabla u\in L^2(\partial\Omega) \\ &\Rightarrow \mathcal{N}v\in L^p(\partial\Omega), \ \text{ some } \ p>1, \end{aligned} \tag{24}$$

whenever Ω is a UR domain, by (17), and as shown in [HMT],

$$\begin{aligned} f\in L^2(\partial\Omega) &\Rightarrow |\nabla u|^2\in L^q(\Omega) \\ &\Rightarrow \operatorname{div} v\in L^q(\Omega), \ \text{ some } \ q>1, \end{aligned} \tag{25}$$

whenever Ω is Ahlfors regular. Also u and v are continuous on the interior regions Ω and Ω^-.

Applicable Gauss-Green Theorem
For $p \in [1, \infty)$, set

$$\mathfrak{L}^p = \{v \in C(\Omega, TM) : \mathcal{N}v \in L^p(\partial\Omega), \text{and} \quad \exists \text{ nontangential limit } v_b,\ \sigma\text{-a.e.}\}. \tag{26}$$

Theorem 3 ([HMT]). Assume $\partial\Omega$ is Ahlfors regular and

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0. \tag{27}$$

Assume that, for some $p > 1$,

$$v \in \mathfrak{L}^p, \quad \text{and } \operatorname{div} v \in L^1(\Omega). \tag{28}$$

Then

$$\int\limits_{\Omega} \operatorname{div} v \, dx = \int\limits_{\partial\Omega} \nu \cdot v_b \, d\sigma. \tag{29}$$

The proof of Theorem 3 uses the results of De Giorgi and Federer that (29) holds for $v \in \operatorname{Lip}(\overline{\Omega})$. (See below.) In addition, it looks at $\chi_\delta v$ when χ_δ is a family of cutoffs, and it uses the following results:

$$\frac{1}{\delta} \int\limits_{\mathcal{O}_\delta} |v| \, dx \le C \|\mathcal{N}v\|_{L^1(\partial\Omega)}, \quad \forall\, v \in \mathfrak{L}^1, \tag{30}$$

where

$$\mathcal{O}_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \le \delta\}, \tag{31}$$

and, for $p \in (1, \infty)$,

$$\begin{aligned} &\text{If } v \in \mathfrak{L}^p,\ \exists w \in \mathfrak{L}^1,\ w_b = v_b, \\ &\text{and } \exists w_k \in \operatorname{Lip}(\overline{\Omega}),\ \|\mathcal{N}(w - w_k)\|_{L^1(\partial\Omega)} \to 0. \end{aligned} \tag{32}$$

The proof of (30) uses a covering argument. The proof of (32) uses an explicit operator resembling a Poisson kernel, acting on v_b. Hardy-Littlewood maximal function estimates make an appearance (which is partly why we need $p > 1$).

Remark. For Ω Lipschitz, (29) is due to Verchota.

Finite Perimeter Domains. DeGiorgi-Federer Results
Definition. Open $\Omega \subset \mathbb{R}^n$ has locally finite perimeter provided

$$\nabla \chi_\Omega = \mu, \tag{33}$$

a locally finite $\mathbb{R}^n$-valued measure. Radon-Nikodym $\Rightarrow$

$$\mu = -\nu\, \sigma, \tag{34}$$

σ locally finite positive measure, $\nu \in L^\infty(\partial\Omega, \sigma)$, $|\nu(x)| = 1,\ \sigma$-a.e. Besicovitch $\Rightarrow$

$$\lim_{r\to 0} \frac{1}{\sigma(B_r(x))} \int\limits_{B_r(x)} \nu\, d\sigma = \nu(x), \tag{35}$$

for σ-a.e. x.
Distribution theory $\Rightarrow$ given $v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$ (vector field),

$$\langle \operatorname{div} v, \chi_\Omega \rangle = -\langle v, \nabla \chi_\Omega \rangle, \tag{36}$$

so (33)–(34) equivalent to

$$\int\limits_\Omega \operatorname{div} v\, dx = \int\limits_{\partial\Omega} \nu \cdot v\, d\sigma. \tag{37}$$

DeGiorgi-Federer results:

$$\sigma = \mathcal{H}^{n-1} \lfloor \partial^* \Omega, \tag{38}$$

where $\partial^*\Omega \subset \partial\Omega$ **(reduced boundary)** consists of $x \in \partial\Omega$ where (35) holds, with $|\nu(x)| = 1$.
Also $\partial^*\Omega$ is **countably rectifiable:**

$$\partial^*\Omega = \bigcup_k M_k \cup N, \tag{39}$$

M_k compact subset of C^1 hypersurface, $\mathcal{H}^{n-1}(N) = 0$.
Measure-theoretic boundary $\partial_*\Omega \subset \partial\Omega$:

$$x \in \partial_*\Omega \Leftrightarrow \limsup_{r\to 0}\, r^{-n} \mathcal{L}^n(B_r(x) \cap \Omega^\pm) > 0, \tag{40}$$

where $\Omega^+ = \Omega,\ \Omega^- = \mathbb{R}^n \setminus \Omega$. Federer proved that $\partial^*\Omega \subset \partial_*\Omega$ and

$$\mathcal{H}^{n-1}(\partial_*\Omega \setminus \partial^*\Omega) = 0, \tag{42}$$

so (37) can be written

$$\int_{\Omega} \operatorname{div} v\, dx = \int_{\partial_* \Omega} \nu \cdot v\, d\mathcal{H}^{n-1}. \tag{43}$$

So far, we have (37) and (43) for $v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$.
Want such identities for more general v.
Easy extension to $v \in C_0^1(\mathbb{R}^n, \mathbb{R}^n)$, even to Lipschitz v.

Better extension (still easy). If Ω has locally finite perimeter, then (37) holds for v in

$$\mathcal{D} = \{v \in C_0^0(\mathbb{R}^n, \mathbb{R}^n) : \operatorname{div} v \in L^1(\mathbb{R}^n)\}. \tag{44}$$

Proof. Apply (37) to mollifications $v_k = \varphi_k * v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and pass to the limit. We have $\operatorname{div} v_k \to \operatorname{div} v$ in $L^1(\mathbb{R}^n)$ and $\nu \cdot v_k \to \nu \cdot v$ uniformly on $\partial\Omega$.

Drawback. Want to treat functions defined on Ω, not on a neighborhood of $\overline{\Omega}$.

Still better extension (though not implying Theorem 3)
We say Ω has a **tame interior approximation** $\{\Omega_k : k \in \mathbb{N}\}$ provided open $\Omega_k \subset \overline{\Omega}_k \subset \Omega_{k+1} \nearrow \Omega$ with

$$\|\nabla \chi_{\Omega_k}\|_{\mathrm{TV}(B_R)} \le C(R) < \infty, \quad \forall\, k. \tag{45}$$

For such Ω, (37) holds for v in

$$\widetilde{\mathcal{D}} = \{v \in C_0^0(\overline{\Omega}, \mathbb{R}^n) : \operatorname{div} v \in L^1(\Omega)\}. \tag{46}$$

Proof. Use previous extension to get

$$\int_{\Omega_k} \operatorname{div} v\, dx = -\langle v, \nabla \chi_{\Omega_k} \rangle, \tag{47}$$

and examine limit as $k \to \infty$, using bounds (45).

Result of Federer (1952). If Ω is bounded and $\mathcal{H}^{n-1}(\partial\Omega) < \infty$, then (37) holds provided $v \in C(\overline{\Omega})$ and each term $\partial_j v_j$ in $\operatorname{div} v$ belongs to $L^1(\Omega)$.

Examples of locally finite perimeter sets. Assume

$$A \in C(\mathbb{R}^{n-1}), \quad \nabla A \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}). \tag{48}$$

With $x = (x', x_n)$, set

$$\Omega = \{x \in \mathbb{R}^n : x_n > A(x')\}. \tag{49}$$

Proposition A. Such Ω has locally finite perimeter.

Given this, we can consider

$$\Big\{x \in \mathbb{R}^n : x_n > A(x') + \frac{1}{k}\Big\} \tag{50}$$

and deduce:

Corollary B. Such Ω has a tame interior approximation.

Proof of Prop. A. Use a mollifier to produce $A_k \in C^\infty(\mathbb{R}^{n-1})$ such that

$$A_k \to A \ \text{ in } \ C(\mathbb{R}^{n-1}), \quad \nabla A_k \to \nabla A \ \text{ in } \ L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}). \tag{51}$$

Set

$$\Omega_k = \{x \in \mathbb{R}^n : x_n > A_k(x')\}. \tag{52}$$

Then

$$\begin{aligned} \chi_{\Omega_k} &\to \chi_\Omega \ \text{ in } \ L^1_{\mathrm{loc}}(\mathbb{R}^n), \ \text{ so} \\ \nabla\chi_{\Omega_k} &\to \nabla\chi_\Omega \ \text{ in } \ \mathcal{D}'(\mathbb{R}^n). \end{aligned} \tag{53}$$

For Ω_k, Gauss-Green formula is elementary:

$$\nabla\chi_{\Omega_k} = -\nu_k\, \sigma_k, \tag{54}$$

σ_k surface area on graph of A_k, given in x'-coordinates by

$$d\sigma_k(x') = \sqrt{1 + |\nabla A_k(x')|^2}\, dx'. \tag{55}$$

Hypothesis (48) $\Rightarrow$ $\{\nu_k\sigma_k : k \in \mathbb{N}\}$ bounded set of $\mathbb{R}^n$-valued measures, on each set $B_R = \{x \in \mathbb{R}^n : |x| \le R\}$.
So passing to limit in (53) gives

$$\nabla\chi_\Omega = \mu, \tag{55}$$

locally finite $\mathbb{R}^n$-valued measure.
Hence (37) holds, i.e., for $v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$,

$$\int_\Omega \operatorname{div} v\, dx = \int_{\partial\Omega} \nu \cdot v\, d\sigma. \tag{56}$$

Going further, the Gauss-Green formula for Ω_k gives

$$\int_{\Omega_k} \operatorname{div} v\, dx = \int_{\mathbb{R}^{n-1}} (\nabla A_k(x'), -1) \cdot v(x', A_k(x'))\, dx', \tag{57}$$

valid for $v \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$. As $k \to \infty$,

$$\begin{aligned} &\text{LHS (57)} \to \text{LHS (56)} \\ &\text{RHS (57)} \to \int_{\mathbb{R}^{n-1}} (\nabla A(x'), -1) \cdot v(x', A(x'))\, dx'. \end{aligned} \tag{58}$$

Hence

$$\int \nu \cdot v\, d\sigma = \int_{\mathbb{R}^{n-1}} \tilde{\nu}(x') \cdot v(x', A(x'))\, d\sigma(x'), \tag{59}$$

with

$$\begin{aligned} \tilde{\nu}(x') &= \frac{(\nabla A(x'), -1)}{\sqrt{1 + |\nabla A(x')|^2}}, \\ d\sigma(x') &= \sqrt{1 + |\nabla A(x')|^2}\, dx'. \end{aligned} \tag{60}$$

Remark. If A is Lipschitz, one easily has $\partial_* \Omega = \partial \Omega$. More generally:

Proposition C. For Ω given by (48)–(49),

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^*\Omega) = 0. \tag{61}$$

Proof. Uses results of Tompson (1954) and Federer (1960).
Given $K \subset \mathbb{R}^{n-1}$, set

$$\Sigma_K = \{(x', A(x')) : x' \in K\}. \tag{62}$$

Tompson proved the first identity in

$$\begin{aligned} \mathcal{I}^{n-1}(\Sigma_K) &= \int_K \sqrt{1 + |\nabla A(x')|^2}\, dx' \\ &= \sigma(\Sigma_K), \end{aligned} \tag{63}$$

the second identity holding by (59)–(60). Federer proved

$$\mathcal{H}^{n-1}(\Sigma_K) = \mathcal{I}^{n-1}(\Sigma_K). \tag{64}$$

So

$$\mathcal{H}^{n-1}(\Sigma_K) = \sigma(\Sigma_K) = \mathcal{H}^{n-1}(\Sigma_K \cap \partial^*\Omega), \tag{65}$$

the last identity by (38). This gives (61).