Singular integrals and elliptic boundary problems on regular Semmes-Kenig-Toro domains

Steve Hofmann
Mathematics Department, University of Missouri

Marius Mitrea
Mathematics Department, University of Missouri,

Michael Taylor
Mathematics Department, University of North Carolina

Abstract

We develop the theory of layer potentials and related singular integral operators as a tool to study a variety of elliptic boundary problems on a family of domains introduced by Semmes [101]–[102] and Kenig and Toro [64]–[66], which we call regular Semmes-Kenig-Toro domains. This extends the classic work of Fabes, Jodeit, and Rivière in several ways. For one, the class of domains considered contains the class of VMO_1 domains, which in turn contains the class of C^1 domains. In addition we study not only the Dirichlet and Neumann boundary problems, but also a variety of others. Furthermore, we treat not only constant coefficient operators, but also operators with variable coefficients, including operators on manifolds.

Contents

1. Introduction

2. Finite perimeter domains, Ahlfors regular domains, and BMO_1 domains
- 2.1. Ahlfors regular domains and nontangential maximal functions
- 2.2. Finite perimeter domains and Green's formula
- 2.3. Green's formula on Ahlfors regular domains
- 2.4. Analysis on spaces of homogeneous type
- 2.5. Ahlfors regularity of BMO_1 domains

3. Singular integrals on UR domains
- 3.1. Countably rectifiable sets and uniformly rectifiable sets
- 3.2. First estimates on layer potentials
- 3.3. Boundary behavior of Newtonian layer potentials
- 3.4. General odd, homogeneous layer potentials
- 3.5. The variable coefficient case
- 3.6. Singular integrals on Sobolev spaces

4. Semmes-Kenig-Toro domains, Poincaré inequalities, and singular integrals
- 4.1. Reifenberg flat domains, SKT domains, and regular SKT domains
- 4.2. A Poincaré type inequality, Semmes decomposition, and consequences

4.3. Sobolev spaces revisited
4.4. Compactness of double layer-like operators on VMO_1 domains
4.5. Compactness of double layer-like operators on regular SKT domains
4.6. Characterizations of regular SKT domains via compactness
4.7. Clifford-Szegö projections and regular SKT domains

5. Laplace-Beltrami layer potentials and the Dirichlet and Neumann problems
5.1. Boundedness and jump relations
5.2. Compactness of K
5.3. Green formulas on Riemannian manifolds
5.4. Invertibility of boundary layer potentials
5.5. The Dirichlet and Neumann problems
5.6. Extensions to ε-regular SKT domains

6. Second order elliptic systems on regular SKT domains: set-up
6.1. Examples
6.2. Compactness of layer potential operators on Sobolev spaces
6.3. The invertibility of boundary double layer potentials
6.4. The invertibility of boundary single layer potentials
6.5. The invertibility of the magnetostatic layer potential

7. Second order elliptic systems: specific cases
7.1. Boundary value problems for the Laplacian
7.2. Boundary value problems for the Stokes system
7.3. Boundary value problems for the Lamé system
7.4. Boundary value problems for Maxwell's equations

1 Introduction

The original motivation behind the development of Fredholm theory was to use the compactness of various double layer potential operators on smooth domains, arising in mathematical physics, in order to solve boundary value problems via integral equation methods. Indeed, in his 1898 Ph.D. Thesis, Erik Ivar Fredholm himself pioneered the use of such an approach in the study of a problem in elasticity theory. Shortly thereafter, in a paper based on his doctoral dissertation, which appeared in 1900, Fredholm proved his famous theorems for the integral equations associated with the Dirichlet problem for the Laplacian and, in 1906, he used potential theoretic methods in the study of the first basic problem of elasticity theory (for which he utilized the so-called pseudo-stress elastic double layer – cf. the discussion in §6.3). Through the work of Mikhlin and of Calderón and Zygmund and others, integral operators have played a key role in the study of elliptic boundary problems, first for smooth boundaries, and then more recently for rougher boundaries.

In the classical setting of a bounded domain Ω with smooth boundary, the source of compactness for, say, the harmonic double layer

$$Kf(X) := \lim_{\varepsilon\to 0^+} \frac{1}{\omega_n} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} \frac{\langle \nu(Y), Y-X\rangle}{|X-Y|^{n+1}} f(Y)\, d\sigma(Y), \qquad X \in \partial\Omega, \tag{1.0.1}$$

on $L^p(\partial\Omega, d\sigma)$, $1 < p < \infty$, is the weak singularity of its integral kernel (itself, a consequence of the special algebraic structure of the integral kernel in (1.0.1)). This holds when the boundary has defining functions whose first derivatives have a Hölder, or even Dini, modulus of continuity, but it fails for boundaries that are C^1 or rougher. See also [9], [69], [80] and the references therein for some related, early developments.

Calderón in [11] initiated a breakthrough, proving L^p-bounds for Cauchy integral operators on Lipschitz curves with small Lipschitz constant. This was applied in [37] by Fabes, Jodeit, and Rivière, who showed that K in (1.0.1) is compact on $L^p(\partial\Omega, d\sigma)$ for each $p \in (1, \infty)$ whenever $\partial\Omega$ is a C^1 surface. Since then there have been further developments in a number of directions.

In one direction, Coifman, McIntosh, and Meyer [20] extended Calderón's estimate on Cauchy integrals to general Lipschitz curves, and applied this to such results as boundedness of K in (1.0.1) on $L^p(\partial\Omega, d\sigma)$ for each $p \in (1, \infty)$, whenever $\partial\Omega$ is a strongly Lipschitz surface. Generally, compactness fails here, but other methods have led to invertibility of various layer potentials and applications to the Dirichlet and Neumann problems, beginning in [116], and extended to other settings and other boundary problems in various papers, including [38], [91], and others.

In another direction, Jerison and Kenig [56] showed that the Poisson kernel h of a bounded C^1 domain, given by $h_X = d\omega^X/d\sigma$, the Radon-Nikodym derivative of harmonic measure with pole at X with respect to surface measure, has the property

$$\log h_X \in \mathrm{VMO}(\partial\Omega, d\sigma), \quad \forall\, X \in \Omega. \tag{1.0.2}$$

Then Kenig and Toro [65] demonstrated (1.0.2) for a much larger class of domains, namely for what they called chord-arc domains with vanishing constant (which we call here regular Semmes-Kenig-Toro domains, or, briefly, regular SKT domains), and in [66] they proved the converse.

In a third direction, Hofmann [50] established compactness of K when $\partial\Omega$ is a VMO_1 domain. The work of Kenig and Toro mentioned above leads one to speculate that such compactness might hold on regular SKT domains, and indeed one of the central results of the current paper is that this is true.

Yet another direction has led to L^p-boundedness of such singular integral operators as K on surfaces more general than the boundaries of Lipschitz domains. Works of David [28], [29], [30], of David-Jerison [31], and of David-Semmes [33], [34], and Semmes [100] yield such boundedness when the surface Σ is Ahlfors regular and has "big pieces of Lipschitz surfaces," in a uniform manner; one calls Σ a uniformly rectifiable surface. (See §2 and §3 for definitions of these terms.) This work has interfaced tightly with geometric measure theory, but until now it has not been applied to problems in PDE.

Our aim here is to find the optimal geometric measure theoretic context in which Fredholm theory can be successfully implemented, along the lines of its original development, for solving boundary value problems with L^p data via the method of layer potentials. In the process, we forge new links between the analysis of singular integral operators on uniformly rectifiable surfaces, and in particular on regular SKT domains, and problems in PDE, notably boundary problems for the Laplace operator and other second order elliptic operators, including systems. The following is the structure of the rest of this paper.

Section 2 discusses Ahlfors regular domains. There are several reasons to start here. For one, Ahlfors regularity is the first part of the defining property of uniform rectifiability. For another, it is a natural general setting in which weakly singular integral operators can be shown to be compact (cf. §5.1). Also it is a setting in which the harmonic analysis of [23] applies, which is useful in several respects. In addition, as shown in §2.3, it is a natural setting for a version of Green's formula that will play an important role further on. In §2.4 we gather some results on analysis on spaces of homogeneous type. In §2.5 we show that surfaces that are locally graphs of BMO_1 functions are Ahlfors regular.

In §3 we define uniformly rectifiable sets and UR domains. We discuss sufficient conditions for a domain Ω to be a UR domain, such as conditions involving Ahlfors regularity and the NTA condition, or the more general John condition. We recall basic singular integral estimates of David and colleagues, and supplement them with nontangential maximal function estimates. The major effort in this section is devoted to establishing nontangential convergence at the boundary of appropriate classes of layer potentials applied to elements of $L^p(\partial\Omega, d\sigma)$.

In §4 we define SKT domains (called chord arc domains in [101]–[102] and [64]–[66]) and regular SKT domains, and recall some of their basic properties. We produce further equivalent characterizations of regular SKT domains, based on a Poincaré inequality and a careful analysis of the Semmes decomposition. One notable characterization is that the domain satisfies a two-sided John condition (which is the case if the domain in question is two-sided NTA), is Ahlfors regular, and its unit normal ν belongs to $\mathrm{VMO}(\partial\Omega, d\sigma)$. Making use of the Poincaré inequality of §4.2, we also show that the L^p-Sobolev space $L^p_1(\partial\Omega)$ is isomorphic to the space $W^{p,1}(\partial\Omega)$ defined for general metric measure spaces by Hajłasz [46], which will prove useful. From here we proceed to the main goal in §4, which is the proof of compactness on $L^p(\partial\Omega, d\sigma)$ of a class of operators including K in (1.0.1) in case Ω is a regular SKT domain. We also establish the converse result, that if Ω is a UR domain (satisfying a two-sided John condition) for which such a class of operators (together with a natural class of commutators) is compact, then Ω must be a regular SKT domain, thus completing this circle of compactness results.

In §4 we also define the class of ε-regular SKT domains, replacing the property that $\nu \in \mathrm{VMO}(\partial\Omega, d\sigma)$ by the property

$$\operatorname{dist}\big(\nu, \mathrm{VMO}\,(\partial\Omega, d\sigma)\big) < \varepsilon, \tag{1.0.3}$$

where the distance is measured in the BMO-norm. The compactness results described above extend to results of the sort that such operators as K have small norm modulo compacts if Ω is an ε-regular SKT domain for small ε.

Sections 5–7 apply these results to boundary problems for second order elliptic PDE on ε-regular SKT domains. Section 5 deals with the Dirichlet and Neumann problem for the Laplace operator. We go beyond the constant coefficient case, and in the spirit of work developed in [91]–[93], work on domains in a manifold, endowed with a Riemannian metric tensor whose components have a certain Dini-type modulus of continuity. In §§6–7 we explore various systems, particularly the Lamé system, the Stokes system, and the Maxwell system, and natural boundary problems that arise for such systems. We present general results (on invertibility of boundary integral equations, etc.) in §6 and concentrate on applications to these specific cases of boundary problems in §7.

2 Finite perimeter domains, Ahlfors regular domains, and BMO_1 domains

As noted in the introduction, we are engaged in analysis on a domain Ω whose boundary $\partial\Omega$ satisfies certain weak forms of regularity. One of the conditions is that $\partial\Omega$ be Ahlfors regular. In §2.1, we define Ahlfors regularity and record a fundamental result on the Hardy-Littlewood maximal function applied to functions on Ahlfors regular surfaces. We then define the nontangential maximal function $\mathcal{N}u$ associated to a function $u \in C^0(\Omega)$ and establish a basic result to the effect that $\|\mathcal{N}u\|_{L^p(\partial\Omega,d\sigma)}$ depends only weakly on the choice of definition of nontangential approach region.

The next two subsections are devoted to versions of the Gauss-Green formula. This formula is crucial in the study of layer potentials for two distinct reasons. One, pursued in §3, is to provide a tool for proving jump relations for layer potentials. The other, pursued in §5 and §6, is to yield a Green formula for certain functions given as layer potentials, which in turn implies injectivity (and hence, in connection with other arguments, invertibility) of certain layer potential operators, key to our attack on elliptic boundary problems. In §2.2 we discuss the validity of a Green formula of the form

$$\int\limits_{\Omega} \operatorname{div} v \, dx = \int\limits_{\partial^*\Omega} \langle \nu, v \rangle \, d\sigma, \tag{2.0.1}$$

for Ω having locally finite perimeter, valid for a compactly supported Lipschitz vector field v. Results here are due to Federer and De Giorgi. These results are adequate for applications in §3. For applications in §5 and §6, we need such an identity for a larger class of v. We establish such an identity in §2.3, in the case when Ω is Ahlfors regular.

Section 2.4 discusses general results on spaces of homogeneous type, including Ahlfors regular surfaces. This includes discussions of Hardy spaces and spaces BMO and VMO on such surfaces. In §2.5 we introduce BMO_1 domains and show they are Ahlfors regular.

2.1 Ahlfors regular domains and nontangential maximal functions

A closed set $\Sigma \subset \mathbb{R}^{n+1}$ is said to be Ahlfors regular provided there exist $0 < a \leq b < \infty$ such that

$$a\, r^n \leq \mathcal{H}^n\big(B(X,r) \cap \Sigma\big) \leq b\, r^n, \tag{2.1.1}$$

for each $X \in \Sigma$, $r \in (0,\infty)$ (if Σ is unbounded), where $\mathcal{H}^n$ denotes n-dimensional Hausdorff measure and $B(X,r) := \{Y \in \mathbb{R}^{n+1} : |X-Y| < r\}$. If Σ is compact, we require (2.1.1) only for $r \in (0,1]$. Nonetheless, (2.1.1) continues to hold in this case (albeit with possibly different constants) for each $0 < r < \operatorname{diam}\Sigma$. It should be pointed out that Ahlfors regularity is not a regularity property per se, but rather a scale-invariant way of expressing the fact that the set in question is n-dimensional.

An open set $\Omega \subset \mathbb{R}^{n+1}$ is said to be an *Ahlfors regular domain* provided $\partial\Omega$ is Ahlfors regular. Most of our analysis will be done on Ahlfors regular domains. Note that if (2.1.1) holds then (cf. Theorem 4 on p. 61 in [36]),

$$\sigma := \mathcal{H}^n \lfloor \Sigma \text{ is a Radon, doubling measure.} \tag{2.1.2}$$

Hence fundamental results of [23] (cf. the discussion on p. 624) yield the following.

Proposition 2.1.1 *An Ahlfors regular surface $\Sigma \subset \mathbb{R}^{n+1}$ is a space of homogeneous type (in the sense of Coifman-Weiss), when equipped with the Euclidean distance and the measure $\sigma = \mathcal{H}^n\lfloor\Sigma$. In particular, the associated Hardy-Littlewood maximal operator*

$$\mathcal{M}f(X) := \sup_{r>0} ⨍_{Y\in\Sigma:\,|X-Y|<r} |f(Y)| \, d\sigma(Y), \qquad X \in \Sigma, \tag{2.1.3}$$

is bounded on $L^p(\Sigma, d\sigma)$ for each $p \in (1,\infty)$. Here and elsewhere, the barred integral denotes averaging (with the convention that this is zero if the set in question has zero measure). Furthermore, there exists $C = C(\Sigma) \in (0,\infty)$ such that

$$\sigma\Big(\{X \in \Sigma : \mathcal{M}f(X) > \lambda\}\Big) \leq C\,\lambda^{-1}\|f\|_{L^1(\Sigma,d\sigma)}, \tag{2.1.4}$$

for every $f \in L^1(\Sigma, d\sigma)$ and $\lambda > 0$.

We will say more about spaces of homogeneous type in §2.4. We now turn to the notion of the non-tangential maximal operator, applied to functions on an open set $\Omega \subset \mathbb{R}^{n+1}$. To define this, fix $\alpha > 0$ and for each boundary point $Z \in \partial\Omega$ introduce the non-tangential approach region

$$\Gamma(Z) := \Gamma_\alpha(Z) := \{X \in \Omega : |X - Z| < (1+\alpha)\,\mathrm{dist}\,(X, \partial\Omega)\}. \tag{2.1.5}$$

It should be noted that, under the current hypotheses, it could happen that $\Gamma(Z) = \emptyset$ for points $Z \in \partial\Omega$. (This point will be discussed further in §2.3.)

Next, for $u : \Omega \to \mathbb{R}$, we define the non-tangential maximal function of u by

$$\mathcal{N}u(Z) := \mathcal{N}_\alpha u(Z) := \sup\{|u(X)| : X \in \Gamma_\alpha(Z)\}, \qquad Z \in \partial\Omega. \tag{2.1.6}$$

Here and elsewhere in the sequel, we make the convention that $\mathcal{N}u(Z) = 0$ whenever $Z \in \partial\Omega$ is such that $\Gamma(Z) = \emptyset$.

The following result implies that the choice of α plays a relatively minor role when measuring the size of the nontangential maximal function in $L^p(\partial\Omega, d\sigma)$.

Proposition 2.1.2 *Assume $\Omega \subset \mathbb{R}^{n+1}$ is open and Ahlfors regular. Then for every $\alpha, \beta > 0$ and $0 < p < \infty$ there exist $C_0, C_1 > 0$ such that*

$$C_0\|\mathcal{N}_\alpha u\|_{L^p(\partial\Omega,d\sigma)} \leq \|\mathcal{N}_\beta u\|_{L^p(\partial\Omega,d\sigma)} \leq C_1\|\mathcal{N}_\alpha u\|_{L^p(\partial\Omega,d\sigma)}, \tag{2.1.7}$$

for each function u.

Proof. We adapt a well-known point-of-density argument of Fefferman and Stein [42] (cf. also [108]). Specifically, fix $\lambda > 0$ and consider the open subset of $\partial\Omega$ given by

$$\mathcal{O}_\alpha := \Big\{X \in \partial\Omega : \sup\{|u(Y)| : Y \in \Gamma_\alpha(X)\} > \lambda\Big\}. \tag{2.1.8}$$

As a consequence, $A := \partial\Omega \setminus \mathcal{O}_\alpha$ is closed. For each $\gamma \in (0,1)$ we then set

$$A^*_\gamma := \{X \in \partial\Omega : \sigma(A \cap \Delta(X,r)) \geq \gamma\sigma(\Delta(X,r)),\ \ \forall\, r > 0\}, \tag{2.1.9}$$

where $\Delta(X,r) := B(X,r) \cap \partial\Omega$. That is, A^*_γ is the collection of points of (global) γ-density for the set A.

We now claim that there exists $\gamma \in (0,1)$ such that

$$\mathcal{O}_\beta \subseteq \partial\Omega \setminus A^*_\gamma. \tag{2.1.10}$$

To justify this inclusion, fix an arbitrary point $X \in \mathcal{O}_\beta$. Then there exists $Y \in \Gamma_\beta(X)$ such that $|u(Y)| > \lambda$ and we select $\bar{Y} \in \partial\Omega$ such that $|Y - \bar{Y}| = \operatorname{dist}(Y, \partial\Omega)$. We now make two observations of geometrical nature. First,

$$Z \in \Delta(\bar{Y}, \alpha|Y - \bar{Y}|) \Longrightarrow Y \in \Gamma_\alpha(Z). \tag{2.1.11}$$

Indeed, if $Z \in \partial\Omega$ and $|Z - \bar{Y}| < \alpha|Y - \bar{Y}|$ then

$$|Z - Y| \leq |Z - \bar{Y}| + |\bar{Y} - Y| < \alpha|Y - \bar{Y}| + |Y - \bar{Y}| = (1 + \alpha) \operatorname{dist}(Y, \partial\Omega), \tag{2.1.12}$$

i.e., $Y \in \Gamma_\alpha(Z)$, as desired. Our second observation is that

$$\Delta(\bar{Y}, \alpha|Y - \bar{Y}|) \subseteq \Delta(X, (2 + \alpha + \beta)|Y - \bar{Y}|). \tag{2.1.13}$$

To see this, we note that if $Z \in \partial\Omega$ and $|Z - \bar{Y}| < \alpha|Y - \bar{Y}|$ then

$$\begin{aligned} |X - Z| &\leq |X - Y| + |Y - \bar{Y}| + |\bar{Y} - Z| \\ &\leq (1 + \beta) \operatorname{dist}(Y, \partial\Omega) + (1 + \alpha)|Y - \bar{Y}| = (2 + \alpha + \beta)|Y - \bar{Y}|. \end{aligned} \tag{2.1.14}$$

In concert, (2.1.11), (2.1.13) and the fact that $|u(Y)| > \lambda$ yield

$$\Delta(\bar{Y}, \alpha|Y - \bar{Y}|) \subseteq \mathcal{O}_\alpha \cap \Delta(X, (2 + \alpha + \beta)|Y - \bar{Y}|), \tag{2.1.15}$$

so that, thanks to the estimate (2.1.1) defining Ahlfors regularity,

$$\begin{aligned} \frac{\sigma\Big(\mathcal{O}_\alpha \cap \Delta(X, (2 + \alpha + \beta)|Y - \bar{Y}|)\Big)}{\sigma\Big(\Delta(X, (2 + \alpha + \beta)|Y - \bar{Y}|)\Big)} &\geq \frac{\sigma\Big(\Delta(\bar{Y}, \alpha|Y - \bar{Y}|)\Big)}{\sigma\Big(\Delta(X, (2 + \alpha + \beta)|Y - \bar{Y}|)\Big)} \\ &\geq c\left(\frac{\alpha}{2 + \alpha + \beta}\right)^n, \end{aligned} \tag{2.1.16}$$

where c is a small, positive constant which depends only on Ω and n. In particular, if we set $r := (2 + \alpha + \beta)|Y - \bar{Y}|$, then

$$\frac{\sigma\Big(A \cap \Delta(X, r)\Big)}{\sigma\Big(\Delta(X, r)\Big)} \leq 1 - c\left(\frac{\alpha}{2 + \alpha + \beta}\right)^n. \tag{2.1.17}$$

Thus, if we select γ such that $1 - c(\frac{\alpha}{2+\alpha+\beta})^n < \gamma < 1$, then (2.1.17) entails $X \notin A^*_\gamma$. This proves the claim (2.1.10).

Let $\mathcal{M}$ be the Hardy-Littlewood maximal operator associated as in (2.1.3) to $\Sigma := \partial\Omega$. Then, based on (2.1.10) and (2.1.4), we may write

$$
\begin{aligned}
\sigma(\mathcal{O}_\beta) &\le \sigma(\partial\Omega \setminus A^*_\gamma) = \sigma\Big(\{X \in \partial\Omega : \mathcal{M}(\mathbf{1}_{\partial\Omega\setminus A})(X) > 1-\gamma\}\Big) \\
&\le \frac{C}{1-\gamma}\sigma(\partial\Omega \setminus A) = C(\Omega,\gamma)\sigma(\mathcal{O}_\alpha). \qquad (2.1.18)
\end{aligned}
$$

Hence,

$$
\sigma(\mathcal{O}_\beta) \le C(\Omega,\alpha,\beta)\sigma(\mathcal{O}_\alpha), \qquad (2.1.19)
$$

and (2.1.7) readily follows from this. □

2.2 Finite perimeter domains and Green's formula

Let $\Omega \subset \mathbb{R}^m$ be open. We say Ω has locally finite perimeter provided

$$
\mu := \nabla \mathbf{1}_\Omega \qquad (2.2.1)
$$

is a locally finite $\mathbb{R}^m$-valued measure. It follows from the Radon-Nikodym theorem that $\mu = -\nu\,\sigma$, where σ is a locally finite positive measure, supported on $\partial\Omega$, and $\nu \in L^\infty(\partial\Omega,\sigma)$ is an $\mathbb{R}^m$-valued function, satisfying $|\nu(x)| = 1$, σ-a.e. It then follows from the Besicovitch differentiation theorem that

$$
\lim_{r\to 0} \frac{1}{\sigma(B_r(x))} \int\limits_{B_r(x)} \nu\, d\sigma = \nu(x) \qquad (2.2.2)
$$

for σ-a.e. x.

Via distribution theory, we can restate (2.2.1) as follows. Take a vector field $v \in C_0^\infty(\mathbb{R}^m,\mathbb{R}^m)$. Then

$$
\langle \operatorname{div} v, \mathbf{1}_\Omega\rangle = -\langle v, \nabla \mathbf{1}_\Omega\rangle. \qquad (2.2.3)
$$

Hence (2.2.1) is equivalent to

$$
\int\limits_\Omega \operatorname{div} v\, dx = \int\limits_{\partial\Omega} \langle \nu, v\rangle\, d\sigma, \qquad \forall\, v \in C_0^\infty(\mathbb{R}^m,\mathbb{R}^m). \qquad (2.2.4)
$$

Works of Federer and of De Giorgi produced the following results on the structure of σ, when Ω has locally finite perimeter. First,

$$
\sigma = \mathcal{H}^{m-1} \lfloor \partial^*\Omega, \qquad (2.2.5)
$$

where $\mathcal{H}^{m-1}$ is $(m-1)$-dimensional Hausdorff measure and $\partial^*\Omega \subset \partial\Omega$ is the reduced boundary of Ω, defined as

$$\partial^*\Omega := \Big\{x : \text{(2.2.2) holds, with } |\nu(x)| = 1\Big\}. \tag{2.2.6}$$

(It follows from the remarks leading up to (2.2.2) that σ is supported on $\partial^*\Omega$.) Second, $\partial^*\Omega$ is countably rectifiable; it is a countable disjoint union

$$\partial^*\Omega = \bigcup_k M_k \cup N, \tag{2.2.7}$$

where each M_k is a compact subset of an $(m-1)$-dimensional C^1 surface (to which ν is normal in the usual sense), and $\mathcal{H}^{m-1}(N) = 0$. Given (2.2.5), the identity (2.2.4) yields the Gauss-Green formula

$$\int\limits_{\Omega} \operatorname{div} v\, dx = \int\limits_{\partial^*\Omega} \langle \nu, v\rangle\, d\mathcal{H}^{m-1}, \tag{2.2.8}$$

for $v \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^m)$. Third, there exist constants $C_m \in (1, \infty)$ such that

$$C_m^{-1} \le \liminf_{r\to 0^+} r^{-(m-1)}\sigma(B_r(x)) \le \limsup_{r\to 0^+} r^{-(m-1)}\sigma(B_r(x)) \le C_m \tag{2.2.9}$$

for each $x \in \partial^*\Omega$ (which, informally speaking, can be thought of as an infinitesimal Ahlfors regularity conditions).

It is also useful to record some results on sets $\partial_*\Omega \supset \partial_0\Omega \supset \partial^*\Omega$, which will be formally introduced shortly (cf. (2.2.14) and (2.2.12) below). Good references for this material, as well as the results stated above, are [41], [36], and [118]. First, given a unit vector ν_E and $x \in \partial\Omega$, set

$$H^{\pm}_{\nu_E}(x) = \{y \in \mathbb{R}^m : \pm\langle \nu_E, y - x\rangle \ge 0\}. \tag{2.2.10}$$

Then (cf. [36], p. 203), for $x \in \partial^*\Omega$, $\Omega^+ := \Omega$, $\Omega^- := \mathbb{R}^m \setminus \Omega$, one has

$$\lim_{r\to 0} r^{-m}\mathcal{L}^m\big(B_r(x) \cap \Omega^{\pm} \cap H^{\pm}_{\nu_E}(x)\big) = 0, \tag{2.2.11}$$

when $\nu_E = \nu(x)$ is given by (2.2.2). Here $\mathcal{L}^m$ denotes the Lebesgue measure on $\mathbb{R}^m$. More generally, a unit vector ν_E for which (2.2.11) holds is called the measure-theoretic outer normal to Ω at x. It is easy to show that if such ν_E exists it is unique. With $\nu_E(x)$ denoting the measure-theoretic outer normal, if we now define

$$\partial_0\Omega := \{x \in \partial\Omega : \text{(2.2.11) holds}\}, \tag{2.2.12}$$

we may then conclude that

$$\partial_0\Omega \supset \partial^*\Omega \quad \text{and} \quad \nu_E(x) = \nu(x) \quad \text{on} \quad \partial^*\Omega. \tag{2.2.13}$$

Next, we define the measure-theoretic boundary of Ω by

$$\partial_*\Omega := \Big\{x \in \partial\Omega : \limsup_{r\to 0} r^{-m}\mathcal{L}^m\big(B_r(x)\cap\Omega^{\pm}\big) > 0\Big\}. \tag{2.2.14}$$

It is clear that $\partial_*\Omega \supset \partial_0\Omega$. Furthermore (cf. [36], p. 208) one has

$$\mathcal{H}^{m-1}(\partial_*\Omega \setminus \partial^*\Omega) = 0. \tag{2.2.15}$$

Consequently the Green formula (2.2.8) can be rewritten

$$\int_\Omega \operatorname{div} v\, dx = \int_{\partial_*\Omega} \langle \nu, v\rangle\, d\mathcal{H}^{m-1}, \tag{2.2.16}$$

for $v \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^m)$. The advantage of (2.2.16) is that the definition of $\partial_*\Omega$ is more straightforward and geometrical than is that of $\partial^*\Omega$. Note that $\partial_*\Omega$ is well defined whether or not Ω has locally finite perimeter. It is known that

$$\Omega \text{ has locally finite perimeter} \Longleftrightarrow \mathcal{H}^{m-1}(\partial_*\Omega\cap\mathcal{K}) < \infty,\ \forall\,\mathcal{K}\subset\mathbb{R}^m \text{ compact}. \tag{2.2.17}$$

Cf. [36], p. 222. In general $\partial\Omega\setminus\partial_*\Omega$ can be quite large. It is of interest to know conditions under which $\mathcal{H}^{m-1}(\partial\Omega\setminus\partial_*\Omega) = 0$. We will comment further on this later on.

We next discuss an important class of domains with locally finite perimeter. Let $\Omega \subset \mathbb{R}^m$ be the region over the graph of a function $A : \mathbb{R}^{m-1} \to \mathbb{R}$:

$$\Omega = \{x \in \mathbb{R}^m : x_m > A(x')\}, \tag{2.2.18}$$

where $x = (x', x_m)$. We have:

Proposition 2.2.1 *Given a function*

$$A \in C^0(\mathbb{R}^{m-1}), \quad \nabla A \in L^1_{loc}(\mathbb{R}^{m-1}), \tag{2.2.19}$$

then Ω defined as in (2.2.18) has locally finite perimeter.

For the reader's convenience we include a proof of this result, which is more than sufficient for use on BMO_1 domains. A more elaborate result, treating graphs of BV functions, is given in [41], §4.5.9.

Proof. Pick $\psi \in C_0^\infty(\mathbb{R}^{m-1})$ such that $\int \psi(x')\,dx' = 1$, define $\psi_k(x') = k^{m-1}\psi(kx')$, $k \in \mathbb{N}$, and set $A_k = \psi_k * A$,

$$\Omega_k = \{x \in \mathbb{R}^m : x_m > A_k(x')\}. \tag{2.2.20}$$

Clearly

$$A_k \longrightarrow A, \quad \text{locally, uniformly,} \tag{2.2.21}$$

so

$$\mathbf{1}_{\Omega_k} \longrightarrow \mathbf{1}_\Omega \ \text{ in } \ L^1_{loc}(\mathbb{R}^m). \tag{2.2.22}$$

Hence $\nabla \mathbf{1}_{\Omega_k} \to \nabla \mathbf{1}_\Omega$ in $\mathcal{D}'(\mathbb{R}^m)$. Also

$$\nabla \mathbf{1}_{\Omega_k} = -\nu_k \, \sigma_k, \tag{2.2.23}$$

where σ_k is surface area on

$$\Sigma_k = \{x \in \mathbb{R}^m : x_m = A_k(x')\}, \tag{2.2.24}$$

given in x'-coordinates by

$$d\sigma_k(x') = \sqrt{1 + |\nabla A_k(x')|^2} \, dx', \tag{2.2.25}$$

and ν_k is the downward-pointing unit normal to the surface Σ_k. The hypothesis (2.2.19) implies that $\{\nu_k \sigma_k : k \geq 1\}$ is a bounded set of $\mathbb{R}^m$-valued measures on each set $B_R = \{x \in \mathbb{R}^m : |x| \leq R\}$, so passing to the limit gives

$$\nabla \mathbf{1}_\Omega = \mu \tag{2.2.26}$$

where μ is a locally finite $\mathbb{R}^m$-valued measure. This proves the proposition. □

The measure μ in (2.2.26) has the form $\mu = -\nu \, \sigma$, as described after (2.2.1). To obtain a more explicit formula, we invoke (2.2.4),

$$\int_\Omega \operatorname{div} v \, dx = \int_{\partial\Omega} \langle \nu, v \rangle \, d\sigma, \tag{2.2.27}$$

together with the elementary identity

$$\int_{\Omega_k} \operatorname{div} v \, dx = \int_{\mathbb{R}^{m-1}} \langle (\nabla A_k(x'), -1), v(x', A_k(x')) \rangle \, dx', \tag{2.2.28}$$

valid for each $v \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^m)$ and $k \in \mathbb{N}$. As $k \to \infty$, the left side of (2.2.28) converges to the left side of (2.2.27), while the right side of (2.2.28) converges to

$$\int_{\mathbb{R}^{m-1}} \langle (\nabla A(x'), -1), v(x', A(x')) \rangle \, dx'. \tag{2.2.29}$$

Hence

$$\int\limits_{\partial\Omega} \langle \nu, v\rangle \, d\sigma = \int\limits_{\mathbb{R}^{m-1}} \langle \tilde{\nu}(x'), v(x', A(x'))\rangle \, d\sigma(x'), \tag{2.2.30}$$

where

$$\tilde{\nu}(x') := \frac{(\nabla A(x'), -1)}{\sqrt{1 + |\nabla A(x')|^2}}, \quad d\sigma(x') := \sqrt{1 + |\nabla A(x')|^2}\, dx'. \tag{2.2.31}$$

The formula (2.2.30) is valid for all $v \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^m)$, hence for all $v \in C_0^0(\mathbb{R}^m, \mathbb{R}^m)$.

Formulas (2.2.30)-(2.2.31) identify the integral of a class of functions (of the form $\langle \nu, v\rangle$) against $d\sigma$. Given that v can be any vector field in $C_0^0(\mathbb{R}^m, \mathbb{R}^m)$, they amount to an identity of two vector measures, namely

$$\nu \, d\sigma = \nu' \, d\sigma' \tag{2.2.32}$$

where $\nu'(X) = (\nabla A(x'), -1)/\sqrt{1 + |\nabla A(x')|^2}$ if $X = (x', A(x')) \in \partial\Omega$, and we have temporarily denoted by σ' the push-forward of $\sqrt{1 + |\nabla A|^2}\, dx'$ to the boundary of Ω via the mapping $x' \mapsto (x', A(x'))$. Since the total variation measure of the left side of (2.2.32) is σ and the total variation measure of the right side is σ', one arrives at

$$\sigma = \sigma', \qquad \text{and} \qquad \nu = \nu' \quad \sigma\text{-a.e.} \tag{2.2.33}$$

In particular,

$$\sigma(\{(x', A(x')) : x' \in \mathcal{O}\}) = \int_{\mathcal{O}} \sqrt{1 + |\nabla A(x')|^2}\, dx', \quad \text{for every Borel set } \ \mathcal{O} \subseteq \mathbb{R}^{m-1}. \tag{2.2.34}$$

Remark. At this point we can invoke (2.2.5), to get

$$\int\limits_{\partial^*\Omega} \langle \nu, v\rangle \, d\mathcal{H}^{m-1} = \int\limits_{\mathbb{R}^{m-1}} \langle \tilde{\nu}(x'), v(x', A(x'))\rangle \, d\sigma(x'), \tag{2.2.35}$$

for each $v \in C_0^0(\mathbb{R}^m, \mathbb{R}^m)$.

It is also of interest to see how the decomposition (2.2.7), asserting countable rectifiability, arises in the context of (2.2.18)–(2.2.19). For simplicity, assume A has compact support. Then set

$$f := |A| + |\nabla A|, \qquad g := \mathcal{M}f, \tag{2.2.36}$$

the latter being the Hardy-Littlewood maximal function in $\mathbb{R}^{m-1}$, and for $\lambda > 0$ take

$$R^\lambda := \{x \in \mathbb{R}^{m-1} : \, g(x) \leq \lambda\}. \tag{2.2.37}$$

Then $\mathcal{L}^{m-1}(\mathbb{R}^{m-1} \setminus R^\lambda) \le C\lambda^{-1}\|f\|_{L^1}$, and an argument involving the Poincaré inequality yields

$$x, y \in R^\lambda \Longrightarrow |A(x)| \le \lambda \ \text{ and } \ |A(x) - A(y)| \le C\lambda|x - y|. \tag{2.2.38}$$

Using this one writes $\partial\Omega = \cup_k L_k \cup \widetilde{N}$, where each L_k is a Lipschitz graph and $\sigma(\widetilde{N}) = 0$. Passing to (2.2.7) is then done by decomposing each Lipschitz graph into a countable union of C^1 graphs plus a negligible remainder, via Rademacher's theorem and Whitney's theorem. See §6.6 of [36] for details.

Regarding the issue of $\partial\Omega$ versus $\partial_*\Omega$, it is clear that $\partial\Omega = \partial_*\Omega$ whenever A is locally Lipschitz. For more general A satisfying (2.2.19), we have the following, which is a consequence of the main results of [114] and [40].

Proposition 2.2.2 *If Ω is the region in $\mathbb{R}^m$ over the graph of a function A satisfying (2.2.19), then*

$$\mathcal{H}^{m-1}(\partial\Omega \setminus \partial_*\Omega) = 0. \tag{2.2.39}$$

Proof. Given a "rectangle" $Q = I_1 \times \cdots \times I_{m-1} \subset \mathbb{R}^{m-1}$, a product of compact intervals, set $K_Q = Q \times \mathbb{R}$. Given the formula (2.2.31) for σ, it follows from Theorem 3.17 of [114] that

$$\sigma(\partial\Omega \cap K_Q) = \mathcal{I}^{m-1}(\partial\Omega \cap K_Q), \tag{2.2.40}$$

where $\mathcal{I}^{m-1}$ denotes $(m-1)$-dimensional integral-geometric measure. Furthermore, it is shown in [40] that

$$\mathcal{H}^{m-1}(\partial\Omega \cap K_Q) = \mathcal{I}^{m-1}(\partial\Omega \cap K_Q). \tag{2.2.41}$$

On the other hand, we have from (2.2.5) that $\sigma(\partial\Omega \cap K_Q) = \mathcal{H}^{m-1}(\partial^*\Omega \cap K_Q)$, so (2.2.39) follows. □

It is useful to note explicitly the following consequence of the preceding arguments.

Proposition 2.2.3 *Assume that Ω and A are as in (2.2.18)–(2.2.19), that $\mathcal{O} \subset \mathbb{R}^{m-1}$ is a Borel set and that $M := \{(x', A(x')) : x' \in \mathcal{O}\}$. Then*

$$\mathcal{H}^{m-1}(M) = \int_{\mathcal{O}} \sqrt{1 + |\nabla A(x')|^2}\, dx'. \tag{2.2.42}$$

Proof. It follows from (2.2.34) that the right side of (2.2.42) is equal to $\sigma(M)$. That $\sigma(M) = \mathcal{H}^{m-1}(M)$ follows from the proof of Proposition 2.2.2, namely from (2.2.40)–(2.2.41). □

So far we have discussed the Green formula for $v \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^m)$. A simple limiting argument extends (2.2.4), and hence (2.2.8) and (2.2.16), to $v \in C_0^1(\mathbb{R}^m, \mathbb{R}^m)$; [41] emphasizes that (2.2.8) is true for compactly supported Lipschitz v. The smooth case will be adequate for use in proving jump relations in Section 3 but for other purposes, such as establishing invertibility of certain layer potentials in Sections 5 and 6, further extensions are desirable. We present some preliminary results here, prior to pursuing the matter much further in §2.3. Here is one easy extension.

Proposition 2.2.4 *If $\Omega \subset \mathbb{R}^m$ has locally finite perimeter, then formula (2.2.4) holds for v in*

$$\mathfrak{D} := \{v \in C_0^0(\mathbb{R}^m, \mathbb{R}^m) : \operatorname{div} v \in L^1(\mathbb{R}^m)\}. \tag{2.2.43}$$

Proof. Given $v \in \mathfrak{D}$, take $\varphi \in C_0^\infty(\mathbb{R}^m)$ such that $\int \varphi\, dx = 1$, set $\varphi_k(x) := k^m \varphi(kx)$, and define $v_k := \varphi_k * v \in C_0^\infty(\mathbb{R}^m)$. Then (2.2.4) applies to v_k, i.e.,

$$\int\limits_{\Omega} \operatorname{div} v_k\, dx = \int\limits_{\partial\Omega} \langle \nu, v_k \rangle\, d\sigma. \tag{2.2.44}$$

Meanwhile, $\operatorname{div} v_k = \varphi_k * (\operatorname{div} v)$ implies $\operatorname{div} v_k \to \operatorname{div} v$ in $L^1(\mathbb{R}^m)$, and $\langle \nu, v_k \rangle \to \langle \nu, v \rangle$ uniformly on $\partial\Omega$, so as $k \to \infty$, the left side of (2.2.44) converges to the left side of (2.2.4), while the right side of (2.2.44) converges to the right side of (2.2.4). □

In many cases one deals with functions defined only on $\overline{\Omega}$, and one would like to avoid assuming they have extensions to $\mathbb{R}^m$ with nice properties. To obtain a result for such functions, we will introduce the following concept. Let open sets Ω_k satisfy $\overline{\Omega}_k \subset \Omega$, $\Omega_k \subset \Omega_{k+1}$, and $\Omega_k \nearrow \Omega$. We say $\{\Omega_k : k \geq 1\}$ is a tame interior approximation to Ω if in addition there exists $C(R) < \infty$ such that, for $R \in (0, \infty)$,

$$\|\nabla \mathbf{1}_{\Omega_k}\|_{\mathrm{TV}(B_R)} \leq C(R), \quad \forall\, k \geq 1. \tag{2.2.45}$$

Here TV stands for the total variation norm of a vector measure.

To give an example, take $A : \mathbb{R}^{m-1} \to \mathbb{R}$, satisfying (2.2.19), and let Ω be given by (2.2.18). We have seen that Ω has locally finite perimeter. The arguments proving Proposition 2.2.1 also imply that

$$\Omega_k = \{(x', x_m) \in \mathbb{R}^m : x_m > A(x') + k^{-1}\} \tag{2.2.46}$$

is a tame interior approximation to Ω. The following is a partial extension of Proposition 2.2.4.

Proposition 2.2.5 *Assume $\Omega \subset \mathbb{R}^m$ has locally finite perimeter and a tame interior approximation. Then (2.2.4) holds for v in*

$$\widetilde{\mathfrak{D}} := \{v \in C_0^0(\overline{\Omega}, \mathbb{R}^m) : \operatorname{div} v \in L^1(\Omega)\}. \tag{2.2.47}$$

Proof. Let $\{\Omega_k\}_k$ denote a tame interior approximation. Pick $\varphi_k \in C_0^\infty(\Omega)$ to be $\equiv 1$ on a neighborhood of $\overline{\Omega}_k \cap \operatorname{supp} v$, set $v_k = \varphi_k v$, and apply Proposition 2.2.1 with Ω replaced by Ω_k and v by v_k, noting that $\operatorname{div} v_k = \varphi_k \operatorname{div} v + \langle \nabla \varphi_k, v \rangle$. We have

$$\int\limits_{\Omega_k} \operatorname{div} v\, dx = -\langle v, \nabla \mathbf{1}_{\Omega_k} \rangle. \tag{2.2.48}$$

As $k \to \infty$, the left side of (2.2.48) converges to the left side of (2.2.4). Meanwhile, we can take $w \in C_0^0(\mathbb{R}^m, \mathbb{R}^m)$, equal to v on $\overline{\Omega}$, and the right side of (2.2.48) is equal to $-\langle w, \nabla \mathbf{1}_{\Omega_k} \rangle$. Now $\mathbf{1}_{\Omega_k} \to \mathbf{1}_\Omega$ in $L^1_{loc}(\mathbb{R}^m)$, so $\nabla \mathbf{1}_{\Omega_k} \to \nabla \mathbf{1}_\Omega$ in $\mathcal{D}'(\mathbb{R}^m)$, and hence

$$\langle w, \nabla \mathbf{1}_{\Omega_k} \rangle \longrightarrow \langle w, \nabla \mathbf{1}_\Omega \rangle \tag{2.2.49}$$

for each $w \in C_0^\infty(\mathbb{R}^m, \mathbb{R}^m)$. The bounds (2.2.45) then imply that (2.2.49) holds for each $w \in C_0^0(\mathbb{R}^m, \mathbb{R}^m)$. Hence the right side of (2.2.48) converges to

$$-\langle w, \nabla \mathbf{1}_\Omega \rangle = \int\limits_{\partial\Omega} \langle \nu, v \rangle \, d\sigma, \tag{2.2.50}$$

which is the right side of (2.2.4). □

Remark. Proposition 2.2.5 can be compared with the following result, given in [39], p. 314. Let $\Omega \subset \mathbb{R}^m$ be a bounded open set such that $\mathcal{H}^{m-1}(\partial\Omega) < \infty$. Fix $j \in \{1, \dots, m\}$ and take f such that

$$f \in C(\overline{\Omega}), \quad \partial_j f \in L^1(\Omega). \tag{2.2.51}$$

Then

$$\int\limits_{\Omega} \partial_j f \, dx = \int\limits_{\partial_0\Omega} \langle e_j, \nu \rangle \, f \, d\mathcal{H}^{m-1}, \tag{2.2.52}$$

where $\partial_0\Omega$ has been introduced in (2.2.12), and e_j is the jth standard basis vector of $\mathbb{R}^m$. In light of (2.2.15) one could replace $\partial_0\Omega$ by $\partial^*\Omega$ or by $\partial_*\Omega$ in (2.2.52). This leads to the identity (2.2.16) for a vector field $v \in C(\overline{\Omega})$ provided each term $\partial_j v_j$ in $\operatorname{div} v$ belongs to $L^1(\Omega)$. However, the vector fields arising in the applications of Green's formula needed in §5–§6 need not have this additional structure, so (2.2.52) is not applicable.

We also mention results given in §2 of [16], dealing with a vector field $v \in L^p(\mathcal{O})$ such that $\operatorname{div} v$ is a measure on $\mathcal{O}$, and $\overline{\Omega} \subset \mathcal{O}$. These results also extend Proposition 2.2.4, but they do not imply Proposition 2.2.5, nor the results given in the next subsection.

Further results related to the last two propositions can be found in [95].

We next recall a Green formula for

$$\int\limits_{\Omega \cap B_r} \operatorname{div} v \, dx, \tag{2.2.53}$$

where Ω has locally finite perimeter and $B_r := \{x \in \mathbb{R}^m : |x| < r\}$. This classical result will be of direct use in our proof of jump relations for layer potentials.

Assume $v \in C_0^{0,1}(\mathbb{R}^m, \mathbb{R}^m)$. Given $\varepsilon \in (0, r)$, set

$$\psi_\varepsilon(x) := \begin{cases} 1 & \text{for } |x| \le r - \varepsilon, \\ 1 - \frac{1}{\varepsilon}(|x| - r + \varepsilon) & \text{for } r - \varepsilon \le |x| \le r, \\ 0 & \text{for } |x| \ge r. \end{cases} \tag{2.2.54}$$

Then, with $\nabla \mathbf{1}_\Omega = -\nu \, \sigma$, we have

$$
\begin{aligned}
\int\limits_{\Omega\cap B_r} \operatorname{div} v\, dx &= \lim_{\varepsilon\searrow 0} \int\limits_{\Omega} \psi_\varepsilon \operatorname{div} v\, dx \\
&= \lim_{\varepsilon\searrow 0} \int\limits_{\Omega} [\operatorname{div} \psi_\varepsilon v - \langle v, \nabla\psi_\varepsilon\rangle]\, dx \\
&= \lim_{\varepsilon\searrow 0} \Big(\int \langle \nu, \psi_\varepsilon v\rangle\, d\sigma - \int\limits_{\Omega} \langle v, \nabla\psi_\varepsilon\rangle\, dx \Big) \\
&= \int\limits_{B_r} \langle \nu, v\rangle\, d\sigma + \lim_{\varepsilon\searrow 0} \frac{1}{\varepsilon} \int\limits_{\Omega\cap S_\varepsilon} \langle n, v\rangle\, dx, \qquad (2.2.55)
\end{aligned}
$$

where n is the outward unit normal to B_r and

$$
S_\varepsilon := B_r \setminus B_{r-\varepsilon}. \qquad (2.2.56)
$$

Consequently,

$$
\int\limits_{\Omega\cap B_r} \operatorname{div} v\, dx = \int\limits_{B_r} \langle \nu, v\rangle\, d\sigma + D_r^- \Phi(r), \qquad (2.2.57)
$$

where D_r^- indicates differentiation from the left with respect to r, and

$$
\Phi(r) := \int\limits_{\Omega\cap B_r} \langle n, v\rangle\, dx. \qquad (2.2.58)
$$

Note that, by the change of variable formula and Fubini's theorem,

$$
\Phi(r) = \int_0^r \int\limits_{\Omega\cap\partial B_s} \langle n, v\rangle\, d\mathcal{H}^{m-1}\, ds, \qquad (2.2.59)
$$

so

$$
D_r^- \Phi(r) = \int\limits_{\Omega\cap\partial B_r} \langle n, v\rangle\, d\mathcal{H}^{m-1}, \qquad \text{for } \mathcal{L}^1\text{-a.e. } r > 0. \qquad (2.2.60)
$$

It is of interest to note that $D_r^- \Phi(r)$ exists for *all* $r \in (0, \infty)$ (under our standing hypothesis on Ω), though the identity (2.2.60) is valid perhaps not for each r, but just for a.e. r.

Remark. Having (2.2.57), one can bring in (2.2.5) and write

$$
\int\limits_{\Omega\cap B_r} \operatorname{div} v\, dx = \int\limits_{B_r\cap\partial^*\Omega} \langle \nu, v\rangle\, d\mathcal{H}^{m-1} + D_r^- \Phi(r). \qquad (2.2.61)
$$

It is useful to note that (2.2.5) is not needed to prove (2.2.57), since (2.2.57) plays a role in proofs of (2.2.5). (Cf. [36].) It is also useful to put together (2.2.60)–(2.2.61), to write

$$\int\limits_{\Omega\cap B_r} \operatorname{div} v\, dx = \int\limits_{B_r\cap\partial^*\Omega} \langle \nu, v\rangle\, d\sigma + \int\limits_{\Omega\cap\partial B_r} \langle n, v\rangle\, d\mathcal{H}^{m-1}, \qquad \text{for } \mathcal{L}^1\text{-a.e. } r>0. \tag{2.2.62}$$

While the result (2.2.62) is well known (cf. Lemma 1 on p. 195 of [36]), we think it useful to include a proof, not only for the reader's convenience, but also to emphasize two points: first that one does not need the relatively advanced co-area formula in the proof, and second that the formulation (2.2.57) of the result actually holds for all r, not merely almost all r.

2.3 Green's formula on Ahlfors regular domains

Assume that $\Omega\subset\mathbb{R}^{n+1}$ is a bounded open set whose boundary is Ahlfors regular and satisfies

$$\mathcal{H}^n(\partial\Omega\setminus\partial_*\Omega)=0. \tag{2.3.1}$$

Note that, by (2.2.17), these conditions entail that Ω is of finite perimeter. In keeping with earlier conventions, we denote by ν the measure theoretic outward unit normal to $\partial\Omega$ and set $\sigma:=\mathcal{H}^n\lfloor\partial\Omega$. We wish to study the validity of a version of Green's formula, i.e.,

$$\int\limits_{\Omega} \operatorname{div} v\, dX = \int\limits_{\partial\Omega} \langle \nu, v\big|_{\partial\Omega}\rangle\, d\sigma, \tag{2.3.2}$$

for vector fields $v\in C^0(\Omega)$ for which

$$\begin{aligned} &\operatorname{div} v\in L^1(\Omega), \quad \mathcal{N}v\in L^p(\partial\Omega, d\sigma) \text{ for some } p\in[1,\infty] \\ &\text{and the pointwise nontangential trace } v\big|_{\partial\Omega} \text{ exists } \sigma\text{-a.e.} \end{aligned} \tag{2.3.3}$$

In the case when Ω is Lipschitz, a convenient approach is to approximate it by a nested family of nice domains $\Omega_j\nearrow\Omega$, write Green's formula in each Ω_j and then obtain (2.3.2) by passing to the limit in j. See [116]. For Reifenberg flat domains, an approximation result of this nature has been proved by C. Kenig and T. Toro in Appendix A.1 of [66]. This is not entirely satisfactory since one needs to impose a "flatness" condition on Ω, which is not natural in this context. Our goal is to present a new approach to proving (2.3.2), which does not require this condition.

Our main result in this regard is the following.

Theorem 2.3.1 *Let $\Omega\subset\mathbb{R}^{n+1}$ be a bounded open set whose boundary is Ahlfors regular and satisfies (2.3.1) (hence, in particular, is of finite perimeter). As usual, set $\sigma:=\mathcal{H}^n\lfloor\partial\Omega$ and denote by ν the measure theoretic outward unit normal to $\partial\Omega$. Then Green's formula (2.3.2) holds for each vector field $v\in C^0(\Omega)$ that satisfies the conditions in (2.3.3), with $p\in(1,\infty)$.*

One clarification is in order here. Generally speaking, given a domain $\Omega\subset\mathbb{R}^{n+1}$, $\alpha>0$ and a function $u:\Omega\to\mathbb{R}$, we set

$$u\big|_{\partial\Omega}(Z) := \lim_{\substack{X\to Z\\ X\in\Gamma_\alpha(Z)}} u(X), \qquad Z\in\partial\Omega, \tag{2.3.4}$$

whenever the limit exists. For this definition to be pointwise σ-a.e. meaningful, it is necessary that

$$Z \in \overline{\Gamma_\alpha(Z)} \quad \text{for } \sigma\text{-a.e.} \quad Z \in \partial\Omega. \tag{2.3.5}$$

We shall call a domain Ω satisfying (2.3.5) above *weakly accessible* and our first order of business is to show that any domain as in the statement of Theorem 2.3.1 is weakly accessible.

To get started, fix $\alpha \in (0, \infty)$ and, for each $\delta > 0$, introduce

$$\mathcal{O}_\delta := \{X \in \Omega : \operatorname{dist}(X, \partial\Omega) \leq \delta\}, \quad \forall\, \delta > 0. \tag{2.3.6}$$

Also, for $Z \in \partial\Omega$, set

$$\begin{aligned} \Gamma_\alpha^\delta(Z) &:= \Gamma_\alpha(Z) \cap \mathcal{O}_\delta \\ &= \{X \in \Omega : \operatorname{dist}(X, \partial\Omega) \leq \delta \text{ and } |X - Z| \leq (1+\alpha)\operatorname{dist}(X, \partial\Omega)\}, \end{aligned} \tag{2.3.7}$$

and define

$$\mathcal{I}^\delta := \{Z \in \partial\Omega : \Gamma_\alpha^\delta(Z) = \emptyset\}, \qquad \mathcal{I} := \bigcup_{\delta > 0} \mathcal{I}^\delta. \tag{2.3.8}$$

Clearly, $\mathcal{I}^\delta$ is relatively closed in $\partial\Omega$, so that $\mathcal{I}$ is a Borel set. Also, $\mathcal{I}^\delta \nearrow \mathcal{I}$ as $\delta \searrow 0$. Then Ω is weakly accessible provided $\sigma(\mathcal{I}) = 0$ or, equivalently, provided $\sigma(\mathcal{I}^\delta) = 0$ for each $\delta > 0$.

Proposition 2.3.2 *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with an Ahlfors regular boundary $\partial\Omega$. Assume that $\mathcal{H}^n(\partial\Omega \setminus \partial_*\Omega) = 0$ (so that, in particular, Ω is of locally finite perimeter; cf. (2.2.17)). Then Ω is a weakly accessible domain.*

As a preliminary result, we shall establish an estimate which will also be useful in several other instances later on. To be definite, for each $X \in \partial\Omega$ set

$$\Gamma(X) := \{Y \in \Omega : |X - Y| \leq 10 \operatorname{dist}(Y, \partial\Omega)\}. \tag{2.3.9}$$

Proposition 2.3.3 *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with Ahlfors regular boundary. Then there exists $C > 0$ depending only on n and the Ahlfors regularity constant on Ω such that*

$$\frac{1}{\delta} \int_{\mathcal{O}_\delta} |v|\, dX \leq C \|\mathcal{N} v\|_{L^1(\partial\Omega, d\sigma)}, \qquad 0 < \delta \leq \mathit{diam}\,\Omega, \tag{2.3.10}$$

for any measurable $v : \Omega \to \mathbb{R}$.

Proof. We first note that it suffices to prove that

$$\frac{1}{\delta} \int_{\widetilde{\mathcal{O}}_\delta} |v|\, dX \leq C \|\mathcal{N} v\|_{L^1(\partial\Omega, d\sigma)}, \quad \text{where } \widetilde{\mathcal{O}}_\delta := \mathcal{O}_\delta \setminus \mathcal{O}_{\delta/2}, \tag{2.3.11}$$

since (2.3.10) then follows by applying (2.3.11) with δ replaced by $2^{-j}\delta$ and summing over $j \in \mathbb{Z}_+$. We now bring in the following lemma.

Lemma 2.3.4 *There exists $K = K_n \in \mathbb{N}$ with the following property. For each $\delta \in (0, (diam\,\Omega)/10]$, there exists a covering of $\partial\Omega$ by a collection*

$$\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_K \tag{2.3.12}$$

of balls of radius δ, centered in $\partial\Omega$, such that for each $k \in \{1, \ldots, K\}$, if B and B' are distinct balls in $\mathcal{C}_k$, their centers are separated by a distance $\geq 10\delta$.

We postpone the proof of Lemma 2.3.4, and show how it is used to finish the proof of Proposition 2.3.3.

To begin, take a collection $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_K$ of balls of radius δ covering $\partial\Omega$, with the properties stated above. Then $\mathcal{C}^\# = \mathcal{C}_1^\# \cup \cdots \cup \mathcal{C}_K^\#$, consisting of balls concentric with those of $\mathcal{C}$ with radius 2δ, covers $\mathcal{O}_\delta$. Furthermore, there exists $A = A_n \in (0, \infty)$ such that one can cover each ball $B \in \mathcal{C}_k^\#$ by balls $B_1, \ldots, B_A$ of radius $\delta/8$, centered at points in B. Now form collections of balls $\mathcal{C}_{k\ell}^\#$, $1 \leq k \leq K$, $1 \leq \ell \leq A$, with each of the balls $B_1, \ldots, B_A$ covering $B \in \mathcal{C}_k^\#$, described above, put into a different one of the collections $\mathcal{C}_{k\ell}^\#$. Throw away some balls from $\mathcal{C}_{k\ell}^\#$, thinning them out to a *minimal* collection

$$\widetilde{\mathcal{C}} = \bigcup_{k \leq K, \ell \leq A} \widetilde{\mathcal{C}}_{k\ell} \tag{2.3.13}$$

covering $\widetilde{\mathcal{O}}_\delta$. For each (k, ℓ), any two distinct balls in $\widetilde{\mathcal{C}}_{k\ell}$ have centers separated by a distance $\geq 7\delta$. Each ball $B \in \widetilde{\mathcal{C}}_{k\ell}$ has radius $\delta/8$ and each point $Q \in B$ has distance from $\partial\Omega$ lying between $\delta/4$ and $5\delta/4$. For each such B, we will compare $\int_B |v|\,dX$ with the integral of $\mathcal{N}v$ over a certain set $\widetilde{\mathfrak{A}}(B) \subset \partial\Omega$, which we proceed to define.

Given $Y \in \Omega$, set $d(Y) := \text{dist}\,(Y, \partial\Omega)$ and consider

$$\mathfrak{A}(Y) := \{X \in \partial\Omega : \, Y \in \Gamma(X)\}. \tag{2.3.14}$$

There exists $Q \in \partial\Omega$ such that $|Y - Q| = d(Y)$, and certainly $Q \in \mathfrak{A}(Y)$. Also, if (2.3.9) holds, then

$$\mathfrak{A}(Y) \supset B(Q, 9d(Y)) \cap \partial\Omega. \tag{2.3.15}$$

Now, for a ball $B \in \widetilde{\mathcal{C}}_{k\ell}$, set

$$\mathfrak{A}(B) := \{X \in \partial\Omega : \, B \subset \Gamma(X)\}. \tag{2.3.16}$$

If B is centered at Y and $Q \in \partial\Omega$ is closest to Y, then for each $Y' \in B$, $d(Y') \geq \delta/4$ and $|Y' - Q| \leq d(Y) + \delta/8$. Now $d(Y) \leq (9/8)\delta$, so $|Y' - Q| \leq (5/4)\delta \leq 5\,d(Y')$ and, hence,

$$\widetilde{\mathfrak{A}}(B) := B(Q, d(Y)) \cap \partial\Omega \subset \mathfrak{A}(B). \tag{2.3.17}$$

The use of $\widetilde{\mathfrak{A}}(B)$ in establishing (2.3.11) arises from the estimates

$$\sup_B |v| \leq \inf_{\widetilde{\mathfrak{A}}(B)} \mathcal{N}v, \quad \text{and } \mathcal{H}^n(\widetilde{\mathfrak{A}}(B)) \geq C_1\delta^n, \tag{2.3.18}$$

the latter estimate due to the hypothesis that $\partial\Omega$ is Ahlfors regular. Hence

$$\frac{1}{\delta}\int_B |v|\, dY \leq C_n\delta^n \inf_{\widetilde{\mathfrak{A}}(B)} \mathcal{N}v \leq C \int_{\widetilde{\mathfrak{A}}(B)} \mathcal{N}v\, d\sigma. \tag{2.3.19}$$

Furthermore, the separation properties established for balls in each collection $\widetilde{\mathcal{C}}_{k\ell}$ yield

$$B \neq B' \in \widetilde{\mathcal{C}}_{k\ell} \Longrightarrow \widetilde{\mathfrak{A}}(B) \cap \widetilde{\mathfrak{A}}(B') = \emptyset, \tag{2.3.20}$$

so for each $k \leq K$, $\ell \leq A$,

$$\frac{1}{\delta}\sum_{B\in\widetilde{\mathcal{C}}_{k\ell}} \int_B |v|\, dX \leq C \int_{\cup\widetilde{\mathfrak{A}}(B),\, B\in\widetilde{\mathcal{C}}_{k\ell}} \mathcal{N}v\, d\sigma \leq C\|\mathcal{N}v\|_{L^1(\partial\Omega,d\sigma)}. \tag{2.3.21}$$

Consequently,

$$\frac{1}{\delta}\int_{\widetilde{\mathcal{O}}_\delta} |v|\, dX \leq \frac{1}{\delta}\sum_{\ell=1}^{A}\sum_{k=1}^{K}\sum_{B\in\widetilde{\mathcal{C}}_{k\ell}} \int_B |v|\, dX \leq CAK\, \|\mathcal{N}v\|_{L^1(\partial\Omega,d\sigma)}, \tag{2.3.22}$$

and Proposition 2.3.3 is established, modulo the proof of Lemma 2.3.4, to which we now turn. $\Box$

Proof of Lemma 2.3.4. This can be proved in several ways. One approach starts by applying Besicovitch's Covering Theorem (cf. Theorem 2 on p. 30 in [36]) to the family of balls $\mathcal{F} := \{B(Q,\delta) : Q \in \partial\Omega\}$. This yields some $N = N_n \in \mathbb{N}$ and $\mathcal{G}_1, ..., \mathcal{G}_N \subset \mathcal{F}$ such that each $\mathcal{G}_j$, $j = 1, ..., N$, is a countable collection of disjoint balls in $\mathcal{F}$ and

$$\partial\Omega \subset \bigcup_{j=1}^{N} \bigcup_{B\in\mathcal{G}_j} B. \tag{2.3.23}$$

To finish the proof, we need to further subdivide each $\mathcal{G}_j$ into finitely many subclasses, say

$$\mathcal{G}_j = \bigcup_{k=1}^{N} \bigcup_{B\in\mathcal{G}_{jk}} B, \tag{2.3.24}$$

with the property that if B and B' are distinct balls in $\mathcal{G}_{jk}$ then their centers are separated by a distance $\geq 10\delta$. We then relabel $\{\mathcal{G}_{jk} : 1 \leq j, k \leq N\}$ as $\{\mathcal{C}_j : 1 \leq j \leq K\}$, where $K := N^2$. To construct such a family $\{\mathcal{G}_{jk}\}_{1\leq k\leq N}$ for each j, we once again apply Besicovitch's Covering Theorem to the family $\{B(Q, 5\delta) : B(Q,\delta) \in \mathcal{G}_j\}$. This readily yields the desired conclusion. $\Box$

Before proceeding further, let us record a byproduct of (2.3.10) which has intrinsic interest.

Proposition 2.3.5 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set with an Ahlfors regular boundary, and fix $\alpha > 0$. Then there exists $C > 0$ depending only on n, α, and the Ahlfors regularity constant on Ω such that*

$$\frac{1}{\delta} \int\limits_{\mathcal{O}_\delta} |v| \, dX \leq C \|\mathcal{N}^\delta v\|_{L^1(\partial\Omega, d\sigma)}, \qquad 0 < \delta \leq diam\,(\Omega), \tag{2.3.25}$$

for any measurable function $v : \Omega \to \mathbb{R}$, where

$$\mathcal{N}^\delta v(X) := \sup\{|v(Y)| : \, Y \in \Gamma_\alpha(X), \, |X - Y| \leq 2\delta\}, \tag{2.3.26}$$

with the convention that $\mathcal{N}^\delta v(X) := 0$ whenever the supremum in the right-hand side of (2.3.26) is taken over the empty set. Consequently, there exists $C > 0$ with the property that for any measurable function $v : \Omega \to \mathbb{R}$ and any measurable set $E \subseteq \Omega$,

$$\int\limits_E |v| \, dX \leq C\delta \int\limits_{\mathcal{U}(E)} \mathcal{N}^\delta v \, d\sigma, \qquad \delta := diam\,E + dist\,(E, \partial\Omega), \tag{2.3.27}$$

where $\mathcal{U}(E) := \{X \in \partial\Omega : \, \Gamma_\alpha(X) \cap E \neq \emptyset\}$.

Proof. To begin with, (2.3.25) follows from directly from (2.3.10) and a simple cutoff argument. As for (2.3.27), we use (2.3.25) to write

$$\int\limits_E |v| \, dX = \int\limits_{\mathcal{O}_\delta} |\mathbf{1}_E \, v| \, dX \leq C\delta \int\limits_{\partial\Omega} \mathcal{N}^\delta(\mathbf{1}_E \, v) \, d\sigma \leq C\delta \int\limits_{\mathcal{U}(E)} \mathcal{N}^\delta v \, d\sigma, \tag{2.3.28}$$

as desired. $\square$

Having established (2.3.25), we are now in a position to carry out the

Proof of Proposition 2.3.2. To see this, take $\delta_0 > 0$ and let $K \subset \mathcal{I}^{\delta_0}$ be an arbitrary compact set. We want to show that $\sigma(K) = 0$. Fix $\varepsilon > 0$ and define

$$K^\varepsilon := \{Z \in \mathbb{R}^{n+1} : \, \text{dist}\,(Z, K) \leq \varepsilon\}, \quad K_\varepsilon := K^\varepsilon \cap \partial\Omega = \{Z \in \partial\Omega : \, \text{dist}\,(Z, K) \leq \varepsilon\}, \tag{2.3.29}$$

so that

$$K_\varepsilon \searrow K \;\; \text{as} \;\; \varepsilon \searrow 0. \tag{2.3.30}$$

Let us also take $f_\varepsilon \in C^0(\partial\Omega)$ such that

$$|f_\varepsilon| \leq 1, \quad \text{supp}\, f_\varepsilon \subset K_{\varepsilon/2}, \tag{2.3.31}$$

and consider an extension u_ε of f_ε satisfying

$$u_\varepsilon \in C^0(\bar\Omega), \quad \operatorname{supp} u_\varepsilon \subset K^\varepsilon \cap \bar\Omega, \quad |u_\varepsilon| \le 1, \quad u_\varepsilon\big|_{\partial\Omega} = f_\varepsilon. \tag{2.3.32}$$

In this case, (2.3.25) gives

$$\frac{1}{\delta}\int\limits_{\mathcal{O}_\delta} |u_\varepsilon|\, dX \le C\|\mathcal{N}^\delta u_\varepsilon\|_{L^1(\partial\Omega, d\sigma)}, \tag{2.3.33}$$

with C independent of ε and δ. Note that, for $\delta < \delta_0/4$,

$$Z \in K \Longrightarrow \mathcal{N}^\delta u_\varepsilon(Z) = 0 \tag{2.3.34}$$

since $\Gamma_\alpha^{\delta_0}(Z)$ (defined as in (2.3.7)) is the empty set in this case. Also, elementary geometrical considerations (and earlier conventions) entail

$$Z \in \partial\Omega \setminus K_{\varepsilon(2+\alpha)} \Longrightarrow \Gamma_\alpha(Z) \cap K^\varepsilon = \emptyset \Longrightarrow \mathcal{N} u_\varepsilon(Z) = 0. \tag{2.3.35}$$

Hence, (2.3.33)-(2.3.35) imply

$$\limsup_{\delta\to 0} \frac{1}{\delta}\int\limits_{\mathcal{O}_\delta} |u_\varepsilon|\, dX \le C\sigma(K_{\varepsilon(2+\alpha)} \setminus K). \tag{2.3.36}$$

To proceed, for each $\delta > 0$ set $\Omega_\delta := \{X \in \Omega : \operatorname{dist}(X, \partial\Omega) \ge \delta\}$, let $\varphi_\delta(X) := \operatorname{dist}(X, \partial\Omega_{\delta/2})$ and introduce

$$\chi_\delta(X) := \begin{cases} 1 & \text{if } X \in \Omega_\delta, \\ 2\delta^{-1}\,\varphi_\delta(X) & \text{if } X \in \widetilde{\mathcal{O}}_\delta := \mathcal{O}_\delta \setminus \mathcal{O}_{\delta/2}, \\ 0 & \text{if } X \in \mathcal{O}_{\delta/2} \cup (\mathbb{R}^{n+1} \setminus \Omega). \end{cases} \tag{2.3.37}$$

We have $\chi_\delta \to \chi_\Omega$ in $L^1_{loc}(\mathbb{R}^{n+1})$ as $\delta \to 0$, hence

$$\nabla\chi_\delta \longrightarrow \nabla\chi_\Omega = \nu\,\sigma \quad \text{ in } \mathcal{D}'(\mathbb{R}^{n+1}). \tag{2.3.38}$$

On the other hand, $\chi_\delta \in \operatorname{Lip}(\mathbb{R}^{n+1})$ with $\|\nabla\chi_\delta\|_{L^\infty(\mathbb{R}^{n+1})} \le 2/\delta$ and $\operatorname{supp}(\nabla\chi_\delta) \subset \widetilde{\mathcal{O}}_\delta$. Also, (2.3.25) implies the following bound on the total variation of the measure $\nabla\chi_\delta$ in K^ε:

$$\|\nabla\chi_\delta\|_{TV(K^\varepsilon)} = \sup_{\|v\|_{L^\infty(K^\varepsilon)}\le 1} \left|\int v\,\nabla\chi_\delta\, dX\right| \le \frac{2}{\delta}\int\limits_{\mathcal{O}_\delta} |v|\, dX \le C\|\mathcal{N}^\delta v\|_{L^1(\partial\Omega, d\sigma)} \le C, \tag{2.3.39}$$

uniformly in δ. Thus, we see that for each coordinate vector e_j,

$$e_j \cdot \nabla\chi_\delta \longrightarrow (e_j\cdot\nu)\,\sigma, \quad \text{ weak}^* \text{ as Radon measures in } K^\varepsilon, \tag{2.3.40}$$

as $\delta \to 0$. Consequently, for each u_ε as in (2.3.32),

$$\lim_{\delta\to 0} \langle u_\varepsilon, e_j \cdot \nabla\chi_\delta\rangle = \langle u_\varepsilon, (e_j \cdot \nu)\sigma\rangle = \int\limits_{\partial\Omega} f_\varepsilon(e_j \cdot \nu)\, d\sigma. \tag{2.3.41}$$

Since

$$|\langle u_\varepsilon, e_j \cdot \nabla\chi_\delta\rangle| \leq \frac{2}{\delta} \int\limits_{\mathcal{O}_\delta} |u_\varepsilon|\, dX, \tag{2.3.42}$$

(2.3.36) implies that

$$\limsup_{\delta\to 0} |\langle u_\varepsilon, e_j \cdot \nabla\chi_\delta\rangle| \leq C\sigma(K_{\varepsilon(2+\alpha)} \setminus K). \tag{2.3.43}$$

Comparison with (2.3.41) gives

$$\Big|\int\limits_{\partial\Omega} f_\varepsilon\, (e_j \cdot \nu)\, d\sigma\Big| \leq C\sigma(K_{\varepsilon(2+\alpha)} \setminus K). \tag{2.3.44}$$

Now the supremum of the left side of (2.3.44) over the set of all $f_\varepsilon \in C^0(\partial\Omega)$ satisfying (2.3.31) is equal to the total variation of $(e_j \cdot \nu)\sigma$ restricted to $K_{\varepsilon/2}$, which in turn is $\geq \|(e_j \cdot \nu)\sigma\|_{TV(K)}$, so we have

$$\|(e_j \cdot \nu)\sigma\|_{TV(K)} \leq C\sigma(K_{\varepsilon(2+\alpha)} \setminus K), \qquad 1 \leq j \leq n+1. \tag{2.3.45}$$

Taking $\varepsilon \searrow 0$ gives

$$\sigma(K) = 0, \tag{2.3.46}$$

on account of (2.3.30), since σ is a Radon measure (cf. (2.1.2) and Theorem 4 on p. 8 in [36]), proving Proposition 2.3.2. □

In order to properly set up the proof of Theorem 2.3.1, we continue our discussion of a number of preliminary results. Concretely, let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded open set with finite perimeter for which (2.3.1) holds, and let ν, σ be as stated in the opening paragraph of § 2.3. For $p \in [1, \infty)$, set

$$\mathfrak{L}^p := \Big\{ v \in C^0(\Omega) : \mathcal{N}v \in L^p(\partial\Omega, d\sigma), \text{ and } \exists \text{ nontangential limit } v\big|_{\partial\Omega}\ \sigma\text{-a.e.}\Big\}. \tag{2.3.47}$$

Our strategy is to first establish a Green formula for vector fields $v \in \mathfrak{L}^p$ with divergence in $L^1(\Omega)$, provided we have the following:

$$\begin{aligned} &\text{if } v \in \mathfrak{L}^p,\ \exists\, w \in \mathfrak{L}^1 \text{ with } w\big|_{\partial\Omega} = v\big|_{\partial\Omega} \text{ and } \exists\, w_k \in \mathrm{Lip}\,(\bar{\Omega}) \\ &\text{such that } \|\mathcal{N}(w - w_k)\|_{L^1(\partial\Omega, d\sigma)} \longrightarrow 0 \quad \text{as } k \to \infty. \end{aligned} \tag{2.3.48}$$

Then we will show that (2.3.48) holds whenever $\partial\Omega$ is Ahlfors regular, and $p \in (1,\infty)$.

Before we state our first result, a comment is in order. In condition (2.3.48) it would be equivalent to demand merely that $w_k \in C^0(\bar{\Omega})$, since elements of $C^0(\bar{\Omega})$ are easily uniformly approximated by Lipschitz functions (e.g., via the Stone-Weierstrass theorem). Here is our first result.

Proposition 2.3.6 *Pick $p \in [1,\infty)$ and assume Ω is a bounded open set with Ahlfors regular boundary, satisfying (2.3.1) as well as (2.3.48). Then*

$$\int_{\Omega} \operatorname{div} v \, dX = \int_{\partial\Omega} \langle \nu, v\big|_{\partial\Omega} \rangle \, d\sigma, \tag{2.3.49}$$

whenever the vector field v satisfies

$$v \in \mathfrak{L}^p \quad \textit{and} \quad \operatorname{div} v \in L^1(\Omega). \tag{2.3.50}$$

Proof. For each $\delta > 0$ let χ_δ be as in (2.3.37) so that, clearly, $\chi_\delta \in \operatorname{Lip}(\bar{\Omega})$. Also, if v satisfies (2.3.50), then $\chi_\delta v \in C^0_0(\Omega)$ and $\operatorname{div}(\chi_\delta v) \in L^1(\Omega)$, so it is elementary that

$$\int_{\Omega} \operatorname{div}(\chi_\delta v) \, dX = 0. \tag{2.3.51}$$

Hence,

$$\int_{\Omega} \chi_\delta \operatorname{div} v \, dX = -\int_{\Omega} \langle \nabla \chi_\delta, v \rangle \, dX = \frac{2}{\delta} \int_{\widetilde{\mathcal{O}}_\delta} \langle \nu_\delta, v \rangle \, dX, \tag{2.3.52}$$

where

$$\nu_\delta := -\nabla \varphi_\delta \quad \text{and} \quad \widetilde{\mathcal{O}}_\delta := \mathcal{O}_\delta \setminus \mathcal{O}_{\delta/2}. \tag{2.3.53}$$

The first integral in (2.3.52) converges to the left side of (2.3.49) as $\delta \to 0$, whenever $\operatorname{div} v \in L^1(\Omega)$. Hence (2.3.49) is true provided

$$\frac{2}{\delta} \int_{\widetilde{\mathcal{O}}_\delta} \langle \nu_\delta, v \rangle \, dX \longrightarrow \int_{\partial\Omega} \langle \nu, v\big|_{\partial\Omega} \rangle \, d\sigma \qquad \text{as} \ \ \delta \to 0. \tag{2.3.54}$$

Of course, by (2.3.52), the left side of (2.3.54) does converge as $\delta \to 0$, namely to the left side of (2.3.49). Hence (2.3.54) is true whenever (2.3.49) is true. In particular, since Ω has finite perimeter, (2.3.54) is true whenever $v \in \operatorname{Lip}(\overline{\Omega})$.

More generally, if $v \in \mathfrak{L}^p$, take w, w_k as in (2.3.48). Under our hypotheses, we have (2.3.10), hence

$$\Big| \frac{2}{\delta} \int_{\widetilde{\mathcal{O}}_\delta} \Big(\langle \nu_\delta, w_k \rangle - \langle \nu_\delta, w \rangle \Big) \, dX \Big| \leq C \|\mathcal{N}(w - w_k)\|_{L^1(\partial\Omega, d\sigma)}. \tag{2.3.55}$$

The weak accessibility result (Proposition 2.3.2) implies

$$\int_{\partial\Omega} |v|\, d\sigma \leq \int_{\partial\Omega} |\mathcal{N} v|\, d\sigma, \quad \forall\, v \in \mathfrak{L}^1, \tag{2.3.56}$$

so we also have

$$\Big| \int_{\partial\Omega} \Big(\langle \nu, w_k \rangle - \langle \nu, w\big|_{\partial\Omega} \rangle \Big)\, d\sigma \Big| \leq \|\mathcal{N}(w_k - w)\|_{L^1(\partial\Omega, d\sigma)}. \tag{2.3.57}$$

Thus, since (2.3.54) holds for w_k, we have

$$\frac{2}{\delta} \int_{\widetilde{\mathcal{O}}_\delta} \langle \nu_\delta, w \rangle\, dX \longrightarrow \int_{\partial\Omega} \langle \nu, w\big|_{\partial\Omega} \rangle\, d\sigma = \int_{\partial\Omega} \langle \nu, v\big|_{\partial\Omega} \rangle\, d\sigma \quad \text{ as } \ \delta \to 0. \tag{2.3.58}$$

Thus, to obtain (2.3.54) for each $v \in \mathfrak{L}^p$, it suffices to show that

$$\frac{2}{\delta} \int_{\widetilde{\mathcal{O}}_\delta} \Big(\langle \nu_\delta, v \rangle - \langle \nu_\delta, w \rangle \Big)\, dX \longrightarrow 0 \quad \text{ as } \ \delta \to 0. \tag{2.3.59}$$

Hence, it suffices to show that

$$u \in \mathfrak{L}^1,\ u\big|_{\partial\Omega} = 0 \Longrightarrow \frac{2}{\delta} \int_{\widetilde{\mathcal{O}}_\delta} |u|\, dX \longrightarrow 0 \quad \text{ as } \ \delta \to 0. \tag{2.3.60}$$

Recalling that (2.3.10) implies (2.3.25), we see that it suffices to show that

$$u \in \mathfrak{L}^1,\ u\big|_{\partial\Omega} = 0 \Longrightarrow \|\mathcal{N}^\delta u\|_{L^1(\partial\Omega, d\sigma)} \longrightarrow 0 \quad \text{ as } \ \delta \to 0. \tag{2.3.61}$$

Indeed, the hypotheses of (2.3.61) yield $(\mathcal{N}^\delta u)(X) \to 0$ for σ-a.e. $X \in \partial\Omega$ and since $\mathcal{N}^\delta u \leq \mathcal{N} u$ for each $\delta > 0$, (2.3.61) follows from the Dominated Convergence Theorem. Proposition 2.3.6 is therefore proven. □

We next show that Ahlfors regularity implies (2.3.48), for $p \in (1, \infty)$.

Proposition 2.3.7 *If $\Omega \subset \mathbb{R}^{n+1}$ is a bounded open set satisfying (2.3.1) and whose boundary is Ahlfors regular, then (2.3.48) holds for each $p \in (1, \infty)$.*

Proof. Fix $p \in (1, \infty)$. For $f \in L^p(\partial\Omega, d\sigma)$ and $X \in \Omega$, set

$$\Psi f(X) := \frac{1}{V(X)} \int_{\partial\Omega} \psi(X, Y) f(Y)\, d\sigma(Y), \tag{2.3.62}$$

where

$$\psi(X,Y) := \Big(1 - \frac{|X-Y|}{2\,\mathrm{dist}\,(X,\partial\Omega)}\Big)_+ \quad \text{and} \quad V(X) := \int_{\partial\Omega} \psi(X,Y)\,d\sigma(Y). \tag{2.3.63}$$

Then $\Psi 1 = 1$. Also, it is readily checked that

$$\Psi : C(\partial\Omega) \longrightarrow C(\bar\Omega) \quad \text{and} \quad \Psi f\Big|_{\partial\Omega} = f, \quad \forall\, f \in C^0(\partial\Omega). \tag{2.3.64}$$

Furthermore, given that $\partial\Omega$ is Ahlfors regular,

$$\mathcal{N}(\Psi f) \leq C\mathcal{M}f, \quad \forall\, f \in L^1(\partial\Omega, d\sigma), \tag{2.3.65}$$

where $\mathcal{M}f$ is the Hardy-Littlewood maximal function of f. Hence,

$$\|\mathcal{N}(\Psi f)\|_{L^p(\partial\Omega,d\sigma)} \leq C_p\|f\|_{L^p(\partial\Omega,d\sigma)}, \quad 1 < p < \infty. \tag{2.3.66}$$

We now claim that for each $p \in (1,\infty)$,

$$\Psi : L^p(\partial\Omega, d\sigma) \longrightarrow \mathfrak{L}^p, \quad (\Psi f)\Big|_{\partial\Omega} = f \quad \sigma\text{-a.e.} \tag{2.3.67}$$

In light of (2.3.66), only the nontangential convergence of Ψf to f remains to be justified (that this issue is meaningful, to begin with, is ensured by Proposition 2.3.2). However, this follows from the second assertion in (2.3.64), the denseness of $C(\partial\Omega)$ in $L^p(\partial\Omega,\sigma)$, and the maximal function estimate (2.3.66).

Now, to establish (2.3.48), we argue as follows. Take $v \in \mathfrak{L}^p$ and set $w := \Psi(v\big|_{\partial\Omega})$. By (2.3.67), $w \in \mathfrak{L}^p$ and $w\big|_{\partial\Omega} = v\big|_{\partial\Omega}$. Then take $f_k \in C^0(\partial\Omega)$ such that $f_k \to v\big|_{\partial\Omega}$ in $L^p(\partial\Omega, d\sigma)$ and set $\tilde w_k := \Psi f_k$. By (2.3.64), each $\tilde w_k \in C^0(\bar\Omega)$, and by (2.3.66),

$$\|\mathcal{N}(\tilde w_k - w)\|_{L^p(\partial\Omega,d\sigma)} \leq C_p\|f_k - v\big|_{\partial\Omega}\|_{L^p(\partial\Omega,d\sigma)} \to 0, \tag{2.3.68}$$

which is stronger than the L^1 estimate demanded in (2.3.48). As mentioned in the paragraph after (2.3.48), having such continuous functions suffices, since they are easily approximated by Lipschitz functions. This finishes the proof of Proposition 2.3.7. □

At this stage, it is straightforward to present the final arguments in the

Proof of Theorem 2.3.1. This is a direct consequence of Proposition 2.3.6, Proposition 2.3.7, Proposition 2.3.3 and Proposition 2.3.2. □

2.4 Analysis on spaces of homogeneous type

Let us first recall the definition of a space of homogeneous type, as introduced by R. Coifman and G. Weiss in [23]. Assume that Σ is a set equipped with a quasi-distance, i.e., a function $d : \Sigma \times \Sigma \to [0,\infty)$ satisfying

$$
\begin{aligned}
&d(X,Y)=0 \Leftrightarrow X=Y, \qquad d(X,Y)=d(Y,X), \\
&d(X,Y)\leq \kappa(d(X,Z)+d(Z,Y)), \quad \forall\, X,Y,Z\in\Sigma,
\end{aligned}
\tag{2.4.1}
$$

where $\kappa\geq 1$ is a fixed constant. In turn, a choice of a quasi-distance naturally induced a topology on Σ for which the balls $B_d(X,r):=\{Y\in\Sigma:\, d(X,Y)<r\}$ (while not necessarily open when $\kappa>1$) form a base.

A space of homogeneous type is a structure (Σ,d,μ), where d is a quasi-distance on the set Σ and μ is a measure satisfying the doubling condition

$$
0<\mu(B_d(X,2r))\leq C_o\mu(B_d(X,r))<+\infty, \quad \forall\, X\in\Sigma,\ \forall\, r>0,
\tag{2.4.2}
$$

for some $C_o\geq 1$. The number $D:=\log_2 C_o\geq 0$ is called the doubling order of μ. Iterating (2.4.2) then gives

$$
\frac{\mu(B_1)}{\mu(B_2)}\leq C\Big(\frac{\text{radius of } B_1}{\text{radius of } B_2}\Big)^D, \qquad \text{for all balls } \ B_2\subseteq B_1.
\tag{2.4.3}
$$

As a consequence, whenever f is a nonnegative, measurable function on Σ,

$$
⨍_{B_2} f\,d\mu\leq C\Big(\frac{\text{radius of } B_1}{\text{radius of } B_2}\Big)^D ⨍_{B_1} f\,d\mu, \qquad \text{for all balls } \ B_2\subseteq B_1.
\tag{2.4.4}
$$

Let us also point out that

$$
\Sigma \ \text{ bounded } \iff \mu(\Sigma)<+\infty,
\tag{2.4.5}
$$

and denote by $L^p_{loc}(\Sigma,d\mu)$, $0<p<\infty$, the class of measurable functions f on Σ, having the property that $\int_E |f|^p\,d\mu<+\infty$ whenever $E\subseteq\Sigma$ is a bounded, measurable set.

R.A. Macias and C. Segovia have proved in [74] that, given a space of homogeneous type (Σ,d,μ), there exists a quasi-metric d' which is equivalent with d, in the sense that $C^{-1}d'(X,Y)\leq d(X,Y)\leq Cd'(X,Y)$ for all $X,Y\in\Sigma$, and which satisfies the additional property

$$
|d'(X,Y)-d'(Z,Y)|\leq C\,d'(X,Z)^\theta\Big(d'(X,Y)+d'(Z,Y)\Big)^{1-\theta}, \quad \forall\, X,Y,Z\in\Sigma,
\tag{2.4.6}
$$

for some finite $C>0$ and $\theta\in(0,1)$. Using (2.4.6) it can then be verified that there exists $\varepsilon>0$ such that $d'(X,Y)^\varepsilon$ is a metric on Σ. Furthermore, the balls $B_{d'}(X,R)$ associated with d' are open. It has also been shown in [74] that

$$
\delta(X,Y):=\inf\{\mu(B):\ B\ d\text{-ball containing } X \text{ and } Y\}, \qquad X,Y\in\Sigma,
\tag{2.4.7}
$$

is a quasi-metric yielding the same topology on Σ as d. In addition, there exist $C_1,C_2>0$ such that

$$C_1 R \leq \mu(B_\delta(X,R)) \leq C_2 R, \qquad \forall\, X \in \Sigma, \quad \mu(\{X\}) < R < \mu(\Sigma). \tag{2.4.8}$$

Let us also point out here that

$$\delta(X,Y) \approx \mu(B_d(X, d(X,Y))), \qquad \text{uniformly for } X, Y \in \Sigma, \;\; R > 0. \tag{2.4.9}$$

The index θ is indicative of the amount of smoothness, measured on Hölder scales, functions defined on Σ can display. For example, the Hölder space $\mathcal{C}^\alpha(\Sigma, d)$ defined as the collection of all real-valued functions f on Σ for which $\|f\|_{\mathcal{C}^\alpha(\Sigma,d)} < +\infty$, where

$$\|f\|_{\mathcal{C}^\alpha(\Sigma,d)} := \begin{cases} \sup_{X \neq Y \in \Sigma} \frac{|f(X)-f(Y)|}{d(X,Y)^\alpha}, & \text{if } \;\mu(\Sigma) = +\infty, \\ \left|\int_\Sigma f\, d\mu\right| + \sup_{X \neq Y \in \Sigma} \frac{|f(X)-f(Y)|}{d(X,Y)^\alpha}, & \text{if } \;\mu(\Sigma) < +\infty, \end{cases} \tag{2.4.10}$$

is non-trivial whenever $\alpha \in (0,\theta)$. Indeed, if ψ is a nice bump function on the real line and $X_o \in \Sigma$ is fixed, then $\psi(d'(\cdot, X_o))$ belongs to $\mathcal{C}^\alpha(\Sigma)$ for every $\alpha \in (0,\theta)$.

A related smoothness space is $\mathcal{C}^\alpha(\Sigma, \delta)$, whose significance is apparent from the following observation. Let $\theta \in (0,1)$ be the Hölder exponent associated with the quasi-distance δ as in (2.4.6). If $\frac{1}{1+\theta} < p < 1$ and $\alpha = 1/p - 1 \in (0,\theta)$, we define the Hardy space $H^p_{at}(\Sigma, d\mu)$ as the collection of all functionals f in $\Big(\mathcal{C}^\alpha(\Sigma,\delta)\Big)^*$ which possess an atomic decomposition $f = \sum_j \lambda_j a_j$, with convergence in $\Big(\mathcal{C}^\alpha(\Sigma,\delta)\Big)^*$, where $\{\lambda_j\}_j \in \ell^p$ and each a_j is a p-atom, i.e. satisfies

$$\operatorname{supp} a \subseteq B_d(X_o, r), \quad \|a\|_{L^2(\Sigma,d\mu)} \leq \mu(B_d(X_o,r))^{1/2-1/p}, \quad \int_\Sigma a\, d\mu = 0. \tag{2.4.11}$$

When $\mu(\Sigma) < +\infty$, the constant function $a(X) = \mu(\Sigma)^{-1/p}$, $X \in \Sigma$, is also considered to be a p-atom. We then set

$$\|f\|_{H^p_{at}(\Sigma)} := \inf\Big\{ \Big(\sum_j |\lambda_j|^p\Big)^{1/p} \;:\; f = \sum_j \lambda_j a_j, \;\; \text{with each } a_j \text{ a } p\text{-atom}\Big\}. \tag{2.4.12}$$

The space $H^1_{at}(\Sigma, d\mu)$ is defined analogously, the sole exception being that the series $f = \sum_j \lambda_j a_j$ is assumed to converge in $L^1(\Sigma, d\mu)$. As is well-known ([23]), we have

$$(H^p_{at}(\Sigma, d\mu))^* = \mathcal{C}^\alpha(\Sigma, \delta), \qquad \text{if } \;\alpha = 1/p - 1 \in (0,\theta). \tag{2.4.13}$$

Also, corresponding to $p = 1$,

$$(H^1_{at}(\Sigma, d\mu))^* = \mathrm{BMO}(\Sigma, d\mu), \tag{2.4.14}$$

where $\mathrm{BMO}(\Sigma, d\mu)$ consists of functions $f \in L^1_{loc}(\Sigma, d\mu)$ for which $\|f\|_{\mathrm{BMO}(\Sigma,d\mu)} < +\infty$. As usual, we have set

$$
\|f\|_{\mathrm{BMO}(\Sigma,d\mu)} := \begin{cases} \sup_{R>0} M_1(f;R) & \text{if } \mu(\Sigma) = +\infty, \\ \left|\int_\Sigma f\,d\mu\right| + \sup_{R>0} M_1(f;R) & \text{if } \mu(\Sigma) < +\infty, \end{cases} \tag{2.4.15}
$$

where, for $p \in [1,\infty)$, we have set

$$
\begin{aligned}
&M_p(f;R) := \sup_{X\in\Sigma} \sup_{r\in(0,R]} \Big(⨍_{B_d(X,r)} \Big| f - ⨍_{B_d(X,r)} f\,d\mu \Big|^p d\mu \Big)^{1/p}, \\
&\text{and} \qquad ⨍_{B_d(X,r)} f\,d\mu := \tfrac{1}{\mu(B_d(X,r))} \int_{B_d(X,r)} f\,d\mu.
\end{aligned} \tag{2.4.16}
$$

Note that Hölder's inequality gives

$$
\begin{aligned}
M_1(f;R) &\le M_p(f;R) \\
&\approx \sup_{X\in\Sigma} \sup_{r\in(0,R]} \Big(⨍_{B_d(X,r)} ⨍_{B_d(X,r)} |f(Y) - f(Z)|^p \, d\mu(Z) d\mu(Y) \Big)^{1/p},
\end{aligned} \tag{2.4.17}
$$

uniformly for $f \in \mathrm{BMO}(\Sigma, d\mu)$. Also, the John-Nirenberg inequality ensures that, for each fixed $p \in [1,\infty)$,

$$
\|f\|_{\mathrm{BMO}(\Sigma,d\mu)} \approx \begin{cases} \sup_{R>0} M_p(f;R) & \text{if } \mu(\Sigma) = +\infty, \\ \left|\int_\Sigma f\,d\mu\right| + \sup_{R>0} M_p(f;R) & \text{if } \mu(\Sigma) < +\infty, \end{cases} \tag{2.4.18}
$$

uniformly for $f \in \mathrm{BMO}(\Sigma, d\mu)$.

Moving on, if as in [23] we set

$$
\begin{aligned}
\mathrm{VMO}_0(\Sigma, d\mu) \;\; := \;\; & \text{the closure in } \mathrm{BMO}(\Sigma, d\mu) \text{ of the space of continuous} \\
& \text{functions with bounded support on } \Sigma,
\end{aligned} \tag{2.4.19}
$$

then

$$
(\mathrm{VMO}_0(\Sigma, d\mu))^* = H^1_{at}(\Sigma, d\mu). \tag{2.4.20}
$$

For our purposes, the space $\mathrm{VMO}_0(\Sigma, d\mu)$ is inadequate, so we shall consider a related version of it. Specifically, following [98], if $\mathrm{UC}(\Sigma)$ stands for the space of uniformly continuous functions on Σ, we introduce $\mathrm{VMO}(\Sigma, d\mu)$, the space of functions of vanishing mean oscillations on Σ, as

$$
\mathrm{VMO}(\Sigma, d\mu) := \text{the closure of } \mathrm{UC}(\Sigma) \cap \mathrm{BMO}(\Sigma, d\mu) \text{ in } \mathrm{BMO}(\Sigma, d\mu). \tag{2.4.21}
$$

Note that $\mathrm{VMO}_0(\Sigma, d\mu) = \mathrm{VMO}(\Sigma, d\mu)$ if Σ is compact. In this latter setting, we also have the following useful equivalent characterization of $\mathrm{VMO}(\Sigma, d\mu)$.

Proposition 2.4.1 *Assume that (Σ, d, μ) is a compact space of homogeneous type. Then*

$$\mathrm{VMO}(\Sigma, d\mu) \text{ is the closure of } \mathcal{C}^\alpha(\Sigma, d) \text{ in } \mathrm{BMO}(\Sigma, d\mu), \tag{2.4.22}$$

for every $\alpha \in (0, \theta)$.

Proof. This is seen from (2.4.21) and the fact that $\mathcal{C}^\alpha(\Sigma, d) \hookrightarrow C^0(\Sigma)$ densely in the uniform norm, as a simple application of the Stone-Weierstrass theorem shows. Indeed, $\mathcal{C}^\alpha(\Sigma, d)$ is a sub-algebra of $C^0(\Sigma)$ which separates the points on Σ. The latter claim is readily checked by observing that, if $X_1, X_2 \in \Sigma$, $X_1 \neq X_2$, then $d'(X_1, \cdot) \in \mathcal{C}^\alpha(\Sigma, d)$ satisfies $d'(X_1, X_1) = 0$ and $d'(X_1, X_2) \neq 0$. $\square$

Given a space of homogeneous type (Σ, d, μ), call a real-valued function ω defined on Σ a weight if it is non-negative and measurable. If $1 < p < \infty$, a weight w belongs to the Muckenhoupt class A_p if

$$[w]_{A_p} := \sup_{B \text{ ball}} \Big(\frac{1}{\mu(B)} \int_B w \, d\mu\Big)\Big(\frac{1}{\mu(B)} \int_B w^{-1/(p-1)} \, d\mu\Big)^{p-1} < +\infty. \tag{2.4.23}$$

Corresponding to $p = 1$, the class A_1 is then defined as the collection of all weights w for which

$$[w]_{A_1} := \sup_{B \text{ ball}} \Big(\operatorname*{ess\,inf}_B w\Big)^{-1}\Big(\frac{1}{\mu(B)} \int_B w \, d\mu\Big) < +\infty. \tag{2.4.24}$$

In particular,

$$\frac{1}{\mu(B)} \int_B w \, d\mu \leq [w]_{A_1} \, w(X) \quad \text{for } \mu\text{-a.e. } X \in B, \tag{2.4.25}$$

for every ball $B \subset \Sigma$. See, e.g., [110] for a more detailed discussion, including basic properties. Below we summarize a number of well-known facts which are relevant for us here. To state them, denote by $\mathcal{M}$ the Hardy-Littlewood maximal function on Σ,

$$\mathcal{M}f(X) = \sup_{B \ni X} \frac{1}{\mu(B)} \int_B |f| \, d\mu, \tag{2.4.26}$$

and, for any weight ω on Σ, abbreviate $L^p(\omega) := L^p(\Sigma, \omega\, d\mu)$. Then the following hold:

(1) If $1 < p < \infty$ and $\omega \in A_p$ then $\mathcal{M} : L^p(\omega) \to L^p(\omega)$ is bounded with norm $\leq C(\Sigma, [w]_{A_p})$;

(2) $\omega \in A_p$ if and only if $\omega^{1-p'} \in A_{p'}$, where $1/p + 1/p' = 1$, and $[\omega^{1-p'}]_{A_{p'}} = [\omega]_{A_p}^{1/(p-1)}$;

(3) If $\omega_1, \omega_2 \in A_1$, then $\omega_1 \, \omega_2^{1-p} \in A_p$ and $[\omega_1 \, \omega_2^{1-p}]_{A_p} \leq [\omega_1]_{A_1}^{1/(p-1)} [\omega_2]_{A_1}^{p-1}$;

(4) If w is a weight for which $\mathcal{M}(w) \leq C\, w$ on Σ then $w \in A_1$ and $[w]_{A_1} \leq C$.

The following is the weighted, homogeneous space version of the commutator theorem of Coifman-Rochberg-Weiss [21].

Theorem 2.4.2 *Let (Σ, d, μ) be a space of homogeneous type and let $p_0 \in (1,\infty)$. Assume that T is a linear operator with the property that for any weight $\omega \in A_{p_0}$, T maps $L^{p_0}(\omega)$ boundedly into itself, with norm controlled solely in terms of $[\omega]_{A_{p_0}}$.*

Then for every $b \in \mathrm{BMO}(\Sigma, d\mu)$, the commutator $[M_b, T]$ between T and the operator of multiplication by b is bounded on $L^p(\omega)$, with norm $\leq C(p, [\omega]_{A_p})\|b\|_{\mathrm{BMO}(\Sigma,d\mu)}$, for each $p \in (1,\infty)$ and $\omega \in A_p$.

In essence, this is known, and various related versions can be found in e.g., [8], [3]. Given that this result plays an important role in this paper, we chose to present a proof based on Muckenhoupt theory of weights and Rubio de Francia's extrapolation theory, adapted to spaces of homogeneous type. We begin by discussing the latter, by closely paralleling the approach recently developed in [26] (we wish to use the opportunity to thank Chema Martell for calling this reference to our attention).

Proposition 2.4.3 *Let f, g be two nonnegative, measurable functions with the property that there exist $p_0 \in (1,\infty)$ such that for every $w \in A_{p_0}$*

$$\int_\Sigma f^{p_0}\, w\, d\mu \leq C \int_\Sigma g^{p_0}\, w\, d\mu, \tag{2.4.27}$$

where the constant C depends only on $[w]_{A_{p_0}}$. Then for each $1 < p < \infty$ and $\omega \in A_p$,

$$\int_\Sigma f^p \omega\, d\mu \leq C \int_\Sigma g^p \omega\, d\mu, \tag{2.4.28}$$

where the constant C depends only on p and $[\omega]_{A_p}$.

Proof. The proof is divided into several steps, starting with:

`Step I.` *Rubio de Francia's construction.* Assume that $1 < p < \infty$ and that a weight function ω has been fixed. Given a sublinear operator $A : L^p(\omega) \to L^p(\omega)$, with $\|A\|_{L^p(\omega)\to L^p(\omega)} := \sup\{\|Af\|_{L^p(\omega)} : \|f\|_{L^p(\omega)} = 1\} < +\infty$, define

$$T_A f := \sum_{j=0}^{\infty} \frac{A^j f}{2^j \|A\|^j_{L^p(\omega)\to L^p(\omega)}}, \qquad \text{whenever } \ f \in L^p(\omega), \ \ f \geq 0, \tag{2.4.29}$$

where $A^j := A \circ \cdots \circ A$ (j factors) if $j \in \mathbb{N}$, and $A^0 := I$, the identity operator. For any $f \in L^p(\omega)$, $f \geq 0$, the following properties are then easily checked:

$$f \leq T_A f, \quad \|T_A f\|_{L^p(\omega)} \leq 2\|f\|_{L^p(\omega)}, \quad A(T_A f) \leq 2\|A\|_{L^p(\omega)\to L^p(\omega)}\, T_A f. \tag{2.4.30}$$

`Step II.` *Construction adapted to $\mathcal{M}$.* Thanks to (1) above, the construction in Step I can be applied to $\mathcal{M}$ whenever $\omega \in A_p$. Assuming that this is the case, for any $h \in L^p(\omega)$, $h \geq 0$, we then obtain

(i) $h \leq T_{\mathcal{M}} h$;

(ii) $\|T_{\mathcal{M}} h\|_{L^p(\omega)} \leq 2\|h\|_{L^p(\omega)}$;

(iii) $\mathcal{M}(T_{\mathcal{M}}h) \leq 2\|\mathcal{M}\|_{L^p(\omega)\to L^p(\omega)}\, T_{\mathcal{M}}h$.

In particular, by (4) above, we have

$$T_{\mathbb{M}}h \in A_1. \tag{2.4.31}$$

`Step III.` *Construction adapted to* $\mathcal{M}'$. Let p' be such that $\frac{1}{p}+\frac{1}{p'}=1$, and consider the sublinear operator

$$\mathcal{M}' : L^{p'}(\omega) \longrightarrow L^{p'}(\omega), \qquad \mathcal{M}'f := \frac{\mathcal{M}(f\omega)}{\omega}. \tag{2.4.32}$$

By (2) above, $\mathcal{M}'$ is bounded provided $\omega \in A_p$. If that is the case, then the construction in Step I can be applied to $\mathcal{M}'$. This shows that, if $\omega \in A_p$, then for any nonnegative function $h \in L^{p'}(\omega)$

(i)' $h \leq T_{\mathcal{M}'}h$;

(ii)' $\|T_{\mathcal{M}'}h\|_{L^{p'}(\omega)} \leq 2\|h\|_{L^{p'}(\omega)}$;

(iii)' $\mathcal{M}'(T_{\mathcal{M}'}h) \leq 2\|T_{\mathcal{M}'}\|_{L^{p'}(\omega)\to L^{p'}(\omega)} T_{\mathcal{M}'}h$.

The last estimate entails $\mathcal{M}(\omega\, T_{\mathcal{M}'}h) \leq C\,\omega\, T_{\mathcal{M}'}h$. Hence, as before,

$$\omega\, T_{\mathcal{M}'}h \in A_1. \tag{2.4.33}$$

`Step IV.` *Proof of the extrapolation estimate.* Fix $(f,g) \in F$ and $1 < p, p' < \infty$ with $1/p + 1/p' = 1$. Then there exists a nonnegative function $h \in L^{p'}(\omega)$ with $\|h\|_{L^{p'}(\omega)} = 1$ and for which $\|f\|_{L^p(\omega)} = \int_\Sigma f h\,\omega\, d\mu$. Making also use of (i)' and Hölder's inequality (with indices p_0, p_0' and measure $T_{\mathcal{M}'}h\,\omega\, d\mu$), we can then write

$$\begin{aligned}
\|f\|_{L^p(\omega)} &\leq \int_\Sigma f\,(T_{\mathcal{M}'}h)\,\omega\, d\mu \\
&= \int_\Sigma f\Big(T_{\mathcal{M}}g\Big)^{-1/p_0'}\Big(T_{\mathcal{M}}g\Big)^{1/p_0'}(T_{\mathcal{M}'}h)\,\omega\, d\mu \\
&\leq \Big(\int_\Sigma f^{p_0}(T_{\mathcal{M}}g)^{1-p_0}T_{\mathcal{M}'}h\,\omega\, d\mu\Big)^{1/p_0}\Big(\int_\Sigma T_{\mathcal{M}}g\, T_{\mathcal{M}'}h\,\omega\, d\mu\Big)^{1/p_0'} =: A\cdot B.
\end{aligned} \tag{2.4.34}$$

To proceed, set $\omega_1 := T_{\mathcal{M}}g$, $\omega_2 := T_{\mathcal{M}'}h\,\omega$ and $w := \omega_1^{1-p_0}\omega_2$. Then (iii) and (iii)' ensure that $\omega_1, \omega_2 \in A_1$. Moreover, $w \in A_{p_0}$ by (3). Using these and (2.4.27), the first factor in the rightmost side of (2.4.34) can be estimated as follows

$$\begin{aligned}
A &= \Big(\int_\Sigma f^{p_0}\, w\, d\mu\Big)^{1/p_0} \leq C\Big(\int_\Sigma g^{p_0}\, w\, d\mu\Big)^{1/p_0} \\
&= C\Big(\int_\Sigma g^{p_0}(T_{\mathcal{M}}g)^{1-p_0}(T_{\mathcal{M}'}h)\omega\, d\mu\Big)^{1/p_0} \\
&\leq C\Big(\int_\Sigma T_{\mathcal{M}}g\,(T_{\mathcal{M}'}h)\omega\, d\mu\Big)^{1/p_0},
\end{aligned} \tag{2.4.35}$$

where the last inequality in (2.4.35) is based on (i). By combining (2.4.34) and (2.4.35), applying Hölder's inequality, then using (ii) and (ii)' we arrive at

$$\begin{aligned}\|f\|_{L^p(\omega)} &\leq C\Big(\int_\Sigma T_{\mathcal{M}}g\,T_{\mathcal{M}'}h\,\omega\,d\mu\Big)^{1/p_0+1/p_0'} = C\int_\Sigma T_{\mathcal{M}}g\,T_{\mathcal{M}'}h\,\omega\,d\mu \\ &\leq C\|T_{\mathcal{M}}g\|_{L^p(\omega)}\|T_{\mathcal{M}'}h\|_{L^{p'}(\omega)} \leq C\|g\|_{L^p(\omega)}\|h\|_{L^{p'}(\omega)} = C\|g\|_{L^p(\omega)}, \qquad (2.4.36)\end{aligned}$$

since $\|h\|_{L^{p'}(\omega)} = 1$. Hence, (2.4.27) is proved. □

Having established Proposition 2.4.3, we are prepared to present the

Proof of Theorem 2.4.2. We shall closely follow [58], [91]. First, from simple homogeneity considerations, there is no loss of generality in assuming that $\|b\|_{\mathrm{BMO}\,(\Sigma,d\mu)} = 1$. From Proposition 2.4.3, T maps any $L^p(\omega)$ boundedly into itself, $1 < p < \infty$, $\omega \in A_p$, with norm controlled by p and $[\omega]_{A_p}$. Fix now $p \in (1,\infty)$, $b \in \mathrm{BMO}\,(\Sigma, d\mu)$, $\omega \in A_p$, and let $\varepsilon > 0$ be sufficiently small so that, for any complex number z with $|z| \leq \varepsilon$,

$$\omega\, e^{(\mathrm{Re}\, z)b} \in A_p \qquad (2.4.37)$$

with A_p norm controlled by $C(p,\omega)$ uniformly in z (cf., e.g., [58] pp. 32–33 for a proof in the Euclidean setting which easily adapts to spaces of homogeneous type). The idea is now to observe that, for an eventually smaller ε, the analytic mapping

$$\Phi : \{z \in \mathbb{C} :\, |z| < \varepsilon\} \longrightarrow \mathcal{L}(L^p(\omega)), \quad \Phi(z) := M_{e^{zb}}TM_{e^{-zb}} \qquad (2.4.38)$$

satisfies

$$\|\Phi(z)\|_{\mathcal{L}(L^p(\omega))} \leq C \text{ for } |z| < \varepsilon, \quad \text{and} \quad [T, M_b] = \Phi'(0). \qquad (2.4.39)$$

The estimate $\|[T, M_b]\|_{L^p(\omega)\to L^p(\omega)} \leq C$ now follows from elementary considerations involving (2.4.39) and Cauchy's reproducing formula. □

We next discuss the connection between Theorem 2.4.2 and Calderón-Zygmund operators. A linear, continuous operator $T : \mathcal{C}^\alpha(\Sigma, d) \to \Big(\mathcal{C}^\alpha(\Sigma, d)\Big)^*$, $\alpha \in (0, \theta)$, is said to be associated with the kernel $K \in L^1_{loc}(\Sigma \times \Sigma \setminus \mathrm{diag})$ if

$$\langle Tf, g\rangle = \int_\Sigma\int_\Sigma K(X,Y)f(Y)g(X)\,d\mu(X)d\mu(Y), \qquad (2.4.40)$$

whenever $f, g \in \mathcal{C}^\alpha(\Sigma, d)$ have bounded, disjoint supports. If, in addition, there exist a finite $C > 0$ and a small $\varepsilon > 0$ such that

$$|K(X,Y)| \leq \frac{C}{\mu(B_d(X, d(X,Y)))}, \qquad \forall\, X, Y \in \Sigma, \quad \text{and} \qquad (2.4.41)$$

$$|K(X,Y) - K(X',Y)| + |K(Y,X) - K(Y,X')| \leq \left(\frac{d(X,X')}{d(X,Y')}\right)^\varepsilon \left(\frac{C}{\mu(B_d(X, d(X,Y)))}\right)$$
$$\text{whenever } X, X', Y \in \Sigma \text{ satisfy } d(X,Y) \geq \kappa\, d(X,X'), \qquad (2.4.42)$$

then T is called a *Calderón-Zygmund type operator.*

Corollary 2.4.4 *Assume that (Σ, d, μ) is a space of homogeneous type and that T is a Calderón-Zygmund type operator which is bounded on $L^2(\Sigma, d\mu)$. Then the same conclusions as in Theorem 2.4.2 are valid.*

Proof. This is a direct consequence of Theorem 2.4.2 and the fact that Calderón-Zygmund type operators which are bounded on $L^2(\Sigma, d\mu)$ are also bounded on $L^p(\omega)$ whenever $1 < p < \infty$ and $\omega \in A_p$. See [19] for the Euclidean space and, e.g., [75] for spaces of homogeneous type. $\square$

Corollary 2.4.4 and the characterization (2.4.22) further entail the following.

Theorem 2.4.5 *Suppose that (Σ, d, μ) is a compact space of homogeneous type, and assume that T is a Calderón-Zygmund type operator which is bounded on $L^2(\Sigma, d\mu)$. Then for every $p \in (1, \infty)$ there exists $C > 0$ such that, for every $b \in \mathrm{BMO}(\Sigma, d\mu)$,*

$$\inf_K \|[M_b, T] - K\|_{L^p(\Sigma,d\mu)\to L^p(\Sigma,d\mu)} \le C \,\mathrm{dist}\,(b, \mathrm{VMO}(\Sigma, d\mu)), \tag{2.4.43}$$

where the infimum is taken over all compact operators K on $L^p(\Sigma, d\mu)$, and the distance is measured in $\mathrm{BMO}(\Sigma, d\mu)$.

As a consequence, if T is as above and $b \in \mathrm{VMO}(\Sigma, d\mu)$, then the commutator $[M_b, T]$ between T and the operator of multiplication by b is compact on $L^p(\Sigma, d\mu)$ for each $p \in (1, \infty)$.

Proof. Once the compactness of $[M_b, T]$ for each $b \in \mathrm{VMO}(\Sigma, d\mu)$ has been established, estimate (2.4.43) follows readily from the operator bound in Corollary 2.4.4. In concert with Proposition 2.4.1, Corollary 2.4.4 also allows one to prove compactness of $[M_b, T]$ for each $b \in \mathrm{VMO}(\Sigma, d\mu)$ from such compactness when $b \in \mathcal{C}^\alpha(\Sigma, d)$, for some small $\alpha > 0$. In such a case, we have

$$[M_b, T]f(X) = \int_\Sigma k(X, Y) f(Y)\, d\mu(Y), \tag{2.4.44}$$

with

$$|k(X, Y)| \le C \frac{d(X, Y)^\alpha}{\mu(B_d(X, d(X, Y)))}, \tag{2.4.45}$$

and this implies the desired compactness result by virtue of Lemma 2.4.6 below. $\square$

In fact, we give a result in a natural level of generality, which establishes such asserted compactness and which will also prove useful in §5.

Lemma 2.4.6 *Suppose (Σ, d, μ) is a space of homogeneous type such that $\mu(\Sigma) < \infty$. Let $k(X, Y)$ be a real-valued, measurable function on $\Sigma \times \Sigma$ satisfying*

$$|k(X, Y)| \le \frac{\psi(d(X, Y))}{\mu(B_d(X, d(X, Y)))}, \tag{2.4.46}$$

where $\psi(t)$ is monotone increasing and slowly varying, with

$$\int_0^1 \frac{\psi(t)}{t}\, dt < \infty. \tag{2.4.47}$$

Consider

$$Kf(X) = \int_{\Sigma} k(X,Y)f(Y)\,d\mu(Y). \tag{2.4.48}$$

Then $K : L^p(\Sigma, d\mu) \to L^p(\Sigma, d\mu)$ is compact, for each $p \in (1, \infty)$.

Proof. Take $\varepsilon > 0$ and set $K = K^{\#} + K^b$ with integral kernels $k(X,Y) = k^{\#}(X,Y) + k^b(X,Y)$, where

$$\begin{aligned} k^{\#}(X,Y) = k(X,Y) \qquad &\text{for} \qquad d(X,Y) \leq \varepsilon \\ 0 \qquad &\text{for} \qquad d(X,Y) > \varepsilon. \end{aligned} \tag{2.4.49}$$

If for each integer j we now set

$$\Delta_j(X) = \{Y \in \Sigma :\, e^{-j-1} \leq d(X,Y) < e^{-j}\}, \tag{2.4.50}$$

we may then compute

$$\begin{aligned} \int_{\Sigma} |k^{\#}(X,Y)|\,d\mu(Y) \quad &\leq \quad C \sum_{j \geq \log 1/\varepsilon} \int_{\Delta_j(X)} \frac{\psi(d(X,Y))}{\mu(B_d(X, d(X,Y)))}\,d\mu(Y) \\ &\leq \quad C \sum_{j \geq \log 1/\varepsilon} \psi(e^{-j}) \\ &\leq \quad C \int_0^{\varepsilon} \frac{\psi(t)}{t}\,dt =: \delta(\varepsilon). \end{aligned} \tag{2.4.51}$$

There is a similar estimate for $\int_{\Sigma} |k^{\#}(X,Y)|\,d\mu(X)$, since the doubling hypothesis allows us to switch the roles of X and Y in (2.4.46). Therefore, if $\mathcal{L}(L^p)$ denotes the Banach space of bounded linear operators on $L^p(\Sigma, d\mu)$, then Schur's lemma gives

$$\|K^{\#}\|_{\mathcal{L}(L^p)} \leq \delta(\varepsilon), \tag{2.4.52}$$

and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Thus it remains to show that K^b is compact on each L^p space, for $p \in (1, \infty)$, under the hypothesis that $k^b(X,Y)$ is bounded. First note that K^b is compact on $L^2(\Sigma, d\mu)$, since it is Hilbert-Schmidt, due to the fact that $\mu(\Sigma) < \infty$. The compactness of K^b on $L^p(\Sigma, d\mu)$ for each $p \in (1, \infty)$ then follows from an interpolation theorem of Krasnoselski (see, e.g., [6], Theorem 2.9, p. 203). This finishes the proof of the lemma. □

We now record a result proved by M. Christ in [18] which provides an analogue of the grid of Euclidean dyadic cubes on a space of homogeneous type.

Proposition 2.4.7 *Let (Σ, d, μ) be a space of homogeneous type. Then there exist a collection $\mathcal{Q} = \{Q_\alpha^k :\, k \in \mathbb{Z},\, \alpha \in I_k\}$ of open subsets of Σ, where I_k is some (possibly finite) index set, along with constants $\eta \in (0,1)$ and $C_1, C_2 > 0$ such that the following hold:*

(i) for each fixed $k \in \mathbb{Z}$, $\mu(\Sigma \setminus \cup_{\alpha \in I_k} Q^k_\alpha) = 0$ and $Q^k_\alpha \cap Q^k_\beta = \emptyset$ whenever $\alpha \neq \beta$;

(ii) for any $k, \ell \in \mathbb{Z}$ with $\ell \geq k$ and any $\alpha \in I_k$, $\beta \in I_\ell$, either $Q^\ell_\beta \subseteq Q^k_\alpha$ or $Q^\ell_\beta \cap Q^k_\alpha = \emptyset$;

(iii) for each $k \in \mathbb{Z}$, $\alpha \in I_k$, and each $\ell \in \mathbb{Z}$ with $\ell < k$, there exists a unique $\beta \in I_\ell$ such that $Q^k_\alpha \subseteq Q^\ell_\beta$;

(iv) for each $k \in \mathbb{Z}$ and $\alpha \in I_k$, the set Q^k_α has at least one child, i.e., there exists some $\beta \in I_{k+1}$ such that $Q^{k+1}_\beta \subseteq Q^k_\alpha$;

(v) for each $k \in \mathbb{Z}$ and $\alpha \in I_k$, there exists $X^k_\alpha \in Q^k_\alpha$ (referred to as the center of Q^k_α) such that

$$B_d(X^k_\alpha, C_1\eta^k) \subseteq Q^k_\alpha \subseteq B_d(X^k_\alpha, C_2\eta^k). \tag{2.4.53}$$

By a slight abuse of terminology, we shall refer to the sets Q^k_α as dyadic cubes on Σ, and call $\mathcal{Q} = \{Q^k_\alpha : k \in \mathbb{Z}, \alpha \in I_k\}$ dyadic grid on Σ. Also, the index k will be referred to as the generation of the dyadic cube Q^k_α. In the context of (iii) above, the dyadic cube Q^ℓ_β will be referred to as an ancestor of Q^k_α. Corresponding to the case when $k = \ell + 1$, we shall call $Q^{\ell+1}_\alpha$ the parent of Q^ℓ_α. Let us also note here that there there exists a small constant $c > 0$ such that

$$Q^{k+1}_\beta \text{ is a child of } Q^k_\alpha \Longrightarrow \mu(Q^{k+1}_\beta) \geq c\eta^D \mu(Q^k_\alpha). \tag{2.4.54}$$

Indeed, since the diameter of Q^k_α is $\leq C_o\eta^k$, and if $X^{k+1}_\beta \in Q^{k+1}_\beta \subseteq Q^k_\alpha$ is the center of Q^{k+1}_β, then $Q^k_\alpha \subseteq B_d(X^{k+1}_\beta, C_o\eta^k)$ and, hence, $\mu(Q^k_\alpha) \leq \mu(B_d(X^{k+1}_\beta, C_o\eta^k)) \leq C\eta^{-D}\mu(B_d(X^{k+1}_\beta, C_1\eta^{k+1})) \leq C\eta^{-D}\mu(Q^{k+1}_\beta)$, justifying (2.4.54). In particular, the number of children of a dyadic cube is always $\leq C\eta^{-D}$.

Assume next that $\Sigma \subset \mathbb{R}^{n+1}$ is a closed set which is Ahlfors regular (i.e., there exist two constants $0 < a \leq b < \infty$ such that condition (2.1.1) is satisfied). When equipped with the measure $\mu := \mathcal{H}^n \lfloor \Sigma$ and the distance $d(X, Y) := |X - Y|$, the set Σ becomes a space of homogeneous type. In this scenario, $\delta(X, Y) \approx |X - Y|^n$ and $\theta := 1/n$. In particular, the Hardy space $H^p_{at}(\Sigma)$ is well-defined whenever $\frac{n}{n+1} < p \leq 1$.

Proposition 2.4.8 *Let (Σ, d, μ) be as above. Then for each $p \in [1, \infty)$,*

$$\begin{aligned} \operatorname{dist}(f, \operatorname{VMO}(\Sigma, d\mu)) &\approx \limsup_{r \to 0^+} \left\{ \sup_{X \in \Sigma} ⨍_{B_d(X,r)} ⨍_{B_d(X,r)} |f(Y) - f(Z)|^p \, d\mu(Y)\, d\mu(Z) \right\}^{1/p} \\ &\approx \limsup_{r \to 0^+} \left\{ \sup_{X \in \Sigma} ⨍_{B_d(X,r)} \left| f - ⨍_{B_d(X,r)} f \, d\mu \right|^p d\mu \right\}^{1/p}, \end{aligned} \tag{2.4.55}$$

uniformly for $f \in \operatorname{BMO}(\Sigma, d\mu)$ (i.e., the constants do not depend on f), where the distance is measured in the BMO norm. In particular, for each $p \in [1, \infty)$,

$$\operatorname{dist}(f, \operatorname{VMO}(\Sigma, d\mu)) \approx \lim_{R \to 0^+} M_p(f; R), \quad \textit{uniformly for } f \in \operatorname{BMO}(\Sigma, d\mu), \tag{2.4.56}$$

where $M_p(f;R)$ is defined as in (2.4.16). Moreover, for each function $f \in \mathrm{BMO}(\Sigma, d\mu)$ and each $p \in [1,\infty)$,

$$f \in \mathrm{VMO}(\Sigma, d\mu) \iff \lim_{r\to 0^+} \left\{ \sup_{X\in\Sigma} ⨍_{B_d(X,r)} \Big| f - ⨍_{B_d(X,r)} f\, d\mu \Big|^p d\mu \right\}^{1/p} = 0. \tag{2.4.57}$$

Proof. It is clear that the two upper-limits in (2.4.55) have comparable sizes, uniformly for $f \in \mathrm{BMO}(\Sigma, d\mu)$, and that in turn, each is dominated by a fixed multiple of $\mathrm{dist}\,(f,\, \mathrm{VMO}\,(\Sigma, d\mu))$. Thus, by Hölder's inequality, (2.4.55) is proved as soon as we show that there exists $C > 0$ such that

$$\mathrm{dist}\,(f,\, \mathrm{VMO}\,(\Sigma, d\mu)) \le C \lim_{R\to 0^+} M_1(f;R), \qquad \forall\, f \in \mathrm{BMO}\,(\Sigma, d\mu). \tag{2.4.58}$$

To this end, we shall employ an approximation to the identity adapted to the scale η intervening in the statement of Proposition 2.4.7. Concretely, it is possible to construct a sequence of functions $\{p_k(X,Y)\}_{k\in\mathbb{Z}}$ on $\Sigma\times\Sigma$ for which there exist $C, c > 0$ and $\theta > 0$ such that the following properties hold for every integer $k \in \mathbb{Z}$:

(i) $p_k(X,Y) = 0$ whenever $X, Y \in \Sigma$ are such that $d(X,Y) \ge c\eta^k$;

(ii) $|p_k(X,Y)| \le C\eta^{-nk}$, for every $X, Y \in \Sigma$;

(iii) $|p_k(X,Y) - p_k(X',Y)| \le C\eta^{-k(n+\theta)} d(X,X')^\theta$, for every $X, X', Y, \in \Sigma$;

(iv) $\int_\Sigma p_k(X,Y)\, d\mu(Y) = 1$ for every $X \in \Sigma$.

The construction of the such an approximation to the identity follows closely the outline in [32]. More specifically, pick a smooth function $h : \mathbb{R}_+ \to \mathbb{R}_+$ which is identically 1 on $(0,1)$ and identically zero on $(2,\infty)$, and let T_k be the integral operator with kernel $\eta^{-kn}h(\eta^{-k}|X-Y|)$. Then there exists $C > 1$ such that $C^{-1} \le T_k 1 \le C$ for every k. Let M_k and W_k be the operators of multiplication by $(T_k 1)^{-1}$ and $(T_k((T_k 1)^{-1}))^{-1}$, respectively, and set $P_k := M_k T_k W_k T_k M_k$. Then $p_k(X,Y)$, the integral kernel of P_k, satisfies (i)-(iv) above.

Going further, we note that as a consequence of (i) and (ii) we have

$$\sup_k \sup_{X\in\Sigma} \int_\Sigma |p_k(X,Y)|\, d\mu(Y) < +\infty. \tag{2.4.59}$$

Next, fix $f \in \mathrm{BMO}\,(\Sigma)$ and a for each positive integer k define

$$g_k(X) := \int_\Sigma p_k(X,Y) f(Y)\, d\mu(Y), \qquad X \in \Sigma. \tag{2.4.60}$$

Properties (i) and (iii) of the function $p_k(X,Y)$ then imply that

$$|g_k(X) - g_k(X')| \le C_1 \left(\eta^{-k}\, d(X,X')\right)^\theta M_1(f; C_2\eta^k), \quad \text{if} \;\; X, X' \in \Sigma, \quad d(X,X') < C\eta^k. \tag{2.4.61}$$

In particular, $g_k \in \mathrm{UC}(\Sigma)$ for each k.

We now claim that $C_1, C_2 > 0$ can be chosen, independent of $f \in \mathrm{BMO}(\Sigma, d\mu)$ and the integer k, such that

$$\sup_{R>0} M_1(f-g_k; R) \leq C_1\, M_1(f; C_2\eta^k). \tag{2.4.62}$$

To justify this inequality, fix an integer k along with an arbitrary point $X \in \Sigma$ and number $R > 0$. In the case when $0 < R < \eta^k$, we estimate

$$\begin{aligned} &⨍_{B_d(X,R)} \Big|(f-g_k) - ⨍_{B_d(X,R)} (f-g_k)\, d\mu\Big|\, d\mu \\ &\qquad \leq ⨍_{B_d(X,R)} \Big|f - ⨍_{B_d(X,R)} f\, d\mu\Big|\, d\mu + ⨍_{B_d(X,R)} \Big|g_k - ⨍_{B_d(X,R)} g_k\, d\mu\Big|\, d\mu \\ &\qquad =: I + II. \end{aligned} \tag{2.4.63}$$

On the one hand, from definitions it follows that $I \leq M_1(f; \eta^k)$. On the other hand,

$$II \leq ⨍_{B_d(X,R)} ⨍_{B_d(X,R)} |g_k(Y) - g_k(Z)|\, d\mu(Y)\, d\mu(Z) \leq C_1\, M_1(f; C_2\eta^k), \tag{2.4.64}$$

by (2.4.61) and the assumption on R. In summary,

$$0 < R < \eta^k \Longrightarrow ⨍_{B_d(X,R)} \Big|(f-g_k) - ⨍_{B_d(X,R)} (f-g_k)\, d\mu\Big|\, d\mu \leq C_1\, M_1(f; C_2\eta^k). \tag{2.4.65}$$

In the case when $R \geq \eta^k$, we make the observation that $Q^k_\alpha \cap B_d(X,R) \neq \emptyset$ forces $Q^k_\alpha \subseteq B_d(X, C_o R)$, for some C_o independent of k and R. In particular,

$$\bigcup_{\alpha \in J} Q^k_\alpha \subseteq B_d(X, C_o R). \tag{2.4.66}$$

Thus, for a sufficiently large constant $C > 0$, we may write

$$\begin{aligned} ⨍_{B_d(X,R)} &\Big|(f-g_k) - ⨍_{B_d(X,R)} (f-g_k)\, d\mu\Big|\, d\mu \leq 2 ⨍_{B_d(X,R)} |f(Y) - g_k(Y)|\, d\mu(Y) \\ &\leq \frac{2}{\mu(B_d(X,R))} \sum_{\alpha \in J} \int_{Q^k_\alpha} \Big| \Big(f(Y) - ⨍_{B_d(X^k_\alpha, C\eta^k)} f\, d\mu\Big) \\ &\qquad\qquad - \int_\Sigma p_k(Y,Z) \Big(f(Z) - ⨍_{B_d(X^k_\alpha, C\eta^k)} f\, d\mu\Big)\, d\mu(Z)\Big|\, d\mu(Y) \\ &\leq C \sum_{\alpha \in J} \frac{\mu(B_d(X^k_\alpha, C\eta^k))}{\mu(B_d(X,R))} ⨍_{B_d(X^k_\alpha, C\eta^k)} \Big|f - ⨍_{B_d(X^k_\alpha, C\eta^k)} f\, d\mu\Big|\, d\mu \\ &\leq C\, M_1(f; C_2\eta^k) \Big(\sum_{\alpha \in J} \frac{\mu(Q^k_\alpha)}{\mu(B_d(X,R))}\Big) = C\, M_1(f; C_2\eta^k) \frac{\mu\Big(\cup_{\alpha \in J} Q^k_\alpha\Big)}{\mu(B_d(X,R))} \\ &\leq C\, M_1(f; C_2\eta^k) \frac{\mu(B_d(X, C_o R))}{\mu(B_d(X,R))} \leq C\, M_1(f; C_2\eta^k). \end{aligned} \tag{2.4.67}$$

Hence, the implication in (2.4.65) also holds when $R \geq \eta^k$ and this completes the proof of (2.4.62).

In turn, when $\mu(\Sigma) = +\infty$, the estimate (2.4.62) implies

$$\operatorname{dist}(f,\, \mathrm{VMO}\,(\Sigma, d\mu)) \leq \|f - g_k\|_{\mathrm{BMO}(\Sigma,d\mu)} \leq C_1\, M_1(f; C_2\eta^k), \tag{2.4.68}$$

which readily yields (2.4.58).

When $\mu(\Sigma) < +\infty$, the same type of argument applies as soon as we show that

$$\lim_{k\to\infty} \Big| \int_\Sigma (f - g_k)\, d\mu \Big| = 0. \tag{2.4.69}$$

To justify this, by Hölder's inequality it suffices to prove that

$$T_k f \to f \text{ in } L^2(\Sigma, d\mu) \text{ as } k \to \infty, \text{ where } T_k f(X) := \int_\Sigma p_k(X,Y) f(Y)\, d\mu(Y). \tag{2.4.70}$$

Since the integral kernel of the operator T_k is an approximation to the identity, we have that

$$\begin{aligned} &T_k f(X) \to f(X) \text{ as } k \to \infty, \text{ whenever } f \in \mathrm{Lip}\,(\Sigma). \\ &\text{and } \ |T_k f(X)| \leq C \mathcal{M}_\Sigma f(X), \quad \text{ for every } X \text{ and } k, \end{aligned} \tag{2.4.71}$$

where $\mathcal{M}_\Sigma$ denotes the Hardy-Littlewood maximal function on Σ. Hence, the claim in (2.4.70) follows that (2.4.71), Lebesgue's Dominated Convergence Theorem, the boundedness of $\mathcal{M}_\Sigma$ on $L^2(\Sigma, d\mu)$, and the density result contained in Lemma 2.4.9 below. This finishes the proof of (2.4.55). Finally, (2.4.56) and (2.4.57) are direct consequences of (2.4.55) and definitions. $\square$

We now record a density result, already invoked above, and which is also going to be useful later on.

Lemma 2.4.9 *Assume that Σ is a locally compact metric space and that σ is a locally finite Borel measure on Σ. Also, denote by $\mathrm{Lip}_o\,(\Sigma)$ the space of compactly supported, Lipschitz functions on Σ. Then for every $p \in [1, \infty)$ the inclusion*

$$\mathrm{Lip}_o\,(\Sigma) \hookrightarrow L^p(\Sigma, d\sigma) \tag{2.4.72}$$

has dense range.

Proof. Pick $\varphi \in \mathrm{Lip}_o([0,\infty))$, monotone decreasing, with $\varphi(0) = 1$. Let $K \subset \Sigma$ be compact, and for large m consider the compactly supported Lipschitz functions

$$f_{K,m}(x) = \varphi\big(m \operatorname{dist}(x, K)\big). \tag{2.4.73}$$

Let $\mathbf{1}_K$ denote the characteristic function of K. For each $p \in [1, \infty)$ one has $|f_{K,m}(x) - \mathbf{1}_K(x)|^p \searrow 0$, so by the Monotone Convergence Theorem

$$f_{K,m} \longrightarrow \mathbf{1}_K \text{ in } L^p\text{--norm}, \tag{2.4.74}$$

as $m \to \infty$, for each $p \in [1, \infty)$. Next, let $S \subset \Sigma$ be a Borel set of finite measure. Since a locally finite Borel measure on a locally compact metric space is regular, there is a sequence of compact sets $K_m \nearrow S_0 \subset S$ such that $\sigma(S \setminus K_m) \searrow 0$. Hence the closure in L^p-norm of $\mathrm{Lip}_o(\Sigma)$ contains the space of finite linear combinations of the characteristic functions of such sets S, i.e., the space of simple functions. It is standard that the space of simple functions is dense in $L^p(\Sigma, d\sigma)$ for each $p \in [1, \infty)$, so the proof is done. $\square$

For future purposes, we find it convenient to restate (2.4.56) in a slightly different form. More specifically, in the context of Proposition 2.4.8, given $f \in L^2_{loc}(\Sigma, d\mu)$, $X \in \Sigma$ and $R > 0$, we set

$$\|f\|_*(B_d(X,R)) := \sup_{B \subset B_d(X,R)} \Big(\fint_B |f - f_B|^2 \, d\mu \Big)^{1/2}, \tag{2.4.75}$$

where the supremum is taken over all (metric) balls B included in $B_d(X,R)$, and $f_B := \mu(B)^{-1} \int_B f \, d\mu$. It is then clear from definitions that

$$\sup_{X \in \Sigma} \|f\|_*(B_d(X,R)) \approx M_2(f;R). \tag{2.4.76}$$

Consequently, (2.4.56) yields:

Corollary 2.4.10 *With the above notation and conventions,*

$$\lim_{R \to 0^+} \Big[\sup_{X \in \Sigma} \|f\|_*(B_d(X,R)) \Big] \approx \mathrm{dist}\,(f, \, \mathrm{VMO}\,(\Sigma, d\mu)), \tag{2.4.77}$$

uniformly for $f \in \mathrm{BMO}(\Sigma, d\mu)$.

2.5 Ahlfors regularity of BMO_1 domains

Consider a function $A \in \mathrm{BMO}_1(\mathbb{R}^n)$, i.e., the components of ∇A are functions of bounded mean oscillation. More specifically, we assume that

$$A : \mathbb{R}^n \to \mathbb{R} \text{ is locally integrable, with } \nabla A \in L^1_{\mathrm{loc}} \text{ and} \tag{2.5.1}$$

$$\|\nabla A\|_* := \sup_{B \,\mathrm{ball}} \fint_B \Big| \nabla A(x) - \Big(\fint_B \nabla A(y)\, dy \Big) \Big| \, dx < \infty. \tag{2.5.2}$$

From this and the John-Nirenberg inequality it follows that

$$A \text{ as in } (2.5.1) - (2.5.2) \Longrightarrow A \in \bigcap_{1<p<\infty} W^{1,p}_{loc}(\mathbb{R}^n). \tag{2.5.3}$$

Consequently a function satisfying (2.5.1)–(2.5.2) is continuous. Furthermore, by the Calderon-Rademacher theorem (cf. [111], Proposition 11.6),

$$A \text{ as in } (2.5.1) - (2.5.2) \Longrightarrow A \text{ is differentiable at almost every point in } \mathbb{R}^n. \tag{2.5.4}$$

Given such a function A, the domain

$$\Omega := \{X = (x, x_{n+1}) \in \mathbb{R}^{n+1} : \, x \in \mathbb{R}^n, \; x_{n+1} > A(x)\}, \tag{2.5.5}$$

i.e., the domain above the graph of A, is called a BMO_1 domain. The main result of this section is the following, on the regularity of surface measure of the boundary of a BMO_1 domain.

Proposition 2.5.1 *Let A be as in (2.5.1)–(2.5.2). There exists $\kappa = \kappa(\|\nabla A\|_*, n) > 1$ such that*

$$\kappa^{-1} r^n \leq \int\limits_{\substack{|x-y|^2+(A(x)-A(y))^2<r^2 \\ y\in\mathbb{R}^n}} \sqrt{1+|\nabla A(y)|^2}\, dy \leq \kappa r^n, \tag{2.5.6}$$

for every $x \in \mathbb{R}^n$ and $r > 0$. Consequently, the domain Ω defined by (2.5.5) is Ahlfors regular.

Proof. First, we note that the surface area σ on $\partial\Omega$ defined by (2.5.6) coincides with n-dimensional Hausdorff measure on $\partial\Omega$. This is a consequence of the fact that $\sigma = \mathcal{H}^n \lfloor \partial_*\Omega$ (discussed in §2.2) together with Proposition 2.2.2, which gives $\mathcal{H}^n(\partial\Omega \setminus \partial_*\Omega) = 0$. Actually, in this case a stronger result holds, namely $\partial\Omega = \partial_*\Omega$. In fact, as shown in [55], each BMO_1 domain Ω satisfies the following "corkscrew condition." There are constants $M, R \in (0,1)$ such that for each $x \in \partial\Omega$ and each $r \in (0, R]$ there are balls $B_{Mr}(y_1) \subset \Omega$ and $B_{Mr}(y_2) \subset \mathbb{R}^{n+1} \setminus \Omega$ of radius Mr, with $|x - y_k| \leq r$. It remains to prove (2.5.6).

To proceed, fix $x \in \mathbb{R}^n$, $r > 0$, $B = B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$, $\eta \in C_0^\infty(B(x,3r))$ with $\eta \equiv 1$ on $B(x,2r)$, and set

$$m(x,r) := ⨍_B \nabla A(y)\, dy, \quad A_B := (A - A(x))\eta, \quad \tilde{A}_B(z) := A_B(z) - \langle m(x,4r), z\rangle, \;\; z \in \mathbb{R}^n. \tag{2.5.7}$$

In particular,

$$\begin{aligned} A_B(x) - A_B(y) &= A(x) - A(y) \text{ for } y \in B(x,2r), \\ \nabla\tilde{A}_B(z) &= \nabla A(z) - m(x,4r) \text{ for } z \in B(x,2r). \end{aligned} \tag{2.5.8}$$

In the sequel, we shall write $|E|$ for the Euclidean measure of a (measurable) set $E \subset \mathbb{R}^n$.

For an arbitrary $y \in B(x,r)$ and two fixed parameters, $\varepsilon \in (0,1/2)$ and $p > n$, we use the Mary Weiss Lemma (cf. Lemma 1.4 on p. 144 in [14]) in concert with the John-Nirenberg inequality and a well-known property of averages of functions in BMO in order to estimate

$$\begin{aligned} \frac{|\tilde{A}_B(x) - \tilde{A}_B(y)|}{|x-y|} &\leq C_{p,n}\Big(⨍_{|x-z|\leq 2|x-y|} |\nabla\tilde{A}_B(z)|^p\, dz\Big)^{1/p} \\ &\leq C_{p,n}\Big(⨍_{|x-z|\leq 2|x-y|} |\nabla A(z) - m(x, 2|x-y|)|^p\, dz\Big)^{1/p} \\ &\quad + C_{p,n}\big|m(x,2|x-y|) - m(x,4r)\big| \\ &\leq C_{p,n}\|\nabla A\|_*\Big\{1 + \log\Big(\frac{2r}{|x-y|}\Big)\Big\} \\ &\leq C_0\|\nabla A\|_*\Big(\frac{r}{|x-y|}\Big)^{\varepsilon}, \end{aligned} \tag{2.5.9}$$

where we shall take C_0 to be a sufficiently large constant which depends only on p, n and ε.

In order to continue, introduce

$$\Delta(x,r) := \{y \in \mathbb{R}^n : |x-y|^2 + (A(x) - A(y))^2 < r^2\} \subseteq B(x,r), \tag{2.5.10}$$

and decompose $B(x,r)$ as $Y_1 \cup Y_2$ where

$$
\begin{aligned}
Y_1 &:= \Big\{ y \in B(x,r) : \Big| \Big\langle \frac{x-y}{|x-y|}\,,\, m(x,r) \Big\rangle \Big| < 4C_0(1+\|\nabla A\|_*)\Big(\frac{r}{|x-y|} \Big)^{\varepsilon} \Big\}, &(2.5.11)\\
Y_2 &:= \Big\{ y \in B(x,r) : \Big| \Big\langle \frac{x-y}{|x-y|}\,,\, m(x,r) \Big\rangle \Big| \geq 4C_0(1+\|\nabla A\|_*)\Big(\frac{r}{|x-y|} \Big)^{\varepsilon} \Big\}. &(2.5.12)
\end{aligned}
$$

Then, assuming that

$$
m(x,r) \neq 0 \tag{2.5.13}
$$

we may estimate, writing $\omega := (x-y)/|x-y|$ and $\rho := |x-y|$,

$$
\begin{aligned}
|Y_1| &\leq \int_0^r \rho^{n-1} \Big(\int_{\big|\big\langle \omega\,,\, \frac{m(x,r)}{|m(x,r)|} \big\rangle\big| < \frac{4C_0(1+\|\nabla A\|_*)}{|m(x,r)|} \left(\frac{r}{\rho}\right)^{\varepsilon}} d\omega \Big) d\rho \\
&\leq 4c_n C_0 (1+\|\nabla A\|_*) \frac{r^n}{|m(x,r)|}. &(2.5.14)
\end{aligned}
$$

Next, since for each $y \in B(x,r)$,

$$
\begin{aligned}
\frac{A(x)-A(y)}{|x-y|} &= \frac{\tilde{A}_B(x)-\tilde{A}_B(y)}{|x-y|} + \Big\langle \frac{x-y}{|x-y|}\,,\, m(x,4r)-m(x,r) \Big\rangle \\
&\quad + \Big\langle \frac{x-y}{|x-y|}\,,\, m(x,r) \Big\rangle, &(2.5.15)
\end{aligned}
$$

and, as is well-known,

$$
|m(x,4r)-m(x,r)| \leq c_n \|\nabla A\|_*, \tag{2.5.16}
$$

it follows from (2.5.15) and (2.5.16) that

$$
y \in Y_2 \Longrightarrow \frac{|A(x)-A(y)|}{|x-y|} \geq \frac{1}{2} \Big| \Big\langle \frac{x-y}{|x-y|}\,,\, m(x,r) \Big\rangle \Big|. \tag{2.5.17}
$$

As a consequence, since $|A(x)-A(y)| < r$ for each $y \in \Delta(x,r)$, we may write

$$
y \in Y_2 \cap \Delta(x,r) \Longrightarrow \Big| \Big\langle \frac{x-y}{|x-y|}\,,\, m(x,r) \Big\rangle \Big| < \frac{2r}{|x-y|}. \tag{2.5.18}
$$

Hence, by once again passing to polar coordinates ($\omega := (x-y)/|x-y|$ and $\rho := |x-y|$),

$$
\begin{aligned}
|Y_2 \cap \Delta(x,r)| &\leq \int_0^r \rho^{n-1} \Big(\int_{\big|\big\langle \omega\,,\, \frac{m(x,r)}{|m(x,r)|} \big\rangle\big| < \frac{2r}{\rho|m(x,r)|}} d\omega \Big) d\rho &(2.5.19)\\
&\leq c_n \frac{r^n}{|m(x,r)|}, &(2.5.20)
\end{aligned}
$$

assuming that $n \geq 2$. Thus, altogether, (2.5.14) and (2.5.19) yield

$$
|\Delta(x,r)| \leq |Y_1| + |Y_2 \cap \Delta(x,r)| \leq c_n (1+\|\nabla A\|_*) \frac{r^n}{|m(x,r)|}. \tag{2.5.21}
$$

We next seek a similar bound from below. To get started, we note that (2.5.9) implies the existence of some positive, finite, universal constant C_1 such that for any $y \in B(x,r)$ we have

$$
\begin{aligned}
|x-y|^2 + (A(x)-A(y))^2 &= |x-y|^2\Big(1 + \frac{|A_B(x)-A_B(y)|^2}{|x-y|^2}\Big) \qquad (2.5.22)\\
&\leq |x-y|^2\Big(1 + 2\frac{|\tilde{A}_B(x)-\tilde{A}_B(y)|^2}{|x-y|^2} + 2\Big|\Big\langle \frac{x-y}{|x-y|}\,,\, m(x,4r)\Big\rangle\Big|^2\Big)\\
&\leq |x-y|^2\Big(1 + C_1\|\nabla A\|_*^2\Big(\frac{r}{|x-y|}\Big)^{2\varepsilon} + 2\Big|\Big\langle \frac{x-y}{|x-y|}\,,\, m(x,r)\Big\rangle\Big|^2\Big),
\end{aligned}
$$

where in the last step we have used (2.5.16). Thus, for each $\delta \in (0,1)$,

$$
y \in Y_1 \cap B(x,\delta r) \Longrightarrow |x-y|^2 + (A(x)-A(y))^2 < r^2\Big\{\delta^2 + C_2\delta^{2(1-\varepsilon)}\|\nabla A\|_*^2\Big\} \qquad (2.5.23)
$$

where $C_2 := C_1 + 32C_0^2$. In particular,

$$
\delta = \frac{1}{1+C_2\|\nabla A\|_*^2} \text{ and } y \in Y_1 \cap B(x,\delta r) \Longrightarrow |x-y|^2 + (A(x)-A(y))^2 < r^2, \qquad (2.5.24)
$$

i.e.,

$$
\delta = \frac{1}{1+C_2\|\nabla A\|_*^2} \Longrightarrow Y_1 \cap B(x,\delta r) \subseteq \Delta(x,r). \qquad (2.5.25)
$$

Thus, for this choice of δ,

$$
\begin{aligned}
|m(x,r)| &< 4C_0(1+\|\nabla A\|_*) \Longrightarrow Y_1 = B(x,r) \Longrightarrow B(x,\delta r) \subseteq \Delta(x,r)\\
&\Longrightarrow |\Delta(x,r)| \geq c_n\delta^n r^n \geq \frac{c_n r^n}{(1+\|\nabla A\|_*^2)^n}. \qquad (2.5.26)
\end{aligned}
$$

On the other hand, since the set Y_1 (introduced in (2.5.11)) is star-like with respect to the point x, in the case when $|m(x,r)| \geq 4C_0(1+\|\nabla A\|_*)$ we have

$$
\begin{aligned}
|\Delta(x,r)| &\geq |Y_1 \cap B(x,\delta r)| \geq \delta^n \cdot r \cdot \Big[\text{surface measure of } \overline{Y_1} \cap \partial B(x,r)\Big]\\
&\geq c_n\delta^n r^n \int_{\left|\left\langle \omega\,,\,\frac{m(x,r)}{|m(x,r)|}\right\rangle\right| < \frac{4C_0(1+\|\nabla A\|_*)}{|m(x,r)|}} d\omega\\
&\geq 4C_0c_n(1+\|\nabla A\|_*)\delta^n \frac{r^n}{|m(x,r)|} \geq \frac{C(n)}{(1+\|\nabla A\|_*)^{2n}}\frac{r^n}{|m(x,r)|}. \qquad (2.5.27)
\end{aligned}
$$

Hence, at this stage we have proved that there exist three finite dimensional constants, $C_0, C_1, C_2 > 0$ such that

$$|\Delta(x,r)| \le C_1(1+\|\nabla A\|_*)\frac{r^n}{|m(x,r)|}, \tag{2.5.28}$$

$$|\Delta(x,r)| \ge \frac{C_2}{(1+\|\nabla A\|_*)^{2n}}\frac{r^n}{|m(x,r)|} \quad \text{if} \quad |m(x,r)| > 4C_0(1+\|\nabla A\|_*), \tag{2.5.29}$$

$$|\Delta(x,r)| \ge \frac{C_2 r^n}{(1+\|\nabla A\|_*)^{2n}} \quad \text{if} \quad |m(x,r)| < 4C_0(1+\|\nabla A\|_*). \tag{2.5.30}$$

After this preamble, we shall show that there exists $C = C(\|\nabla A\|_*, n) > 0$ such that

$$\int_{\Delta(x,r)} (1+|\nabla A(y)|)\, dy \ge Cr^n. \tag{2.5.31}$$

Thanks to (2.5.29)–(2.5.30), it is enough to consider the case when $|m(x,r)|$ is very large, say

$$|m(x,r)| > M(1+\|\nabla A\|_*), \tag{2.5.32}$$

with $M > 4C_0$ to be specified later. Since $\Delta(x,r) \subset B(x,r)$, Hölder's and John-Nirenberg inequalities, along with the estimates (2.5.29), (2.5.32), give that

$$\begin{aligned}
\int_{\Delta(x,r)} |\nabla A(y) - m(x,r)|\, dy &\le |\Delta(x,r)|^{1/2}\Big(\int_{\Delta(x,r)} |\nabla A(y) - m(x,r)|^2\, dy\Big)^{1/2} \\
&\le |\Delta(x,r)|^{1/2}\Big(\int_{B(x,r)} |\nabla A(y) - m(x,r)|^2\, dy\Big)^{1/2} \\
&\le C_0^{1/2}(1+\|\nabla A\|_*)^{1/2}\frac{r^n}{|m(x,r)|^{1/2}}\Big(\fint_{B(x,r)} |\nabla A(y) - m(x,r)|^2\, dy\Big)^{1/2} \\
&\le c_n C_0^{1/2}(1+\|\nabla A\|_*)^{1/2}\|\nabla A\|_*\frac{r^n}{|m(x,r)|^{1/2}} \\
&\le c_n C_0^{1/2}\|\nabla A\|_* M^{-1/2} r^n.
\end{aligned} \tag{2.5.33}$$

Consequently,

$$\begin{aligned}
\int_{\Delta(x,r)} (1+|\nabla A(y)|)\, dy &\ge \int_{\Delta(x,r)} |\nabla A(y)|\, dy \\
&\ge |\Delta(x,r)||m(x,r)| - \int_{\Delta(x,r)} |\nabla A(y) - m(x,r)|\, dy \\
&\ge \frac{C_2 r^n}{(1+\|\nabla A\|_*)^{2n}} - c_n C_0^{1/2}\|\nabla A\|_* M^{-1/2} r^n \\
&\ge Cr^n,
\end{aligned} \tag{2.5.34}$$

if M is large enough. For instance,

$$M > \max\{C_0, 4c_n^2 C_0 C_2^{-2}\}(1+\|\nabla A\|_*)^{4n+2} \tag{2.5.35}$$

will do.

The proof of (2.5.6) will therefore be completed as soon as we show that there exists $C = C(\|\nabla A\|_*, n) > 0$ such that

$$\int_{\Delta(x,r)} (1 + |\nabla A(y)|)\, dy \leq C r^n. \tag{2.5.36}$$

However, based on the inclusion $\Delta(x,r) \subseteq B(x,r)$ and the inequality (2.5.28) we may write

$$\begin{aligned} \int_{\Delta(x,r)} (1 + |\nabla A(y)|)\, dy &\leq \int_{B(x,r)} dy + \int_{B(x,r)} |\nabla A(y) - m(x,r)|\, dy + |m(x,r)| \cdot |\Delta(x,r)| \\ &\leq c_n r^n + c_n r^n \|\nabla A\|_* + C_1(1 + \|\nabla A\|_*) r^n, \end{aligned} \tag{2.5.37}$$

as desired. This finishes the proof of the proposition in the case when (2.5.13) holds.

Finally, if $m(x,r) = 0$, then (2.5.25) gives $B(x, \delta r) \subset \Delta(x,r)$ for some $\delta = \delta(\|\nabla A\|_*, n) > 0$ sufficiently small, which in turn readily yields (2.5.31). Also, (2.5.36) follows much as in (2.5.37), so (2.5.6) holds in this case as well. □

In concert with Proposition 2.2.3, Proposition 2.5.1 yields the following.

Corollary 2.5.2 *Let A be as in (2.5.1)–(2.5.2), define Ω as in (2.5.5) and set $\sigma := \mathcal{H}^n \lfloor \partial\Omega$. Then $\partial\Omega$ is Ahlfors regular, with constants depending only on n and $\|A\|_*$.*

Define $\mathrm{VMO}_1(\mathbb{R}^n)$ as the (closed) subspace of $\mathrm{BMO}_1(\mathbb{R}^n)$ consisting of functions $A : \mathbb{R}^n \to \mathbb{R}$ for which $\partial_j A \in \mathrm{VMO}(\mathbb{R}^n)$ for every $j \in \{1, ..., n+1\}$. The latter space consists of functions $f \in \mathrm{BMO}(\mathbb{R}^n)$ for which

$$\lim_{\delta \to 0} \left(\sup_{\substack{Q \subset \mathbb{R}^{n+1} \ cube \\ |Q| \leq \delta}} \fint_Q \Big| f(x) - \fint_Q f \Big|\, dx \right) = 0. \tag{2.5.38}$$

We would now like to discuss a natural sufficient condition guaranteeing that a BMO_1 domain Ω has a unit normal of small BMO-norm. In particular, it is of interest to know whether ν is in VMO whenever Ω is a VMO_1 domain. These issues are addressed in the proposition below, which should be compared with the (proof of) Corollary 5.4 in [64].

Proposition 2.5.3 *Let A be as in (2.5.1)-(2.5.2), and let Σ be the graph of A. Also, denote by ν and σ the outer normal unit and the surface measure on Σ, respectively. Then there exists a finite constant $C = C(n) > 0$ such that*

$$\|\nu\|_{\mathrm{BMO}(\Sigma, d\sigma)} \leq C \|\nabla A\|_* (1 + \|\nabla A\|_*), \tag{2.5.39}$$

and

$$\mathrm{dist}\,(\nu\,,\ \mathrm{VMO}\,(\Sigma, d\sigma)) \leq C(1 + \|\nabla A\|_*)\, \mathrm{dist}(\nabla A\,,\ \mathrm{VMO}\,(\mathbb{R}^n)), \tag{2.5.40}$$

where the distances are measured in $\mathrm{BMO}(\Sigma, d\sigma)$ *and* $\mathrm{BMO}(\mathbb{R}^n)$*, respectively. As a consequence,*

$$A \in \mathrm{VMO}_1(\mathbb{R}^n) \Longrightarrow \nu \in \mathrm{VMO}(\Sigma, d\sigma). \tag{2.5.41}$$

Proof. As far as (2.5.39) is concerned, we note that by (2.4.17)-(2.4.18) and (2.5.6) it suffices to prove that

$$\int_{\Delta(Z,r)}\int_{\Delta(Z,r)}|\nu(X)-\nu(Y)|\,d\sigma(X)\,d\sigma(Y)\leq C\,r^{2n}\|\nabla A\|_*(1+\|\nabla A\|_*), \tag{2.5.42}$$

for every $r>0$ and $Z=(z,A(z))\in\Sigma$. Using the notation in (2.5.10), we may estimate the left-hand side of (2.5.42) by

$$\begin{aligned}
&\int_{\Delta(z,r)}\int_{\Delta(z,r)}\left|\frac{(\nabla A(x),-1)}{\sqrt{1+|\nabla A(x)|^2}}-\frac{(\nabla A(y),-1)}{\sqrt{1+|\nabla A(y)|^2}}\right|\sqrt{1+|\nabla A(x)|^2}\sqrt{1+|\nabla A(y)|^2}\,dx\,dy\\
&\quad\leq 2\int_{\Delta(z,r)}\int_{\Delta(z,r)}\sqrt{1+|\nabla A(y)|^2}|\nabla A(x)-\nabla A(y)|\,dx\,dy\\
&\quad\leq 2\Big(\int_{\Delta(z,r)}\int_{\Delta(z,r)}|\nabla A(x)-\nabla A(y)|^2\,dx\,dy\Big)^{1/2}\Big(\int_{\Delta(z,r)}\int_{\Delta(z,r)}[1+|\nabla A(y)|^2]\,dx\,dy\Big)^{1/2}\\
&\quad=:2\,I\cdot II.
\end{aligned}\tag{2.5.43}$$

Now,

$$I\leq\Big(\int_{B(z,r)}\int_{B(z,r)}|\nabla A(x)-\nabla A(y)|^2\,dx\,dy\Big)^{1/2}\leq Cr^n\|\nabla A\|_* \tag{2.5.44}$$

by (2.5.10), (2.4.17)-(2.4.18) and the John-Nirenberg inequality. Upon recalling notation introduced in (2.5.7) and the estimate (2.5.28), the John-Nirenberg inequality also gives

$$\begin{aligned}
II &\leq Cr^n+C|\Delta(z,r)|^{1/2}\Big(\int_{\Delta(z,r)}|\nabla A(y)|^2\,dy\Big)^{1/2}\\
&\leq Cr^n+C|\Delta(z,r)|^{1/2}\Big(\int_{\Delta(z,r)}|\nabla A(y)-m(z,r)|^2\,dy\Big)^{1/2}\\
&\quad+C|\Delta(z,r)||m(z,r)|\\
&\leq Cr^n+Cr^{n/2}\Big(\int_{B(z,r)}|\nabla A(y)-m(z,r)|^2\,dy\Big)^{1/2}\\
&\quad+Cr^n(1+\|\nabla A\|_*)\\
&\leq Cr^n(1+\|\nabla A\|_*),
\end{aligned}\tag{2.5.45}$$

for some purely dimensional constants. Now, (2.5.42) follows easily from (2.5.44) and (2.5.45).

Note that the above argument and (2.4.17) also gives that for every $r>0$ and $Z=(z,A(z))\in\Sigma$,

$$\begin{aligned}
&\int_{\Delta(Z,r)}\int_{\Delta(Z,r)}|\nu(X)-\nu(Y)|\,d\sigma(X)d\sigma(Y)\\
&\qquad\leq C\,r^n(1+\|\nabla A\|_*)\Big(\int_{B(z,r)}\int_{B(z,r)}|\nabla A(x)-\nabla A(y)|^2\,dx\,dy\Big)^{1/2}\\
&\qquad\leq C\,r^{2n}(1+\|\nabla A\|_*)M_2(\nabla A;r).
\end{aligned}\tag{2.5.46}$$

Thus, (2.5.40) readily follows from this, (2.4.17) and (2.4.56). $\square$

In order to continue, we make the following definition.

Definition 2.5.4 *(i) Let Ω be a nonempty, proper open subset of $\mathbb{R}^{n+1}$. Call Ω a* BMO$_1$*-domain if for every compact set $\mathcal{K} \subset \mathbb{R}^{n+1}$ there exist $b, c > 0$ such that the following hold. For every $X_0 \in \partial\Omega \cap \mathcal{K}$ there exists an n-plane $H \subset \mathbb{R}^{n+1}$ passing through X_0, a choice N of the unit normal to H, and an open cylinder $\mathcal{C}_{b,c} := \{X + tN : X \in H, |X - X_0| < b, |t| < c\}$ such that*

$$\mathcal{C}_{b,c} \cap \Omega = \mathcal{C}_{b,c} \cap \{X + tN : X \in H, t > \varphi(X)\}, \tag{2.5.47}$$

$$\mathcal{C}_{b,c} \cap \partial\Omega = \mathcal{C}_{b,c} \cap \{X + tN : X \in H, t = \varphi(X)\}, \tag{2.5.48}$$

$$\mathcal{C}_{b,c} \cap \overline{\Omega}^c = \mathcal{C}_{b,c} \cap \{X + tN : X \in H, t < \varphi(X)\}, \tag{2.5.49}$$

for some function $\varphi : H \to \mathbb{R}$ satisfying

$$\varphi \in \mathrm{BMO}_1(H), \quad \varphi(X_0) = 0 \quad \text{and} \quad |\varphi(X)| < c \quad \text{if} \quad |x' - x_0| \le b. \tag{2.5.50}$$

(ii) It is said that the BMO$_1$*-domain Ω has constant $\le \delta$ if it is always (i.e., for every choice of the compact $\mathcal{K}$ and boundary point X_0) possible to ensure that $\|\nabla\varphi\|_{\mathrm{BMO}(H)} \le \delta$.*

(iii) The classes of VMO$_1$*-domains and Lipschitz domains in $\mathbb{R}^{n+1}$ are defined analogously, demanding that $\varphi \in \mathrm{VMO}_1(H)$ and $\varphi \in \mathrm{Lip}\,(H)$ in place of $\varphi \in \mathrm{BMO}_1(H)$.*

Parenthetically, we note that conditions (2.5.47)-(2.5.49) are not independent since, in fact, (2.5.47) implies (2.5.48)-(2.5.49). In this vein, let us also mention that, (2.5.48) implies (2.5.47), (2.5.49) (up to changing N into $-N$) if it is known a priori that

$$\partial\Omega = \partial\overline{\Omega}. \tag{2.5.51}$$

Theorem 2.5.5 *If $\Omega \subset \mathbb{R}^{n+1}$ is a bounded* BMO$_1$*-domain with constant $\le \delta$ then $\partial\Omega$ is Ahlfors regular and*

$$\mathrm{dist}\,(\nu\,,\,\mathrm{VMO}\,(\partial\Omega, d\sigma)) \le C\,\delta \tag{2.5.52}$$

where C depends only on n, and the distance is measured in BMO$(\partial\Omega, d\sigma)$*. In particular,*

$$\Omega \text{ bounded } \mathrm{VMO}_1 \text{ domain} \Longrightarrow \nu \in \mathrm{VMO}\,(\partial\Omega, d\sigma). \tag{2.5.53}$$

Proof. This is a consequence of Corollary 2.5.2, Proposition 2.5.3 and Definition 2.5.4. $\square$

3 Singular integrals on UR domains

In this section we study various layer potentials on an open set $\Omega \subset \mathbb{R}^{n+1}$ whose boundary is uniformly rectifiable (a UR domain). In §3.1 we recall the notion of uniform rectifiability, introduced by G. David and S. Semmes, and discuss several classes of domains that have the UR property, including Ahlfors regular NTA domains, and more generally Ahlfors regular John domains. In

§3.2 we record some fundamental estimates for a broad class of layer potentials on UR domains. Sections 3.3 and 3.4 establish the existence of nontangential limits a.e. of layer potentials applied to elements of $L^p(\partial\Omega, d\sigma)$, first in the case of Newtonian potentials and then more generally. Tools include nontangential maximal estimates from §3.2, the Green formula discussed in §2.2 and, in §3.4, some Clifford analysis. In §3.5 we extend the results of §3.4 to the "variable coefficient" setting, which will be useful for the treatment of variable coefficient PDE in §5. In §3.6 we obtain boundedness results on L^p-Sobolev spaces.

3.1 Countably rectifiable sets and uniformly rectifiable sets

Let $\Sigma \subset \mathbb{R}^{n+1}$ be closed. We say that Σ is countably rectifiable (of dimension n) provided it can be written as a countable union

$$\Sigma = \bigcup_k L_k \cup \widetilde{N}, \quad \mathcal{H}^n(\widetilde{N}) = 0, \tag{3.1.1}$$

where each L_k is the image of a compact subset of $\mathbb{R}^n$ under a Lipschitz map. As is well known, via Rademacher's theorem and Whitney's extension theorem we can then write $\Sigma = \cup_k M_k \cup N$ where $\mathcal{H}^n(N) = 0$ and each M_k is a compact subset of an n-dimensional C^1 submanifold of $\mathbb{R}^{n+1}$. (This characterization was used in (2.2.7).)

A countably rectifiable set $\Sigma \subset \mathbb{R}^{n+1}$ need not have tangent planes in the ordinary sense, but it will have approximate tangent planes. By definition, an n-plane $\pi \subset \mathbb{R}^{n+1}$ passing through $X_0 \in \Sigma$ is called the approximate tangent n-plane to Σ at X_0 provided

$$\limsup_{R\searrow 0} R^{-n}\, \mathcal{H}^n\big(\Sigma \cap B_R(X_0)\big) > 0 \tag{3.1.2}$$

and

$$\limsup_{R\searrow 0} R^{-n}\, \mathcal{H}^n\big(\{X \in \Sigma \cap B_R(X_0) : \operatorname{dist}(X, \pi) > \lambda|X - X_0|\}\big) = 0, \quad \forall\, \lambda > 0. \tag{3.1.3}$$

The conditions (3.1.2) and (3.1.3) together imply that if such an n-plane π exists, then it is unique. The following result is contained in Theorem 3.2.19 of [41].

Theorem 3.1.1 *Assume $\Sigma \subset \mathbb{R}^{n+1}$ is $\mathcal{H}^n$-measurable, of locally finite Hausdorff measure. If Σ is countably rectifiable then there exists an approximate tangent n-plane to Σ at $\mathcal{H}^n$-almost every point in Σ.*

If $\Omega \subset \mathbb{R}^{n+1}$ is an open set of locally finite perimeter, satisfying $\mathcal{H}^n(\partial\Omega \setminus \partial_*\Omega) = 0$, so $\Sigma = \partial\Omega$ is countably rectifiable, we denote $\partial_T\Omega$ the set of points $X_0 \in \partial\Omega$ with an approximate tangent plane, π_{X_0}. We note that

$$\partial^*\Omega \subset \partial_T\Omega. \tag{3.1.4}$$

This follows from Theorem 5.6.5 of [118], plus Lemma 5.5.4 of [118], to cover the property (3.1.2) (as long as $\mathcal{H}^n(\partial\Omega \setminus \partial^*\Omega) = 0$).

We compare π_{X_0} with the exterior normal $\nu(X_0)$, given $X_0 \in \partial^*\Omega \subset \partial_0\Omega$. Recall that for such X_0

$$\lim_{R\to 0} R^{-(n+1)}\mathcal{H}^{n+1}\big(B_R(X_0)\cap\Omega^{\pm}\cap H^{\pm}_{\nu(X_0)}(X_0)\big)=0, \tag{3.1.5}$$

when $\Omega^+ := \Omega$, $\Omega^- := \mathbb{R}^{n+1}\setminus\Omega$, and

$$H^{\pm}_{\nu(X_0)}(X_0)=\{Y\in\mathbb{R}^{n+1}:\ \pm\langle\nu(X_0),Y-X_0\rangle\geq 0\}. \tag{3.1.6}$$

Let us also set

$$H^{0}_{\nu(X_0)}(X_0):=\{Y\in\mathbb{R}^{n+1}:\ \langle\nu(X_0),Y-X_0\rangle=0\}. \tag{3.1.7}$$

Results just described plus comparison of (2.2.2) with (3.1.3) yield the following.

Proposition 3.1.2 *Let $\Omega\subset\mathbb{R}^{n+1}$ be an open set with locally finite perimeter, satisfying $\mathcal{H}^n(\partial\Omega\setminus\partial_*\Omega)=0$, and suppose $X_0\in\partial^*\Omega$. Then $X_0\in\partial_T\Omega$ and $\nu(X_0)\perp\pi_{X_o}$ or, equivalently,*

$$\pi_{X_0}=H^{0}_{\nu(X_0)}(X_0). \tag{3.1.8}$$

We record some notation that will be useful later in this section. Given $X_0\in\partial^*\Omega\subset\partial_T\Omega,\ R\in(0,\infty)$, set

$$\pi^{\pm}_{X_0}=H^{\pm}_{\nu(X_0)}(X_0),\quad \partial^{\pm}B_R(X_0)=\partial B_R(X_0)\cap\pi^{\pm}_{X_0}, \tag{3.1.9}$$

and

$$W(X,R)=\partial^{-}B_R(X)\triangle\Big[\partial B_R(X)\cap\Omega\Big], \tag{3.1.10}$$

where $U\triangle V$ denotes the symmetric difference $(U\setminus V)\cup(V\setminus U)$. The following result will be useful in §§3.3–3.4.

Proposition 3.1.3 *In the setting of Proposition 3.1.2, if $X_0\in\partial^*\Omega$, there is a set $\mathcal{O}\subset[0,1]$ of density 1 at 0 such that*

$$R^{-n}\mathcal{H}^n(W(X_0,R))\longrightarrow 0 \tag{3.1.11}$$

as $R\to 0$ in $\mathcal{O}$.

Proof. We have

$$\begin{aligned}\int_0^R \mathcal{H}^n(W(X_0,r))\,dr &= \mathcal{H}^{n+1}\big(B_R(X_0)\cap(\Omega\triangle\pi^{-}_{X_0})\big)\\ &= \mathcal{H}^{n+1}\big(B_R(X_0)\cap\Omega^{+}\cap\pi^{+}_{X_0}\ \cup\ B_R(X_0)\cap\Omega^{-}\cap\pi^{-}_{X_0}\big) \qquad (3.1.12)\\ &= o(R^{n+1}),\end{aligned}$$

as $R\to 0$, and this implies (3.1.11). $\square$

For the purposes we have in mind, the purely qualitative concept of countable rectifiability is too weak and should be replaced by *uniform rectifiability.* Following G. David and S. Semmes [33] we make the following.

Definition 3.1.4 *Call* $\Sigma \subset \mathbb{R}^{n+1}$ `uniformly rectifiable` *provided it is Ahlfors regular and the following holds. There exist* ε, $M \in (0,\infty)$ *(called the UR constants of* Σ*) such that for each* $x \in \Sigma$, $R > 0$*, there is a Lipschitz map* $\varphi : B^n_R \to \mathbb{R}^{n+1}$ *(where* B^n_R *is a ball of radius* R *in* $\mathbb{R}^n$*) with Lipschitz constant* $\leq M$*, such that*

$$\mathcal{H}^n\big(\Sigma \cap B_R(x) \cap \varphi(B^n_R)\big) \geq \varepsilon R^n. \tag{3.1.13}$$

If Σ *is compact, this is required only for* $R \in (0,1]$.

Any uniformly rectifiable set Σ is countably rectifiable. To see this, let $(x_j)_{j\in\mathbb{N}}$ be a countable, dense subset of Σ, and consider $(R_k)_{k\in\mathbb{N}}$ an enumeration of $\mathbb{Q}_+$ (or $(0,1)\cap\mathbb{Q}_+$ if Σ is compact). For each $j,k \in \mathbb{N}$ set $\Delta_{jk} := \Sigma \cap B(x_j, R_k)$ and $L_{jk} := \varphi_{jk}(B^n_{jk})$, where B^n_{jk} is an n-dimensional ball of radius R_k and $\varphi_{jk} : B^n_{jk} \to \mathbb{R}^{n+1}$ is a Lipschitz function for which

$$\mathcal{H}^n\big(\Sigma \cap B(x_j, R_k) \cap \varphi(B^n_{jk})\big) \geq \varepsilon R^n_k, \tag{3.1.14}$$

for some $\varepsilon > 0$ is a fixed constant, independent of j,k. Put $E := \bigcup_{j,k\in\mathbb{N}}(L_{jk}\cap\Sigma)$ and $N := \Sigma \setminus E$, so that

$$\Sigma = \Big(\bigcup_{j,k\in\mathbb{N}} (L_{jk}\cap\Sigma)\Big) \cup N. \tag{3.1.15}$$

Then, using (3.1.14) and the fact that Σ is Ahlfors regular, we may write

$$⨍_{\Delta_{jk}} \mathbf{1}_E \, d\mathcal{H}^n \geq ⨍_{\Delta_{jk}} \mathbf{1}_{L_{jk}\cap\Sigma} \, d\mathcal{H}^n \geq \frac{\mathcal{H}^n(L_{jk}\cap\Delta_{jk})}{\mathcal{H}^n(\Delta_{jk})} \geq \frac{\mathcal{H}^n(L_{jk}\cap\Delta_{jk})}{C\,R^n_k} \geq \varepsilon/C, \tag{3.1.16}$$

for every $j,k \in \mathbb{N}$ which, by density, further entails

$$⨍_{B(x,R)\cap\Sigma} \mathbf{1}_E \, d\mathcal{H}^n \geq \varepsilon/C, \qquad \forall\, x \in \Sigma,\ \forall\, R > 0. \tag{3.1.17}$$

Hence, by Lebesgue-Besicovitch Differentiation Theorem (cf., e.g., Theorem 1 on p. 43 in [36]), $\mathbf{1}_E(x) > 0$ at $\mathcal{H}^n$-a.e. point $x \in \Sigma$ which proves that $\mathcal{H}^n(N) = 0$. In turn, this and (3.1.15) show that (3.1.1) holds, thus Σ is countably rectifiable (of dimension n), as claimed earlier.

There are alternative characterizations of uniform rectifiability, discussed at length in the monographs [33] and [34]. We mention one here. We say a mapping $\psi : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is ω-regular if $|\nabla\psi| \leq C\omega^{1/n}$ and

$$\sup_{x\in\mathbb{R}^{n+1}} \sup_{R>0} R^{-n} \int\limits_{\psi^{-1}(B_R(x))} \omega(y)\,dy < \infty. \tag{3.1.18}$$

Then, as shown in [33], Σ is uniformly rectifiable if and only if it is Ahlfors regular and there exists an A_1-weight ω and an ω-regular mapping $\psi : \mathbb{R}^n \to \mathbb{R}^{n+1}$ whose image contains Σ.

An important class of uniformly rectifiable sets was identified in [31], where G. David and D. Jerison proved the following result.

Proposition 3.1.5 *Let $\Sigma \subset \mathbb{R}^{n+1}$ be closed and Ahlfors regular. Assume Σ satisfies the following "two disks" condition. There exists $C_0 \in (0,\infty)$ such that for each $x \in \Sigma$ and $r > 0$, there exist two n-dimensional disks, with centers at a distance $\leq r$ from x, radius r/C_0, contained in two different connected components of $\mathbb{R}^{n+1} \setminus \Sigma$. (If Σ is compact, one can pick $R_0 \in (0,\infty)$ and restrict attention to $r \in (0, R_0]$.) Then Σ is uniformly rectifiable.*

The somewhat more restrictive case of Proposition 3.1.5 where the disks are replaced by balls has been established earlier by S. Semmes in [104]. As pointed out on p. 844 in [31], the same conclusion holds if the two disks can be replaced by bi-Lipschitz images of disks. What David and Jerison actually prove is that any set Σ as in the statement of Proposition 3.1.5 contains "big pieces of Lipschitz graphs" (cf. Theorem 1 on p. 840 loc. cit.). There is, in fact, a more precise version of this statement, which is implicit in the discussion following formula (10) on p. 842 of [31]. To state this result, we bring in the notion of the *Corkscrew condition*, which is that there are constants $M > 1$ and $R > 0$ (called the corkscrew constants of Ω) such that for each $X \in \partial\Omega$ and $r \in (0, R)$ there exists $Y = Y(X, r)$, called the corkscrew point relative to X, such that $|X - Y| < r$ and $\mathrm{dist}(Y, \partial\Omega) > r/M$. Here is the result of [31].

Proposition 3.1.6 *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, with an Ahlfors regular boundary $\partial\Omega$ which satisfies the "two disks" condition, for every $r \in (0, R_0)$, as in the previous proposition (here R_0 may be infinite in the case that $\partial\Omega$ is unbounded). Suppose also that Ω satisfies the Corkscrew condition for every $r \in (0, R_0)$. Then Ω contains "big pieces of Lipschitz domains" that is, there exist $c_1, c_2 \geq 1$ such that for every $X \in \partial\Omega$, and every $r \in (0, R_0)$, one can find a Lipschitz domain $D \subset \mathbb{R}^{n+1}$ for which:*

(i) $D \subset \Omega \cap B(X, 10r)$;

(ii) In a new system of coordinates (which is a rigid motion of the original one) one has $X = (x', x_{n+1})_{1 \leq j \leq n+1} \in \mathbb{R}^n \times \mathbb{R}$ and

$$D = \{Y = (y', y_{n+1}) :\ \psi(y') < y_{n+1} < x_{n+1} + r/2,\ |x' - y'| < r/(2c_1)\}, \tag{3.1.19}$$

where $\psi : \mathbb{R}^n \to \mathbb{R}$ is a function satisfying

$$|\psi(y') - \psi(z')| \leq c_1 |y' - z'| \ \text{ for all }\ y', z' \in \mathbb{R}^n \ \text{ and }\ \|\psi\|_{L^\infty(\mathbb{R}^n)} \leq |x_{n+1}| + r/2, \tag{3.1.20}$$

as well as

$$\mathcal{H}^n\Big(\{(y', \psi(y')) :\ y' \in \mathbb{R}^n,\ |x' - y'| < r/(2c_1)\} \cap \partial\Omega\Big) \geq r^n/c_2. \tag{3.1.21}$$

Moving on, we make the following

Definition 3.1.7 *A nonempty, proper open subset Ω of $\mathbb{R}^{n+1}$ is called a* `UR domain` *provided $\partial\Omega$ is uniformly rectifiable and also*

$$\mathcal{H}^n(\partial\Omega \setminus \partial_*\Omega) = 0. \tag{3.1.22}$$

We impose the last condition to eliminate such cases as a slit disk. Let us emphasize that, by definition, a UR domain Ω has an Ahlfors regular boundary.

One important class of UR domains is the class of Ahlfors regular domains with the NTA property, introduced in [55]. We recall the definition here, since these domains will play a role in subsequent sections.

Definition 3.1.8 *A nonempty, proper open subset Ω of $\mathbb{R}^{n+1}$ is called an* `NTA (nontangentially accessible) domain` *provided*

$$\Omega \ \textit{satisfies a two-sided corkscrew condition}, \tag{3.1.23}$$

and

$$\Omega \ \textit{satisfies a Harnack chain condition.} \tag{3.1.24}$$

As mentioned right above Proposition 3.1.6, the (interior) corkscrew condition on Ω is that there are constants $M > 1$ and $R > 0$ (called the corkscrew constants of Ω) such that for each $X \in \partial\Omega$ and $r \in (0, R)$ there exists $Y = Y(X, r) \in \Omega$, called corkscrew point relative to X, such that $|X - Y| < r$ and $\mathrm{dist}(Y, \partial\Omega) > M^{-1}r$. Next, we shall say that Ω satisfies an exterior corkscrew condition if $\mathbb{R}^{n+1} \setminus \Omega$ has the (interior) corkscrew condition, and say that Ω satisfies a two-sided corkscrew condition if Ω has both the interior and the exterior corkscrew condition.

The Harnack chain condition is defined as follows (with reference to M and R as above). First, given $X_1, X_2 \in \Omega$, a Harnack chain from X_1 to X_2 in Ω is a sequence of balls $B_1, \dots, B_K \subset \Omega$ such that $X_1 \in B_1$, $X_2 \in B_K$ and $B_j \cap B_{j+1} \neq \emptyset$ for $1 \leq j \leq K - 1$, and such that each B_j has a radius r_j satisfying $M^{-1}r_j < \mathrm{dist}(B_j, \partial\Omega) < Mr_j$. The length of the chain is K.

Then the Harnack chain condition on Ω is that if $\varepsilon > 0$ and $X_1, X_2 \in \Omega \cap B_{r/4}(Q)$ for some $Q \in \partial\Omega$, $r \in (0, R)$, and if $\mathrm{dist}(X_j, \partial\Omega) > \varepsilon$ and $|X_1 - X_2| < 2^k\varepsilon$, then there exists a Harnack chain $B_1, \dots, B_K$ from X_1 to X_2, of length $K \leq Mk$, having the further property that the diameter of each ball B_j is $\geq M^{-1} \min\big(\mathrm{dist}(X_1, \partial\Omega), \mathrm{dist}(X_2, \partial\Omega)\big)$.

If Ω is unbounded, the NTA condition also requires that $R = \infty$ and that $\mathbb{R}^{n+1} \setminus \partial\Omega$ has exactly two connected components, Ω and $\mathbb{R}^{n+1} \setminus \overline{\Omega}$.

Finally, call $\Omega \subset \mathbb{R}^{n+1}$ a `two-sided NTA domain` if both Ω and $\mathbb{R}^{n+1} \setminus \bar{\Omega}$ are nontangentially accessible domains.

Let us remark that

$$\Omega \ \text{ satisfies (3.1.23)} \Longrightarrow \partial\Omega = \partial_*\Omega, \tag{3.1.25}$$

$$\Omega \ \text{ satisfies an exterior corkscrew condition} \Longrightarrow \partial\Omega = \partial\overline{\Omega}. \tag{3.1.26}$$

The fact that Ahlfors regular domains satisfying the condition (3.1.23) are UR domains is a special case of Proposition 3.1.5; this class of domains was also investigated in [100]. As a consequence of these observations and (2.2.17), we have the following.

Corollary 3.1.9 *If $\Omega \subseteq \mathbb{R}^{n+1}$ is a domain satisfying a two-sided corkscrew condition and whose boundary is Ahlfors regular, then Ω is a UR domain.*

When discussing the boundary behavior of layer potential operators from either side of the boundary, the following observation (whose simple proof is omitted) will be important.

Proposition 3.1.10 *If $\Omega \subset \mathbb{R}^{n+1}$ is a UR domain for which $\partial\Omega = \partial\bar{\Omega}$ then $\mathbb{R}^{n+1} \setminus \bar{\Omega}$ is also a UR domain.*

As already pointed out in (3.1.26), $\partial\Omega = \partial\bar{\Omega}$ is automatically satisfied if Ω has the exterior corkscrew property. Parenthetically, we wish to point out that while being an Ahlfors regular domain does not, generally speaking, imply the corkscrew condition, the following related result can be established (by arguing as in [10]).

Proposition 3.1.11 *If $\Omega \subset \mathbb{R}^{n+1}$ is an open set with an Ahlfors regular boundary then there exists $\gamma \in (0,1)$ (depending only on the Ahlfors regularity constants of Ω) with the property that*

$$\forall\, X \in \partial\Omega,\ \forall\, R > 0,\ \exists\, Q \in \mathbb{R}^{n+1} \text{ with } B(Q, \gamma R) \subset B(X, R) \setminus \partial\Omega. \tag{3.1.27}$$

For later purposes, it is useful to introduce a certain scale invariant local connectivity condition for domains in $\mathbb{R}^{n+1}$. To put this in the proper context, recall that a bounded open set $\Omega \subset \mathbb{R}^{n+1}$ is called a *John domain* if there exist $X^* \in \Omega$, called the (global) John center, and $C_o > 1$ such that for every point $X \in \Omega$ there exists a rectifiable curve (called John path) $\gamma : [0, \ell] \to \Omega$, parametrized by the arc-length $s \in [0, \ell]$, such that $\gamma(0) = X$, $\gamma(\ell) = X^*$ and

$$\text{dist}\,(\gamma(s), \partial\Omega) > s/C_o, \qquad \forall\, s \in (0, \ell]. \tag{3.1.28}$$

This terminology has been introduced in [77], in homage of F. John who has first used such a condition in his work in elasticity [57].

Assume that Ω is a John domain with center X^* and consider an arbitrary point $Q \in \partial\Omega$. Pick a sequence $X_j \in \Omega$ converging to Q and denote by $\gamma_j : [0, \ell_j] \to \mathbb{R}^{n+1}$ the John path joining X_j with X^* in Ω. Note that, from (3.1.28), $0 < \ell_j \leq C_o \text{dist}\,(X^*, \partial\Omega)$ for every j. Thus, by passing to a subsequence, it can be assumed that $\ell_j \to \ell$ as $j \to \infty$. If we now renormalize each γ_j to $\tilde{\gamma}_j : [0,1] \to \mathbb{R}^{n+1}$, $\tilde{\gamma}_j(s) := \gamma_j(\ell_j s)$, a simple application of the Arzela-Ascoli compactness criterion then shows that, by eventually passing to a subsequence, the $\tilde{\gamma}_j$'s converge uniformly to a rectifiable path γ_Q joining the boundary point Q with the John center X^* in Ω and such that $\text{dist}\,(Z, \partial\Omega) \geq \theta|Z - Q|$ for each $Z \in \gamma_Q$, where $\theta = \theta(\Omega, X^*, Q) > 0$. For our purposes, we shall need a local, scale invariant version of this property, which we term *local John condition.* This is made precise in the definition below.

Definition 3.1.12 *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. This is said to satisfy a* `local John condition` *if there exist $\theta \in (0,1)$ and $R > 0$ (required to be ∞ if $\partial\Omega$ is unbounded), called the John constants of Ω, with the following significance. For every $Q \in \partial\Omega$ and $r \in (0, R)$ one can find $Q_r \in B(Q, r) \cap \Omega$, called John center relative to $\Delta(Q, r) := B(Q, r) \cap \partial\Omega$, such that $B(Q_r, \theta r) \subset \Omega$ and with the property that for each $X \in \Delta(Q, r)$ one can find a rectifiable path $\gamma_X : [0,1] \to \bar{\Omega}$, whose length is $\leq \theta^{-1} r$ and such that*

$$\gamma_X(0) = X, \quad \gamma_X(1) = Q_r, \quad \text{dist}\,(\gamma_X(t), \partial\Omega) > \theta\,|\gamma_X(t) - X| \qquad \forall\, t > 0. \tag{3.1.29}$$

Finally, Ω is said to satisfy a `two-sided local John condition` *if both Ω and $\mathbb{R}^{n+1} \setminus \bar{\Omega}$ satisfy a local John condition.*

Clearly, any domain satisfying a local John condition also satisfies a corkscrew condition. In the opposite direction, we wish to point out that any NTA domain satisfies a local John condition. In fact, the following stronger result holds.

Lemma 3.1.13 *Let $\Omega \subset \mathbb{R}^{n+1}$ be an NTA domain with constants M, R. Suppose that $X \in \Omega$, $Y \in \partial\Omega$ and that $r \in (0, R)$, $C > 1$ are such that $B(X, r) \subset B(Y, Cr) \cap \Omega$. Then there exists $C_o > 1$ which depends only on C and the NTA constants of Ω along with a rectifiable path $\gamma(X, Y)$ of length $\leq C_o r$, joining X with Y in Ω, and such that dist$(Z, \partial\Omega) \geq C_o^{-1}|Z - Y|$ for each point $Z \in \gamma(X, Y)$.*

In particular, any NTA domain satisfies a local John condition.

Proof. To justify the existence of such a path, set $X_0 := X$ and, for $j = 1, 2, ...,$ let X_j denote a corkscrew point relative to Y at scale $\approx 2^{-j}r$. It is then not difficult to check, with the help of the Harnack chain condition, that there exist a number $N_o \in \mathbb{N}$ and a constant $C_1 > 1$ (both depending only on the NTA constants of Ω and the constant C in the statement of the lemma) with the property that, for each $j \in \mathbb{N}$, one can find a family of balls $\{B_k\}_{1 \leq k \leq N}$, with $N \leq N_o$, of radii $\approx 2^{-j}r$ such that $CB_k \subset \Omega$, $X_{j-1} \in B_1$, $X_j \in B_N$, and $B_k \cap B_{k+1} \neq \emptyset$ for $k \in \{1, ..., N-1\}$. Consequently, there exists a polygonal path γ_j from X_{j-1} to X_j which stays roughly at distance $2^{-j}r$ from $\partial\Omega$, and has length $\leq C_2 2^{-j}r$, for some $C_2 = C_2(M, R, C) > 1$. If we now take $\gamma(X, Y)$ to be the union of the paths γ_j, $j \in \mathbb{N}$, it follows that $\gamma(X, Y)$ is rectifiable, and has length $\leq C_3 \sum_{j=1}^{\infty} 2^{-j}r = C_3 r$, for some geometrical constant $C_3 > 1$. Furthermore, if $Z \in \gamma(X, Y)$, say $Z \in \gamma_j$ for some $j \in \mathbb{N}$, then on the one hand $\mathrm{dist}\,(Z, \partial\Omega) \geq C_4 2^{-j}r$, while on the other hand $|Z - Y| \leq |Z - X_j| + |X_j - Y| \leq \mathrm{length}\,(\gamma_j) + C_5\,\mathrm{dist}\,(X_j, \partial\Omega) \leq C_6 2^{-j}r$. Altogether, $|Z - Y| \leq C_o\,\mathrm{dist}\,(Z, \partial\Omega)$ for some $C_o > 1$, finishing the proof of the lemma. □

From Definition 3.1.12 and Corollary 3.1.9 we also have:

Corollary 3.1.14 *Let $\Omega \subseteq \mathbb{R}^{n+1}$ be a domain satisfying a two-sided local John condition and whose boundary is Ahlfors regular. Then Ω is a UR domain, of locally finite perimeter.*

In the last part of this subsection we shall show that any BMO_1 domain is NTA, which appears to be folklore. To set the stage, recall that a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ belongs to Zygmund's $\Lambda_*(\mathbb{R}^n)$ class if

$$\|\varphi\|_{\Lambda_*(\mathbb{R}^n)} := \sup_{x,h \in \mathbb{R}^n} \frac{|\varphi(x+h) + \varphi(x-h) - 2\varphi(x)|}{|h|} < \infty. \tag{3.1.30}$$

A typical example of a function in $\Lambda_*(\mathbb{R}^n)$ is Weierstrass' nowhere differentiable function

$$\sum_{j=0}^{\infty} \frac{\sin(\pi 2^j x)}{2^j}, \qquad x \in \mathbb{R}. \tag{3.1.31}$$

Going further, Zygmund's $\lambda_*(\mathbb{R}^n)$ class is the collection of functions $\varphi \in \Lambda_*(\mathbb{R}^n)$ satisfying

$$\lim_{h \to 0} \frac{|\varphi(x+h) + \varphi(x-h) - 2\varphi(x)|}{|h|} = 0 \tag{3.1.32}$$

uniformly in x. The space $\lambda_*(\mathbb{R}^n)$ contains functions which are quite irregular, such as

$$\sum_{j=1}^{\infty} \frac{\cos(2^j x)}{2^j \sqrt{j}}, \qquad x \in \mathbb{R}, \tag{3.1.33}$$

which is almost everywhere not differentiable (cf. p. 47 in [119]).

For a function $\varphi : \mathbb{R}^n \to \mathbb{R}$, consider next the following conditions

$$\sup_{B \text{ ball}} \Big(\inf_{L \text{ affine}} \frac{\|\varphi - L\|_{L^\infty(B)}}{|B|^{1/n}} \Big) < \infty, \tag{3.1.34}$$

$$\lim_{r\to 0} \Big\{ \sup_{B \text{ ball of radius } \leq r} \Big(\inf_{L \text{ affine}} \frac{\|\varphi - L\|_{L^\infty(B)}}{|B|^{1/n}} \Big) \Big\} = 0. \tag{3.1.35}$$

It is then clear that (3.1.34) implies (3.1.30), and that (3.1.34) together with (3.1.35) imply (3.1.32). In fact, the opposite implications are also valid. Indeed, a good affine approximation to the graph of φ near $(x, \varphi(x))$ at scale $r > 0$ can be constructed as follows. Fix $\eta \in C_0^\infty(\mathbb{R}^n)$, nonnegative, even, with $\int \eta = 1$, and consider $L(z) := \varphi(x) + \langle \nabla(\varphi * \eta_r)(x), z - x\rangle$, $z \in \mathbb{R}^n$. Then, according to Lemma 3.7 on p. 94 in [55] and its proof,

$$\sup_{z\in B(x,r)} \frac{|\varphi(z) - L(z)|}{r} \leq C \sup_{z\in B(x,r)} \sup_{|h|<r} \frac{|\varphi(z+h) + \varphi(z-h) - 2\varphi(z)|}{|h|} \leq C\|\varphi\|_{\Lambda_*(\mathbb{R}^n)}. \tag{3.1.36}$$

The desired conclusions follow from this double inequality.

For $1 \leq p < \infty$, define the local L^p-oscillations of a function $f \in L^1_{loc}(\mathbb{R}^n)$ as

$$\{f\}_{p,r} := \sup_{B \text{ ball of radius } \leq r} \Big(⨍_B \Big| f(x) - \Big(⨍_B f \Big) \Big|^p \, dx \Big)^{1/p}, \qquad r > 0, \tag{3.1.37}$$

so that, for each fixed $p \in [1, \infty)$, we have $\|f\|_* \approx \sup_{r>0} \{f\}_{p,r}$ by the John-Nirenberg inequality. Also, using this and the fact that $\mathrm{VMO}(\mathbb{R}^n)$ is the closure in $\mathrm{BMO}(\mathbb{R}^n)$ of the space of uniformly continuous functions belonging to $\mathrm{BMO}(\mathbb{R}^n)$, it can be checked that $f \in \mathrm{VMO}(\mathbb{R}^n)$ if and only if $f \in \mathrm{BMO}(\mathbb{R}^n)$ and $\lim_{r\to 0} \{f\}_{p,r} = 0$. Let us also note here a well-known estimate relating integral averages over concentric balls of different radii in concert with Hölder's inequality imply

$$\Big| ⨍_{B(x,R)} f(y)\, dy - ⨍_{B(x,r)} f(y)\, dy \Big| \leq C_n \Big(1 + \log\Big(\frac{R}{r}\Big)\Big) \{f\}_{p,R}, \tag{3.1.38}$$

whenever $x \in \mathbb{R}^n$, $0 < r < R < \infty$ and $1 \leq p < \infty$.

Proposition 3.1.15 *The following inclusions*

$$\mathrm{BMO}_1(\mathbb{R}^n) \hookrightarrow \Lambda_*(\mathbb{R}^n), \tag{3.1.39}$$

$$\mathrm{VMO}_1(\mathbb{R}^n) \hookrightarrow \lambda_*(\mathbb{R}^n), \tag{3.1.40}$$

are well-defined and continuous.

Proof. Assume that $\varphi \in \mathrm{BMO}_1(\mathbb{R}^n)$ and fix $x \in \mathbb{R}^n$, $r > 0$, arbitrary. Set

$$m(x,r) := ⨍_{B(x,r)} \nabla\varphi(y)\, dy, \qquad L(y) := \varphi(x) + \langle m(x, 4r), y - x\rangle, \quad y \in \mathbb{R}^n. \tag{3.1.41}$$

Next, fix two parameters, $\varepsilon \in (0,1)$ and $p > n$. In an analogous fashion to (2.5.9), for any $y \in B(x,r)$ we may then estimate

$$\begin{aligned}
\frac{|\varphi(y) - L(y)|}{r} &= \frac{|\varphi(y) - \varphi(x) - \langle m(x,4r), y-x\rangle|}{r} \\
&\leq C_{p,n}\frac{|x-y|}{r}\Big(⨍_{|x-z|\leq 2|x-y|} |\nabla\varphi(z) - m(x,4r)|^p\,dz\Big)^{1/p} \\
&\leq C_{p,n}\frac{|x-y|}{r}\Big(⨍_{|x-z|\leq 2|x-y|} |\nabla\varphi(z) - m(x,2|x-y|)|^p\,dz\Big)^{1/p} \\
&\quad + C_{p,n}\frac{|x-y|}{r}|m(x,2|x-y|) - m(x,4r)| \\
&\leq C_{p,n}\frac{|x-y|}{r}\{\nabla\varphi\}_{p,4r}\Big(1 + \log\Big(\frac{2r}{|x-y|}\Big)\Big) \\
&\leq C_{p,n,\varepsilon}\{\nabla\varphi\}_{p,4r}\Big(\frac{|x-y|}{r}\Big)^{1-\varepsilon} \leq C_{p,n,\varepsilon}\{\nabla\varphi\}_{p,4r}, \qquad (3.1.42)
\end{aligned}$$

by using Mary Weiss's lemma (cf. [50], Lemma 2.10) and referring (twice) to (3.1.38). Hence, $|\varphi(y) - L(y)| \leq Cr\|\nabla\varphi\|_*$ and, further,

$$\sup_{B \text{ ball of radius } r}\left(\inf_{L \text{ affine}} \frac{\|\varphi - L\|_{L^\infty(B)}}{r}\right) \leq C\|\nabla\varphi\|_*. \qquad (3.1.43)$$

This proves (3.1.39). Since with the help of (3.1.42) one can readily check that (3.1.35) holds, we may conclude that the inclusion (3.1.40) is also well-defined and bounded. □

According to Proposition 3.6 on p. 94 in [55], any Λ_*-domain, i.e., a set of the form

$$\Omega := \{X = (x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > \varphi(x)\} \qquad (3.1.44)$$

with $\varphi \in \Lambda_*(\mathbb{R}^n)$, is NTA. As a result of this and Proposition 3.1.15, any BMO_1 domain is NTA.

We conclude this section with a review of sufficient conditions guaranteeing that the harmonic measure of a domain is absolutely continuous with respect to its surface measure.

Proposition 3.1.16 *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, with an Ahlfors regular boundary $\partial\Omega$ which satisfies the "two disks" condition introduced in Proposition 3.1.5. In addition, assume that Ω satisfies an interior corkscrew condition and the Harnack chain condition (cf. Definition 3.1.8). Fix $X_o \in \Omega$ and denote by ω^{X_o} the harmonic measure on $\partial\Omega$ (relative to Ω) with pole at X_o.*

Then ω^{X_o} belongs to the Muckenhoupt class A_∞ with respect to $\sigma := \mathcal{H}^n\lfloor\partial\Omega$. In particular, ω^{X_o} and σ are mutually absolutely continuous.

With the two disks condition replaced by a two-sided corkscrew condition and when $\mathbb{R}^n \setminus \partial\Omega$ has precisely two connected components, this has been first obtained in [104]. In the current format, the above result appears as Theorem 2 on p. 842 in [31]. A result of a similar flavor, when the Harnack chain condition is suppressed, has been established by B. Bennewitz and J. Lewis in [5]. Their Theorem 1 entails the following:

Proposition 3.1.17 *In the context of Proposition 3.1.16, the mutual absolute continuity of ω^{X_o} and σ remains valid even when the assumption that Ω satisfies the Harnack chain condition is dropped.*

3.2 First estimates on layer potentials

The purpose of this subsection is to provide nontangential maximal function estimates for a class of layer potentials. To be concrete, take a function

$$k \in C^N(\mathbb{R}^{n+1} \setminus \{0\}) \quad \text{with} \quad k(-X) = -k(X) \quad \text{and} \quad k(\lambda X) = \lambda^{-n} k(X) \ \ \forall\, \lambda > 0, \tag{3.2.1}$$

and define the singular integral operator

$$\mathcal{T} f(X) := \int_{\partial\Omega} k(X - Y) f(Y)\, d\sigma(Y), \qquad X \in \Omega, \tag{3.2.2}$$

as well as

$$T_* f(X) := \sup_{\varepsilon > 0} |T_\varepsilon f(X)|, \qquad X \in \partial\Omega, \quad \text{where} \tag{3.2.3}$$

$$T_\varepsilon f(X) := \int_{\substack{Y \in \partial\Omega \\ |X-Y| > \varepsilon}} k(X - Y) f(Y)\, d\sigma(Y), \qquad X \in \partial\Omega. \tag{3.2.4}$$

The following result was established in Proposition 4 bis of [28].

Proposition 3.2.1 *Assume $\Omega \subset \mathbb{R}^{n+1}$ is a UR domain. Take $p \in (1, \infty)$. There exist $N \in \mathbb{Z}_+$ and $C \in (0, \infty)$, each depending only on p along with the Ahlfors regularity and UR constants of $\partial\Omega$, with the following property. If k satisfies (3.2.1), then*

$$\|T_* f\|_{L^p(\partial\Omega, d\sigma)} \le C \|k|_{S^n}\|_{C^N} \|f\|_{L^p(\partial\Omega, d\sigma)} \tag{3.2.5}$$

for each $f \in L^p(\partial\Omega, d\sigma)$.

Remark. We often refer to "geometrical characteristics" of Ω as the collection of Ahlfors regularity and UR constants, and use the notation $G(\Omega)$, so C above has the form $C = C(G(\Omega), p)$. In other settings, $G(\Omega)$ might involve other geometrical characteristics, such as those appearing to define the John condition or the NTA condition.

To help put matters into proper perspective it is worth recalling that, with Ahlfors regularity as a background assumption, the validity of (3.2.5) for all kernels k as in (3.2.1) is actually equivalent to $\partial\Omega$ being uniformly rectifiable. See the theorem on pp. 10-14 in [33]. In this connection, a significant open problem is to show that, for the left-to-right implication, it suffices to consider only the Riesz kernels, i.e. $k_j(X) := x_j / |X|^{n+1}$, $1 \le j \le n+1$. Regarding the nature of the principal value integrals associated with the Riesz kernels, a recent result from [113] states that if $E \subset \mathbb{R}^{n+1}$ has $\mathcal{H}^n(E) < \infty$ then

$$E \text{ is countably rectifiable (of dimension } n) \iff \qquad (3.2.6)$$

$$\lim_{\varepsilon\to 0^+} \int\limits_{Y\in E:\,|Y-X|>\varepsilon} \frac{x_j-y_j}{|X-Y|^{n+1}}\,d\mathcal{H}^n(Y) \text{ exists for } \mathcal{H}^n\text{-a.e. } X\in E.$$

We will concern ourselves with issues pertaining to the existence of principal value singular integral operators later on, in § 3.3-§ 3.4.

Moving on, let $L^{1,\infty}(\partial\Omega, d\sigma)$ stand for the weak-L^1 space on $\partial\Omega$, i.e. the collection of all σ-measurable functions f on $\partial\Omega$ for which $\|f\|_{L^{1,\infty}(\partial\Omega,d\sigma)} := \sup_{\lambda>0}\Big[\lambda\,\sigma(\{X\in\partial\Omega:\,|f(X)|>\lambda\})\Big]$ is finite. Corresponding to the case $p=1$ in (3.2.5), we have the following.

Proposition 3.2.2 *In the context of Proposition 3.2.1, there also holds*

$$\|T_*f\|_{L^{1,\infty}(\partial\Omega,d\sigma)} \leq C(\Omega,k)\|f\|_{L^1(\partial\Omega,d\sigma)} \qquad (3.2.7)$$

for each $f\in L^1(\partial\Omega,d\sigma)$.

Proof. Essentially, this is a consequence of Proposition 3.2.1 and standard Calderón-Zygmund theory. For the benefit of the reader, we include a brief sketch. The departure point is the estimate

$$\|T_\varepsilon f\|_{L^{1,\infty}(\partial\Omega,d\sigma)} \leq C(\Omega,k)\|f\|_{L^1(\partial\Omega,d\sigma)}, \qquad (3.2.8)$$

uniformly for $\varepsilon>0$, itself a consequence of Proposition 3.2.1 and the Calderón-Zygmund decomposition lemma (that the latter continues to hold in the context of spaces of homogeneous type is well-known; see, e.g., [23]). Next, a variant of the classical Cotlar lemma gives the following. For each $\gamma\in(0,1)$ there exists a constant C, depending only on $\partial\Omega$, k and γ with the property that, for each $f\in L^1(\partial\Omega,d\sigma)$ and $X\in\partial\Omega$,

$$T_{*,\varepsilon}f(X) \leq C\mathcal{M}f(X) + C\mathcal{M}_\gamma(T_{\varepsilon_0}f)(X), \qquad \forall\,\varepsilon,\,\varepsilon_0, \text{ with } \varepsilon>\varepsilon_0>0, \qquad (3.2.9)$$

where $\mathcal{M}$ is the usual Hardy-Littlewood maximal function,

$$T_{*,\varepsilon}f(X) := \sup_{\varepsilon'>\varepsilon}|T_{\varepsilon'}f(X)|, \qquad X\in\partial\Omega, \qquad (3.2.10)$$

and

$$\mathcal{M}_\gamma f(X) := \sup_R\Big(⨍_{\Delta(X,R)}|f|^\gamma\,d\sigma\Big)^{1/\gamma}, \qquad X\in\partial\Omega. \qquad (3.2.11)$$

With (3.2.8)-(3.2.9) in hand, (3.2.5) follows from the mapping properties of the Hardy-Littlewood maximal function; see, e.g., pp. 250-251 in [82] for a related discussion. □

We further complement Proposition 3.2.1 with the following nontangential maximal function estimate.

Proposition 3.2.3 *In the setting of Proposition 3.2.1, for each $\alpha > 0$ there exists a finite constant $C > 0$, depending only on p, α, as well as the Ahlfors regularity and UR constants of $\partial\Omega$, such that*

$$\|\mathcal{N}(\mathcal{T}f)\|_{L^p(\partial\Omega,d\sigma)} \leq C\|k|_{S^n}\|_{C^N}\|f\|_{L^p(\partial\Omega,d\sigma)}. \tag{3.2.12}$$

Moreover, corresponding to $p = 1$,

$$\|\mathcal{N}(\mathcal{T}f)\|_{L^{1,\infty}(\partial\Omega,d\sigma)} \leq C(\Omega,k,\alpha)\|f\|_{L^1(\partial\Omega,d\sigma)}. \tag{3.2.13}$$

Proof. To begin, assume that $X \in \Omega$ and $Z \in \partial\Omega$ are fixed points such that

$$|X - Z| < (1+\alpha)\,\mathrm{dist}\,(X,\partial\Omega). \tag{3.2.14}$$

Set $\varepsilon := |X - Z|$ and estimate

$$\begin{aligned}
|\mathcal{T}f(X) - T_{2\varepsilon}f(Z)| &= \Big|\int_{\partial\Omega} k(X-Y)f(Y)\,d\sigma(Y) - \int_{\substack{Y\in\partial\Omega \\ |Z-Y|>2\varepsilon}} k(Z-Y)f(Y)\,d\sigma(Y)\Big| \\
&\leq \Big|\int_{\substack{Y\in\partial\Omega \\ |Z-Y|<2\varepsilon}} k(X-Y)f(Y)\,d\sigma(Y)\Big| \\
&\quad + \Big|\int_{\substack{Y\in\partial\Omega \\ |Z-Y|>2\varepsilon}} \Big(k(X-Y) - k(Z-Y)\Big)f(Y)\,d\sigma(Y)\Big| \\
&=: I + II.
\end{aligned} \tag{3.2.15}$$

Since

$$|X - Y| \geq \mathrm{dist}\,(X,\partial\Omega) > \frac{|X-Z|}{1+\alpha} = \frac{\varepsilon}{1+\alpha}, \tag{3.2.16}$$

we have

$$|I| \leq \frac{C(\alpha,n)}{\varepsilon^n} \int_{\substack{Y\in\partial\Omega \\ |Z-Y|<2\varepsilon}} |f(Y)|\,d\sigma(Y) \leq C(\alpha,\Omega)\mathcal{M}f(Z), \tag{3.2.17}$$

by Ahlfors regularity. On the other hand, if $X \in \Gamma(Z)$, $Y \in \partial\Omega$ and $|Y - Z| > 2\varepsilon = 2|X - Z|$, we also have $|tX + (1-t)Z - Y| \geq C|Z - Y|$ for $t \in [0,1]$, and hence the Mean Value Theorem gives

$$\begin{aligned}
|k(X-Y) - k(Z-Y)| &\leq |X - Z| \sup_{0\leq t\leq 1} |\nabla k(tX + (1-t)Z - Y)| \\
&\leq C\frac{\varepsilon}{|Z-Y|^{n+1}}.
\end{aligned} \tag{3.2.18}$$

Consequently, a familiar argument gives

$$
\begin{aligned}
|II| &\leq \int_{\substack{Y\in\partial\Omega \\ |Z-Y|>2\varepsilon}} \frac{\varepsilon}{|Z-Y|^{n+1}}|f(Y)|\,d\sigma(Y) \\
&= c_n\sum_{j=1}^{\infty}\int_{\substack{Y\in\partial\Omega \\ 2^j\varepsilon<|Z-Y|<2^{j+1}\varepsilon}} \frac{\varepsilon}{|Z-Y|^{n+1}}|f(Y)|\,d\sigma(Y) \\
&\leq c_n\sum_{j=1}^{\infty}2^{-j}(2^{j+1}\varepsilon)^{-n}\int_{\substack{Y\in\partial\Omega \\ |Z-Y|<2^{j+1}\varepsilon}}|f(Y)|\,d\sigma(Y) \\
&\leq C(\Omega)\sum_{j=1}^{\infty}2^{-j}\mathcal{M}f(Z)=C(\Omega)\mathcal{M}f(Z),
\end{aligned}
\tag{3.2.19}
$$

where the next-to-the-last step utilizes Ahlfors regularity.

In summary, the above analysis proves that for any two points, $X\in\Omega$ and $Z\in\partial\Omega$, such that (3.2.14) holds we have

$$
|\mathcal{T}f(X)-T_{2\varepsilon}f(Z)|\leq C(\alpha,\Omega)\mathcal{M}f(Z),
\tag{3.2.20}
$$

which further entails

$$
|\mathcal{T}f(X)|\leq|T_*f(Z)|+C\mathcal{M}f(Z).
\tag{3.2.21}
$$

For each $Z\in\partial\Omega$ fixed, with the property that $\Gamma(Z)\neq\emptyset$, by taking the supremum in $X\in\Gamma(Z)$ in (3.2.21) we arrive at

$$
\mathcal{N}(\mathcal{T}f)(Z)\leq T_*f(Z)+C\mathcal{M}f(Z).
\tag{3.2.22}
$$

Since the above estimate trivially valid when $\Gamma(Z)=\emptyset$, it follows that (3.2.22) holds for all $Z\in\partial\Omega$.

Now (3.2.12) is a consequence of this, (3.2.5) and Proposition 2.1.1. Finally, (3.2.13) follows from (3.2.22), (2.1.4) and Proposition 3.2.2. □

Recall next the Hardy spaces introduced in §2.4.

Proposition 3.2.4 *In the context of Proposition 3.2.1, for each* $p\in(\frac{n}{n+1},1]$ *and* $\alpha>0$ *there exists a finite constant* $C=C(p,\alpha,\Omega)$ *such that*

$$
\|\mathcal{N}(\mathcal{T}f)\|_{L^p(\partial\Omega,d\sigma)}\leq C\|k|_{S^n}\|_{C^N}\|f\|_{H^p_{at}(\partial\Omega,d\sigma)}.
\tag{3.2.23}
$$

Proof. It suffices to estimate $\mathcal{N}(\mathcal{T}a)$ when a is a p-atom, i.e. it satisfies (2.4.11). First, for any $\kappa>1$, the estimate (3.2.12) with $p=2$ gives

$$\int_{B(X_o,\kappa r)\cap\partial\Omega} |\mathcal{N}(\mathcal{T}a)|^p\, d\sigma \leq \|\mathcal{N}(\mathcal{T}a)\|^p_{L^2(\partial\Omega,d\sigma)}\cdot\sigma(\Delta(X_o,\kappa r))^{1-2/p}$$

$$\leq C\|a\|^p_{L^2(\partial\Omega,d\sigma)}\cdot\sigma(\Delta(X_o,\kappa r))^{1-2/p}$$

$$\leq C(p,\kappa,\Omega) < +\infty, \tag{3.2.24}$$

by (2.4.11) and Ahlfors regularity.

On the other hand, if $X\in\Omega$ is such that $|X-X_o|\geq 2r$ then, based on the vanishing moment condition for the atom and the Mean Value Theorem, we may write

$$|\mathcal{T}a(X)| \leq \int_{B(X_o,r)} |k(X-Y)-k(X-X_o)||a(Y)|\, d\sigma(Y)$$

$$\leq C\frac{r}{|X-X_o|^{n+1}}\sigma(\Delta(X_o,r))^{1-1/p}. \tag{3.2.25}$$

Now, if $Y\in\partial\Omega$, $X\in\Omega$ are such that

$$|Y-X_o|\geq 2(2+\alpha)r \quad\text{and}\quad |X-Y|<(1+\alpha)\mathrm{dist}\,(X,\partial\Omega) \tag{3.2.26}$$

then

$$2(2+\alpha)r\leq|Y-X_o|\leq|Y-X|+|X-X_o|\leq(2+\alpha)|X-X_o|. \tag{3.2.27}$$

In particular,

$$|X-X_o|\geq 2r \quad\text{and}\quad |X-X_o|\geq\frac{1}{2+\alpha}|Y-X_o|, \tag{3.2.28}$$

so that (3.2.25) gives

$$|\mathcal{N}(\mathcal{T}a)(Y)|\leq C\frac{r}{|Y-X_o|^{n+1}}\sigma(\Delta(X_o,r))^{1-1/p}, \qquad \forall\, Y\in\partial\Omega\setminus\Delta(X_o,(2+\alpha)r). \tag{3.2.29}$$

Consequently, if we set $R:=(2+\alpha)r$ then (3.2.29) and the defining condition for Ahlfors regularity give

$$\int_{\partial\Omega\setminus\Delta(X_o,R)} |\mathcal{N}(\mathcal{T}a)(Y)|^p\, d\sigma(Y)$$

$$\leq Cr^p\sigma(\Delta(X_o,r))^{p-1}\sum_{j=0}^{\infty}\int_{\Delta(X_o,2^{j+1}R)\setminus\Delta(X_o,2^jR)}\frac{d\sigma(Y)}{|Y-X_o|^{p(n+1)}}$$

$$\leq Cr^p\sigma(\Delta(X_o,r))^{p-1}\sum_{j=0}^{\infty}(2^jR)^n\cdot(2^jR)^{-p(n+1)}$$

$$\leq C(p,\alpha,\Omega)<+\infty, \tag{3.2.30}$$

since $p(n+1) > n$. If we now choose $\kappa := 2+\alpha$, it follows that $\|\mathcal{N}(\mathcal{T}a)\|_{L^p(\partial\Omega,d\sigma)} \leq C$ for some finite constant $C > 0$, independent of the atom. This readily yields (3.2.23), finishing the proof of the proposition. $\square$

It is also useful to have estimates of the form

$$\mathcal{T} : L^p(\partial\Omega, d\sigma) \longrightarrow L^r(\Omega), \tag{3.2.31}$$

when $\mathcal{T}$ has the form (3.2.2). We present two ways to do this. One is to combine Proposition 3.2.3 with a result of the form

$$\|u\|_{L^r(\Omega)} \leq C\|\mathcal{N}u\|_{L^p(\partial\Omega,d\sigma)}. \tag{3.2.32}$$

We give such an estimate in Proposition 3.2.7 below. First, we tackle (3.2.31) by interpolating between cases $p = 1$ and $p = \infty$. This method is elementary and while not quite as sharp as the result derivable from Proposition 3.2.7, it applies to a larger class of domains, which will be useful in Sections 5–7.

Proposition 3.2.5 *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded, open, Ahlfors regular domain, with the property that $\mathcal{H}^n(\partial\Omega\backslash\partial_*\Omega) = 0$. Let $K : \overline{\Omega}\times\partial\Omega \to \mathbb{R}$ be continuous on the complement of $\{(X,X) : X \in \partial\Omega\}$ and satisfy*

$$|K(X,Y)| \leq C|X-Y|^{-n}, \tag{3.2.33}$$

and set

$$\mathcal{T}f(X) = \int_{\partial\Omega} K(X,Y)f(Y)\,d\sigma(Y), \quad X \in \Omega. \tag{3.2.34}$$

Then, for $p \geq 1$, (3.2.31) holds for all $r < p(n+1)/n$.

Proof. It is elementary from bounds

$$\|K(\cdot,Y)\|_{L^q(\Omega)} \leq C_q, \quad q < \frac{n+1}{n} \tag{3.2.35}$$

that

$$\mathcal{T} : L^1(\partial\Omega, d\sigma) \longrightarrow L^q(\Omega), \quad \forall\, q < \frac{n+1}{n}. \tag{3.2.36}$$

For this one needs only Ω bounded and $\mathcal{H}^n(\partial\Omega) < \infty$. We will show that

$$\mathcal{T} : L^\infty(\partial\Omega, d\sigma) \longrightarrow L^s(\Omega), \quad \forall\, s < \infty. \tag{3.2.37}$$

Then (3.2.31) follows by interpolation from (3.2.36)–(3.2.37).

To prove (3.2.37), we will establish an estimate of the form

$$\int_{\partial\Omega} |K(X,Y)|\,d\sigma(Y) \leq \gamma(X), \quad \gamma \in L^s(\Omega), \quad \forall\, s < \infty. \tag{3.2.38}$$

Then

$$\Big|\int_{\partial\Omega} K(X,Y)f(Y)\,d\sigma(Y)\Big| \leq \|f\|_{L^\infty(\partial\Omega,d\sigma)}\,\gamma(X), \tag{3.2.39}$$

and (3.2.37) follows. Here is part of (3.2.38).

Lemma 3.2.6 *In the setting of Proposition 3.2.5,*

$$\int_{\partial\Omega} |K(X,Y)|\, d\sigma(Y) \le C \,\log \frac{2M}{\mathrm{dist}(X,\partial\Omega)}, \tag{3.2.40}$$

with $M = \mathrm{diam}\,\Omega$.

Proof. Given $X \in \Omega$, $\delta = \mathrm{dist}(X, \partial\Omega)$, take $P \in \partial\Omega$ with $|X - P| = \delta$ and set

$$\begin{aligned} \mathcal{A}_0 &:= \{X' \in \partial\Omega :\, |X' - P| \le 2\delta\}, \\ \mathcal{A}_k &:= \{X' \in \partial\Omega :\, |X' - P| \in (2^k\delta, 2^{k+1}\delta]\}, \quad k \ge 1. \end{aligned} \tag{3.2.41}$$

Ahlfors regularity of $\partial\Omega$ yields

$$\mathcal{H}^n(\mathcal{A}_k) \le C(2^k\delta)^n, \tag{3.2.42}$$

so

$$\int_{\mathcal{A}_k} |K(X,Y)|\, d\sigma(Y) \le C\frac{(2^k\delta)^n}{(2^k\delta)^n} = C. \tag{3.2.43}$$

Summing (3.2.43) over $k \ge 0$ such that $2^k\delta \le M$ gives (3.2.40). $\square$

We now present the

End of proof of Proposition 3.2.5. The fact that $\gamma(X) = C\,\log 2M/\mathrm{dist}(X,\partial\Omega)$ belongs to $L^s(\Omega)$ for all $s < \infty$ follows immediately from the estimate

$$\mathrm{vol}\,(\mathcal{O}_\delta) \le C\delta, \tag{3.2.44}$$

for

$$\mathcal{O}_\delta = \{X \in \overline{\Omega} :\, \mathrm{dist}\,(X, \partial\Omega) \le \delta\}, \tag{3.2.45}$$

valid when Ω is bounded and Ahlfors regular. The estimate (3.2.44) has a short, elementary proof, but rather than give it we note that it follows upon taking $v \equiv 1$ in (2.3.10). $\square$

We now present a sharp estimate of the form (3.2.32). Specifically, we have

Proposition 3.2.7 *Let* $\Omega \subset \mathbb{R}^{n+1}$ *be an Ahlfors regular domain which is either bounded, or has an unbounded boundary. Then for each* $p \in (0,\infty)$*, (3.2.32) holds with* $r = p(n+1)/n$*, i.e.,*

$$\|u\|_{L^{p(n+1)/n}(\Omega)} \le C\|\mathcal{N}u\|_{L^p(\partial\Omega, d\sigma)}, \tag{3.2.46}$$

for some geometrical constant C*, independent of the function* $u : \Omega \to \mathbb{R}$.

Proof. To prove the proposition, fix $p \in (0, \infty)$ and assume that u is such that $\|\mathcal{N}u\|_{L^p(\partial\Omega, d\sigma)} < \infty$. We will first show that there exists a geometric constant $C > 0$ with the property that

$$|\{X \in \Omega : |u(X)| > \lambda\}| \leq C\sigma(\{Q \in \partial\Omega : \mathcal{N}u(Q) > \lambda\})^{(n+1)/n}, \qquad \forall\, \lambda > 0, \tag{3.2.47}$$

where $|E|$ denotes the $(n+1)$-dimensional Euclidean measure of a measurable set $E \subseteq \mathbb{R}^{n+1}$. The strategy for proving (3.2.47) is to work with "tent" regions

$$T(\mathcal{O}) := \Omega \setminus \Big[\bigcup_{P \in \partial\Omega \setminus \mathcal{O}} \Gamma_\alpha(P)\Big] \tag{3.2.48}$$

associated with arbitrary open subsets $\mathcal{O}$ of $\partial\Omega$. The ingredients going into the proof of (3.2.46) are the simple inclusion (itself a consequence of (3.2.48))

$$\{X \in \Omega :\, |u(X)| > \lambda\} \subseteq T(\{Q \in \partial\Omega : \mathcal{N}u(Q) > \lambda\}), \qquad \forall\, \lambda > 0, \tag{3.2.49}$$

and the geometric estimate

$$|T(\mathcal{O})| \leq C\sigma(\mathcal{O})^{(n+1)/n}, \quad \forall\, \mathcal{O} \text{ proper open subset of } \partial\Omega. \tag{3.2.50}$$

Then (3.2.47) follows writing (3.2.50) for $\mathcal{O} := \{Q \in \partial\Omega : \mathcal{N}u(Q) > \lambda\}$, which, given that $\mathcal{N}u \in L^p(\partial\Omega, d\sigma)$, is a proper open set of $\partial\Omega$ when $\partial\Omega$ is unbounded. When Ω is bounded, we can ensure that (3.2.50) also holds in the case when $\mathcal{O} = \partial\Omega$ by taking $C > \sigma(\partial\Omega)^{-(n+1)/n}|\Omega|$.

Our next task is to prove (3.2.50). To do this, fix a proper open subset $\mathcal{O}$ of $\partial\Omega$ and decompose $\mathcal{O}$ into a finite-overlap family of Whitney surface balls $\{\Delta_k\}$ (considering $\partial\Omega$ as a space of homogeneous type; see Theorem 3.1 and the footnote on p. 71 of [22] for details). Also, for each surface ball $\Delta := B_R(Q) \cap \partial\Omega$, $Q \in \partial\Omega$, $0 < R < \operatorname{diam}\Omega$, consider the Carleson region

$$\mathcal{C}_t(\Delta) := B_{tR}(Q) \cap \Omega, \tag{3.2.51}$$

where $t > 0$ is a large constant, to be specified later. We now claim that t can be chosen so that

$$T(\mathcal{O}) \subset \bigcup_k \mathcal{C}_t(\Delta_k). \tag{3.2.52}$$

In order to justify (3.2.52) we note that the definition (3.2.48) can be rephrased as

$$T(\mathcal{O}) = \{X \in \Omega :\, \operatorname{dist}(X, \mathcal{O}) \leq (1+\alpha)^{-1} \operatorname{dist}(X, \partial\Omega \setminus \mathcal{O})\}. \tag{3.2.53}$$

Let now X be an arbitrary point in $T(\mathcal{O})$ and, for some small $\varepsilon > 0$, pick $X^* \in \mathcal{O}$ such that

$$|X - X^*| \leq (1+\varepsilon) \operatorname{dist}(X, \mathcal{O}). \tag{3.2.54}$$

Then there exists an index k for which $X^* \in \Delta_k$ and we shall show that ε and t can be chosen so as to guarantee that

$$X \in \mathcal{C}_t(\Delta_k). \tag{3.2.55}$$

Indeed, assume $\Delta_k = B_{R_k}(Q_k) \cap \partial\Omega$ for some $Q_k \in \partial\Omega$ and $R_k \in (0, \operatorname{diam}\Omega)$, and write

$$
\begin{aligned}
|X-X^*| &\leq (1+\varepsilon)\,\mathrm{dist}\,(X,\mathcal{O}) \leq \frac{1+\varepsilon}{1+\alpha}\,\mathrm{dist}\,(X,\partial\Omega\setminus\mathcal{O}) \\
&\leq \frac{1+\varepsilon}{1+\alpha}\Big(|X-X^*|+\mathrm{dist}\,(X^*,\partial\Omega\setminus\mathcal{O})\Big) \\
&\leq \frac{1+\varepsilon}{1+\alpha}\Big(|X-X^*|+CR_k\Big).
\end{aligned}
\tag{3.2.56}
$$

Choosing $\varepsilon\in(0,\alpha)$, this now yields

$$
|X-X^*|\leq C\frac{1+\varepsilon}{\alpha-\varepsilon}R_k,
\tag{3.2.57}
$$

so that (3.2.55) holds provided we take $t>1+C(1+\varepsilon)/(\alpha-\varepsilon)$ to begin with.

Having established (3.2.52), we can finish the proof of (3.2.50) by estimating

$$
\begin{aligned}
|T(\mathcal{O})| &\leq \sum_k|\mathcal{C}_t(\Delta_k)|\leq C\sum_k R_k^{n+1}\leq C\sum_k\sigma(\Delta_k)^{(n+1)/n} \\
&\leq C\Big[\sum_k\sigma(\Delta_k)\Big]^{(n+1)/n}\leq C\sigma(\mathcal{O})^{(n+1)/n},
\end{aligned}
\tag{3.2.58}
$$

where the third inequality is based on the Ahlfors regularity of $\partial\Omega$. This justifies (3.2.50).

There remains to show how to use (3.2.47) in the derivation of (3.2.46). First, Chebysheff's inequality gives

$$
\lambda^p\,\sigma(\{Q\in\partial\Omega:\,\mathcal{N}u(Q)>\lambda\})\leq\|\mathcal{N}u\|^p_{L^p(\partial\Omega,d\sigma)},\qquad\forall\,\lambda>0.
\tag{3.2.59}
$$

Second, from (3.2.50) and (3.2.59), for each $\lambda>0$ we have

$$
\begin{aligned}
&\lambda^{-1+p(n+1)/n}\,|\{X\in\Omega:\,|u(X)|>\lambda\}| \\
&\qquad\leq C\lambda^{-1+p+p/n}\,\sigma(\{Q\in\partial\Omega:\,\mathcal{N}u(Q)>\lambda\})^{1+1/n} \\
&\qquad\leq C\|\mathcal{N}u\|^{p/n}_{L^p(\partial\Omega,d\sigma)}\,\lambda^{p-1}\,\sigma(\{Q\in\partial\Omega:\,\mathcal{N}u(Q)>\lambda\}).
\end{aligned}
\tag{3.2.60}
$$

At this stage, integrating the extreme sides of (3.2.60) over $\lambda\in(0,\infty)$ yields (3.2.46). $\square$

Remark. If Ω is a smoothly bounded domain in $\mathbb{R}^{n+1}$ and u is a harmonic function with, say $\mathcal{N}u\in L^2(\partial\Omega,d\sigma)$, then the global Sobolev regularity of u in Ω is, generally speaking, no better than $H^{1/2}(\Omega)$. This space further embeds into $L^{2(n+1)/n}(\Omega)$. Similar considerations apply to other values of $p\neq 2$, and, in this sense, the result above is sharp. Given its significance when Ω is bounded and $u\equiv 1$, the estimate (3.2.46) can be thought of as a *weighted isoperimetric inequality*, in which the functions u, $\mathcal{N}u$ play the role of weights.

Having established Proposition 3.2.7, we can now present a version of Theorem 2.3.1 which applies to domains which are not necessarily bounded.

Theorem 3.2.8 *Let $\Omega\subset\mathbb{R}^{n+1}$ be an open set which is either bounded or has an unbounded boundary. Assume that $\partial\Omega$ is Ahlfors regular and satisfies (2.3.1) (thus, in particular, Ω is of locally* *finite perimeter). As before, set $\sigma := \mathcal{H}^n\lfloor\partial\Omega$ and denote by ν the measure theoretic outward unit normal to $\partial\Omega$. Then Green's formula (2.3.2) holds for each vector field $v \in C^0(\Omega)$ that satisfies*

$$\begin{aligned} &\operatorname{div} v \in L^1(\Omega), \quad \mathcal{N}v \in L^1(\partial\Omega, d\sigma) \cap L^p_{loc}(\partial\Omega, d\sigma) \textit{ for some } p \in (1,\infty), \\ &\textit{and the pointwise nontangential trace } v\big|_{\partial\Omega} \textit{ exists } \sigma\textit{-a.e. on } \partial\Omega. \end{aligned} \tag{3.2.61}$$

Proof. Assume that $\partial\Omega$ is unbounded. As a first step we note that, as is apparent from a careful inspection of the proof of Theorem 2.3.1, formula (2.3.2) continues to hold for vector fields $v \in C^0(\Omega)$ satisfying

$$\begin{aligned} &\operatorname{div} v \in L^1(\Omega), \quad \mathcal{N}v \in L^p(\partial\Omega, d\sigma) \text{ for some } p \in (1,\infty), \\ &\text{the pointwise nontangential trace } v\big|_{\partial\Omega} \text{ exists } \sigma\text{-a.e.} \\ &\text{and } \operatorname{supp} v \text{ is a bounded subset of } \Omega. \end{aligned} \tag{3.2.62}$$

To continue, pick a function $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$ satisfying $\varphi \equiv 1$ on $B(0,1)$, $\varphi \equiv 0$ outside $B(0,2)$, and for each $R > 0$ set $\varphi_R(X) := \varphi(X/R)$. Hence, $\varphi_R \equiv 1$ on $B(0,R)$, $\varphi_R(X) \equiv 0$ outside $B(0,2R)$, $|\varphi_R(X)| \leq C$ and $|\nabla\varphi_R(X)| \leq C/R$ with $C > 0$ independent of X and R. Thanks to Proposition 3.2.7 and the fact that, by Proposition 2.3.2, Ω is weakly accessible, if $v \in C^0(\Omega)$ is a vector field satisfying (3.2.61) we have

$$\|v\|_{L^{(n+1)/n}(\Omega)} + \|v|_{\partial\Omega}\|_{L^1(\partial\Omega,d\sigma)} \leq C\|\mathcal{N}v\|_{L^1(\partial\Omega,d\sigma)} < +\infty. \tag{3.2.63}$$

Also, $\varphi_R v$ is as in (3.2.62) so for every $R > 0$ we may write

$$\int_{\partial\Omega} \langle \nu, v\rangle \varphi_R\, d\sigma = \int_\Omega \operatorname{div}(\varphi_R v)\, dX = \int_\Omega \Big(\varphi_R \operatorname{div} v + \langle \nabla\varphi_R, v\rangle\Big)\, dX. \tag{3.2.64}$$

It is then clear from (3.2.63) and hypotheses that

$$\int_{\partial\Omega} \langle \nu, v\rangle \varphi_R\, d\sigma \to \int_{\partial\Omega} \langle \nu, v\rangle\, d\sigma \quad \text{and} \quad \int_\Omega \varphi_R \operatorname{div} v\, dX \to \int_\Omega \operatorname{div} v\, dX \text{ as } R \to \infty. \tag{3.2.65}$$

Set $\Omega_R := \Omega \cap (B(0,2R) \setminus B(0,R))$ so that $\operatorname{supp}(\nabla\varphi_R) \subseteq \Omega_R$ and $|\Omega_R| \leq CR^{n+1}$, then estimate

$$\begin{aligned} \Big|\int_\Omega \langle \nabla\varphi_R, v\rangle\, dX\Big| &\leq \frac{C}{R}\int_{\Omega_R} |v(X)|\, dX \leq \frac{C}{R}\Big(\int_{\Omega_R} |v(X)|^{(n+1)/n}\, dX\Big)^{\frac{n}{n+1}} \cdot R^{(n+1)(1-\frac{n}{n+1})} \\ &= C\Big(\int_{\Omega_R} |v(X)|^{(n+1)/n}\, dX\Big)^{\frac{n}{n+1}} \longrightarrow 0, \quad \text{as } R \to \infty, \end{aligned} \tag{3.2.66}$$

by (3.2.63). Hence, passing to limit $R \to \infty$ in (3.2.64) yields (2.3.2). $\square$

For scalar-valued functions, the following version of Theorem 3.2.8 holds.

Corollary 3.2.9 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set which is either bounded or has an unbounded boundary. In addition, suppose that $\partial\Omega$ is Ahlfors regular and satisfies (2.3.1) Set $\sigma := \mathcal{H}^n\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, ..., \nu_{n+1})$ the measure theoretic outward unit normal to $\partial\Omega$. Then*

$$\int_\Omega (\partial_j u)(X)v(X)\,dX = -\int_\Omega (\partial_j v)(X)u(X)\,dX + \int_{\partial\Omega} \nu_j uv\,d\sigma \tag{3.2.67}$$

for each $j \in \{1, ..., n+1\}$ and each scalar-valued functions $u, v \in C^0(\Omega)$ with $\partial_j u, \partial_j v \in L^1_{loc}(\Omega)$ and which also satisfy

$$u\partial_j v,\ v\partial_j u \in L^1(\Omega), \quad \mathcal{N}(uv) \in L^1(\partial\Omega, d\sigma) \ \text{ and } \ (uv)\big|_{\partial\Omega} \text{ exists } \sigma\text{-a.e. on } \partial\Omega. \tag{3.2.68}$$

Proof. Granted the current hypotheses, it can be checked that in the sense of distributions

$$\partial_j(uv) = (\partial_j u)v + (\partial_j v)u \quad \text{ in } \ \Omega. \tag{3.2.69}$$

See the discussion on p. 210 in [2]. Then (3.2.67) follows as soon as we prove that

$$\int_\Omega (\partial_j w)(X)\,dX = \int_{\partial\Omega} \nu_j w\,d\sigma \tag{3.2.70}$$

for each $j \in \{1, ..., n+1\}$ and each scalar-valued function $w \in C^0(\Omega)$ which satisfies

$$\partial_j w \in L^1(\Omega), \quad \mathcal{N}w \in L^1(\partial\Omega, d\sigma) \ \text{ and } \ w\big|_{\partial\Omega} \text{ exists } \sigma\text{-a.e. on } \partial\Omega. \tag{3.2.71}$$

Indeed, it suffices to take $w := uv$ and invoke (3.2.69)-(3.2.70). To justify (3.2.70), pick a function $\varphi \in C_0^\infty((-2,2))$ with $\varphi \equiv 1$ on $(-1,1)$, and set $\psi(t) := \int_0^t \varphi(s)\,ds$, $t \in \mathbb{R}$. Finally, define $\psi_R(t) := R\psi(t/R)$, $R > 0$, and consider $v_R(X) := \psi_R(v(X))$, $X \in \Omega$. Then $v_R \in C^0(\Omega) \cap L^\infty(\Omega)$ and, in the sense of distributions,

$$\partial_j v_R = (\psi_R' \circ v)\,(\partial_j v) \in L^1(\Omega). \tag{3.2.72}$$

Then (3.2.71) follows by writing the version of Green's formula established in Theorem 3.2.8 for the vector field $v_R e_j$ and then letting $R \to \infty$. □

3.3 Boundary behavior of Newtonian layer potentials

We need to complement the estimates of §3.2 with results on the limiting behavior of $\mathcal{T}f(X)$ as $X \to Z \in \partial\Omega$. In this section we accomplish this for the double layer Newtonian potential and variants, before tackling more general cases in the following section.

Recall the (harmonic) double layer potential operator associated with Ω

$$\mathcal{D}f(X) := \frac{1}{\omega_n}\int_{\partial\Omega} \frac{\langle \nu(Y), Y - X\rangle}{|X - Y|^{n+1}} f(Y)\,d\sigma(Y), \qquad X \in \Omega, \tag{3.3.1}$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^{n+1}$, as well as its principal-value version

$$Kf(X) := \lim_{\varepsilon\to 0^+} K_\varepsilon f(X), \quad X \in \partial\Omega \quad \text{where} \tag{3.3.2}$$

$$K_\varepsilon f(X) := \frac{1}{\omega_n} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} \frac{\langle \nu(Y), Y-X\rangle}{|X-Y|^{n+1}} f(Y)\, d\sigma(Y), \qquad X \in \partial\Omega. \tag{3.3.3}$$

Here, $\nu(Y)$ denotes the outward normal to $\partial\Omega$ at Y. In analogy with (3.2.3), we also set

$$K_* f(X) := \sup_{\varepsilon>0} |K_\varepsilon f(X)|, \qquad X \in \partial\Omega. \tag{3.3.4}$$

Then Propositions 3.2.1–3.2.3 show that if $\alpha > 0$, $p \in (1, \infty)$, and if Ω is a UR domain, then there exist $C_1 = C_1(p, \Omega)$ and $C_2 = C_2(p, \alpha, \Omega)$ such that

$$\|K_* f\|_{L^p(\partial\Omega, d\sigma)} \le C_1 \|f\|_{L^p(\partial\Omega, d\sigma)} \tag{3.3.5}$$

and

$$\|\mathcal{N}(\mathcal{D}f)\|_{L^p(\partial\Omega, d\sigma)} \le C_2 \|f\|_{L^p(\partial\Omega, d\sigma)} \tag{3.3.6}$$

for every function $f \in L^p(\partial\Omega, d\sigma)$. Corresponding to $p = 1$ we also have

$$\|K_* f\|_{L^{1,\infty}(\partial\Omega, d\sigma)} \le C \|f\|_{L^1(\partial\Omega, d\sigma)}, \tag{3.3.7}$$

$$\|\mathcal{N}(\mathcal{D}f)\|_{L^{1,\infty}(\partial\Omega, d\sigma)} \le C \|f\|_{L^1(\partial\Omega, d\sigma)}. \tag{3.3.8}$$

Our next goal is to revisit the issue of a.e. pointwise existence of (3.3.2) and to prove a jump-formula for the double layer potential (3.3.1) operator of the form

$$\mathcal{D}f\Big|_{\partial\Omega} = (\tfrac{1}{2}I + K)f, \qquad \forall\, f \in L^p(\partial\Omega, d\sigma), \tag{3.3.9}$$

for each $p \in [1, \infty)$, where I is the identity operator and the boundary trace is taken in the sense of (2.3.4). Note that, in order for this to be pointwise σ-a.e. meaningful, it is required that Ω is weakly accessible (cf. (2.3.5)). Of course, any domain satisfying a corkscrew condition is weakly accessible. As an immediate consequence of Proposition 2.3.2 and Definition 3.1.7 we also obtain the following useful result.

Proposition 3.3.1 *Every UR domain $\Omega \subset \mathbb{R}^{n+1}$ is weakly accessible.*

We start our boundary analysis by establishing the following.

Proposition 3.3.2 *Let Ω be a UR domain. Then for each $f \in L^p(\partial\Omega, d\sigma)$, $p \in [1, \infty)$, the limit in (3.3.2) exists at almost every point $X \in \partial\Omega$, and the operators*

$$K : L^p(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma), \qquad 1 < p < \infty, \tag{3.3.10}$$

$$K : L^1(\partial\Omega, d\sigma) \longrightarrow L^{1,\infty}(\partial\Omega, d\sigma), \tag{3.3.11}$$

are well-defined and bounded. Furthermore, the double layer potential (3.3.1) has the property that

$$\lim_{\substack{Z\to X\\ Z\in\Gamma(X)}} \mathcal{D}f(Z) = (\tfrac{1}{2}I + K)f(X) \qquad \textit{for } \sigma\textit{-a.e. } X \in \partial\Omega, \tag{3.3.12}$$

for every $f \in L^p(\partial\Omega, d\sigma)$, $1 \le p < \infty$.

Proof. Given that, thanks to (3.3.6) and (3.3.7), the maximal operator associated with the type of limit considered in (3.3.2) is bounded on $L^p(\partial\Omega, d\sigma)$ if $1 < p < \infty$ and from $L^1(\partial\Omega, d\sigma)$ into $L^{1,\infty}(\partial\Omega, d\sigma)$, a familiar argument shows that it is sufficient to prove to prove the a.e. existence of the limit in (3.3.2) for f in a dense subspace of $L^p(\partial\Omega, d\sigma)$, say $f \in \mathrm{Lip}_o(\partial\Omega)$ (here, Lemma 2.4.9 is used). In this scenario, matters can be further reduced to proving that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\omega_n} \int\limits_{\substack{\varepsilon<|X-Y|<R\\ Y\in\partial\Omega}} \frac{\langle \nu(Y), Y-X\rangle}{|X-Y|^{n+1}} f(Y)\, d\sigma(Y) \tag{3.3.13}$$

exists at almost every $X \in \partial\Omega$. Replace $f(Y)$ by $[f(Y) - f(X)] + f(X)$. Note that, by Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned} \lim_{\varepsilon\to 0} \frac{1}{\omega_n} \int\limits_{\substack{\varepsilon<|X-Y|<R\\ Y\in\partial\Omega}} & \frac{\langle \nu(Y), Y-X\rangle}{|X-Y|^{n+1}} [f(Y) - f(X)]\, d\sigma(Y) \\ &= \frac{1}{\omega_n} \int\limits_{Y\in\partial\Omega,\, |X-Y|<R} \frac{\langle \nu(Y), Y-X\rangle}{|X-Y|^{n+1}} [f(Y) - f(X)]\, d\sigma(Y), \end{aligned} \tag{3.3.14}$$

since $\int_{Y\in\partial\Omega,\,|X-Y|<R} |X-Y|^{1-n}\, d\sigma(Y) < +\infty$ as can be seen by decomposing the domain of integration in dyadic annuli and using the Ahlfors regularity condition on $\partial\Omega$. Hence, it suffices to show that this limit (3.3.13) exists with $f(Y)$ replaced by $f(X)$ or, equivalently, with $f(Y)$ replaced by 1.

To proceed, we can use the harmonicity of the kernel and integrate by parts, based on the Green formula discussed at the end of §2.2, thus obtaining, for almost all ε and R,

$$\begin{aligned} &\int\limits_{\substack{\varepsilon<|X-Y|<R\\ Y\in\partial\Omega}} \frac{\langle \nu(Y), Y-X\rangle}{|X-Y|^{n+1}}\, d\sigma(Y) \\ &= \int\limits_{\substack{|X-Y|=R\\ Y\in\mathbb{R}^{n+1}\setminus\Omega}} \frac{\langle \nu(Y), Y-X\rangle}{|X-Y|^{n+1}}\, d\sigma(Y) + \int\limits_{\substack{|X-Y|=\varepsilon\\ Y\in\mathbb{R}^{n+1}\setminus\Omega}} \frac{\langle \nu(Y), Y-X\rangle}{|X-Y|^{n+1}}\, d\sigma(Y) \\ &= -\Big[\text{surface measure of } \partial B(X,\varepsilon)\cap\Omega^c\Big]\cdot R^{-n} + \Big[\text{surface measure of } \partial B(X,\varepsilon)\cap\Omega^c\Big]\cdot \varepsilon^{-n}. \end{aligned} \tag{3.3.15}$$

Thus, if we assume that $X \in \partial^*\Omega \subset \partial_T\Omega$, it follows from Proposition 3.1.3 that the limit as $\varepsilon \to 0^+$ of the last term above is $\omega_n/2$, hence the limit in (3.3.13) exists at each such point X, at least as $\varepsilon \to 0$ on a set of density 1 at 0. However, elementary estimates apply to the integral (3.3.13) over a shell $\varepsilon_1 < |X-Y| < \varepsilon_2$ with $\varepsilon_2 - \varepsilon_1 << \varepsilon_1$, giving convergence as asserted. This proves a.e. convergence in (3.3.2). At this point, Proposition 3.2.1 and Proposition 3.2.2 also give (3.3.10)-(3.3.11).

Turning our attention to jump-formulas, we first note that Proposition 3.3.1 ensures that it is meaningful to consider the limit in (3.3.12). Also, by (3.3.6) and (3.3.8), the non-tangential maximal operator associated with the type of limit implicit in (3.3.12) is bounded (on $L^p(\partial\Omega, d\sigma)$ if $1 < p < \infty$, and from $L^1(\partial\Omega, d\sigma)$ into $L^{1,\infty}(\partial\Omega, d\sigma)$). Since by (3.3.10)-(3.3.11) the operator in the right-hand side of (3.3.12) is also bounded in the same context, much as before, it is sufficient to prove (3.3.12) for $f \in \mathrm{Lip}_o(\partial\Omega)$. Assume that this is the case and write

$$\begin{aligned}
\lim_{\substack{Z\in\Gamma(X)\\ Z\to X}} \mathcal{D}f(Z) &= \lim_{\varepsilon\to 0^+} \lim_{\substack{Z\in\Gamma(X)\\ Z\to X}} \frac{1}{\omega_n} \int_{\substack{|X-Y|>\varepsilon\\ Y\in\partial\Omega}} \frac{\langle \nu(Y), Y-Z\rangle}{|Z-Y|^{n+1}} f(Y)\, d\sigma(Y) \\
&\quad + \lim_{\varepsilon\to 0^+} \lim_{\substack{Z\in\Gamma(X)\\ Z\to X}} \frac{1}{\omega_n} \int_{\substack{|X-Y|<\varepsilon\\ Y\in\partial\Omega}} \frac{\langle \nu(Y), Y-Z\rangle}{|Z-Y|^{n+1}} [f(Y)-f(X)]\, d\sigma(Y) \\
&\quad + f(X)\left(\lim_{\varepsilon\to 0^+} \lim_{\substack{Z\in\Gamma(X)\\ Z\to X}} \frac{1}{\omega_n} \int_{\substack{|X-Y|<\varepsilon\\ Y\in\partial\Omega}} \frac{\langle \nu(Y), Y-Z\rangle}{|Z-Y|^{n+1}}\, d\sigma(Y) \right) \\
&=: I_1 + I_2 + I_3. \qquad (3.3.16)
\end{aligned}$$

For each fixed $\varepsilon > 0$, Lebesgue's Dominated Convergence Theorem applies to the limit as $\Gamma(X) \ni Z \to X$ in I_1 and yields

$$I_1 = \lim_{\varepsilon\to 0^+} \frac{1}{\omega_n} \int_{\substack{|X-Y|>\varepsilon\\ Y\in\partial\Omega}} \frac{\langle \nu(Y), Y-X\rangle}{|X-Y|^{n+1}} f(Y)\, d\sigma(Y) = Kf(X). \qquad (3.3.17)$$

To handle I_2, we first observe that for every $X, Y \in \partial\Omega$ and $Z \in \Gamma(X)$,

$$\begin{aligned}
|X-Y| &\leq |Z-Y| + |Z-X| \leq |Z-Y| + (1+\alpha)\,\mathrm{dist}\,(Z,\partial\Omega) \\
&\leq |Z-Y| + (1+\alpha)|Z-Y| = (2+\alpha)|Z-Y|. \qquad (3.3.18)
\end{aligned}$$

Hence, since f is Lipschitz,

$$\left| \frac{\langle \nu(Y), Y-Z\rangle}{|Z-Y|^{n+1}} \right| |f(Y)-f(X)| \leq C(\alpha, f) \frac{1}{|X-Y|^{n-1}}. \qquad (3.3.19)$$

so that, once again using Lebesgue's Dominated Convergence Theorem, we obtain that

$$I_2 = 0. \qquad (3.3.20)$$

As for I_3 in (3.3.16), for each fixed $\varepsilon > 0$ and $Z \in \Gamma(X)$, we use the harmonicity of $|Z - \cdot|^{-n+1}$ in $\mathbb{R}^{n+1} \setminus \{Z\}$ in order to change the contour of integration as follows:

$$\frac{1}{\omega_n}\int\limits_{\substack{|X-Y|<\varepsilon\\ Y\in\partial\Omega}}\frac{\langle\nu(Y),Y-Z\rangle}{|Z-Y|^{n+1}}\,d\sigma(Y)=\frac{1}{\omega_n}\int\limits_{\substack{|X-Y|=\varepsilon\\ Y\in\Omega^c}}\frac{\langle\nu(Y),Y-Z\rangle}{|Z-Y|^{n+1}}\,d\sigma(Y),\tag{3.3.21}$$

for a.e $\varepsilon>0$. This step relies on Green's formula from 2.2. Consequently,

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\lim_{\substack{Z\in\Gamma(X)\\ Z\to X}}\frac{1}{\omega_n}\int\limits_{\substack{|X-Y|<\varepsilon\\ Y\in\partial\Omega}}\frac{\langle\nu(Y),Y-Z\rangle}{|Z-Y|^{n+1}}\,d\sigma(Y)\\
&\qquad=\lim_{\varepsilon\to 0^+}\frac{1}{\omega_n}\int\limits_{\substack{|X-Y|=\varepsilon\\ Y\in\mathbb{R}^{n+1}\setminus\Omega}}\frac{\langle\nu(Y),Y-X\rangle}{|X-Y|^{n+1}}\,d\sigma(Y)\\
&\qquad=\tfrac{1}{2},
\end{aligned}\tag{3.3.22}$$

by reasoning as in (3.3.15). Thus, the limit in the left-hand side of (3.3.16) exists and matches $(\frac{1}{2}I+K)f(X)$. This finishes the proof of the proposition. □

For $1\le i,k\le n+1$, consider now the operators

$$\mathcal{R}_{jk}f(X):=\int_{\partial\Omega}\Big[\nu_j(Y)(\partial_k E)(X-Y)-\nu_k(Y)(\partial_j E)(X-Y)\Big]f(Y)\,d\sigma(Y),\qquad X\in\Omega,\tag{3.3.23}$$

where

$$E(X):=\begin{cases}\dfrac{1}{\omega_n(1-n)}\dfrac{1}{|X|^{n-1}}, & \text{if } n\ge 2,\\[2mm] \dfrac{1}{2\pi}\log|X|, & \text{if } n=1,\end{cases}\qquad X\in\mathbb{R}^{n+1}\setminus\{0\},\tag{3.3.24}$$

is the fundamental solution for the Laplacian in $\mathbb{R}^{n+1}$. Also, for $X\in\partial\Omega$, set

$$R_{jk}f(X):=\lim_{\varepsilon\to 0^+}\int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}}\Big[\nu_j(Y)(\partial_k E)(X-Y)-\nu_k(Y)(\partial_j E)(X-Y)\Big]f(Y)\,d\sigma(Y),\tag{3.3.25}$$

$$(R_{jk})_*f(X):=\sup_{\varepsilon>0}\Big|\int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}}\Big[\nu_j(Y)(\partial_k E)(X-Y)-\nu_k(Y)(\partial_j E)(X-Y)\Big]f(Y)\,d\sigma(Y)\Big|.\tag{3.3.26}$$

Then, thanks to Proposition 3.2.1, under the assumptions of Proposition 3.3.2 we have

$$\|\mathcal{N}(\mathcal{R}_{jk}f)\|_{L^p(\partial\Omega,d\sigma)}\le C\|f\|_{L^p(\partial\Omega,d\sigma)},\quad 1<p<\infty,\tag{3.3.27}$$

$$\|\mathcal{N}(\mathcal{R}_{jk}f)\|_{L^{1,\infty}(\partial\Omega,d\sigma)}\le C\|f\|_{L^1(\partial\Omega,d\sigma)},\tag{3.3.28}$$

and

$$\|(R_{jk})_* f\|_{L^p(\partial\Omega,d\sigma)} \le C\|f\|_{L^p(\partial\Omega,d\sigma)}, \quad 1 < p < \infty, \tag{3.3.29}$$

$$\|(R_{jk})_* f\|_{L^{1,\infty}(\partial\Omega,d\sigma)} \le C\|f\|_{L^1(\partial\Omega,d\sigma)}. \tag{3.3.30}$$

We wish to complement these estimates with the following result.

Proposition 3.3.3 *Let Ω be a UR domain. Then for each $p \in [1,\infty)$ and $f \in L^p(\partial\Omega, d\sigma)$, the limit in (3.3.25) exists at almost every $X \in \partial\Omega$ and*

$$\mathcal{R}_{jk} f\Big|_{\partial\Omega} = R_{jk} f. \tag{3.3.31}$$

Proof. Essentially, the same argument as in the proof of Proposition 3.3.2 applies. This time, we only need to observe that for any reasonable domain $D \subset \mathbb{R}^{n+1}$ which does not contain the point X,

$$\begin{aligned}\int_{\partial D} \Big[\nu_j(Y)(\partial_k E)(X-Y) - \nu_k(Y)(\partial_j E)(X-Y)\Big]\, d\sigma(Y) \\ = \int_D \Big[(\partial_j\partial_k E)(X-Y) - (\partial_k\partial_j E)(X-Y)\Big]\, dY = 0\end{aligned} \tag{3.3.32}$$

and

$$(\nu_j(Y)\partial_{y_k} - \nu_k(Y)\partial_{y_j})E(X-Y) = 0 \quad \text{for} \quad Y \in \partial B(X,\varepsilon), \tag{3.3.33}$$

since $\nu_j\partial_k - \nu_k\partial_j$ is a tangential derivative and $E(X-\cdot)$ is constant on $\partial B(X,\varepsilon)$. $\square$

In order to proceed, introduce the (harmonic) single layer potential operator

$$\mathcal{S}f(X) := \int_{\partial\Omega} E(X-Y)f(Y)\, d\sigma(Y), \qquad X \in \Omega, \tag{3.3.34}$$

and denote by K^* the adjoint of the principal value double layer operator introduced in (3.3.2)-(3.3.3). Also, for further reference, let us also introduce here the boundary version of (3.3.34), i.e.,

$$Sf(X) := \int_{\partial\Omega} E(X-Y)f(Y)\, d\sigma(Y), \qquad X \in \partial\Omega. \tag{3.3.35}$$

As a direct consequence of Proposition 3.2.1-Proposition 3.2.2, we have the estimates

$$\|\mathcal{N}(\nabla\mathcal{S}f)\|_{L^p(\partial\Omega,d\sigma)} \le C\|f\|_{L^p(\partial\Omega,d\sigma)}, \quad 1 < p < \infty, \tag{3.3.36}$$

$$\|\mathcal{N}(\nabla\mathcal{S}f)\|_{L^{1,\infty}(\partial\Omega,d\sigma)} \le C\|f\|_{L^1(\partial\Omega,d\sigma)}, \tag{3.3.37}$$

for some $C > 0$ depending only on p, α (entering the definition of $\mathcal{N}$) as well as the Ahlfors regularity and UR constants of $\partial\Omega$.

Proposition 3.3.4 *Let Ω be a UR domain. Then for every $f \in L^p(\partial\Omega, d\sigma)$, $1 \le p < \infty$, and each $j \in \{1, ..., n+1\}$, the limit*

$$\lim_{\varepsilon \to 0^+} \int\limits_{\substack{|X-Y|>\varepsilon \\ Y \in \partial\Omega}} (\partial_j E)(X-Y) f(Y)\, d\sigma(Y), \tag{3.3.38}$$

exists at almost every $X \in \partial\Omega$. Furthermore,

$$\lim_{\substack{Z \to X \\ Z \in \Gamma(X)}} \partial_j \mathcal{S} f(Z) = -\tfrac{1}{2}\nu_j(X) f(X) + \lim_{\varepsilon \to 0^+} \int\limits_{\substack{|X-Y|>\varepsilon \\ Y \in \partial\Omega}} (\partial_j E)(X-Y) f(Y)\, d\sigma(Y), \tag{3.3.39}$$

at almost every $X \in \partial\Omega$.

Parenthetically, let us note that (3.3.39) implies

$$\partial_\nu \mathcal{S} f(X) := \Big\langle \nu(X), \lim_{\substack{Z \to X \\ Z \in \Gamma(X)}} \nabla \mathcal{S} f(Z) \Big\rangle = (-\tfrac{1}{2} I + K^*) f(X), \tag{3.3.40}$$

at almost every $X \in \partial\Omega$.

Proof of Proposition 3.3.4. With $\partial_\nu := \langle \nu, \nabla \rangle$ denoting the normal derivative, the idea is to write

$$\begin{aligned}(\partial_j E)(X-Y) &= -\nu_j(Y)\partial_{\nu(Y)}[E(X-Y)] \\ &\quad + \nu_k(Y)\Big[\nu_k(Y)(\partial_j E)(X-Y) - \nu_j(Y)(\partial_k E)(X-Y)\Big]\end{aligned} \tag{3.3.41}$$

so Proposition 3.3.3 and Proposition 3.3.2 can be used (with f replaced by $-\nu_j f$ and $\nu_k f$, respectively) in order to ensure that (3.3.38) exists a.e. on $\partial\Omega$. Since the decomposition (3.3.41) also entails

$$\partial_j \mathcal{S} f = -\mathcal{D}(\nu_j f) + \mathcal{R}_{jk}(\nu_k f), \tag{3.3.42}$$

we can once again invoke Proposition 3.3.3 and Proposition 3.3.2 in order to justify (3.3.39). $\square$

3.4 General odd, homogeneous layer potentials

Our goal in this subsection is to extend Proposition 3.3.4 to the setting of operators $\mathcal{T}$ treated in Proposition 3.2.1. One tool used in this extension is Clifford analysis, which we will review.

To begin, the Clifford algebra with $n+1$ imaginary units is the minimal enlargement of $\mathbb{R}^{n+1}$ to a unitary real algebra $(\mathcal{C}\ell_{n+1}, +, \odot)$, which is not generated (as an algebra) by any proper subspace of $\mathbb{R}^{n+1}$ and such that

$$X \odot X = -|X|^2 \quad \text{for any } X \in \mathbb{R}^{n+1}. \tag{3.4.1}$$

This identity readily implies that, if $\{e_j\}_{j=1}^{n+1}$ is the standard orthonormal basis in $\mathbb{R}^{n+1}$, then

$$e_j \odot e_j = -1 \quad \text{and} \quad e_j \odot e_k = -e_k \odot e_j \quad \text{for any } 1 \le j \ne k \le n+1. \tag{3.4.2}$$

In particular, we identify the canonical basis $\{e_j\}_j$ from $\mathbb{R}^{n+1}$ with the $n+1$ imaginary units generating $\mathcal{C}\ell_{n+1}$, so that we have the embedding

$$\mathbb{R}^{n+1} \hookrightarrow \mathcal{C}\ell_{n+1}, \qquad \mathbb{R}^{n+1} \ni X = (x_1, ..., x_{n+1}) \equiv \sum_{j=1}^{n+1} x_j e_j \in \mathcal{C}\ell_{n+1}. \tag{3.4.3}$$

Also, any element $u \in \mathcal{C}\ell_{n+1}$ can be uniquely represented in the form

$$u = \sum_{l=0}^{n+1} {\sum_{|I|=l}}' u_I\, e_I, \quad u_I \in \mathbb{R}. \tag{3.4.4}$$

Here e_I stands for the product $e_{i_1} \odot e_{i_2} \odot \cdots \odot e_{i_l}$ if $I = (i_1, i_2, \ldots, i_l)$ and $e_0 := e_\emptyset := 1$ is the multiplicative unit. Also, $\sum'$ indicates that the sum is performed only over strictly increasing multi-indices, i.e. $I = (i_1, i_2, ..., i_l)$ with $1 \leq i_1 < i_2 < \cdots < i_l \leq n+1$. We endow $\mathcal{C}\ell_{n+1}$ with the natural Euclidean metric $|u| := \Big[\sum_I |u_I|^2\Big]^{1/2}$, if $u = \sum_I u_I e_I \in \mathcal{C}\ell_{n+1}$. Next, recall the *Dirac operator*

$$D := \sum_{j=1}^{n+1} e_j \partial_j. \tag{3.4.5}$$

In the sequel, we shall use D_L and D_R to denote the action of D on a C^1 function $u : \Omega \to \mathcal{C}\ell_{n+1}$ (where Ω is an open subset of $\mathbb{R}^{n+1}$) from the left and from the right, respectively. For a sufficiently nice domain Ω with outward unit normal $\nu = (\nu_1, ..., \nu_{n+1})$ (identified with the $\mathcal{C}\ell_{n+1}$-valued function $\nu = \sum_{j=1}^{n+1} \nu_j e_j$) and surface measure σ, and for any two reasonable $\mathcal{C}\ell_{n+1}$-valued functions u, v in Ω, the following integration by parts formula holds:

$$\int_{\partial\Omega} u(X) \odot \nu(X) \odot v(X)\, d\sigma(X) = \int_\Omega \Big[(D_R u)(X) \odot v(X) + u(X) \odot (D_L v)(X)\Big]\, dX. \tag{3.4.6}$$

More detailed accounts of these and related matters can be found in [7] and [88]. Another simple but useful observation in this context is that, for any $1 \leq p \leq \infty$,

$$\nu\odot : L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1} \longrightarrow L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1} \quad \text{is an isomorphism.} \tag{3.4.7}$$

Indeed, by (3.4.1), its inverse is $-\nu\odot$.

Let $[\cdot]_j$ denote the projection onto the j-th Euclidean coordinate, i.e., $[X]_j := x_j$ if $X = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1}$. The following lemma of S. Semmes (cf. [100]) will play an important role for us.

Lemma 3.4.1 *For any odd, harmonic, homogeneous polynomial $P(X)$, $X \in \mathbb{R}^{n+1}$, of degree $l \geq 3$, there exist a family $P_{ij}(X)$, $1 \leq i, j \leq n+1$, of harmonic, homogeneous polynomials of degree $l-2$, as well as family of odd, C^∞ functions $k_{ij} : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1} \hookrightarrow \mathcal{C}\ell_{n+1}$, $1 \leq i, j \leq n+1$, which are homogeneous of degree $-n$, satisfying the following properties:*

$$\forall\, X \in \mathbb{R}^{n+1} \setminus \{0\} \Longrightarrow \frac{P(X)}{|X|^{n+l}} = C_{n,l} \sum_{i,j=1}^{n+1} [k_{ij}(X)]_j \quad \textit{and} \tag{3.4.8}$$

$$(D_R k_{ij})(X) = \frac{\partial}{\partial x_i}\left(\frac{P_{ij}(X)}{|X|^{n+l-2}}\right), \quad 1 \leq i, j \leq n+1, \tag{3.4.9}$$

for some constant $C_{n,l}$ depending only on n and l.

As a consequence of (3.4.9) and (3.4.6), if we set

$$k^{ij}(X) := P_{ij}(X)/|X|^{n+l-2} \quad \text{for} \quad X \in \mathbb{R}^{n+1} \setminus \{0\}, \quad 1 \leq i, j \leq n+1, \tag{3.4.10}$$

then for any finite perimeter domain $\Omega \subset \mathbb{R}^{n+1}$ such that $\mathcal{H}^n(\partial\Omega \setminus \partial_*\Omega) = 0$, with surface measure σ, outward unit normal $\nu = (\nu_k)_{1 \leq k \leq n+1}$, and such that $0 \notin \overline{\Omega}$, we have

$$\int_{\partial\Omega} k_{ij}(X) \odot \nu(X)\, d\sigma(X) = \int_{\partial\Omega} k^{ij}(X)\nu_i(X)\, d\sigma(X), \qquad 1 \leq i, j \leq n+1. \tag{3.4.11}$$

To state one of our main results in this section, we let "hat" denote the Fourier transform in $\mathbb{R}^{n+1}$.

Theorem 3.4.2 *Let Ω be a UR domain, and let $P(X)$ be an odd, harmonic, homogeneous polynomial of degree $l \geq 1$ in $\mathbb{R}^{n+1}$. Also, set $k(X) := P(X)/|X|^{n+l}$ for $X \in \mathbb{R}^{n+1} \setminus \{0\}$ and recall the operators $\mathcal{T}$ and T_ε associated with this kernel as in (3.2.2), (3.3.2).*

Then, for each $p \in [1, \infty)$, $f \in L^p(\partial\Omega, d\sigma)$, the limit

$$Tf(X) := \lim_{\varepsilon \to 0^+} T_\varepsilon f(X) \tag{3.4.12}$$

exists for a.e. $X \in \partial\Omega$. Also, the induced operators

$$T : L^p(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma), \quad p \in (1, \infty), \tag{3.4.13}$$

$$T : L^1(\partial\Omega, d\sigma) \longrightarrow L^{1,\infty}(\partial\Omega, d\sigma), \tag{3.4.14}$$

are bounded. Finally, the jump-formula

$$\lim_{\substack{Z \to X \\ Z \in \Gamma(X)}} \mathcal{T}f(Z) = \tfrac{1}{2\sqrt{-1}} \hat{k}(\nu(X))f(X) + Tf(X) \tag{3.4.15}$$

is valid at a.e. $X \in \partial\Omega$, whenever $f \in L^p(\partial\Omega, d\sigma)$, $1 \leq p < \infty$.

Proof. We shall first prove (3.4.12) by induction on l. When $l = 1$, the existence of the limit in (3.4.12) is a consequence of Proposition 3.3.4, so we assume $l \geq 3$ and that this limit exists a.e. on $\partial\Omega$ for any kernel associated with a polynomial of degree $l - 2$ as in the statement of the theorem.

Now, granted the identity (3.4.8), it suffices to treat the case when the operator (3.3.2) is associated with a kernel k_{ij} of the type specified in Lemma 3.4.1. Given that, in this scenario, (3.2.5) holds, there is no loss of generality in assuming that $f \in \nu \odot \Big[\mathrm{Lip}_o(\partial\Omega) \otimes \mathcal{C}\!\ell_{n+1}\Big]$ (here (3.4.7) is used). Assuming that this is the case, it is then easy to show that the limit (3.4.12) exists if and only if

$$\lim_{\varepsilon \to 0^+} \int\limits_{\substack{Y \in \partial\Omega \\ \varepsilon < |X-Y| < 1}} k_{ij}(X - Y) \odot \nu(Y)\, d\sigma(Y) \tag{3.4.16}$$

exists for a.e. $X \in \partial\Omega$. To this end, assume that $X \in \partial^*\Omega \subset \partial_T\Omega$, and for each $\varepsilon \in (0, 1)$ set

$$D_\varepsilon(X) := \Big[B(X,1) \setminus B(X,\varepsilon)\Big] \cap \Omega^c. \tag{3.4.17}$$

We may then compute

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega\\ \varepsilon<|X-Y|<1}} & k_{ij}(X-Y)\odot\nu(Y)\,d\sigma(Y)\\
&= \lim_{\varepsilon\to 0^+} \int\limits_{\partial D_\varepsilon(X)} k_{ij}(X-Y)\odot\nu(Y)\,d\sigma(Y)\\
&\quad - \lim_{\varepsilon\to 0^+} \int\limits_{\partial B(X,\varepsilon)\cap\Omega^c} k_{ij}(X-Y)\odot\nu(Y)\,d\sigma(Y)\\
&\quad - \int\limits_{\partial B(X,1)\cap\Omega^c} k_{ij}(X-Y)\odot\nu(Y)\,d\sigma(Y)\\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{3.4.18}$$

By (3.4.11) written for the domain $D_\varepsilon(X)$, the first limit in the right-hand side of (3.4.18) can be expressed as

$$\begin{aligned}
I_1 &= \lim_{\varepsilon\to 0^+} \int\limits_{\partial D_\varepsilon(X)} k^{ij}(X-Y)\nu_i(Y)\,d\sigma(Y)\\
&= \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega\\ \varepsilon<|X-Y|<1}} k^{ij}(X-Y)\nu_i(Y)\,d\sigma(Y)\\
&\quad + \lim_{\varepsilon\to 0^+} \int\limits_{\partial B(X,\varepsilon)\cap\Omega^c} k^{ij}(X-Y)\nu_i(Y)\,d\sigma(Y)\\
&\quad + \int\limits_{\partial B(X,1)\cap\Omega^c} k^{ij}(X-Y)\nu_i(Y)\,d\sigma(Y)\\
&=: I_{11} + I_{12} + I_{13}.
\end{aligned} \tag{3.4.19}$$

In turn, the existence of the limit I_{11} (for a.e. $X \in \partial\Omega$) is a consequence of the fact that $k^{ij}(X) = P_{ij}(X)/|X|^{n+l-2}$ with $P_{ij}(X)$ odd, homogeneous, harmonic polynomial of degree $l-2$ in $\mathbb{R}^{n+1}$ and the induction hypothesis. Going further, we have

$$I_{12} = \lim_{\varepsilon\to 0^+} \int_{\partial^- B(X,\varepsilon)} k^{ij}(X-Y)\nu_i(Y)\,d\sigma(Y) + \lim_{\varepsilon\to 0^+} \int_{W(X,\varepsilon)} k^{ij}(X-Y)\nu_i(Y)\,d\sigma(Y). \tag{3.4.20}$$

Now, making use of the fact that the kernel k^{ij} is homogeneous of degree $-n$, the first limit above is $\int_{\partial^- B(X,1)} k^{ij}(X-Y)\nu_i(Y)\,d\sigma(Y)$ whereas, by Proposition 3.1.3, the second limit in (3.4.20) is zero, at least as $\varepsilon \to 0$ on a set of density 1 at 0. This finishes the treatment of I_1. The limit I_2 in (3.4.18) is then handled in a similar manner to I_{12}. Altogether, this justifies the existence of the limit in (3.4.16) for a.e. $X \in \partial\Omega$, hence finishing the proof of the fact that the limit in (3.4.12) exists for a.e. $X \in \partial\Omega$. (Passing from the limit on a set thick at $\varepsilon = 0$ to the general limit stated in (3.4.12) is elementary.)

The fact that the operators (3.4.13)-(3.4.14) are bounded follows from what we have proved up to this point and Proposition 3.2.1-Proposition 3.2.2.

As regards (3.4.15), we shall once again proceed by induction on l. The case $l = 1$ has been already dealt with in Proposition 3.3.4, so we may assume that $l \geq 3$ and that the corresponding statement is true for any kernel $k(X)$ associated with a polynomial of degree $l-2$ as in the statement of the theorem. Now, for a given $k(X)$ as in Lemma 3.4.1 we recall the kernels k_{ij} introduced there and set

$$\mathcal{T}_{ij} f(X) := \int_{\partial\Omega} k_{ij}(X - Y) f(Y)\, d\sigma(Y), \quad X \in \Omega. \tag{3.4.21}$$

In particular, the identity (3.4.8) gives

$$\mathcal{T} f = C_{n,l} \sum_{i,j=1}^{n+1} [\mathcal{T}_{ij} f]_j, \qquad \forall\, f \in L^p(\partial\Omega, d\sigma), \tag{3.4.22}$$

and, hence, it suffices to show that

$$\lim_{\substack{Z\to X \\ Z\in\Gamma(X)}} \mathcal{T}_{jk} f(Z) = \tfrac{1}{2\sqrt{-1}} \hat{k}_{ij}(\nu(X)) f(X) + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} k_{ij}(X - Y) f(Y)\, d\sigma(Y), \tag{3.4.23}$$

at almost every $X \in \partial\Omega$.

Much as before, it suffices to prove this identity in the case when $f = \nu \odot g$ for some $g \in \mathrm{Lip}_o(\partial\Omega) \otimes \mathcal{C}\ell_{n+1}$. We shall also assume that $X \in \partial^*\Omega \subset \partial_T\Omega$. In this scenario, by paralleling the treatment of the harmonic double layer in (3.3.16), we decompose

$$\begin{aligned}
\lim_{\substack{Z\in\Gamma(X) \\ Z\to X}} \mathcal{T}_{ij}(\nu \odot g)(Z) \;&=\; \lim_{\varepsilon\to 0^+} \lim_{\substack{Z\in\Gamma(X) \\ Z\to X}} \int\limits_{\substack{|X-Y|>\varepsilon \\ Y\in\partial\Omega}} k_{ij}(Z - Y) \odot \nu(Y) \odot g(Y)\, d\sigma(Y) \\
&\quad + \lim_{\varepsilon\to 0^+} \lim_{\substack{Z\in\Gamma(X) \\ Z\to X}} \frac{1}{\omega_n} \int\limits_{\substack{|X-Y|<\varepsilon \\ Y\in\partial\Omega}} k_{ij}(Z - Y) \odot \nu(Y) \odot [g(Y) - g(X)]\, d\sigma(Y) \\
&\quad + \left(\lim_{\varepsilon\to 0^+} \lim_{\substack{Z\in\Gamma(X) \\ Z\to X}} \int\limits_{\substack{|X-Y|<\varepsilon \\ Y\in\partial\Omega}} k_{ij}(Z - Y) \odot \nu(Y)\, d\sigma(Y) \right) \odot g(X) \\
&=: \; I_1 + I_2 + I_3.
\end{aligned} \tag{3.4.24}$$

As in the case of (3.3.16), we then have

$$I_1 = \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} k_{ij}(X-Y)f(Y)\,d\sigma(Y) \quad\text{and}\quad I_2 = 0. \tag{3.4.25}$$

There remains to treat I_3 and we begin by rewriting the integral there as

$$\begin{aligned} \int\limits_{\substack{|X-Y|<\varepsilon \\ Y\in\partial\Omega}} k_{ij}(Z-Y)\odot\nu(Y)\,d\sigma(Y) &= \int\limits_{\partial(B(X,\varepsilon)\cap\Omega^c)} k_{ij}(Z-Y)\odot\nu(Y)\,d\sigma(Y) \\ &\quad - \int\limits_{\partial B(X,\varepsilon)\cap\Omega^c} k_{ij}(Z-Y)\odot\nu(Y)\,d\sigma(Y). \end{aligned} \tag{3.4.26}$$

As before, based on Proposition 3.1.3 and the homogeneity of the kernel we may then write

$$\lim_{\varepsilon\to 0^+} \lim_{\substack{Z\in\Gamma(X) \\ Z\to X}} \int\limits_{\partial B(X,\varepsilon)\cap\Omega^c} k_{ij}(Z-Y)\odot\nu(Y)\,d\sigma(Y) = \int\limits_{\partial^- B(X,1)} k_{ij}(X-Y)\odot\nu(Y)\,d\sigma(Y). \tag{3.4.27}$$

Now, for the first integral in the right-hand side of (3.4.26) we apply the identity (3.4.11) (for the domain $B(X,\varepsilon)\cap\Omega^c$) and obtain

$$\int\limits_{\partial(B(X,\varepsilon)\cap\Omega^c)} k_{ij}(Z-Y)\odot\nu(Y)\,d\sigma(Y) = \int\limits_{\partial(B(X,\varepsilon)\cap\Omega^c)} k^{ij}(Z-Y)\nu_i(Y)\,d\sigma(Y). \tag{3.4.28}$$

In summary, the above argument gives

$$\begin{aligned} \lim_{\substack{Z\in\Gamma(X) \\ Z\to X}} \mathcal{T}_{ij}(\nu\odot g)(Z) &= \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} k_{ij}(X-Y)\odot\nu(Y)\odot g(Y)\,d\sigma(Y) \\ &\quad + \left(\int\limits_{\partial^- B(X,1)} k_{ij}(X-Y)\odot\nu(Y)\,d\sigma(Y) \right)\odot g(X) \\ &\quad + \left(\lim_{\varepsilon\to 0^+} \lim_{\substack{Z\in\Gamma(X) \\ Z\to X}} \int\limits_{\partial(B(X,\varepsilon)\cap\Omega^c)} k^{ij}(Z-Y)\nu_i(Y)\,d\sigma(Y) \right)\odot g(X). \end{aligned} \tag{3.4.29}$$

Next, we introduce

$$\mathcal{T}^{ij} f(X) := \int_{\partial\Omega} k^{ij}(X-Y)f(Y)\,d\sigma(Y), \quad X\in\Omega, \tag{3.4.30}$$

and note two things about this family of operators. On the one hand, given the nature of the kernels k^{ij}, the induction hypothesis allows us to write that, for every $g\in L^p(\partial\Omega, d\sigma)$,

$$
\begin{aligned}
\lim_{\substack{Z\to X\\ Z\in\Gamma(X)}} \mathcal{T}^{jk}(\nu_i g)(Z) \;&=\; \tfrac{1}{2\sqrt{-1}}\widehat{k^{ij}}(\nu(X))\nu_i(X)g(X) \\
&\quad + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}} k^{ij}(X-Y)\nu_i(Y)g(Y)\,d\sigma(Y) \qquad (3.4.31)
\end{aligned}
$$

at almost every $X\in\partial\Omega$.

On the other hand, by proceeding analogously as in (3.4.24), we also get

$$
\begin{aligned}
\lim_{\substack{Z\to X\\ Z\in\Gamma(X)}} \mathcal{T}^{jk}(\nu_i g)(Z) \;&=\; \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}} k^{ij}(X-Y)\nu_i(Y)g(Y)\,d\sigma(Y) \qquad (3.4.32) \\
&\quad + \left(\,\int\limits_{\partial^- B(X,1)} k^{ij}(X-Y)\nu_i(Y)\,d\sigma(Y)\right) g(X) \\
&\quad + \left(\lim_{\varepsilon\to 0^+}\lim_{\substack{Z\in\Gamma(X)\\ Z\to X}} \int\limits_{\partial(B(X,\varepsilon)\cap\Omega^c)} k^{ij}(Z-Y)\nu_i(Y)\,d\sigma(Y)\right) g(X),
\end{aligned}
$$

a.e. on $\partial\Omega$, for every $g\in L^p(\partial\Omega,d\sigma)$. By comparing (3.4.31) with (3.4.32) we eventually arrive at the conclusion that

$$
\begin{aligned}
\lim_{\varepsilon\to 0^+}\lim_{\substack{Z\in\Gamma(X)\\ Z\to X}} \int\limits_{\partial(B(X,\varepsilon)\cap\Omega^c)} k^{ij}(Z-Y)\nu_i(Y)\,d\sigma(Y) \;&=\; \tfrac{1}{2\sqrt{-1}}\widehat{k^{ij}}(\nu(X))\nu_i(X) \qquad (3.4.33) \\
&\quad - \int\limits_{\partial^- B(X,1)} k^{ij}(X-Y)\nu_i(Y)\,d\sigma(Y).
\end{aligned}
$$

(Note that, strictly speaking, it is this identity that justifies the existence of the last double limit in (3.4.29).) Thus, from this and (3.4.29), we may finally deduce that

$$
\begin{aligned}
\lim_{\substack{Z\in\Gamma(X)\\ Z\to X}} \mathcal{T}_{ij}(\nu\odot g)(Z) \;=\;& \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}} k_{ij}(X-Y)\odot\nu(Y)\odot g(Y)\,d\sigma(Y)\\
&+\left(\;\int\limits_{\partial^- B(X,1)} k_{ij}(X-Y)\odot\nu(Y)\,d\sigma(Y)\right)\odot g(X)\\
&+\tfrac{1}{2\sqrt{-1}}\widehat{k^{ij}}(\nu(X))\nu_i(X)\odot g(X)\\
&-\left(\;\int\limits_{\partial^- B(X,1)} k^{ij}(X-Y)\nu_i(Y)\,d\sigma(Y)\right)g(X)\\
=\;& -\tfrac{1}{2\sqrt{-1}}\widehat{k^{ij}}(\nu(X))\nu_i(X)\nu(X)\odot(\nu\odot g)(X)\\
&-\left(\;\int\limits_{\partial^- B(X,1)} k_{ij}(X-Y)\odot\nu(Y)\,d\sigma(Y)\right)\odot\nu(X)\odot(\nu\odot g)(X)\\
&+\left(\;\int\limits_{\partial^- B(X,1)} k^{ij}(X-Y)\nu_i(Y)\,d\sigma(Y)\right)\nu(X)\odot(\nu\odot g)(X)\\
&+\lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}} k_{ij}(X-Y)\odot(\nu\odot g)(Y)\,d\sigma(Y). \qquad (3.4.34)
\end{aligned}
$$

In short, for every $f\in L^p(\partial\Omega,d\sigma)$, $1\le p<\infty$, we have that at a.e. $X\in\partial\Omega$

$$
\lim_{\substack{Z\in\Gamma(X)\\ Z\to X}} \mathcal{T}_{ij}f(Z)=\alpha_{ij}(X)f(X)+\lim_{\varepsilon\to 0^+}\int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}} k_{ij}(X-Y)f(Y)\,d\sigma(Y), \qquad (3.4.35)
$$

where the coefficient $\alpha_{ij}(X)$ is implicitly defined by (3.4.34). Hence, in order to fully justify (3.4.31), it remains to show that

$$
\alpha_{ij}(X)=\tfrac{1}{2\sqrt{-1}}\hat{k}_{ij}(X). \qquad (3.4.36)
$$

By carrying out a similar analysis but for the domain Ω^c in place of Ω (and keeping in mind that the outward unit normal for Ω^c is $-\nu$), we obtain that for every $f\in L^p(\partial\Omega,d\sigma)$, $1\le p<\infty$, and for a.e. $X\in\partial\Omega$,

$$
\lim_{\substack{Z\in\Gamma_-(X)\\ Z\to X}} \mathcal{T}_{ij}f(Z)=-\alpha_{ij}(X)f(X)+\lim_{\varepsilon\to 0^+}\int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}} k_{ij}(X-Y)f(Y)\,d\sigma(Y), \qquad (3.4.37)
$$

where $\Gamma_-(X)$ stands for the nontangential approach region with vertex at $X \in \partial(\Omega^c)$ corresponding to the domain Ω^c (cf. (2.1.5)), and we have retained the same definition for $\mathcal{T}_{ij}f(X)$ as in (3.4.30) when $X \in \Omega^c$. In particular, for every $f \in L^p(\partial\Omega, d\sigma)$, $1 \leq p < \infty$,

$$\lim_{\substack{Z\in\Gamma(X)\\ Z\to X}} \mathcal{T}_{ij}f(Z) - \lim_{\substack{Z\in\Gamma_-(X)\\ Z\to X}} \mathcal{T}_{ij}f(Z) = 2\alpha_{ij}(X)f(X) \quad \text{at a.e. } X \in \partial\Omega. \tag{3.4.38}$$

Another characteristic of $\alpha_{ij}(X)$, visible from (3.4.34), is that this quantity depends only on the tangent plane to Ω at X and not on Ω itself. Consequently, in order to compute the actual value of $\alpha_{ij}(X)$ using (3.4.38), it suffices to replace Ω by any other (reasonable) domain having the same tangent plane at X, such as a suitably rotated and translated half-space. In this latter scenario, for every $f \in L^p(\partial\Omega, d\sigma)$, $1 \leq p < \infty$, the jump-formula

$$\lim_{\substack{Z\in\Gamma(X)\\ Z\to X}} \mathcal{T}_{ij}f(Z) - \lim_{\substack{Z\in\Gamma_-(X)\\ Z\to X}} \mathcal{T}_{ij}f(Z) = \frac{1}{\sqrt{-1}}\hat{k}_{ij}(\nu(X))f(X) \quad \text{at a.e. } X \in \partial\Omega \tag{3.4.39}$$

is well-known (see, e.g, the discussion in [91]). All in all, from (3.4.39) and (3.4.38) we may conclude that (3.4.36) holds. This justifies (3.4.31) and finishes the proof of the theorem. □

From Theorem 3.4.2 to the general case described by (3.2.1)–(3.2.2) is but a short step. We take care of this in the more general variable coefficient context in the next subsection.

3.5 The variable coefficient case

Our goal in this subsection is to prove a variable coefficient version of Theorem 3.4.2.

Theorem 3.5.1 *If $\Omega \subset \mathbb{R}^{n+1}$ is a UR domain, there exists a positive integer $N = N(n)$ such that if the kernel k is as in (3.2.1) then the limit in (3.4.12) exists and the jump-formula (3.4.15) holds a.e. on $\partial\Omega$ for each $L^p(\partial\Omega, d\sigma)$, $1 \leq p < \infty$.*

Actually, a more general result is true. Namely, there exists a positive integer $M = M(n)$ with the property that if the function $b(X, Z)$ is odd and homogeneous of degree $-n$ in the variable $Z \in \mathbb{R}^{n+1}$, and if $D^\alpha_Z b(X, Z)$ is continuous and bounded on $\mathbb{R}^{n+1} \times S^n$ for $|\alpha| \leq M$, then the limit

$$Bf(X) := \lim_{\varepsilon\to 0^+} B_\varepsilon f(X), \quad \textit{where} \tag{3.5.1}$$

$$B_\varepsilon f(X) := \int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}} b(X, X-Y)f(Y)\, d\sigma(Y), \qquad X \in \partial\Omega, \tag{3.5.2}$$

exists for every $f \in L^p(\partial\Omega, d\sigma)$, $1 \leq p < \infty$, and almost every $X \in \partial\Omega$, and the operator B is bounded on $L^p(\partial\Omega, d\sigma)$ for every $p \in (1, \infty)$, and from $L^1(\partial\Omega, d\sigma)$ into $L^{1,\infty}(\partial\Omega, d\sigma)$. In fact, if

$$B_* f(X) := \sup_{\varepsilon>0} |B_\varepsilon f(X)|, \qquad X \in \partial\Omega, \tag{3.5.3}$$

then, for every $p \in (1, \infty)$,

$$\|B_* f\|_{L^p(\partial\Omega,d\sigma)} \le C(p,\Omega) \sup_{|\alpha|\le M} \|D_Z^\alpha b(X,Z)\|_{L^\infty(\mathbb{R}^{n+1}\times S^n)} \|f\|_{L^p(\partial\Omega,d\sigma)}. \tag{3.5.4}$$

Also, corresponding to $p = 1$, (3.5.4) holds if the weak-L^1 norm is used in the left-hand side.

Furthermore, if

$$\mathcal{B}f(X) := \int_{\partial\Omega} b(X, X-Y) f(Y)\, d\sigma(Y), \qquad X \in \Omega, \tag{3.5.5}$$

then for every $f \in L^p(\partial\Omega, d\sigma)$, $1 \le p < \infty$, at almost every $X \in \partial\Omega$, there holds

$$\lim_{\substack{Z\to X\\ Z\in\Gamma(X)}} \mathcal{B}f(Z) = \tfrac{1}{2\sqrt{-1}} \widehat{b}(X, \nu(X)) f(X) + Bf(X), \tag{3.5.6}$$

where, above, "hat" stands for the Fourier transform in the second variable, and

$$\|\mathcal{N}(\mathcal{B}f)\|_{L^p(\partial\Omega,d\sigma)} \le C(p,\Omega) \sup_{|\alpha|\le M} \|D_Z^\alpha b(X,Z)\|_{L^\infty(\mathbb{R}^{n+1}\times S^n)} \|f\|_{L^p(\partial\Omega,d\sigma)}. \tag{3.5.7}$$

Finally, for $\frac{n}{n+1} < p \le 1$,

$$\|\mathcal{N}(\mathcal{B}f)\|_{L^p(\partial\Omega,d\sigma)} \le C(p,\Omega) \sup_{|\alpha|\le M} \|D_Z^\alpha b(X,Z)\|_{L^\infty(\mathbb{R}^{n+1}\times S^n)} \|f\|_{H^p_{at}(\partial\Omega,d\sigma)}. \tag{3.5.8}$$

Proof. Of course, it suffices to treat the second part of the theorem, in which case we argue as in [91]. For each $X \in \mathbb{R}^{n+1}$, we expand

$$b(X,Z) = \sum_{\ell\in\mathbb{N}} a_\ell(X) \Psi_\ell\Big(\frac{Z}{|Z|}\Big) |Z|^{-n}, \tag{3.5.9}$$

where $\{\Psi_\ell : \ell \in \mathbb{N}\}$ is an orthonormal basis of $L^2(S^n)$ consisting of spherical harmonics. Furthermore, we arrange that each Ψ_ℓ is real, and either even or odd. Consequently, since

$$a_\ell(X) = \int_{S^n} b(X,\omega) \Psi_\ell(\omega)\, d\omega, \tag{3.5.10}$$

it follows that $a_\ell \equiv 0$ whenever Ψ_ℓ is even. Also, from (3.5.10), integrations by parts, and our assumptions on $b(X,Z)$, it follows that the sequence of coefficients $\{a_\ell\}_\ell$ is rapidly decreasing in ℓ, i.e.

$$\forall\, j \in \mathbb{N} \quad \exists C_j > 0 \ \text{ such that } \ \|a_\ell\|_{L^\infty} \le C_j\, \ell^{-j}, \quad \forall\, \ell \in \mathbb{N}. \tag{3.5.11}$$

If for each $\ell \in \mathbb{N}$ for which a_ℓ is not identically zero we now set

$$k_\ell(X) := \Psi_\ell\Big(\tfrac{X}{|X|}\Big)|X|^{-n}, \quad X \in \mathbb{R}^{n+1} \setminus \{0\},$$

$$B_{\ell,\varepsilon}f(X) := \int_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} k_\ell(X-Y)f(Y)\,d\sigma(Y), \quad X \in \partial\Omega, \tag{3.5.12}$$

$$B_{\ell,*}f(X) := \sup_{\varepsilon>0}\Big|\int_{\partial\Omega} k_\ell(X-Y)f(Y)\,d\sigma(Y)\Big|, \quad X \in \partial\Omega,$$

then for each $p \in (1,\infty)$ Proposition 3.2.1 gives

$$\begin{gathered}\|B_{\ell,*}f\|_{L^p(\partial\Omega,d\sigma)} \leq C(n,p,\ell)\|f\|_{L^p(\partial\Omega,d\sigma)}, \quad \text{where} \\ \text{the constant } C(n,p,\ell) \text{ has polynomial growth in } \ell.\end{gathered} \tag{3.5.13}$$

Next, for each $\mu \in \mathbb{N}$, write

$$B_\varepsilon f(X) = \sum_{\ell\leq\mu} a_\ell(X)B_{\ell,\varepsilon}f(X) + \sum_{\ell\geq\mu} a_\ell(X)B_{\ell,\varepsilon}f(X) \tag{3.5.14}$$

and observe that for each $j \in \mathbb{N}$ there exists $C_j > 0$ such that

$$\begin{aligned}\Big\|\sup_{\varepsilon>0}\Big|\sum_{\ell\geq\mu} a_\ell B_{\ell,\varepsilon}f\Big|\Big\|_{L^p(\partial\Omega,d\sigma)} &\leq \sum_{\ell\geq\mu}\|a_\ell\|_{L^\infty}\|B_{\ell,*}f\|_{L^p(\partial\Omega,d\sigma)} \\ &\leq C_j\,\mu^{-j}\|f\|_{L^p(\partial\Omega,d\sigma)}.\end{aligned} \tag{3.5.15}$$

On the other hand, Theorem 3.4.2 gives that, for each fixed μ, the first sum in (3.5.14) converges at almost every $X \in \partial\Omega$. Thus, if we consider the disagreement function

$$\begin{aligned}D_f(X) &:= \limsup_{\varepsilon\to0^+} B_\varepsilon f(X) - \liminf_{\varepsilon\to0^+} B_\varepsilon f(X) \\ &= \limsup_{\varepsilon\to0^+}\Big(\sum_{\ell\geq\mu} a_\ell(X)B_{\ell,\varepsilon}f(X)\Big) - \liminf_{\varepsilon\to0^+}\Big(\sum_{\ell\geq\mu} a_\ell(X)B_\varepsilon f(X)\Big),\end{aligned} \tag{3.5.16}$$

where the second equality holds for every $\mu \in \mathbb{N}$ and almost every $X \in \partial\Omega$, Chebysheff's inequality inequality gives that for every $j \in \mathbb{N}$, $\lambda > 0$ and $f \in L^p(\partial\Omega, d\sigma)$

$$\sigma\Big(\{X \in \partial\Omega :\, D_f(X) > \lambda\}\Big) \leq C_j\mu^{-j}\lambda^{-p}\|f\|^p_{L^p(\partial\Omega,d\sigma)}. \tag{3.5.17}$$

Passing to limit $\mu \to \infty$ then shows that $D_f(X) = 0$ for σ-a.e. $X \in \partial\Omega$ and this concludes the proof of the fact that the limit in (3.5.1) exists at almost every boundary point.

All the other remaining claims in the statement of the theorem can be proved in a similar fashion, using Theorem 3.4.2 and an expansion in spherical harmonics as above. This finishes the proof of Theorem 3.5.1. $\square$

It will also be useful to treat the following variant of (3.5.5):

$$\widetilde{\mathcal{B}}f(X) := \int\limits_{\partial\Omega} b(Y, X-Y) f(Y)\, d\sigma(Y), \quad X \in \Omega. \tag{3.5.18}$$

The same sort of analysis works, with X replaced by Y in the spherical harmonic expansion (3.5.9). We have the following.

Theorem 3.5.2 *In the setting of Theorem 3.5.1, with $\widetilde{\mathcal{B}}$ given by (3.5.18), we have*

$$\|\mathcal{N}\widetilde{\mathcal{B}}f\|_{L^p(\partial\Omega,d\sigma)} \leq C(p,\Omega) \sup_{|\alpha|\leq M} \|D_Z^\alpha b(Y,Z)\|_{L^\infty(\mathbb{R}^{n+1}\times S^n)} \|f\|_{L^p(\partial\Omega,d\sigma)}, \tag{3.5.19}$$

if $1 < p < \infty$, plus a similar estimate (involving replaced by weak-$L^{1,\infty}$ in the left-hand side) when $p = 1$.

On the other hand, if $\frac{n}{n+1} < p < 1$, $r > n(p^{-1}-1)$ and, for each $|\alpha| \leq M$, the function $D_Z^\alpha b(Y,Z)$ is in $C^r(\mathbb{R}^{n+1})$ in the variable Y, uniformly for $|Z| = 1$, then for any compact $\Sigma_o \subset \partial\Omega$,

$$\|\mathcal{N}\widetilde{\mathcal{B}}f\|_{L^p(\Sigma_o,d\sigma)} \leq C(p,\Omega) \sup_{|\alpha|\leq M} \sup_{|Z|=1} \|D_Z^\alpha b(Y,Z)\|_{C^r(\Sigma_o)} \|f\|_{h^p_{at}(\Sigma_o)}, \tag{3.5.20}$$

for f supported on Σ_o.

Finally, given $p \in [1,\infty)$ and $f \in L^p(\partial\Omega, d\sigma)$, then for almost all $X \in \partial\Omega$,

$$\lim_{Z\to X, Z\in\Gamma(X)} \widetilde{\mathcal{B}}f(Z) = \frac{1}{2\sqrt{-1}} \hat{b}(X,\nu(X)) f(X) - B^t f(X), \tag{3.5.21}$$

where B is as in (3.5.1) and the superscript t indicates transposition.

In turn, Theorem 3.5.1 and Theorem 3.5.2 apply to the Schwartz kernels of certain pseudodifferential operators. Recall that a pseudodifferential operator $Q(X,D)$ with symbol $q(X,\xi)$ in Hörmander's class $S^m_{1,0}$ is given by the oscillatory integral

$$\begin{aligned} Q(X,D)u &= (2\pi)^{-(n+1)/2} \int q(X,\xi)\hat{u}(\xi) e^{i\langle X,\xi\rangle}\, d\xi \\ &= (2\pi)^{-(n+1)} \int\int q(X,\xi) e^{i\langle X-Y,\xi\rangle} u(Y)\, dY\, d\xi. \end{aligned} \tag{3.5.22}$$

We are concerned with a smaller class of symbols, $S^m_{\rm cl}$, defined by requiring that (the matrix-valued) function $q(X,\xi)$ has an asymptotic expansion of the form

$$q(X,\xi) \sim q_m(X,\xi) + q_{m-1}(X,\xi) + \cdots, \tag{3.5.23}$$

with q_j smooth in X and ξ and homogeneous of degree j in ξ (for $|\xi| \geq 1$). Call $q_m(X,\xi)$, i.e. the leading term in (3.5.23), the *principal symbol* of $q(X,D)$. In fact, we shall find it convenient to work with classes of symbols which only exhibit a limited amount of regularity in the spatial variable (while still C^∞ in the Fourier variable). Specifically, for each $r \geq 0$ we define

$$C^r S^m_{1,0} := \{q(X,\xi) : \|D_\xi^\alpha q(\cdot,\xi)\|_{C^r} \leq C_\alpha (1+|\xi|)^{m-|\alpha|}, \quad \forall\, \alpha\}. \tag{3.5.24}$$

The class of pseudodifferential operators associated with such symbols will be denoted $\mathrm{OP}C^rS^m_{1,0}$. As before, we write $\mathrm{OP}C^rS^m_{\rm cl}$ for the subclass of *classical* pseudodifferential operators in $\mathrm{OP}C^rS^m_{1,0}$ whose symbols can be expanded as in (3.5.23), where $q_j(X,\xi)\in C^rS^{m-j}_{1,0}$ is homogeneous of degree j in ξ for $|\xi|\geq 1$, $j=m,m-1,\dots$. Finally, we set $\emptyset\mathrm{P}C^rS^m_{\rm cl}$ for the space of all formal adjoints of operators in $\mathrm{OP}C^rS^m_{\rm cl}$.

Given a classical pseudodifferential operator $Q(x,D)\in\mathrm{OP}C^rS^{-1}_{\rm cl}$, we denote by $k_Q(X,Y)$ and $\mathrm{Sym}_Q(X,\xi)$ its Schwartz kernel and its principal symbol, respectively. Next, if $\Omega\subseteq\mathbb{R}^{n+1}$ is a domain with outward unit normal ν and boundary surface measure σ, we can introduce integral operators of layer potential type by formally writing

$$\begin{aligned} K_Qf(X) &:= \mathrm{P.V.}\int_{\partial\Omega}k_Q(X,Y)f(Y)\,d\sigma(Y)\\ &= \lim_{\varepsilon\to0^+}\int_{Y\in\partial\Omega:\,|X-Y|>\varepsilon}k_Q(X,Y)f(Y)\,d\sigma(Y),\qquad X\in\partial\Omega, \end{aligned}\tag{3.5.25}$$

and

$$\mathcal{K}_Qf(X):=\int_{\partial\Omega}k_Q(X,Y)f(Y)\,d\sigma(Y),\qquad X\in\Omega.\tag{3.5.26}$$

In this context, Theorem 3.5.1 and Theorem 3.5.2 yield the following result.

Theorem 3.5.3 *Let $\Omega\subset\mathbb{R}^{n+1}$ be a bounded UR domain. Also let $Q(x,D)\in\mathrm{OP}C^0S^{-1}_{\rm cl}$ be such that $\mathrm{Sym}_Q(X,\xi)$ is odd in ξ and recall the operators (3.5.25)-(3.5.26).*

Then, for each $f\in L^p(\partial\Omega,d\sigma)$, with $1\leq p<\infty$, $K_Qf(X)$ makes sense at almost every boundary point $X\in\partial\Omega$ and

$$K_Q:L^p(\partial\Omega,d\sigma)\to L^p(\partial\Omega,d\sigma),\quad 1<p<\infty,\tag{3.5.27}$$

$$K_Q:L^1(\partial\Omega,d\sigma)\to L^{1,\infty}(\partial\Omega,d\sigma),\tag{3.5.28}$$

are bounded operators. Furthermore,

$$\|\mathcal{N}(\mathcal{K}_Qf)\|_{L^p(\partial\Omega,d\sigma)}\leq C(\Omega,Q,p)\|f\|_{L^p(\partial\Omega,d\sigma)},\qquad\forall f\in L^p(\partial\Omega,d\sigma),\ 1<p<\infty,\tag{3.5.29}$$

plus a similar estimate involving weak-L^1 in the left-hand side when $p=1$.

On the other hand, for $n/(n+1)<p\leq1$,

$$\|\mathcal{N}(\mathcal{K}_Qf)\|_{L^p(\partial\Omega,d\sigma)}\leq C\|f\|_{h^p_{at}(\partial\Omega,d\sigma)},\qquad\forall f\in h^p_{at}(\partial\Omega,d\sigma).\tag{3.5.30}$$

Moreover, if $f\in L^p(\partial\Omega,d\sigma)$ with $1\leq p<\infty$, then $\mathcal{K}_Qf$ has a nontangential boundary trace at almost every boundary point. More specifically,

$$\mathcal{K}_Qf\Big|_{\partial\Omega}=\frac{1}{2\sqrt{-1}}\mathrm{Sym}_Q(\cdot,\nu)f+K_Qf\quad a.e.\ on\ \ \partial\Omega.\tag{3.5.31}$$

Finally, similar results are valid for a pseudodifferential operator $Q(X,D)\in\emptyset\mathrm{P}C^0S^{-1}_{\rm cl}$. In this setting, for the analogue of (3.5.30) we require that $Q\in\emptyset\mathrm{P}C^rS^{-1}_{\rm cl}$ where $r>n(p^{-1}-1)$ and $\frac{n}{n+1}<p<1$.

In fact, since the main claims in Theorem 3.5.3 are local in nature and given the invariance of the class of domains and pseudodifferential operators (along with their Schwartz kernels and principal symbols) under smooth diffeomorphisms, these results can be naturally extended to the setting of domains on manifolds and pseudodifferential operators acting between vector bundles. We shall further elaborate on this aspect in Section 5, where one of the aims is to treat operators with even rougher kernels.

3.6 Singular integrals on Sobolev spaces

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, of locally finite perimeter, whose boundary is Ahlfors regular and satisfies (2.3.1). As in the past, set $\sigma := \mathcal{H}^n \lfloor \partial\Omega$ and denote by $\nu = (\nu_1, ..., \nu_{n+1})$ the measure theoretic outward unit normal to $\partial\Omega$. Next, consider the first-order tangential derivative operators, acting on a compactly supported function φ, of class C^1 in a neighborhood of $\partial\Omega$, by

$$\partial_{\tau_{jk}}\varphi := \nu_j(\partial_k\varphi)\Big|_{\partial\Omega} - \nu_k(\partial_j\varphi)\Big|_{\partial\Omega}, \qquad j,k = 1,\dots,n+1. \tag{3.6.1}$$

We now make the following definition. Given two indices

$$1 < p, p' < \infty, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \tag{3.6.2}$$

set

$$\begin{aligned} L^p_1(\partial\Omega, d\sigma) \quad := \quad & \Big\{ f \in L^p(\partial\Omega, d\sigma) : \exists\, c > 0 \text{ such that if } \varphi \in C^1_0(\mathbb{R}^{n+1}) \\ & \quad \text{then } \sum_{j,k=1}^{n+1} \Big| \int_{\partial\Omega} f\,(\partial_{\tau_{jk}}\varphi)\,d\sigma \Big| \leq c\|\varphi\|_{L^{p'}(\partial\Omega,d\sigma)} \Big\}. \end{aligned} \tag{3.6.3}$$

In order to get a better understanding of the nature of this space, fix $f \in L^p_1(\partial\Omega, d\sigma)$, take $j, k \in \{1, \dots, n+1\}$, and consider the functional Λ_{jk}, defined as follows.

$$\Lambda_{jk} : \{\varphi|_{\partial\Omega} : \varphi \in C^1_0(\mathbb{R}^{n+1})\} \longrightarrow \mathbb{R}, \quad \Lambda_{jk}(\varphi|_{\partial\Omega}) := \int_{\partial\Omega} f(\partial_{\tau_{jk}}\varphi)\,d\sigma. \tag{3.6.4}$$

We claim that Λ_{jk} is unambiguously defined. To justify this claim, assume that $\varphi \in C^1_0(\mathbb{R}^{n+1})$ and $\varphi|_{\partial\Omega} = 0$. Use a mollification argument to produce a sequence $\varphi_\alpha \in C^\infty_0(\mathbb{R}^{n+1})$, $\alpha \in \mathbb{N}$, with the property that $\partial^\gamma\varphi_\alpha$ converges to $\partial^\gamma\varphi$ as $\alpha \to \infty$, uniformly on compact subsets of $\mathbb{R}^{n+1}$, for each multinidex γ of length ≤ 1. Then, for each fixed $\alpha \in \mathbb{N}$ and each $\psi \in C^\infty_0(\mathbb{R}^{n+1})$, based on definitions and repeated integrations by parts we may write

$$
\begin{aligned}
\int_{\partial\Omega}\psi(\partial_{\tau_{jk}}\varphi_\alpha)\,d\sigma &= \int_{\partial\Omega}\psi\big(\nu_j(\partial_k\varphi_\alpha)\big|_{\partial\Omega}-\nu_k(\partial_j\varphi_\alpha)\big|_{\partial\Omega}\big)\,d\sigma \\
&= \int_{\Omega}\big\{\partial_j(\psi\partial_k\varphi_\alpha)-\partial_k(\psi\partial_j\varphi_\alpha)\big\}\,dx \\
&= \int_{\Omega}\big\{\partial_j\psi\partial_k\varphi_\alpha-\partial_k\psi\partial_j\varphi_\alpha\big\}\,dx \\
&= \int_{\Omega}\big\{\partial_k(\partial_j\psi\varphi_\alpha)-\partial_j(\partial_k\psi\varphi_\alpha)\big\}\,dx \\
&= \int_{\partial\Omega}\big(\nu_k(\partial_j\psi)|_{\partial\Omega}-\nu_j(\partial_k\psi)|_{\partial\Omega}\big)(\varphi_\alpha|_{\partial\Omega})\,d\sigma. \qquad (3.6.5)
\end{aligned}
$$

Passing to the limit yields, upon recalling that $\varphi = 0$ on $\partial\Omega$,

$$
\int_{\partial\Omega}\psi(\partial_{\tau_{jk}}\varphi)\,d\sigma = 0. \qquad (3.6.6)
$$

To continue, we remark that thanks to Lemma 2.4.9, the fact that each Lipschitz function initially defined on a subset of $\mathbb{R}^{n+1}$ extends to the entire space (with control of the Lipschitz constant) and a standard mollification argument, we have

$$
\big\{\varphi|_{\partial\Omega} : \varphi \in C_0^\infty(\mathcal{R}^{n+1})\big\} \hookrightarrow L^p(\partial\Omega, d\sigma) \quad \text{densely, for each } p \in (1,\infty). \qquad (3.6.7)
$$

Granted this, (3.6.6) forces $\partial_{\tau_{jk}}\varphi = 0$ on $\partial\Omega$, which concludes the proof that Λ_{jk} is meaningly defined. Having established this, a reference to (3.6.3) allows us to conclude that Λ_{jk} extends to a functional in $(L^{p'}(\partial\Omega, d\sigma))^* = L^p(\partial\Omega, d\sigma)$. Riesz's representation theorem then ensures that there exists $g_{jk} \in L^p(\partial\Omega, d\sigma)$ with the property that

$$
\Lambda_{jk}(\varphi|_{\partial\Omega}) = \int_{\partial\Omega} g_{jk}\varphi\,d\sigma, \quad \forall\,\varphi \in C_0^1(\mathbb{R}^{n+1}). \qquad (3.6.8)
$$

In order to indicate the dependence of g_{jk} on f, from now on we shall denote this function by $\partial_{\tau_{kj}}f$. A calculation very similar in spirit to (3.6.5) then shows that this is compatible with (3.6.1). In summary, we have shown that, given $f \in L_1^p(\partial\Omega, d\sigma)$, for each j,k, there exists a unique function $\partial_{\tau_{kj}}f \in L^p(\partial\Omega, d\sigma)$ with the property that

$$
\int_{\partial\Omega} f(\partial_{\tau_{jk}}\varphi)\,d\sigma = \int_{\partial\Omega}(\partial_{\tau_{kj}}f)\varphi\,d\sigma, \quad \forall\,\varphi \in C_0^1(\mathbb{R}^{n+1}). \qquad (3.6.9)
$$

In particular, $\partial_{\tau_{jk}}f = -\partial_{\tau_{kj}}f$. If for $f \in L^p(\partial\Omega, d\sigma)$ we now define $\partial_{\tau_{jk}}f$ as a functional on $C_0^1(\mathbb{R}^{n+1})\big|_{\partial\Omega}$ by taking (3.6.9) as a definition, the reasoning above also shows that

$$
L_1^p(\partial\Omega, d\sigma) = \Big\{f \in L^p(\partial\Omega, d\sigma) : \partial_{\tau_{jk}}f \in L^p(\partial\Omega, d\sigma), \quad j,k = 1,\dots,n+1\Big\}. \qquad (3.6.10)
$$

When equipped with the natural norm, i.e.,

$$\|f\|_{L^p_1(\partial\Omega,d\sigma)} := \|f\|_{L^p(\partial\Omega,d\sigma)} + \sum_{j,k=1}^{n+1} \|\partial_{\tau_{jk}} f\|_{L^p(\partial\Omega,d\sigma)}, \tag{3.6.11}$$

the space $L^p_1(\partial\Omega, d\sigma)$ is Banach for each $1 < p < \infty$. Henceforth we assume that $\partial\Omega$ is compact. Then $L^p_1(\partial\Omega, d\sigma)$ is, for each $p \in (1,\infty)$, a module over $\{\varphi|_{\partial\Omega} : \varphi \in C^1_0(\mathbb{R}^{n+1})\}$. Moreover, the following Leibnitz rule holds (for each $p \in (1,\infty)$ and $j,k \in \{1,...,n+1\}$)

$$\partial_{\tau_{jk}}(\varphi f) = (\partial_{\tau_{jk}}\varphi)f + \varphi(\partial_{\tau_{jk}} f), \quad \forall\, \varphi \in C^1_0(\mathbb{R}^{n+1}),\ \forall\, f \in L^p_1(\partial\Omega, d\sigma). \tag{3.6.12}$$

Let us also point out here that, since $\{\varphi|_{\partial\Omega} : \varphi \in C^1_0(\mathbb{R}^{n+1})\}$ is a dense subspace of $L^p(\partial\Omega, d\sigma)$ for every $p \in (1,\infty)$, we have

$$L^p_1(\partial\Omega, d\sigma) \hookrightarrow L^p(\partial\Omega, d\sigma) \quad \text{densely, for every } \ p \in (1,\infty). \tag{3.6.13}$$

For each $1 < p < \infty$, let us now set

$$L^p_{-1}(\partial\Omega, d\sigma) := \Big(L^{p'}_1(\partial\Omega, d\sigma)\Big)^*, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \tag{3.6.14}$$

and note that the application

$$J : L^p_1(\partial\Omega, d\sigma) \longrightarrow \Big[L^p(\partial\Omega, d\sigma)\Big]^{1+n(n+1)/2}, \qquad Jf := \Big(f, (\partial_{\tau_{jk}} f)_{1\le j,k\le n+1}\Big), \tag{3.6.15}$$

has the property that Range J is closed, and J an isomorphism onto its range. This shows that $L^p_1(\partial\Omega, d\sigma)$ is a reflexive space, whenever $1 < p < \infty$. In particular, from (3.6.13), we obtain

$$L^p(\partial\Omega, d\sigma) \hookrightarrow L^p_{-1}(\partial\Omega, d\sigma) \quad \text{densely, for every } \ p \in (1,\infty). \tag{3.6.16}$$

Lemma 3.6.1 *Let $\Omega \subset \mathbb{R}^{n+1}$ be a* UR *domain. Also, assume that $u \in C^1(\Omega)$ is such that $\mathcal{N}(\nabla u)$, $\mathcal{N}(u) \in L^p(\partial\Omega, d\sigma)$ for some $p \in (1,\infty)$, and u along with $\partial_j u$, $1 \le j \le n+1$, have nontangential limits at σ-almost every boundary point on $\partial\Omega$. Then*

$$u\Big|_{\partial\Omega} \in L^p_1(\partial\Omega, d\sigma) \quad \textit{and} \quad \partial_{\tau_{jk}}(u|_{\partial\Omega}) = \nu_j(\partial_k u)\Big|_{\partial\Omega} - \nu_k(\partial_j u)\Big|_{\partial\Omega}. \tag{3.6.17}$$

In particular,

$$\sum_{j,k=1}^{n+1} \|\partial_{\tau_{jk}}(u|_{\partial\Omega})\|_{L^p(\partial\Omega,d\sigma)} \le C\|\mathcal{N}(\nabla u)\|_{L^p(\partial\Omega,d\sigma)}. \tag{3.6.18}$$

Proof. For two arbitrary indices $j,k \in \{1,...,n+1\}$ and $\varphi \in C_0^1(\mathbb{R}^{n+1})$ we may write based on Proposition 3.2.5 and Theorem 2.3.1 (or Corollary 3.2.8, if Ω is unbounded) applied twice

$$\begin{aligned}\int_{\partial\Omega} u(\partial_{\tau_{kj}}\varphi)\,d\sigma &= \int_{\partial\Omega} u(\nu_k\partial_j\varphi - \nu_j\partial_k\varphi)\,d\sigma \\ &= \int_{\Omega}\Big(\partial_k u\partial_j\varphi - \partial_j u\partial_k\varphi\Big)\,dX \\ &= \int_{\partial\Omega}\Big(\nu_j(\partial_k u)|_{\partial\Omega} - \nu_k(\partial_j u)|_{\partial\Omega}\Big)\varphi\,d\sigma. \qquad (3.6.19)\end{aligned}$$

Since φ is arbitrary, (3.6.19) shows that $u|_{\partial\Omega} \in L_1^p(\partial\Omega, d\sigma)$ and $\nu_r(\partial_s u)|_{\partial\Omega} - \nu_s(\partial_r u)|_{\partial\Omega} = \partial_{\tau_{rs}}u$, proving (3.6.17). Then (3.6.18) is a consequence of the second formula in (3.6.17). $\square$

Consider now a second-order differential operator (here and below we use the summation convention):

$$Lu := \Big(\partial_r(a_{rs}^{\alpha\beta}\partial_s u_\beta)\Big)_\alpha \qquad (3.6.20)$$

with constant (real) coefficients, which is strongly elliptic in the sense that there exists $\kappa > 0$ such that the following Legendre-Hadamard condition is satisfied:

$$a_{rs}^{\alpha\beta}\xi_r\xi_s\zeta_\alpha\zeta_\beta \geq \kappa|\xi|^2|\zeta|^2, \qquad \forall\,\xi = (\xi_r)_r,\ \forall\,\zeta = (\zeta_\alpha)_\alpha. \qquad (3.6.21)$$

Also, denote by $E \in C^\infty(\mathbb{R}^{n+1}\setminus\{0\})$ a (matrix-valued) fundamental solution for $L^\top$, the adjoint of L, which is even and homogeneous of degree $-(n-1)$, and define the single layer and its boundary version by setting

$$\mathcal{S}f(X) := \int_{\partial\Omega} E(X-Y)f(Y)\,d\sigma(Y), \qquad X \in \mathbb{R}^{n+1}\setminus\partial\Omega, \qquad (3.6.22)$$

$$Sf(X) := \int_{\partial\Omega} E(X-Y)f(Y)\,d\sigma(Y), \qquad X \in \partial\Omega. \qquad (3.6.23)$$

Also, if $E = (E_{\beta\gamma})_{\beta,\gamma}$ and $f = (f_\alpha)_\alpha$ are defined on $\partial\Omega$, introduce the associated double layer and its principal value version by setting

$$\begin{aligned}\mathcal{D}f(X) &:= \Big(-\int_{\partial\Omega}\nu_s(Y)a_{rs}^{\alpha\beta}(\partial_r E_{\gamma\beta})(X-Y)f_\alpha(Y)\,d\sigma(Y)\Big)_\gamma, \quad X \in \mathbb{R}^{n+1}\setminus\partial\Omega, \quad (3.6.24)\\ Kf(X) &:= \Big(-\lim_{\varepsilon\to0^+}\int_{\substack{Y\in\partial\Omega\\|X-Y|>\varepsilon}}\nu_s(Y)a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(X-Y)f_\alpha(Y)\,d\sigma(Y)\Big)_\gamma, \quad X \in \partial\Omega. (3.6.25)\end{aligned}$$

Finally, if $\Omega \subset \mathbb{R}^{n+1}$ is an open set, we define

$$\Omega_+ := \Omega, \qquad \Omega_- := \mathbb{R}^{n+1}\setminus\bar{\Omega}. \qquad (3.6.26)$$

In particular, whenever $1 < p < \infty$,

$$\mathcal{S}f\Big|_{\partial\Omega_\pm} = Sf, \qquad \forall\, f \in L^p(\partial\Omega, d\sigma). \tag{3.6.27}$$

Proposition 3.6.2 *Let $\Omega \subset \mathbb{R}^{n+1}$ be a* UR *domain, and take $p \in (1,\infty)$. Then for each $f \in L^p_1(\partial\Omega, d\sigma)$ the nontangential trace $\partial_j \mathcal{D}f\Big|_{\partial\Omega}$ exists σ-a.e. on $\partial\Omega$, for each $j \in \{1, ..., n+1\}$, and*

$$\|\mathcal{N}(\nabla\mathcal{D}f)\|_{L^p(\partial\Omega,d\sigma)} \leq C\|f\|_{L^p_1(\partial\Omega,d\sigma)}, \tag{3.6.28}$$

for some finite constant $C > 0$ depending only on p, α, as well as the Ahlfors regularity and UR constants of $\partial\Omega$.

Proof. Assume first that Ω is bounded. Then for each index γ, point $X \in \Omega$ and $j \in \{1, ..., n+1\}$, we have

$$\begin{aligned} \partial_j\Big(\mathcal{D}f\Big)_\gamma(X) &= -\int_{\partial\Omega} \nu_s(Y) a^{\beta\alpha}_{rs} (\partial_j\partial_r E_{\gamma\beta})(X-Y) f_\alpha(Y)\, d\sigma(Y) \\ &= -\int_{\partial\Omega} a^{\beta\alpha}_{rs} \partial_{\tau_{js}(Y)}[(\partial_r E_{\gamma\beta})(X-Y)] f_\alpha(Y)\, d\sigma(Y) \\ &= \int_{\partial\Omega} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(X-Y)(\partial_{\tau_{js}} f_\alpha)(Y)\, d\sigma(Y), \end{aligned} \tag{3.6.29}$$

where in the second equality we have used the fact that $a^{\beta\alpha}_{rs}\nu_j(Y)(\partial_s\partial_r E_{\gamma\beta})(X-Y) = 0$ for $X \neq Y$, and we have integrated by parts on the boundary. Then the desired conclusions follow easily from this.

When $\partial\Omega$ is unbounded, more attention should be paid to the third step in (3.6.29), since (3.6.4) no longer directly applies, as $\varphi(Y) := (\partial_r E_{\gamma\beta})(X-Y)$ fails to have compact support. This issue, nonetheless, can be addressed in a straightforward manner, inserting a cut-off factor $\psi(Y/R)$, where $\psi \in C^\infty_0(\mathbb{R}^{n+1})$ satisfies $\psi \equiv 1$ on $B(0,1)$. Similarly to (3.6.12), we express $\psi(Y/R)\partial_{\tau_{js}(Y)}[(\partial_r E_{\gamma\beta})(X-Y)]$ as

$$\partial_{\tau_{js}(Y)}[\psi(Y/R)(\partial_r E_{\gamma\beta})(X-Y)] - (\partial_r E_{\gamma\beta})(X-Y)\partial_{\tau_{js}(Y)}[\psi(Y/R)], \tag{3.6.30}$$

then send $R \to \infty$, and note that residual terms above to converge to zero on account of $|\nabla[\psi(Y/R)]| \leq C/R$. Since $\psi(Y/R) \to 1$ as $R \to \infty$, by relying on Lebesgue's Dominated Convergence Theorem, this ultimately shows that

$$\partial_j\Big(\mathcal{D}f\Big)_\gamma(X) = \int_{\partial\Omega} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(X-Y)(\partial_{\tau_{js}} f_\alpha)(Y)\, d\sigma(Y), \tag{3.6.31}$$

irrespective of whether $\partial\Omega$ is bounded or not. $\square$

Recall next the (principal value) double layer potential operator K introduced in (3.6.25), as well as the (boundary version) single layer S from (3.6.23). We have:

Corollary 3.6.3 *If $\Omega \subset \mathbb{R}^{n+1}$ is a* UR *domain, then*

$$K : L^p_1(\partial\Omega, d\sigma) \longrightarrow L^p_1(\partial\Omega, d\sigma) \tag{3.6.32}$$

is a well-defined, bounded operator for every $p \in (1, \infty)$. In addition, if $\partial\Omega$ is compact, then so is

$$S : L^p(\partial\Omega, d\sigma) \longrightarrow L^p_1(\partial\Omega, d\sigma). \tag{3.6.33}$$

Proof. From Theorem 3.5.2 we have that

$$\mathcal{D}f\Big|_{\partial\Omega} = (\tfrac{1}{2}I + K)f, \qquad \forall\, f \in L^p(\partial\Omega, d\sigma). \tag{3.6.34}$$

Thus, the fact that the double layer potential operator (3.6.32) is well-defined and bounded follows from this and Lemma 3.6.1. The same lemma and Theorem 3.5.2 also prove that the operator (3.6.33) is bounded when $\partial\Omega$ is compact. $\square$

Proposition 3.6.4 *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded* UR *domain. Then for each $1 < p < \infty$, the single layer S, originally defined as in (3.6.33), extends to a bounded operator*

$$S : L^p_{-1}(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma). \tag{3.6.35}$$

Also, there exists a finite constant $C = C(\Omega, p) > 0$ such that

$$\|\mathcal{N}(\mathcal{S}f)\|_{L^p(\partial\Omega, d\sigma)} \le C\|f\|_{L^p_{-1}(\partial\Omega, d\sigma)}, \qquad \forall\, f \in L^p_{-1}(\partial\Omega, d\sigma). \tag{3.6.36}$$

Finally, for each $f \in L^p_{-1}(\partial\Omega, d\sigma)$ the nontangential pointwise trace $\mathcal{S}f\Big|_{\partial\Omega}$ exists at σ-a.e. point on $\partial\Omega$, and in fact

$$\mathcal{S}f\Big|_{\partial\Omega} = Sf, \qquad \forall\, f \in L^p_{-1}(\partial\Omega, d\sigma). \tag{3.6.37}$$

Proof. From (3.6.33) and duality (cf. (3.6.14)), we see that

$$S^* : L^p_{-1}(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma) \tag{3.6.38}$$

is well-defined and bounded for every $p \in (1, \infty)$. To justify dropping the star, we need to show that the action of this operator is compatible with that of S from (3.6.33). However, this is a consequence of (3.6.16), along with the simple observation that the adjoint of $S : L^p(\partial\Omega, d\sigma) \to L^p(\partial\Omega, d\sigma)$ is $S : L^{p'}(\partial\Omega, d\sigma) \to L^{p'}(\partial\Omega, d\sigma)$, $1/p + 1/p' = 1$. This proves that S in (3.6.33) extends uniquely to a bounded, linear operator in the context of (3.6.35).

Next, for $1 < p, p' < \infty$ with $1/p + 1/p' = 1$, we shall prove the following characterization of the space (3.6.14). There exists some $C = C(\Omega, p) > 0$ such that

For all $f \in L^p_{-1}(\partial\Omega, d\sigma)$ there exist $f_0,\ f_{jk} \in L^p(\partial\Omega, d\sigma),\ 1 \le j < k \le n+1,$

$$
\begin{aligned}
&\text{such that } \ \|f_0\|_{L^p(\partial\Omega,d\sigma)} + \sum_{1\le j<k\le n+1} \|f_{jk}\|_{L^p(\partial\Omega,d\sigma)} \le C\|f\|_{L^p_{-1}(\partial\Omega,d\sigma)} \\
&\text{and } \ \langle f, g\rangle = \int_{\partial\Omega}\Big(f_0 g + \sum_{1\le j<k\le n+1} f_{jk}\partial_{\tau_{jk}} g\Big)\, d\sigma \ \text{ for every } \ g \in L^{p'}_1(\partial\Omega, d\sigma).
\end{aligned}
\tag{3.6.39}
$$

To prove this, recall the mapping J from (3.6.15) and consider the composition

$$
f \circ J^{-1} : \mathrm{Range}\, J \longrightarrow \mathbb{R}, \tag{3.6.40}
$$

where $J^{-1} : \mathrm{Range}\, J \to L^{p'}_1(\partial\Omega, d\sigma)$ is an isomorphism, and f is regarded as a functional in $\Big(L^{p'}_1(\partial\Omega, d\sigma)\Big)^*$. Since $\mathrm{Range}\, J$ is a closed subspace of $\Big[L^{p'}(\partial\Omega, d\sigma)\Big]^{1+n(n+1)/2}$, Hahn-Banach's Extension Theorem in concert with Riesz's Representation Theorem ensure the existence of $f_0, f_{jk} \in L^p(\partial\Omega, d\sigma)$, $1 \le j < k \le n+1$, such that the properties listed in (3.6.39) hold.

For each $f \in L^p_{-1}(\partial\Omega, d\sigma)$, $1 < p < \infty$, and $g \in \Big(L^{p'}_1(\partial\Omega, d\sigma)\Big)^*$ we let $f(g) := \langle f, g\rangle$ denote the obvious dual pairing. For such f, we can now (unequivocally) define

$$
\begin{aligned}
\mathcal{S}f(X) &:= f\Big(E(X-\cdot)\Big) = f\circ J^{-1}\circ J\Big(E(X-\cdot)\Big) \\
&= \int_{\partial\Omega} E(X-Y) f_0(Y)\, d\sigma(Y) + \int_{\partial\Omega} \partial_{\tau_{jk}}[E(X-Y)] f_{jk}(Y)\, d\sigma(Y), \quad X \in \Omega,
\end{aligned}
\tag{3.6.41}
$$

where $f_0, f_{jk} \in L^p(\partial\Omega, d\sigma)$, $1 \le j < k \le n+1$, are as in (3.6.39). We observe that even though the functional $f \circ J^{-1}$, acting initially on Range J, may not have a unique extension to all of $\Big[L^{p'}(\partial\Omega, d\sigma)\Big]^{1+n(n+1)/2}$, the expression in (3.6.41) is well-defined, since *every* such extension $\Big(f_0, (f_{jk})_{1\le j,k\le n+1}\Big)$ must agree on Range J. Then (3.6.36) is a consequence of this, (3.6.39) and (3.2.12). Also, the existence of the nontangential pointwise trace $\mathcal{S}f\Big|_{\partial\Omega}$ at σ-a.e. point on $\partial\Omega$ follows from (3.6.41) and Theorem 3.5.2.

To justify (3.6.37) we only need to observe that, from what we have proved to this point, the mappings (3.6.35) and $L^p_{-1}(\partial\Omega, d\sigma) \ni f \mapsto \mathcal{S}f\Big|_{\partial\Omega} \in L^p(\partial\Omega, d\sigma)$ are well-defined, linear and bounded. Since, by (3.6.27), they coincide on $L^p(\partial\Omega, d\sigma)$ which, by (3.6.16), is a dense subspace in their common domain, (3.6.37) follows. This finishes the proof of the proposition. $\square$

Let us now define the tangential gradient operator by setting

$$
\nabla_{\tan} f := \Big(\nu_k \partial_{\tau_{kj}} f\Big)_{1\le j\le n+1}, \qquad \forall\, f \in L^p_1(\partial\Omega, d\sigma). \tag{3.6.42}
$$

Lemma 3.6.5 *Assume that* $\Omega \subset \mathbb{R}^{n+1}$ *is a* UR *domain. Then for each function* $f \in L^p_1(\partial\Omega, d\sigma)$

$$
\partial_{\tau_{jk}} f = \nu_j (\nabla_{\tan} f)_k - \nu_k (\nabla_{\tan} f)_j, \qquad j,k = 1, ..., n+1, \tag{3.6.43}
$$

σ-a.e. on $\partial\Omega$. In particular,

$$\|\nabla_{\tan} f\|_{L^p(\partial\Omega,d\sigma)} \approx \sum_{j,k=1}^{n+1} \|\partial_{\tau_{jk}} f\|_{L^p(\partial\Omega,d\sigma)}, \qquad \forall\, f \in L^p_1(\partial\Omega, d\sigma). \tag{3.6.44}$$

Also, for every $f \in L^p_1(\partial\Omega, d\sigma)$,

$$\langle \nu, \nabla_{\tan} f\rangle = 0 \quad \sigma\text{-a.e. on } \partial\Omega. \tag{3.6.45}$$

Proof. Let $f \in L^p_1(\partial\Omega, d\sigma)$ be arbitrary and select an operator L as in (3.6.20)-(3.6.21) (for example, $L = \Delta$ will do). As before, let E be a fundamental solution of L and construct the double layer $\mathcal{D}$ as in (3.6.24), along with its principal value version (3.6.25). Also, set

$$u^{\pm}(X) := \mathcal{D}f(X) \ \text{ for } \ X \in \Omega_{\pm}. \tag{3.6.46}$$

Thus, by (3.6.31), Theorem 3.4.2, (3.3.9) and Proposition 3.6.2

$$\begin{array}{ll} \text{there exist } \nabla u^{\pm}\Big|_{\partial\Omega}, & u^{\pm}\Big|_{\partial\Omega} = (\pm\tfrac{1}{2}I + K)f \\ \text{and} & \|\mathcal{N}(\nabla u^{\pm})\|_{L^p(\partial\Omega,d\sigma)} \leq C\|f\|_{L^p_1(\partial\Omega,d\sigma)}. \end{array} \tag{3.6.47}$$

Next, for two arbitrary indices $j, k \in \{1, ..., n+1\}$ we decompose

$$\begin{aligned} &\nu_j(\nabla_{\tan} f)_k - \nu_k(\nabla_{\tan} f)_j \\ &\quad = \nu_j\Big(\nabla_{\tan}(\tfrac{1}{2}I + K)f\Big)_k - \nu_k\Big(\nabla_{\tan}(\tfrac{1}{2}I + K)f\Big)_j \\ &\qquad - \nu_j\Big(\nabla_{\tan}(-\tfrac{1}{2}I + K)f\Big)_k + \nu_k\Big(\nabla_{\tan}(-\tfrac{1}{2}I + K)f\Big)_j, \end{aligned} \tag{3.6.48}$$

so (3.6.43) is proved as soon as we show that

$$\nu_j\Big(\nabla_{\tan}(\pm\tfrac{1}{2}I + K)f\Big)_k - \nu_k\Big(\nabla_{\tan}(\pm\tfrac{1}{2}I + K)f\Big)_j = \partial_{\tau_{jk}}(\pm\tfrac{1}{2}I + K)f. \tag{3.6.49}$$

By the trace identity in (3.6.47), this is equivalent to showing that

$$\nu_j\Big(\nabla_{\tan}(u^{\pm}|_{\partial\Omega})\Big)_k - \nu_k\Big(\nabla_{\tan}(u^{\pm}|_{\partial\Omega})\Big)_j = \partial_{\tau_{jk}}(u^{\pm}|_{\partial\Omega}). \tag{3.6.50}$$

Now, by (3.6.42) and Lemma 3.6.1, for each $j, k \in \{1, \dots, n+1\}$ we have

$$\begin{aligned} \nu_j(\nabla_{\tan} u^{\pm}|_{\partial\Omega})_k - \nu_k(\nabla_{\tan} u^{\pm}|_{\partial\Omega})_j &= \nu_j\nu_r\partial_{\tau_{rk}}(u^{\pm}|_{\partial\Omega}) - \nu_k\nu_s\partial_{\tau_{sj}}(u^{\pm}|_{\partial\Omega}) \\ &= \nu_j\nu_r\nu_r(\partial_k u^{\pm})|_{\partial\Omega} - \nu_j\nu_r\nu_k(\partial_s u^{\pm})|_{\partial\Omega} \\ &\quad -\nu_k\nu_s\nu_s(\partial_j u^{\pm})|_{\partial\Omega} + \nu_k\nu_s\nu_j(\partial_s u^{\pm})|_{\partial\Omega} \\ &= \nu_j(\partial_k u^{\pm})|_{\partial\Omega} - \nu_k(\partial_j u^{\pm})|_{\partial\Omega} \\ &= \partial_{\tau_{jk}}(u^{\pm}|_{\partial\Omega}), \end{aligned} \tag{3.6.51}$$

i.e., (3.6.50) holds. This completes the proof of (3.6.43). The identity (3.6.45) is proved in a similar fashion and this finishes the proof of the lemma. $\square$

We conclude this subsection with the following useful remark.

Proposition 3.6.6 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded* UR *domain and that $f \in L^p_1(\partial\Omega, d\sigma)$ for some $p \in (1, \infty)$. Then*

$$f = \textit{locally constant on } \partial\Omega \Longleftrightarrow \nabla_{\tan} f = 0. \tag{3.6.52}$$

Proof. The left-to-right implication is a consequence of (3.6.42) and the fact that the tangential derivatives $\partial_{\tau_{jk}}$ annihilate constants. In the opposite direction, assume that $f \in L^p_1(\partial\Omega, d\sigma)$ has $\nabla_{\tan} f = 0$, and denote by $\mathcal{D}$ the harmonic double layer operator associated with $\partial\Omega$. Then (3.6.31) and (3.6.43) imply that $\mathcal{D}f$ is locally constant both in $\Omega_+ := \Omega$ and its complement, Ω_-, from which we may deduce that $f = \mathcal{D}f|_{\partial\Omega_+} - \mathcal{D}f|_{\partial\Omega_-}$ is locally constant on $\partial\Omega$. $\square$

4 Semmes-Kenig-Toro domains, Poincaré inequalities, and singular integrals

In this section we discuss a special class of UR domains Ω for which certain important layer potentials are not merely bounded on $L^p(\partial\Omega, d\sigma)$ but are actually compact, extending results of [37] that deal with C^1 domains. The domains we identify as having this property are denoted here regular Semmes-Kenig-Toro domains (regular SKT domains for short). We give definitions and basic properties of this class of domains in §4.1. These domains are special cases of what we call SKT domains, which in turn are special cases of Reifenberg flat domains, which we also briefly discuss. We also define the class of ε-regular SKT domains, for $\varepsilon > 0$, a class for which we will show such layer potentials have small norm modulo compacts if ε is small.

As we have mentioned in the Introduction, SKT domains have been called chord arc domains. The notion of chord-arc domains originated in dimension 2, where the defining condition is that the length of a boundary arc between two points should not exceed the length of a chord between these points by too great a factor. The notion in higher dimensions, which is somewhat more sophisticated, originated in S. Semmes [101] and was further developed in [64]–[66]. In higher dimensions, this "chord arc" designation is not so successful in describing the essential features of these domains, so we propose to call them SKT domains. Similarly, we have relabeled what in these papers were called chord arc domains with vanishing constant, calling them regular SKT domains.

In §4.2 we discuss a Poincaré inequality of Semmes and some refinements, define the Semmes decomposition of an SKT domain, and apply the Poincaré inequality to obtain further results on this Semmes decomposition. We use this to obtain further equivalent characterizations of regular SKT domains. In particular we show that an open bounded set $\Omega \subset \mathbb{R}^{n+1}$ is a regular SKT domain if and only if Ω is a two-sided NTA domain that is Ahlfors regular and such that $\nu \in \mathrm{VMO}(\partial\Omega, d\sigma)$.

In §4.3 we make use of the Poincaré inequality of §4.2 to demonstrate that if Ω satisfies a two-sided John condition and is Ahlfors regular, then the Sobolev space $L^p_1(\partial\Omega, d\sigma)$ is isomorphic to the space $W^{p,1}(\partial\Omega)$ defined for general metric measure spaces by Hajłasz [46]. This has a number of useful consequences, one being the denseness in $L^p_1(\partial\Omega, d\sigma)$ of the space of Lipschitz functions on $\partial\Omega$.

§4.5 is the heart of this section. Here we single out a class of layer potentials, which as we will see are of particular interest in the analysis of elliptic boundary problems, and show that they are compact on $L^p(\partial\Omega, d\sigma)$ for $p \in (1, \infty)$ when Ω is a regular SKT domain. A quantitative version of this result, involving the concept of ε-regular SKT domain, is also presented. As a preliminary step, in §4.4 we present a treatment of such compactness in the case of VMO_1 domains, based on work in [50], which also plays a role in the proof of compactness in §4.5, as does the Poincaré inequality established in §4.2.

In §4.6 we show that whenever Ω is a UR domain, satisfying a two-sided John condition, then such compactness (accompanied by compactness of a natural family of commutators) implies that Ω is a regular SKT domain, thus completing the circle of our compactness results. Going further, we estimate the distance from ν to $\mathrm{VMO}\,(\partial\Omega, d\sigma)$ in terms of the distance of a selected family of such operators to the space of compact operators.

In §4.7 we consider "Clifford-Szegö projections", defined a priori on $L^2(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}$, and establish L^p extensions when Ω is a bounded, regular SKT domain. In doing so, we bring in analogues of Kerzman-Stein formulas ([68]).

4.1 Reifenberg flat domains, SKT domains, and regular SKT domains

Here we present definitions and basic properties of Reifenberg-flat domains, SKT domains, and regular SKT domains. Our presentation in this subsection follows closely that in [64]–[66].

Definition 4.1.1 *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a nonempty, locally compact set and let $\delta \in (0, \frac{1}{4\sqrt{2}})$. We say that Σ is δ-`Reifenberg flat` if for each compact set $\mathcal{K} \subset \mathbb{R}^{n+1}$ there exists $R = R(\mathcal{K}) > 0$ such that for every $Q \in \mathcal{K} \cap \Sigma$ and every $r \in (0, R]$ there exists a n-dimensional plane $L(Q, r)$ which contains Q and such that*

$$\frac{1}{r}\, D[\Sigma \cap B(Q,r)\,,\, L(Q,r) \cap B(Q,r)] \leq \delta \tag{4.1.1}$$

where $B(Q,r) := \{X \in \mathbb{R}^{n+1} : |X - Q| < r\}$ and, for each $A, B \subset \mathbb{R}^{n+1}$,

$$D[A,B] := \max\Big\{\sup\{\mathrm{dist}\,(a,B) : a \in A\}\,,\, \sup\{\mathrm{dist}\,(b,A) : b \in B\}\Big\} \tag{4.1.2}$$

is the Hausdorff distance between the sets A, B.

As in [65], for each $Q \in \Sigma$ and $r > 0$, introduce

$$\theta(Q,r) := \inf_{L} \Big\{\frac{1}{r}\, D[\Sigma \cap B(Q,r)\,,\, L \cap B(Q,r)]\Big\}, \tag{4.1.3}$$

where the infimum is taken over all n-planes containing Q, so that condition (4.1.1) becomes

$$\forall\, \mathcal{K} \subset \mathbb{R}^{n+1} \ \text{ compact, } \ \exists\, R > 0 \ \text{ such that } \ \sup_{0<r\leq R}\ \sup_{Q\in\Sigma\cap\mathcal{K}} \theta(Q,r) \leq \delta. \tag{4.1.4}$$

Definition 4.1.2 *We say that $\Sigma \subset \mathbb{R}^{n+1}$ is a* `Reifenberg flat set with vanishing constant` *if it is δ-Reifenberg flat for some $\delta \in (0, \frac{1}{4\sqrt{2}})$ and for each compact set $\mathcal{K} \subset \mathbb{R}^{n+1}$ there holds*

$$\lim_{r\to 0^+}\ \sup_{Q\in\Sigma\cap\mathcal{K}} \theta(Q,r) = 0. \tag{4.1.5}$$

Definition 4.1.3 *We say that $\Omega \subset \mathbb{R}^{n+1}$ has the* `separation property` *if for each compact set $\mathcal{K} \subset \mathbb{R}^{n+1}$ there exists $R > 0$ such that for every $Q \in \partial\Omega \cap \mathcal{K}$ and $r \in (0, R]$ there exists an n-dimensional plane $\mathcal{L}(Q, r)$ containing Q and a choice of unit normal vector to $\mathcal{L}(Q, r)$, $\vec{n}_{Q,r}$, satisfying*

$$\begin{aligned} &\{X + t\,\vec{n}_{Q,r} \in B(Q, r) : \, X \in \mathcal{L}(Q, r), \; t < -\tfrac{r}{4}\} \subset \Omega, \\ &\{X + t\,\vec{n}_{Q,r} \in B(Q, r) : \, X \in \mathcal{L}(Q, r), \; t > \tfrac{r}{4}\} \subset \mathbb{R}^{n+1} \setminus \Omega. \end{aligned} \tag{4.1.6}$$

Moreover, if Ω is unbounded, we also require that $\partial\Omega$ divides $\mathbb{R}^{n+1}$ into two distinct connected components and that $\mathbb{R}^{n+1} \setminus \Omega$ has a non-empty interior.

Note that the separation property clearly implies

$$\partial\Omega = \partial_*\Omega, \tag{4.1.7}$$

i.e., the topological boundary and the measure-theoretic boundary of Ω coincide.

The following result is proved in §3 of [64].

Theorem 4.1.4 *There exists a dimensional constant $\delta_n \in (0, \frac{1}{4\sqrt{2}})$ with the property that any domain $\Omega \subset \mathbb{R}^{n+1}$ that has the separation property and whose boundary is a δ-Reifenberg flat set, $\delta \in (0, \delta_n)$, is an* NTA*-domain.*

Definition 4.1.5 *Let $\Omega \subset \mathbb{R}^{n+1}$ and $\delta \in (0, \delta_n)$. Call Ω a* δ-`Reifenberg flat domain` *if Ω has the separation property and $\partial\Omega$ is δ-Reifenberg flat. Moreover, if Ω is unbounded, we shall also require that*

$$\sup_{r>0} \sup_{Q \in \partial\Omega} \theta(Q, r) < \delta_n. \tag{4.1.8}$$

If Ω is a δ-Reifenberg flat domain and $\partial\Omega$ is Reifenberg-flat with vanishing constant, we say Ω is a Reifenberg flat domain with vanishing constant.

As a consequence of Theorem 4.1.4 and the above definition we have the following.

Corollary 4.1.6 *If the open set $\Omega \subset \mathbb{R}^{n+1}$ is a δ-Reifenberg flat domain with $\delta \in (0, \delta_n)$ then Ω is a two-sided NTA domain.*

Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain of locally finite perimeter, such that $\mathcal{H}^n(\partial\Omega \setminus \partial_*\Omega) = 0$ and $\partial\Omega$ satisfies the Ahlfors-David regularity condition. We denote by ν the measure-theoretic outward unit normal to $\partial\Omega$ and refer to $\sigma := \mathcal{H}^n \lfloor \partial\Omega$ as the surface measure of the boundary of Ω. Then $(\partial\Omega$, Euclidean distance, $\sigma)$ becomes a space of homogeneous type, for which the definitions and the results in § 2.4 apply. In particular, we define the space $\mathrm{BMO}(\partial\Omega, d\sigma)$ as the collection of functions $f \in L^2_{\mathrm{loc}}(\partial\Omega, d\sigma)$ with the property that $\|f\|_* < +\infty$ where

$$\|f\|_* := \sup_{r>0} \sup_{Q \in \partial\Omega} \Big(\frac{1}{\sigma(\Delta(Q, r))} \int_{\Delta(Q,r)} |f - f_{\Delta(Q,r)}|^2 \, d\sigma \Big)^{1/2}, \tag{4.1.9}$$

$$\Delta(Q, r) := \partial\Omega \cap B(Q, r), \qquad f_{\Delta(Q,r)} := \frac{1}{\sigma(\Delta(Q, r))} \int_{\Delta(Q,r)} f \, d\sigma. \tag{4.1.10}$$

As in (2.4.15), the definition of $\|f\|_*$ is slightly adjusted when $\partial\Omega$ is compact, by adding $|\int_{\partial\Omega} f\,d\sigma|$ in the right-hand side of (4.1.9).

Consistent with (2.4.21), we denote by $\mathrm{VMO}(\partial\Omega, d\sigma)$ the closure in $\mathrm{BMO}(\partial\Omega, d\sigma)$ of the space of uniformly continuous real-valued functions belonging to $\mathrm{BMO}(\partial\Omega, d\sigma)$. Then Proposition 2.4.8 provides an alternative description of $\mathrm{VMO}(\partial\Omega, d\sigma)$, as the space of functions $f \in L^2_{\mathrm{loc}}(\partial\Omega, d\sigma)$ with the property that

$$\lim_{r\to 0^+} \sup_{Q\in\partial\Omega} \|f\|_*(\Delta(Q,r)) = 0, \tag{4.1.11}$$

where we have set (compare with (2.4.77))

$$\|f\|_*(\Delta(Q,r)) := \sup_{\Delta\subset\Delta(Q,r)} \Big(\fint_\Delta |f - f_\Delta|^2\,d\sigma\Big)^{1/2}, \tag{4.1.12}$$

with the supremum taken over all surface balls Δ contained in $\Delta(Q,r)$.

As noted in Remark 4.2 on p. 397 of [65], if $\Omega \subset \mathbb{R}^{n+1}$ is a set of locally finite perimeter and a δ-Reifenberg flat domain for some $\delta \in (0,\delta_n)$, then as a consequence of (4.1.7), $\partial\Omega$ and the measure-theoretic boundary of Ω agree. In particular, the the measure-theoretic outward unit normal ν is well-defined σ-a.e. on $\partial\Omega$.

Another observation of interest (cf. Remark 4.1 on pp. 396-397 of [65]) is as follows. If $\Omega \subset \mathbb{R}^{n+1}$ is a δ-Reifenberg flat domain for some $\delta \in (0,\delta_n)$ then given any compact set $\mathcal{K} \subset \mathbb{R}^{n+1}$ there exists $R > 0$ with the property that

$$\sigma(\Delta(Q,r)) \geq (1+\delta)^{-1}\omega_n r^n, \qquad \forall\, Q \in \partial\Omega\cap\mathcal{K},\ \ \forall\, r\in(0,R]. \tag{4.1.13}$$

Definition 4.1.7 *Let $\delta \in (0,\delta_n)$, where δ_n is as in Theorem 4.1.4. A set $\Omega \subset \mathbb{R}^{n+1}$ of locally finite perimeter is said to be a* `δ-SKT domain` *if Ω is a δ-Reifenberg flat domain, $\partial\Omega$ satisfies the Ahlfors-David regularity condition and, for each compact set $\mathcal{K} \subset \mathbb{R}^{n+1}$, there exists $R > 0$ such that*

$$\sup_{Q\in\partial\Omega\cap\mathcal{K}} \|\nu\|_*(\Delta(Q,R)) < \delta, \tag{4.1.14}$$

where, as before, ν is the measure-theoretic outward unit normal to $\partial\Omega$.

Definition 4.1.8 *Call $\Omega \subset \mathbb{R}^{n+1}$ a* `regular SKT domain` *if Ω is a δ-SKT domain for some $\delta \in (0,\delta_n)$ and, in addition, $\nu \in \mathrm{VMO}(\partial\Omega, d\sigma)$.*

For the goals we have in mind, it is natural to finally make the following.

Definition 4.1.9 *An open set $\Omega \subset \mathbb{R}^{n+1}$ is called an* `ε-regular SKT domain` *if Ω is a δ-SKT domain for some $\delta \in (0,\delta_n)$ and, in addition, $\mathrm{dist}\big(\nu, \mathrm{VMO}(\partial\Omega, d\sigma)\big) < \varepsilon$ where the distance is taken in the $\mathrm{BMO}(\partial\Omega, d\sigma)$ norm.*

Definitions 4.1.7 and 4.1.8 are those given in [65] (where the domains were called, respectively δ-chord arc domains and chord arc domains with vanishing constant). In this connection, it is useful to recall Theorem 4.6 of [65], which says that if $\Omega \subset \mathbb{R}^{n+1}$ is a set of locally finite perimeter and also a δ-Reifenberg flat domain for some $\delta \in (0,\delta_n)$, the following statements are equivalent:

(1) Ω is a regular SKT domain;

(2) Ω is a Reifenberg-flat domain with vanishing constant, and for each compact K meeting $\partial\Omega$,

$$\lim_{r\to 0} \sup_{Q\in\partial\Omega\cap K} \frac{\sigma(\Delta(Q,r))}{\omega_n r^n} = 1, \tag{4.1.15}$$

where $\Delta(Q,r) = B_r(Q)\cap\partial\Omega$ and ω_n is the volume of the unit ball in $\mathbb{R}^n$.

(3) For each compact $K\subset\mathbb{R}^{n+1}$ meeting $\partial\Omega$,

$$\lim_{r\to 0} \inf_{Q\in\partial\Omega\cap K} \frac{\sigma(\Delta(Q,r))}{\omega_n r^n} = \lim_{r\to 0} \sup_{Q\in\partial\Omega\cap K} \frac{\sigma(\Delta(Q,r))}{\omega_n r^n} = 1. \tag{4.1.16}$$

From this we also have the following.

Proposition 4.1.10 *Assume that $\Omega\subset\mathbb{R}^{n+1}$ is a regular SKT domain. Then Ω is a δ-SKT domain for each $\delta\in(0,\delta_n)$.*

Proof. As stated above, Ω is a Reifenberg flat domain with vanishing constant. In particular, Ω is δ-Reifenberg flat for each $\delta\in(0,\delta_n)$, and the desired conclusion follows now from definitions. □

Directly from definitions, we also have:

Proposition 4.1.11 *An open set $\Omega\subset\mathbb{R}^{n+1}$ is a regular SKT domain if and only if it is an ε-regular SKT domain for each $\varepsilon>0$.*

The following result is relevant to applying layer potentials.

Proposition 4.1.12 *If $\Omega\subset\mathbb{R}^{n+1}$ is a δ-SKT domain for some $\delta\in(0,\delta_n)$, then Ω is an* UR *domain. In particular, any ε-regular SKT domain is a* UR *domain.*

Proof. The hypotheses imply that $\partial\Omega$ is Ahlfors regular and that Ω is δ-Reifenberg flat for small δ, and hence Ω is an NTA domain. As we saw in §3.1, all Ahlfors regular NTA domains are UR domains. □

4.2 A Poincaré type inequality, Semmes decomposition, and consequences

The following Poincaré inequality will play several important roles, both in further results in this subsection on the Semmes decomposition and in the proof of compactness in §4.5. Recall the local John condition, introduced in Definition 3.1.12.

Proposition 4.2.1 *For each open set $\Omega\subset\mathbb{R}^{n+1}$ which satisfies a two-sided local John condition and whose boundary is Ahlfors regular there exists R_o (which can be taken $+\infty$ if $\partial\Omega$ is unbounded) with the following property. Let $1<p<\infty$, $f\in L^p_1(\partial\Omega,d\sigma)$, $Q\in\partial\Omega$, $R\in(0,R_o)$, $\Delta := B(Q,R)\cap\partial\Omega$. Then there exists $C=C(\Omega,p)>0$ such that*

$$\begin{aligned}\Big[⨍_{\Delta}|f-f_\Delta|^p\,d\sigma\Big]^{1/p} &\leq CR\Big[⨍_{5\Delta}|\nabla_{\tan}f|^p\,d\sigma\Big]^{1/p}\\ &\quad +CR\sum_{j=2}^{\infty}\frac{2^{-j}}{\sigma(2^j\Delta)}\int_{2^j\Delta\setminus 2^{j-1}\Delta}|\nabla_{\tan}f|\,d\sigma.\end{aligned} \tag{4.2.1}$$

In particular, the above estimate holds whenever Ω is a two-sided NTA domain with an Ahlfors regular boundary.

A comment is in order here. In [103] (cf. Lemma 1.1 on p. 406), Semmes derives a Poincaré inequality of the type

$$\fint_{\Delta} |f - f_\Delta|\, d\sigma \leq CR\Big[\fint_{5\Delta} |\nabla_{\tan} f|^2\, d\sigma\Big]^{1/2}, \tag{4.2.2}$$

in the case when the quantity

$$\|\nu\|_{\mathrm{BMO}\,(\partial\Omega, d\sigma)} + \sup_{X\in\partial\Omega}\sup_{0<r\leq R}\sup_{Y\in\Delta(X,R)} R^{-1}|\langle X - Y, \nu_{\Delta(X,R)}\rangle| \tag{4.2.3}$$

is sufficiently small, and when both f and $\partial\Omega$ are smooth (for a constant C which is independent of smoothness). There are several aspects of this result which do not suit the purposes that we have in mind. First, the smallness condition imposed on (4.2.3) is, in effect, an *a priori* flatness assumption on $\partial\Omega$, a hypothesis which we wish to avoid making at this stage. Second, the smoothness assumptions on f and $\partial\Omega$ play a crucial role in Semmes' proof. Among other things, this ensures that (4.2.2) holds, albeit with a 'bad' constant $C = A(\partial\Omega)$ (where $A(\partial\Omega)$ depends on the smoothness of $\partial\Omega$), and Semmes's strategy is to derive an estimate of the form

$$A(\partial\Omega) \leq C_1 A(\partial\Omega) + C_2 \tag{4.2.4}$$

where the constants $C_1, C_2 > 0$ are independent of smoothness, and C_1 is small if (4.2.3) is small. In particular, arranging that $C_1 \in (0, 1/2)$ forces $A(\partial\Omega) \leq 2C_2$, granted that $A(\partial\Omega)$ is known to be finite, to begin with. Such an approach clearly fails if the smoothness assumption on $\partial\Omega$ is dropped.

We remark that in contrast to situations in which an a priori estimate obtained under a hypothesis of smoothness may be used to deduce a general result via a limiting process, in the present setting it is far from clear whether it is possible to construct suitable smooth approximating surfaces in order to remove the regularity assumptions in [103].

For these reasons, we present below a conceptually different proof, which relies on the Calderón-Zygmund theory for singular integrals of layer potential type, developed in earlier chapters.

Proof of Proposition 4.2.1. Assume that $\Omega_+ = \Omega$ and $\Omega_- = \mathbb{R}^{n+1} \setminus \bar{\Omega}$ satisfy a local John condition with constants θ, R_o, and that $\partial\Omega$ is Ahlfors regular. Corollary 3.1.14 guarantees that Ω is a UR domain, of locally finite perimeter. Fix $R \in (0, R_o)$, $Q \in \partial\Omega$, $p \in (1, \infty)$ and $f \in L^p_1(\partial\Omega, d\sigma)$. Next, let $E(X)$ denote the standard (radial) fundamental solution for the Laplacian in $\mathbb{R}^{n+1}$ and, for each $j = 1, \dots, n+1$, define, using the summation convention,

$$g_j(X) := -\int_{\partial\Omega} (\partial_k E)(X - Y)(\partial_{\tau_{jk}} f)(Y)\, d\sigma(Y), \qquad X \in \mathbb{R}^{n+1} \setminus \partial\Omega. \tag{4.2.5}$$

In particular, $g_j(X) = \partial_j \mathcal{D} f(X)$ by (3.6.31). Also, set

$$\vec{g} := (g_j)_{1\leq j\leq n+1}, \qquad \mu_\Delta := \fint_{\Delta} X\, d\sigma(X), \qquad \vec{g}_\Delta := \fint_{\Delta} \vec{g}(X)\, d\sigma(X). \tag{4.2.6}$$

Finally, introduce

$$h_j(X) := -\int_{\partial\Omega} (\partial_k E)(X-Y)(\partial_{\tau_{jk}} f)(Y)\mathbf{1}_{(5\Delta)^c}(Y)\,d\sigma(Y), \tag{4.2.7}$$

and

$$\vec{h} := (h_j)_{1\le j\le n}, \qquad \vec{h}_\Delta = ⨍_\Delta \vec{h}(X)\,d\sigma(X). \tag{4.2.8}$$

With K denoting the principal value version of the harmonic double layer on $\partial\Omega$, we can then write

$$\begin{aligned} f(X) &= \Big(\tfrac{1}{2}I + K\Big)f(X) + \Big(\tfrac{1}{2}I - K\Big)f(X) \\ &= \lim_{Z\in\Gamma^+(X),\, Z\to X} \mathcal{D}^+ f(X) - \lim_{Z\in\Gamma^-(X),\, Z\to X} \mathcal{D}^- f(X), \end{aligned} \tag{4.2.9}$$

where $\mathcal{D}^\pm$ is the harmonic double layer mapping functions on $\partial\Omega$ into $\Omega_\pm$. If we now define

$$u^+(X) := \mathcal{D}^+ f(X) - \langle X, \vec{h}_\Delta\rangle, \quad X\in\Omega_+, \qquad u^-(X) := \mathcal{D}^- f(X) - \langle X, \vec{h}_\Delta\rangle, \quad X\in\Omega_-, \tag{4.2.10}$$

then

$$\begin{aligned} &\Big(⨍_\Delta |f - f_\Delta|^p d\sigma\Big)^{1/p} \\ &\qquad = \Big[⨍_\Delta \Big|\Big(\tfrac{1}{2}I + K\Big)f - ⨍_\Delta \Big(\tfrac{1}{2}I + K\Big)f\,d\sigma - \langle X-\mu_\Delta, \vec{h}_\Delta\rangle \\ &\qquad\qquad + \Big(\tfrac{1}{2}I - K\Big)f - ⨍_\Delta \Big(\tfrac{1}{2}I - K\Big)f\,d\sigma + \langle X-\mu_\Delta, \vec{h}_\Delta\rangle\Big|^p d\sigma\Big]^{1/p} \\ &\qquad = \Big[⨍_\Delta \Big|u^+ - ⨍_\Delta u^+ d\sigma - u^- + ⨍_\Delta u^- d\sigma\Big|^p d\sigma\Big]^{1/p} \\ &\qquad \le \Big[⨍_\Delta \Big|u^+ - ⨍_\Delta u^+ d\sigma\Big|^p d\sigma\Big]^{1/p} + \Big[⨍_\Delta \Big|u^- - ⨍_\Delta u^- d\sigma\Big|^p d\sigma\Big]^{1/p}. \end{aligned} \tag{4.2.11}$$

Let $A_\Delta^\pm \in \Omega_\pm$ be John centers relative to $\Delta(Q,R)$ and note that for almost every $X\in\partial\Omega$,

$$
\begin{aligned}
&\Big[\fint_{\Delta}\Big|u^{\pm}(X)-\fint_{\Delta}u^{\pm}\,d\sigma\Big|^{p}\,d\sigma(X)\Big]^{1/p}\\
&\qquad\qquad=\Big[\fint_{\Delta}\Big|\fint_{\Delta}\Big(u^{\pm}(X)-u^{\pm}(Y)\Big)\,d\sigma(Y)\Big|^{p}\,d\sigma(X)\Big]^{1/p}\\
&\qquad\qquad\leq\fint_{\Delta}\Big[\fint_{\Delta}\Big|u^{\pm}(X)-u^{\pm}(Y)\Big|^{p}\,d\sigma(X)\Big]^{1/p}\,d\sigma(Y)\\
&\qquad\qquad\leq\fint_{\Delta}\Big[\fint_{\Delta}\Big|u^{\pm}(X)-u^{\pm}(A_{\Delta}^{\pm})\Big|^{p}\,d\sigma(X)\Big]^{1/p}\,d\sigma(Y)\\
&\qquad\qquad\quad+\fint_{\Delta}\Big[\fint_{\Delta}\Big|u^{\pm}(Y)-u^{\pm}(A_{\Delta}^{\pm})\Big|^{p}\,d\sigma(X)\Big]^{1/p}\,d\sigma(Y)\\
&\qquad\qquad=\Big[\fint_{\Delta}\Big|u^{\pm}-u^{\pm}(A_{\Delta}^{\pm})\Big|^{p}\,d\sigma\Big]^{1/p}+\fint_{\Delta}\Big|u^{\pm}-u^{\pm}(A_{\Delta}^{\pm})\Big|\,d\sigma\\
&\qquad\qquad\leq 2\Big[\fint_{\Delta}\Big|u^{\pm}-u^{\pm}(A_{\Delta}^{\pm})\Big|^{p}\,d\sigma\Big]^{1/p},
\end{aligned}
\tag{4.2.12}
$$

by Hölder's inequality. Thus, matters are reduced to estimating

$$
\Big[\fint_{\Delta}\Big|u^{+}-u^{+}(A_{\Delta}^{+})\Big|^{p}\,d\sigma\Big]^{1/p},\qquad\Big[\fint_{\Delta}\Big|u^{-}-u^{-}(A_{\Delta}^{-})\Big|^{p}\,d\sigma\Big]^{1/p}. \tag{4.2.13}
$$

We shall indicate how this is done for the first expression above, as the second one can be handled in a similar fashion. To proceed, we recall from Definition 3.1.12 that there exists a rectifiable path γ_X joining X with A_{Δ}^{+}, of length $\leq CR$, and such that $\gamma_X\subset\Gamma_{\kappa}^{+}(X)$ for some geometrical constant κ. Let ds and $Z(s)$ be, respectively, the arc-length element and arc-length parametrization of γ_X. Then

$$
\begin{aligned}
\Big[\fint_{\Delta}\Big|u^{+}-u^{+}(A_{\Delta}^{+})\Big|^{p}\,d\sigma\Big]^{1/p}&=\Big[\fint_{\Delta}\Big|\int_{\gamma_X}\langle\dot{Z}(s),\nabla u^{+}(Z(s))\rangle\,ds\Big|^{p}\,d\sigma(X)\Big]^{1/p}\\
&\leq CR\Big[\fint_{\Delta}\Big|\mathcal{N}(|\nabla u^{+}|\mathbf{1}_{B(Q,2R)\cap\Omega})\Big|^{p}\,d\sigma\Big]^{1/p}.
\end{aligned}
\tag{4.2.14}
$$

On the other hand,

$$
\begin{aligned}
\nabla u^{+}(X)&=\nabla\mathcal{D}^{+}f(X)-\vec{h}_{\Delta}=-\Big(\int_{\partial\Omega}(\partial_k E)(X-Y)(\partial_{\tau_{jk}}f)(Y)\mathbf{1}_{5\Delta}(Y)\,d\sigma(Y)\Big)_{1\leq j\leq n+1}\\
&\quad-\Big(\int_{\partial\Omega}(\partial_k E)(X-Y)(\partial_{\tau_{jk}}f)(Y)\mathbf{1}_{\partial\Omega\setminus 5\Delta}(Y)\,d\sigma(Y)\\
&\qquad-\fint_{\Delta}\int_{\partial\Omega}(\partial_k E)(Z-Y)(\partial_{\tau_{jk}}f)(Y)\mathbf{1}_{\partial\Omega\setminus 5\Delta}(Z)\,d\sigma(Y)\,d\sigma(Z)\Big)_{1\leq j\leq n+1}\\
&=:\vec{a}(X)+\vec{b}(X).
\end{aligned}
\tag{4.2.15}
$$

Observe that

$$\Big[\fint_{\Delta} |\mathcal{N}(\vec{a})|^p \, d\sigma \Big]^{1/p} \leq C \Big[\fint_{5\Delta} |\nabla_{\tan} f|^p \, d\sigma \Big]^{1/p}, \tag{4.2.16}$$

thanks to the estimates on singular integrals from §3.2. Moreover,

$$|\vec{b}(X)| \leq \fint_{\Delta} \Big(\int_{\partial\Omega \setminus 5\Delta} \Big| (\nabla E)(X-Y) - (\nabla E)(Z-Y) \Big| |\nabla_{\tan} f(Y)| \, d\sigma(Y) \Big) \, d\sigma(Z), \tag{4.2.17}$$

so using the Mean Value Theorem we obtain

$$\begin{aligned} \sup_{X \in B(Q,2R) \cap \Omega} |\vec{b}(X)| &\leq C \int_{\partial\Omega \setminus B(Q,2R)} \frac{R}{|Q-Y|^{n+1}} |\nabla_{\tan} f(Y)| \, d\sigma(Y) \\ &\leq C \sum_{j=1}^{\infty} 2^{-j} \fint_{2^{j+1}\Delta \setminus 2^j \Delta} |\nabla_{\tan} f| \, d\sigma. \end{aligned} \tag{4.2.18}$$

Therefore,

$$\Big[\fint_{\Delta} |\mathcal{N}(\vec{b} \mathbf{1}_{B(Q,2R) \cap \Omega})|^p \, d\sigma \Big]^{1/p} \leq C \sum_{j=1}^{\infty} \frac{2^{-j}}{\sigma(2^{j+1}\Delta)} \int_{2^{j+1}\Delta \setminus 2^j \Delta} |\nabla_{\tan} f| \, d\sigma. \tag{4.2.19}$$

Now (4.2.1) follows by combining (4.2.11), (4.2.12), (4.2.14), (4.2.15), (4.2.16) and (4.2.19). $\Box$

This Poincaré lemma enables us to establish the following, which will be of great use both in the proof of Semmes' Decomposition Theorem, to be discussed shortly, as well as later, in §4.5.

Theorem 4.2.2 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set that satisfies a two-sided local John condition and whose boundary is Ahlfors regular (in particular, any two-sided NTA domain with Ahlfors regular boundary will do).*

Then there exist R_o (which can be taken $+\infty$ if Ω is unbounded) and $C = C(\Omega) > 0$ with the property that for each $\varepsilon \in (0,1)$, $X \in \partial\Omega$ and $R \in (0, R_o)$, there holds

$$\sup_{Y \in \Delta(X,2R)} R^{-1} |\langle X - Y, \nu_{\Delta(X,R)} \rangle| \leq C \|\nu\|_* (\Delta(X, 8\varepsilon^{-1} R)) + C\varepsilon, \tag{4.2.20}$$

where $\nu_{\Delta(X,R)} := \fint_{\Delta(X,R)} \nu \, d\sigma$.

It is not too hard to show that if $\Omega \subset \mathbb{R}^{n+1}$ is a set of locally finite perimeter which satisfies an exterior corkscrew condition then

$$\lim_{\substack{Y \to X \\ Y \in \partial\Omega}} \Big\langle \nu(X), \frac{X-Y}{|X-Y|} \Big\rangle = 0, \qquad \forall \, X \in \partial^*\Omega. \tag{4.2.21}$$

This is implicit in the proof of Lemma A.1.3 of [66]. (Let us recall in this context that the exterior corkscrew condition on Ω implies $\partial\Omega = \partial\overline{\Omega}$.) The usefulness of (4.2.20) stems from the fact that this estimate gives, at each fixed scale, a quantitative control of the inner product between the average of unit normal and the (normalized) chord in terms of the corresponding local mean oscillations of the unit normal.

Before presenting the proof of Theorem 4.2.2, we record the following useful corollary.

Corollary 4.2.3 *Granted the geometrical hypotheses on Ω made in the statement of Theorem 4.2.2, there exists a finite geometrical constant $C = C(\Omega) > 0$ such that*

$$\sup_{X\in\partial\Omega}\sup_{R>0}\sup_{Y\in\Delta(X,2R)} R^{-1}|\langle X-Y,\nu_{\Delta(X,R)}\rangle| \leq C\|\nu\|_{\mathrm{BMO}\,(\partial\Omega,d\sigma)}, \tag{4.2.22}$$

and

$$\limsup_{R\to 0^+}\Big[\sup_{X\in\partial\Omega}\sup_{Y\in\Delta(X,2R)} R^{-1}|\langle X-Y,\nu_{\Delta(X,R)}\rangle|\Big] \leq C\,\mathrm{dist}\,(\nu\,,\,\mathrm{VMO}\,(\partial\Omega,d\sigma)). \tag{4.2.23}$$

Proof. First, (4.2.22) follows easily by estimating $\|\nu\|_*(\Delta(X,8\varepsilon^{-1}R)) \leq \|\nu\|_{\mathrm{BMO}\,(\partial\Omega,d\sigma)}$ and then letting $\varepsilon > 0$ approach 0 in (4.2.20). As for (4.2.23), we take the supremum of both sides in (4.2.20) with respect to $X \in \partial\Omega$, then invoke (2.4.77), before making $\varepsilon \to 0^+$. $\square$

Estimate (4.2.22) is the main result in [103] and Semmes establishes this in the case when $\partial\Omega$ is a C^∞ smooth surface using his Poincaré type inequality (Lemma 1.1 on p. 406 of [103]). The necessity that $\partial\Omega$ is smooth in Semmes' argument is inherited from here (see the comments following the statement of Proposition 4.2.1). In our situation, we shall employ our version of Poincaré inequality from Proposition 4.2.1 when dealing with (4.2.20). The appearance of ε in (4.2.20) is an artifact attributed to our Poincaré inequality being weaker than the standard version, due to the presence of the series in the right-hand side of (4.2.1). As already seen in the proof of Corollary 4.2.3, this is not a serious impediment.

Proof of Theorem 4.2.2. Let R_o be such that the Poincaré inequality from Proposition4.2.1 applies on surface balls of radius $\leq CR_o$, where C is a large, suitable geometrical constant.

To justify (4.2.20), fix $X \in \Omega$, $R \in (0,R_o)$, $\varepsilon \in (0,1)$ and abbreviate $\Delta := \Delta(X,R)$. Consider next

$$g_X(Y) := \langle X-Y,\nu_\Delta\rangle, \qquad Y \in \partial\Omega. \tag{4.2.24}$$

For this function, we claim that the following estimate is valid:

$$|g_X(Y)-g_X(Y')| \leq C(\Omega,\alpha)R^{1-\alpha}|Y-Y'|^\alpha\Big\{\|\nu\|_*(\Delta(X,8\varepsilon^{-1}R))+\varepsilon\Big\}, \quad \forall\, Y,Y' \in 2\Delta, \tag{4.2.25}$$

for each $\alpha \in (0,1)$. Granted this, choosing $Y' = X$ yields

$$|g_X(Y)| \leq CR\|\nu\|_*(\Delta(X,8\varepsilon^{-1}R)) + CR\varepsilon, \qquad \forall\, Y \in 2\Delta, \tag{4.2.26}$$

which readily gives (4.2.20). Therefore, there remains to prove (4.2.25). First note that

$$\begin{aligned} |\nabla_{\tan} g_X(Y)| &= |\nabla g_X(Y) - \langle\nabla g_X(Y),\nu(Y)\rangle\nu(Y)| \\ &= |\nu_\Delta - \langle\nu_\Delta,\nu(Y)\rangle\nu(Y)| \\ &= |\nu_\Delta - \nu(Y) - \langle\nu_\Delta-\nu(Y),\nu(Y)\rangle\nu(Y)| \\ &\leq 2|\nu_\Delta-\nu(Y)|. \end{aligned} \tag{4.2.27}$$

Fix an arbitrary surface ball Δ_r or radius r such that $\Delta_r \subset \Delta$ and set $N := [-\log_2 \varepsilon] + 1$. In particular, $2^N \sim \varepsilon^{-1}$ and $2^{-N} \sim \varepsilon$. Then for each $p, q \in (1, \infty)$, the estimate (4.2.27), our Poincaré inequality, John-Nirenberg's inequality and the fact that $|\nu| = 1$ yield

$$\begin{aligned}
\frac{1}{r}\Big(\fint_{\Delta_r} |g_X - (g_X)_{\Delta_r}|^p \, d\sigma\Big)^{1/p} &\leq C\Big(\fint_{4\Delta_r} |\nabla_{\tan} g_X|^p \, d\sigma\Big)^{1/p} + C\sum_{j=1}^{\infty} 2^{-j} \fint_{2^{j+1}\Delta_r} |\nabla_{\tan} g_X| \, d\sigma \\
&\leq C\Big(\fint_{4\Delta_r} |\nu - \nu_\Delta|^p \, d\sigma\Big)^{1/p} + C\sum_{j=1}^{\infty} 2^{-j} \fint_{2^{j+1}\Delta_r} |\nu - \nu_\Delta| \, d\sigma \\
&\leq C\sum_{j=1}^{\infty} 2^{-j} \Big(\fint_{2^{j+1}\Delta_r} |\nu - \nu_\Delta|^{pq} \, d\sigma\Big)^{1/(pq)} \\
&\leq C\sum_{j=1}^{\infty} (2^j r)^{-n/(pq)} 2^{-j} \Big(\int_{2^{j+2}\Delta} |\nu - \nu_\Delta|^{pq} \, d\sigma\Big)^{1/(pq)} \\
&\leq C\Big(\frac{R}{r}\Big)^{n/(pq)} \sum_{j=1}^{\infty} 2^{-j} \Big(\fint_{2^{j+2}\Delta} |\nu - \nu_\Delta|^{pq} \, d\sigma\Big)^{1/(pq)} \\
&\leq C\Big(\frac{R}{r}\Big)^{n/(pq)} \Big\{\sum_{j=1}^{N} 2^{-j} \Big(\fint_{2^{j+2}\Delta} |\nu - \nu_\Delta|^{pq} \, d\sigma\Big)^{1/(pq)} + 2^{-N}\Big\} \\
&\leq C\Big(\frac{R}{r}\Big)^{n/(pq)} \Big\{\Big(\sum_{j=1}^{\infty} j\, 2^{-j}\Big) \|\nu\|_*(\Delta(X, 2^{N+3}R)) + 2^{-N}\Big\} \\
&\leq C\Big(\frac{R}{r}\Big)^{n/(pq)} \Big\{\|\nu\|_*(\Delta(X, 8\varepsilon^{-1}R)) + \varepsilon\Big\}.
\end{aligned} \tag{4.2.28}$$

Given $\alpha \in (0, 1)$, if $p, q \in (1, \infty)$ are chosen such that $\alpha = 1 - n/(pq)$, then the above estimate gives

$$r^{-\alpha}\Big(\fint_{\Delta_r} |g_X - (g_X)_{\Delta_r}|^p \, d\sigma\Big)^{1/p} \leq C(\Omega, p, \alpha) R^{1-\alpha} \Big\{\|\nu\|_*(\Delta(X, 8\varepsilon^{-1}R)) + \varepsilon\Big\}. \tag{4.2.29}$$

Using N. Meyer's criterion for Hölder continuity ([81]) we may then estimate

$$\begin{aligned}
\sup_{Y, Y' \in 2\Delta} \frac{|g_X(Y) - g_X(Y')|}{|Y - Y'|^\alpha} &\leq C \sup_{\Delta_r \subset \Delta} r^{-\alpha} \fint_{\Delta_r} |g_X(Y) - (g_X)_{\Delta_r}| \, d\sigma(Y) \\
&\leq C \sup_{\Delta_r \subset \Delta} r^{-\alpha} \Big(\fint_{\Delta_r} |g_X(Y) - (g_X)_{\Delta_r}|^p \, d\sigma(Y)\Big)^{1/p} \\
&\leq C R^{1-\alpha} \Big\{\|\nu\|_*(\Delta(X, 8\varepsilon^{-1}R)) + \varepsilon\Big\}.
\end{aligned} \tag{4.2.30}$$

Let us remark that while the setting in [81] is that of the ordinary Euclidean space, Meyer's argument carries over to the current setting. Indeed, the key ingredients in the proof are: (i) the classical Calderón-Zygmund lemma, and (ii) the fact that that the estimate under discussion rescales naturally under dilations. That the Calderón-Zygmund lemma continues to be valid in the setting of spaces of homogeneous type is well-known; see, e.g., [22]. As for (ii), the feature which is lost when replacing the Euclidean space $\mathbb{R}^n$ by $\partial\Omega$ is that, as opposed to the latter, the former is stable under dilations. One possible remedy is to consider, in place of just one domain Ω, the entire class of domains whose boundaries are Ahlfors-regular, with fixed Ahlfors regularity constants. This class is then stable under dilations and the same type of argument as in [81] continues to work in this context.

Now, estimate (4.2.30) justifies (4.2.25), thus finishing the proof of the theorem. $\square$

We now turn our attention to an important tool, Semmes' decomposition theorem, which originally appeared in Proposition 5.1 on p. 212 of [101]. In [101], this result was stated for C^2 surfaces, albeit the constants involved were independent of smoothness. A more general formulation, in which the C^2 smoothness assumption is replaced by Reifenberg flatness, appears in Theorem 4.1 on p. 398 of [65] (see also the comments on p. 66 in [15]). Here, however, our goal is to start with a different set of hypotheses which, a priori, do not specifically require the domain in question to be Reifenberg flat. More concretely, we shall ask instead that the domain satisfies a two-sided local John condition, its boundary is Ahlfors regular, and that its unit normal has a small local BMO norm. Cf. Theorem 4.2.4 below. As pointed out in Corollary 4.2.5, the class of domains just described include any two-sided NTA domain with an Ahlfors regular boundary and whose unit normal has a small local BMO norm.

Once Semmes' decomposition theorem is established in this context, we can then show that any domain satisfying the aforementioned hypotheses is necessarily Reifenberg flat (at an appropriate scale). This is accomplished later, in Theorem 4.2.7.

Theorem 4.2.4 *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set that satisfies a two-sided local John condition and whose boundary is Ahlfors regular.*

Then there exists a geometrical constant $C_* > 1$ with the following significance: suppose that there exists $\delta \in (0, 1/(10C_*))$ with the property that for every compact set $\mathcal{K} \subset \mathbb{R}^{n+1}$ there exists $R_{\mathcal{K}} > 0$ for which

$$\sup_{Q \in \mathcal{K} \cap \partial\Omega} \|\nu\|_*(\Delta(Q, R_{\mathcal{K}})) \leq \delta. \tag{4.2.31}$$

Then for every compact set $\mathcal{K} \subset \mathbb{R}^{n+1}$ there exist $C_1, C_2, C_3, C_4 > 0$ depending only on Ω and $\mathcal{K}$, along with $R_* > 0$ depending on Ω, $\mathcal{K}$ and δ, for which the following holds. If $Q \in \mathcal{K} \cap \partial\Omega$ and $0 < r \leq R_*$, then there exists a unit vector $\vec{n}_{Q,r}$ and a Lipschitz function

$$h : H(Q, r) := \langle \vec{n}_{Q,r} \rangle^{\perp} \longrightarrow \mathbb{R}, \qquad \text{with } \|\nabla h\|_{L^\infty} \leq C_3\,\delta, \tag{4.2.32}$$

and whose graph

$$\mathcal{G} := \{X = Q + \zeta + t\,\vec{n}_{Q,r} : \zeta \in H(Q, r),\ t = h(\zeta)\} \tag{4.2.33}$$

(in the coordinate system $X = (\zeta, t) \Leftrightarrow X = Q + \zeta + t\,\vec{n}_{Q,r}$, $\zeta \in H(Q, r)$, $t \in \mathbb{R}$) is a good approximation of $\partial\Omega$ in the cylinder

$$\mathcal{C}(Q, r) := \{Q + \zeta + t\,\vec{n}_{Q,r} : \zeta \in H(Q, r),\ |\zeta| \leq r,\ |t| \leq r\} \tag{4.2.34}$$

in the following sense. With $C_j = C_j(\Omega, \mathcal{K})$, $1 \le j \le 3$, as above,

$$\sigma\Big(\mathcal{C}(Q,r) \cap (\partial\Omega \triangle \mathcal{G})\Big) \le C_1 \omega_n r^n \exp(-C_2/\delta). \tag{4.2.35}$$

Also, there exist two disjoint sets $G(Q,r)$ ('good') and $E(Q,r)$ ('evil') such that

$$\mathcal{C}(Q,r) \cap \partial\Omega = G(Q,r) \cup E(Q,r) \quad \text{with} \;\; G(Q,r) \subset \mathcal{G} \tag{4.2.36}$$

$$\text{and} \quad \sigma(E(Q,r)) \le C_1\, \omega_n r^n \exp(-C_2/\delta). \tag{4.2.37}$$

Moreover, if $\Pi : \mathbb{R}^{n+1} \longrightarrow H(Q,r)$ is defined by $\Pi(X) = \zeta$ if $X = Q + \zeta + t\vec{n}_{Q,r} \in \mathbb{R}^{n+1}$ with $\zeta \in H(Q,r)$ and $t \in \mathbb{R}$, then

$$\Big| X - \Big(Q + \Pi(X) + h(\Pi(X))\vec{n}_{Q,r}\Big)\Big| \le C_3\, \delta \operatorname{dist}(\Pi(X)\,,\, \Pi(G(Q,r))), \quad \forall\, X \in E(Q,r), \tag{4.2.38}$$

and

$$\mathcal{C}(Q,r) \cap \partial\Omega \subseteq \{Q + \zeta + t\,\vec{n}_{Q,r} :\, |t| \le C_3 \delta r,\, \zeta \in H(Q,r)\}, \tag{4.2.39}$$

$$\Pi(\mathcal{C}(Q,r) \cap \partial\Omega) = \{\zeta \in H(Q,r) :\, |\zeta| < r\}. \tag{4.2.40}$$

Finally,

$$(1 - C_4\delta)\omega_n r^n \le \sigma(\Delta(Q,r)) \le (1 + C_4\delta)\omega_n r^n. \tag{4.2.41}$$

Thus, heuristically, the fact that a domain $\Omega \subset \mathbb{R}^{n+1}$ satisfies a two-sided local John condition, has an Ahlfors regular boundary, and its unit normal has small local mean oscillations, implies that, at an appropriate scale (within a cylinder) $\partial\Omega$ agrees with the graph of a function with small Lipschitz constant except for a small bad set, while staying close to the this graph even on the bad set.

Before presenting the proof of this result we record a consequence and a related result, of independent interest. The first one is a direct consequence of Lemma 3.1.13.

Corollary 4.2.5 *Assume that Ω is a two-sided NTA domain in $\mathbb{R}^{n+1}$ with the property that $\partial\Omega$ is Ahlfors regular and such that (4.2.31) holds. Then the conclusions in Theorem 4.2.4 remain valid in this context.*

The second result is readily seen from definitions.

Proposition 4.2.6 *Assume that Ω satisfies a two-sided corkscrew condition and $\partial\Omega$ is Ahlfors regular. Also, suppose that $\partial\Omega$ is δ-Semmes decomposable (i.e., the conclusions in Theorem 4.2.4 hold) for some $\delta > 0$ which is small relative to the corkscrew constant of Ω. Then $\partial\Omega$ is δ_o-Reifenberg flat, with $\delta_o = C\delta$ for some geometric constant $C > 0$.*

We now present the

Proof of Theorem 4.2.4. Let $\delta > 0$ be such that $\delta^2 \in (0, \frac{1}{10C_*})$, with C_* to be specified later, and fix an arbitrary compact $\mathcal{K} \subset \mathbb{R}^{n+1}$. Set $\tilde{\mathcal{K}} := \{X \in \mathbb{R}^{n+1} :\, \operatorname{dist}(X, \mathcal{K}) \le 1\}$. Our hypotheses imply the existence of some $R_{\tilde{\mathcal{K}}} > 0$ such that

$$\|\nu\|_*(\Delta(X, R_{\tilde{\mathcal{K}}})) \leq \delta^2 \quad \text{for each} \quad X \in \tilde{\mathcal{K}} \cap \partial\Omega. \tag{4.2.42}$$

If we now introduce $R_* := \min\{\delta^2 R_{\tilde{\mathcal{K}}}/(8C)\,, R_{\tilde{\mathcal{K}}}/8\,, R_o/100\,, 1\}$, where R_o is the constant used in the statement of Theorem 4.2.2 and $C > 0$ is the constant appearing in (4.2.20), Theorem 4.2.2 gives

$$\sup_{0<r<R_*} \sup_{X \in \tilde{\mathcal{K}} \cap \partial\Omega} \sup_{Y \in \Delta(X,2r)} r^{-1} |\langle X - Y, \nu_{\Delta(X,r)} \rangle| \leq 2\delta^2, \tag{4.2.43}$$

where, as in the past, $\nu_{\Delta(X,r)} := ⨍_{\Delta(X,r)} \nu \, d\sigma$. For $R > 0$ and $X \in \partial\Omega$, set

$$\nu^*_{X,R}(Y) := \sup_{\rho \in (0,R)} ⨍_{\Delta(Y,\rho)} |\nu(Z) - \nu_{\Delta(X,2R)}| \, d\sigma(Z), \qquad Y \in \partial\Omega. \tag{4.2.44}$$

It follows that

$$\nu^*_{X,R}(Y) \leq \mathcal{M}\Big(|\nu - \nu_{\Delta(X,2R)}| \mathbf{1}_{\Delta(X,2R)}\Big)(Y), \qquad \forall\, Y \in \Delta(X, R), \tag{4.2.45}$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal function on $\partial\Omega$. Thus, by (4.2.45), the boundedness of $\mathcal{M}$ on $L^2(\partial\Omega, d\sigma)$, John-Nirenberg's inequality and (4.2.42), we have

$$\Big(⨍_{\Delta(X,R)} |\nu^*_{X,R}(Y)|^2 \, d\sigma(Y)\Big)^{1/2} \leq C\Big(⨍_{\Delta(X,2R)} |\nu(Y) - \nu_{\Delta(X,2R)}|^2 \, d\sigma(Y)\Big)^{1/2} \leq C\delta^2, \tag{4.2.46}$$

whenever $X \in \tilde{\mathcal{K}} \cap \partial\Omega$ and $0 < R < R_{\tilde{\mathcal{K}}}/2$.

Next, fix a point $Q \in \mathcal{K} \cap \partial\Omega$ and choose $C_* := \max\{C, 1\}$, where C is the geometrical constant appearing in (4.2.46). If $0 < \delta^2 < (10C_*)^{-1}$ and $R \in (0, R_{\tilde{\mathcal{K}}}/2)$ it follows from this that there exists $Y_o \in \Delta(Q, R)$ such that $\nu^*_{Q,R}(Y_o) \leq 1/10$. Since matters can be arranged so that $|\nu(Y_o) - \nu_{\Delta(Q,2R)}| \leq \nu^*_{Q,R}(Y_o)$, this forces

$$\tfrac{9}{10} \leq |\nu_{\Delta(Q,2R)}| \leq 1, \qquad \forall\, R \in (0, R_{\tilde{\mathcal{K}}}/2). \tag{4.2.47}$$

In particular, for each $r \in (0, R_*)$,

$$\vec{n}_{Q,r} := \frac{\nu_{\Delta(Q,4r)}}{|\nu_{\Delta(Q,4r)}|} \tag{4.2.48}$$

is a well-defined unit vector in $\mathbb{R}^{n+1}$. Set $H(Q, r) := \{X \in \mathbb{R}^{n+1} : \langle X, \vec{n}_{Q,r} \rangle = 0\}$ and introduce a new system of coordinates in $\mathbb{R}^{n+1}$ by setting

$$X = (\zeta, t) \Longleftrightarrow X = Q + t\,\vec{n}_{Q,r} + \zeta, \quad t \in \mathbb{R}, \ \zeta \in H(Q, r). \tag{4.2.49}$$

Also, define $\Pi : \mathbb{R}^{n+1} \longrightarrow H(Q, r)$ by $\Pi(X) = \zeta$ if $X = (\zeta, t)$, and write $\zeta(X), t(X)$ in place of ζ, t whenever necessary to stress the dependence of the new coordinates on the point $X \in \mathbb{R}^{n+1}$. Finally, consider the cylinder $\mathcal{C}(Q, r)$ defined as in (4.2.34) and introduce

$$G(Q,r) := \{X \in \mathcal{C}(Q,r) \cap \partial\Omega :\, \nu^*_{Q,2r}(X) \leq \delta\}, \qquad E(Q,r) := \Big(\mathcal{C}(Q,r) \cap \partial\Omega\Big) \setminus G(Q,r) \tag{4.2.50}$$

Next, we claim that there exist two geometrical constants $C > 0$, $c > 0$ such that

$$⨍_{\Delta(Q,2r)} \exp\,(c\,\delta^{-2}\,\nu^*_{Q,2r})\,d\sigma \leq C. \tag{4.2.51}$$

Granted this, we may then conclude that

$$\exp\,(c/\delta)\,\frac{\sigma(E(Q,r))}{\sigma(\Delta(Q,2r))} \leq \frac{1}{\sigma(\Delta(Q,2r))}\int_{E(Q,r)} \exp\,(c\,\delta^{-2}\,\nu^*_{Q,2r})\,d\sigma \leq C, \tag{4.2.52}$$

from which the estimate (4.2.37) follows. To justify (4.2.51), set $f := \mathcal{M}(|\nu - \nu_{\Delta(Q,4r)}|\mathbf{1}_{\Delta(Q,4r)})$, so that $\nu^*_{Q,2r}(X) \leq f(X)$ if $X \in \Delta(Q,2r)$. Expanding the exponential function into an infinite power series we may then write

$$\begin{aligned}
⨍_{\Delta(Q,2r)} &\exp\,(c\,\delta^{-2}\,\nu^*_{Q,r})\,d\sigma \leq ⨍_{\Delta(Q,2r)} \exp\,(c\,\delta^{-2}\,f)\,d\sigma \\
&= \frac{1}{\sigma(\Delta(Q,2r))}\int_0^\infty \sigma\Big(\{X \in \Delta(Q,2r) :\, \exp\,(c\,\delta^{-2}\,f(X)) > \lambda\}\Big)\,d\lambda \\
&\leq 1 + \frac{1}{\sigma(\Delta(Q,2r))}\int_1^\infty \sigma\Big(\{X \in \Delta(Q,2r) :\, \exp\,(c\,\delta^{-2}\,f(X)) > \lambda\}\Big)\,d\lambda \\
&\leq 1 + \frac{1}{\sigma(\Delta(Q,2r))}\int_0^\infty \sigma\Big(\{X \in \Delta(Q,2r) :\, c\,\delta^{-2}\,f(X) > s\}\Big)e^s\,ds \\
&\leq e + \frac{1}{\sigma(\Delta(Q,2r))}\sum_{k=0}^\infty \frac{1}{k!}\int_1^\infty \sigma\Big(\{X \in \Delta(Q,2r) :\, f(X) > s\,\delta^2/c\}\Big)s^k\,ds. \qquad (4.2.53)
\end{aligned}$$

To continue, note that for every $p \in [2,\infty)$, the L^p-boundedness of the Hardy-Littlewood maximal operator (with bounds independent of p) gives

$$\begin{aligned}
\frac{\sigma\Big(\{X \in \Delta(Q,2r) :\, f(X) > s\,\delta^2/c\}\Big)}{\sigma(\Delta(Q,2r))} \quad &\leq \quad \Big(\frac{c}{s\,\delta^2}\Big)^p ⨍_{\Delta(Q,2r)} f(X)^p\,d\sigma(X) \qquad (4.2.54) \\
&\leq \quad C\Big(\frac{c}{s\,\delta^2}\Big)^p ⨍_{\Delta(Q,4r)} |\nu(X) - \nu_{\Delta(Q,4r)}|^p\,d\sigma(X),
\end{aligned}$$

and we now claim that

$$⨍_{\Delta(Q,4r)} |\nu(X) - \nu_{\Delta(Q,4r)}|^p\,d\sigma(X) \leq C_1\Gamma(p+1)\Big(C_2\|\nu\|_*(\Delta(Q,4r))\Big)^p, \tag{4.2.55}$$

where $\Gamma(t) := \int_0^\infty \lambda^{t-1}e^{-\lambda}\,d\lambda$ is the Gamma function. Taking this inequality for granted for the time being, we combine (4.2.55) and (4.2.54) and recall that $\|\nu\|_*(\Delta(Q,4r)) \leq \delta$ to obtain

$$\frac{\sigma\Big(\{X\in\Delta(Q,4r):\,f(X)>s\,\delta^2/c\}\Big)}{\sigma(\Delta(Q,4r))}\leq C_1\Gamma(p+1)\Big(\frac{c\,C_2}{s}\Big)^p, \tag{4.2.56}$$

for each $p\in[2,\infty)$. Utilizing (4.2.56), in which we take $p=k+2$, $k=0,1,...$, back into (4.2.53) then yields

$$⨍_{\Delta(Q,2r)}\exp\,(c\,\delta^{-2}\,\nu^*_{Q,4r})\,d\sigma \quad\leq\quad C_o+C_1\sum_{k=0}^{\infty}k^3\int_1^\infty s^k\Big(\frac{c\,C_2}{s}\Big)^{k+2}ds=:C<+\infty, \tag{4.2.57}$$

if $0<c<1/C_2$. This finishes the proof of (4.2.51), modulo that of (4.2.55). As regards the latter, we use following the John-Nirenberg level set estimate with exponential bound

$$\sigma\Big(\{X\in\Delta(Q,4r):\,|\nu(X)-\nu_{\Delta(Q,4r)}|>\lambda\}\Big)\leq C\exp\Big(\frac{-C_2\lambda}{\|\nu\|_*(\Delta(Q,4r))}\Big) \tag{4.2.58}$$

(whose validity in the context of spaces of homogeneous type is well-known; see, e.g., [1], [23], and Theorem 2 on p. 33 in [110]), in order to write

$$\begin{aligned}
⨍_{\Delta(Q,4r)}|\nu(X)-\nu_{\Delta(Q,4r)}|^p\,d\sigma(X) &= p\int_0^\infty\lambda^{p-1}\sigma\Big(\{X\in\Delta(Q,4r):\,|\nu(X)-\nu_{\Delta(Q,4r)}|>\lambda\}\Big)\,d\lambda\\
&\leq C_1p\int_0^\infty\lambda^{p-1}\exp\Big(\frac{-C_2\lambda}{\|\nu\|_*(\Delta(Q,4r))}\Big)\,d\lambda\\
&\leq C_1p\Big(C_2\|\nu\|_*(\Delta(Q,4r))\Big)^p\int_0^\infty t^{p-1}e^{-t}\,dt\\
&= C_1p\Gamma(p)\Big(C_2\|\nu\|_*(\Delta(Q,4r))\Big)^p.
\end{aligned} \tag{4.2.59}$$

Since $p\,\Gamma(p)=\Gamma(p+1)$, this justifies (4.2.55) and concludes the proof of (4.2.51).

We now turn to the task of constructing the Lipschitz function h. As a preliminary matter, we note here that the estimate (4.2.43) gives

$$\begin{aligned}
|\langle X-Y,\nu_{\Delta(Q,4r)}\rangle| &\leq |\langle X-Y,\nu_{\Delta(X,|X-Y|)}\rangle|+|X-Y||\nu_{\Delta(Q,4r)}-\nu_{\Delta(X,|X-Y|)}|\\
&\leq 2\delta^2\,|X-Y|+|X-Y|⨍_{\Delta(X,|X-Y|)}|\nu-\nu_{\Delta(Q,4r)}|\,d\sigma\\
&\leq \Big(2\delta^2+\nu^*_{Q,2r}(X)\Big)|X-Y|,
\end{aligned} \tag{4.2.60}$$

provided $X\in\tilde{\mathcal{K}}\cap\partial\Omega$, $r\in(0,R_*)$ and $Y\in B(X,2r)$. Also, recall from (4.2.49) that

$$t(X)=\langle X-Q,\vec{n}_{Q,r}\rangle,\qquad X\in\mathbb{R}^{n+1}. \tag{4.2.61}$$

This, (4.2.47)-(4.2.48) and (4.2.60) then allow us to control

$$\begin{aligned}|t(X)-t(Y)| &= |\langle X-Y,\vec{n}_{Q,r}\rangle| \\ &\leq \tfrac{10}{9}|\langle X-Y,\nu_{\Delta(Q,4r)}\rangle| \leq 4\delta|X-Y| \end{aligned} \tag{4.2.62}$$

whenever $X \in G(Q,r)$, $Y \in B(X,2r)$, $r \in (0,R_*)$. (Note that $X \in G(Q,r)$ implies $X \in \tilde{\mathcal{K}} \cap \partial\Omega$.) On the other hand, we may write $\zeta(X)-\zeta(Y) = X-Y-(t(X)-t(Y))\vec{n}_{Q,r}$ so that

$$|\zeta(X)-\zeta(Y)| \geq |X-Y| - |t(X)-t(Y)| \geq (1-4\delta)|X-Y|, \tag{4.2.63}$$

granted that X,Y are as above. Combining (4.2.62) and (4.2.63) then gives

$$\begin{gathered}|t(X)-t(Y)| \leq \frac{4\delta}{1-4\delta}|\zeta(X)-\zeta(Y)| \leq C\delta|\zeta(X)-\zeta(Y)|, \\ X \in G(Q,r),\ \ Y \in B(X,2r),\end{gathered} \tag{4.2.64}$$

for some geometrical constant $C>0$. As a consequence, the projection Π is one-to-one on $G(Q,r)$ and, hence, the mapping

$$h : \Pi(G(Q,r)) \longrightarrow \mathbb{R}, \qquad h(\zeta(X)) := t(X), \tag{4.2.65}$$

is well-defined. By (4.2.64), this mapping satisfies a Lipschitz condition with constant $C\delta$. It can be therefore extended as a Lipschitz function, which we continue to denote by h, to the entire hyperplane $H(Q,r)$, with constant $\leq C\delta$. Note that its graph $\mathcal{G}$ (in the (ζ,t)-system of coordinates) contains $\{(\zeta(X),t(X)) : X \in G(Q,r)\} = G(Q,r)$.

The inclusion (4.2.39) is a direct consequence of the convention (4.2.49), formula (4.2.61) and estimate (4.2.43). In turn, this implies that the connected sets

$$\begin{aligned}&\mathcal{C}^+(Q,r) := \{(\zeta,t) : |\zeta| \leq r,\ -r < t < -C_3\delta r\}, \\ &\mathcal{C}^-(Q,r) := \{(\zeta,t) : |\zeta| \leq r,\ C_3\delta r < t < r\},\end{aligned} \tag{4.2.66}$$

do not intersect $\partial\Omega$. Thus, $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^{n+1} \setminus \bar{\Omega}$ form a disjoint, open cover of $\mathcal{C}^\pm(Q,r)$ and since the two-sided corkscrew condition guarantees that $\mathcal{C}^\pm(Q,r) \cap \Omega_\pm \neq \emptyset$, we may finally conclude that

$$\mathcal{C}^+(Q,r) \subseteq \Omega_+ \quad \text{ and } \quad \mathcal{C}^-(Q,r) \subseteq \Omega_-. \tag{4.2.67}$$

Now, clearly, $\Pi(\mathcal{C}(Q,r) \cap \partial\Omega) \subseteq \{\zeta \in H(Q,r),\ |\zeta| \leq r\}$. The opposite inclusion fails only when there exists a line segment parallel to $\vec{n}_{Q,r}$ whose two endpoints belong to $\mathcal{C}^+(Q,r)$ and to $\mathcal{C}^-(Q,r)$, respectively, and which does not intersect $\partial\Omega$. However, (4.2.67) and simple connectivity arguments rule out this scenario, hence (4.2.40) is proved.

Going further, observe that (4.2.40) implies

$$\{\zeta \in H(Q,r),\ |\zeta| \leq r\} \setminus \Pi(G(Q,r)) \subseteq \Pi(E(Q,r)) \tag{4.2.68}$$

so that

$$\mathcal{H}^n\Big(\{\zeta\in H(Q,r),\ |\zeta|\leq r\}\setminus\Pi(G(Q,r))\Big) \leq \mathcal{H}^n\Big(\Pi(E(Q,r))\Big) \leq \mathcal{H}^n(E(Q,r))\leq C_1\,\omega_n r^n\exp\left(-C_2/\delta\right), \tag{4.2.69}$$

using the fact that Π maps balls in $\mathbb{R}^{n+1}$ into balls in hyperplane $H(Q,r)$ or the same radii, and invoking (4.2.37). Now, $\mathcal{C}(Q,r)\cap\Big(\mathcal{G}\setminus\partial\Omega\Big)\subseteq\mathcal{G}\cap\Pi^{-1}\Big(\{\zeta\in H(Q,r),\ |\zeta|\leq r\}\setminus\Pi(G(Q,r))\Big)$ and since $\mathcal{G}$ is the graph of a Lipschitz function, we may deduce that

$$\begin{aligned}
&\mathcal{H}^n\Big(\mathcal{C}(Q,r)\cap\Big(\mathcal{G}\setminus\partial\Omega\Big)\Big)\\
&\quad\leq\mathcal{H}^n\Big(\mathcal{G}\cap\Pi^{-1}\Big(\{\zeta\in H(Q,r),\ |\zeta|\leq r\}\setminus\Pi(G(Q,r))\Big)\Big)\\
&\quad\leq C\,\mathcal{H}^n\Big(\Pi\Big(\mathcal{G}\cap\Pi^{-1}\Big(\{\zeta\in H(Q,r),\ |\zeta|\leq r\}\setminus\Pi(G(Q,r))\Big)\Big)\Big)\\
&\quad\leq C\,\mathcal{H}^n\Big(\{\zeta\in H(Q,r),\ |\zeta|\leq r\}\setminus\Pi(G(Q,r))\Big)\leq C_1\,\omega_n r^n\exp\left(-C_2/\delta\right),
\end{aligned}\tag{4.2.70}$$

by (4.2.69). Keeping in mind that $\mathcal{C}(Q,r)\cap\Big(\partial\Omega\setminus\mathcal{G}\Big)$ is contained in $E(Q,r)$, the estimate (4.2.35) now follows from (4.2.70) and (4.2.37).

As for the proximity condition (4.2.38), fix $X\in\Big(\mathcal{C}(Q,r)\cap\partial\Omega\Big)\setminus G(Q,r)$ and let $X^*\in G(Q,r)$ be arbitrary. Since $X^*\in G(Q,r)$ and $X\in B(X^*,2r)$, estimate (4.2.64) gives

$$|t(X)-h(\Pi(X^*))|=|t(X)-t(X^*)|\leq C\delta|\Pi(X)-\Pi(X^*)|, \tag{4.2.71}$$

for some geometrical constant $C>0$. Consequently,

$$\begin{aligned}
|X-(\Pi(X),h(\Pi(X)))| &= |t(X)-h(\Pi(X))|\\
&\leq |t(X)-h(\Pi(X^*))|+|h(\Pi(X^*))-h(\Pi(X))|\\
&\leq C\delta|\Pi(X)-\Pi(X^*)|,
\end{aligned}\tag{4.2.72}$$

by (4.2.71) and the Lipschitz condition on h. Taking the infimum over $X^*\in G(Q,r)$ now yields (4.2.38).

There remains to prove (4.2.41). Using (4.2.35), (4.2.40) and (4.2.32), we may estimate

$$\begin{aligned}
\sigma(\Delta(Q,r)) &= \mathcal{H}^n(\partial\Omega\cap B(Q,r))\leq\mathcal{H}^n(\mathcal{C}(Q,r)\cap\partial\Omega)\\
&\leq \mathcal{H}^n(\mathcal{C}(Q,r)\cap\mathcal{G})+\mathcal{H}^n\Big(\mathcal{C}(Q,r)\cap(\partial\Omega\setminus\mathcal{G})\Big)\\
&\leq \int_{\zeta\in H(Q,r):\,|\zeta|<r}\sqrt{1+|\nabla h(\zeta)|^2}\,d\mathcal{L}^n(\zeta)+C_1\,\omega_n r^n\exp\left(-C_2/\delta\right)\\
&\leq \Big(1+C_3\delta+C_1\exp\left(-C_2/\delta\right)\Big)\omega_n r^n\leq(1+C_4\delta)\omega_n r^n.
\end{aligned}\tag{4.2.73}$$

Also, from (4.2.35) and (4.2.32),

$$\begin{aligned}
\omega_n r^n &= \mathcal{H}^n\Big(\Pi(\{\zeta\in H(Q,r):\,|\zeta|<r\})\Big)\leq \mathcal{H}^n(\mathcal{C}(Q,r)\cap\partial\Omega)\\
&\leq \mathcal{H}^n(B(Q,r)\cap\partial\Omega)+\mathcal{H}^n\Big((\mathcal{C}(Q,r)\cap\partial\Omega)\setminus B(Q,r)\Big)\\
&\leq \sigma(\Delta(Q,r))+\mathcal{H}^n\Big((\mathcal{C}(Q,r)\cap\mathcal{G})\setminus B(Q,r)\Big)+\mathcal{H}^n\Big(\mathcal{C}(Q,r)\cap(\partial\Omega\setminus\mathcal{G})\Big)\\
&\leq \sigma(\Delta(Q,r))+C_3\,\delta\,\omega_n\,r^n+C_1\,\omega_n r^n\exp\,(-C_2/\delta)\\
&\leq \sigma(\Delta(Q,r))+C_4\,\delta\,\omega_n r^n
\end{aligned}\tag{4.2.74}$$

so that $(1-C_4\,\delta)\,\omega_n\,r^n\leq\sigma(\Delta(Q,r))$. Hence, (4.2.41) follows from this and (4.2.73), completing the proof of the theorem. □

From Definition 4.1.7 and Corollary 4.1.6, it follows that if $\Omega\subseteq\mathbb{R}^{n+1}$ is a δ-SKT domain for some $\delta\in(0,\delta_n)$, then Ω is a two-sided NTA domain, $\partial\Omega$ is Ahlfors regular, and for each compact set $\mathcal{K}\subset\mathbb{R}^{n+1}$, there exists $R>0$ such that (4.1.14) holds. Remarkably, the converse implication is also valid (up to a multiplicative geometrical constant). This is made precise in the theorem below.

Theorem 4.2.7 *Let $\Omega\subseteq\mathbb{R}^{n+1}$ be an open set that satisfies a two-sided local John condition and whose boundary is Ahlfors regular. In the case when Ω is unbounded, it is also required that $\partial\Omega$ divides $\mathbb{R}^{n+1}$ into two distinct connected components.*

Then there exists a geometrical constant $C_o>1$ with the following significance. Assume that there exists $\delta>0$, sufficiently small relative to the John and Ahlfors regularity constants of Ω, with the property that for every compact set $\mathcal{K}\subset\mathbb{R}^{n+1}$ there exists $R_{\mathcal{K}}>0$ such that

$$\sup_{Q\in\mathcal{K}\cap\partial\Omega}\|\nu\|_*(\Delta(Q,R_{\mathcal{K}}))\leq\delta.\tag{4.2.75}$$

Then Ω is a δ_o-SKT domain, with $\delta_o=C_o\delta$. In particular, Ω is a δ_o-Reifenberg flat domain and, hence, a two-sided NTA domain.

Proof. Since Corollary 3.1.14 gives that Ω has locally finite perimeter, it suffices to show that Ω is δ_o-Reifenberg flat if $\delta_o:=C_o\delta$ for some geometrical constant $C_o>1$. The latter is chosen so that the conditions in Definition 4.1.1 are verified (for $\Sigma:=\partial\Omega$ and $C_o\delta$ in place of δ) by the n-plane $L(Q,r):=Q+H(Q,r)$, with $H(Q,r)$ as in Theorem 4.2.4. That this is possible is ensured by (4.2.39). As can be seen from (4.2.66)-(4.2.67), choosing C_o sufficiently large also guarantees that the conditions in Definition 4.1.3, with $\mathcal{L}(Q,r):=Q+H(Q,r)$ and $\vec{n}_{Q,r}$ as in (4.2.48), are verified as well. The desired conclusion follows. □

Theorem 4.2.7 and definitions readily yield the following.

Corollary 4.2.8 *Assume that $\Omega\subseteq\mathbb{R}^{n+1}$ is an open set with compact, Ahlfors regular boundary which satisfies a two-sided John condition and such that* $\mathrm{dist}\,(\nu,\mathrm{VMO}\,(\partial\Omega,d\sigma))<\varepsilon$ *where ν is the unit normal to $\partial\Omega$ and the distance is taken in the* $\mathrm{VMO}\,(\partial\Omega,d\sigma)$ *norm.*

Then there exist ε_o and C_o, depending only on n and the John and Ahlfors regularity constants of Ω, such that if $\varepsilon\in(0,\varepsilon_o)$ then Ω is a δ-SKT domain where $\delta:=C_o\varepsilon$.

Conversely, if the open set $\Omega\subseteq\mathbb{R}^{n+1}$, with compact boundary, is a δ-SKT domain for some $\delta\in(0,\delta_n)$, then Ω is also an ε-regular SKT domain with $\varepsilon:=\delta$.

Theorem 4.2.7 also implies the following equivalent characterizations of bounded, regular SKT domains.

Theorem 4.2.9 *Let $\Omega \subseteq \mathbb{R}^{n+1}$ be an open set with compact boundary and denote by $k_X := k(\cdot, X)$ its Poisson kernel with fixed pole at $X \in \Omega$. Then the following statements are equivalent:*

(i) *Ω satisfies a two-sided local John condition, $\partial\Omega$ is Ahlfors regular, and the unit normal ν of Ω belongs to* $\mathrm{VMO}\,(\partial\Omega, d\sigma)$*;*

(ii) *Ω is a two-sided NTA domain, $\partial\Omega$ is Ahlfors regular, and $\nu \in \mathrm{VMO}\,(\partial\Omega, d\sigma)$;*

(iii) *Ω is a two-sided NTA domain, $\partial\Omega$ is Ahlfors regular,* $\log k_X \in \mathrm{VMO}\,(\partial\Omega, d\sigma)$ *and there exists $\delta > 0$ sufficiently small (relative to the NTA and Ahlfors regularity constants of Ω) with the property that*

$$\mathrm{dist}\,(\nu\,,\ \mathrm{VMO}\,(\partial\Omega, d\sigma)) < \delta; \tag{4.2.76}$$

(iv) *Ω is a regular SKT domain (in particular, Ω is a Reifenberg flat domain with vanishing constant).*

Proof. The equivalence $(i) \Leftrightarrow (ii)$ is covered by Theorem 4.2.7 and Lemma 3.1.13. Next, that $(ii) \Leftrightarrow (iv)$ is a direct consequence of definitions, Theorem 4.2.7 and the comments preceding its statement. In concert with Theorem 4.2.7 and Corollary 2.4.10, conditions in (iii) guarantee that Ω is Reifenberg flat with a sufficiently small constant hence, further, Reifenberg flat with a vanishing constant, by virtue of Theorem 7.36 on p. 139 of [15]. Hence, Ω is a regular SKT domain. That, conversely, $\log k_X \in \mathrm{VMO}\,(\partial\Omega, d\sigma)$ if Ω is a regular SKT domain is part of the main result in [65]. This shows that $(iii) \Leftrightarrow (iv)$, finishing the proof. $\square$

Remark I. We wish to point out that the flatness condition (4.2.76) plays the role of the hypothesis adopted in [65], [66] that Ω is a sufficiently flat Reifenberg domain. Either flatness condition precludes domains such as $\Omega = \{X \in \mathbb{R}^{3+1} : x_4^2 < x_1^2 + x_2^2 + x_3^2\}$ from serving as a counterexample to the claim that:

> a two-sided NTA domain with an Ahlfors regular boundary and for which $\log k_\infty$, the logarithm of the Poisson kernel with pole at infinity, has vanishing mean oscillations is necessarily Reifenberg with vanishing constant.

Indeed, while the boundary of such a domain is the light cone $x_4^2 = x_1^2 + x_2^2 + x_3^2$ in $\mathbb{R}^{3+1}$ and, hence, $k_\infty = d\omega^\infty / d\sigma = c$, constant, by [97] and [64], the domain in question is not δ-Reifenberg and nor is condition (4.2.76) satisfied if $\delta < 1/8$.

Remark II. Characterizations such as those in Theorem 4.2.9 have been used in [53] to show that the class of regular SKT domains is invariant under C^1 diffeomorphisms of the Euclidean space.

4.3 Sobolev spaces revisited

In this subsection we wish to clarify the relationship between our Sobolev spaces $L_1^p(\partial\Omega, d\sigma)$ introduced in § 3.6 and general concept of Sobolev space defined for abstract abstract measure metric spaces. For the reader's convenience, we include a brief review of the latter, based on [46], [48].

Let (Σ, d, μ) be a *measure metric space*, i.e., a metric space (Σ, d) equipped with a doubling, positive Borel measure μ, which is finite on bounded, measurable sets. Given a μ-measurable, real-valued function u on Σ, denote by $D(u)$ the collection of all *generalized gradients of* u, i.e. nonnegative, μ-measurable functions g on Σ with the property that there exists $E \subset \Sigma$ with $\mu(E) = 0$ and such that

$$|u(X) - u(Y)| \leq d(X,Y)(g(X) + g(Y)), \qquad \forall\, X, Y \in \Sigma \setminus E. \tag{4.3.1}$$

As in [46], for each $p \in (1, \infty)$ then define the Sobolev space

$$W^{1,p}(\Sigma) := \{u \in L^p(\Sigma, d\mu) : \, D(u) \cap L^p(\Sigma, d\mu) \neq \emptyset\} \tag{4.3.2}$$

and equip it with the norm

$$\|u\|_{W^{1,p}(\Sigma)} := \|u\|_{L^p(\Sigma, d\mu)} + \inf_{g \in D(u)} \|g\|_{L^p(\Sigma, d\mu)}. \tag{4.3.3}$$

As observed in [46],

$$W^{1,p}(\Sigma) \text{ is a Banach space for each } p \in (1, \infty). \tag{4.3.4}$$

Given $q \in [1, \infty)$ and $f \in L^q_{loc}(\Sigma, d\mu)$, consider the following Calderón-type maximal operator

$$\Lambda_{*,q} f(X) := \sup_{r>0} \left(⨍_{B_d(X,r)} \left| \frac{f(Y) - f_{X,r}}{r} \right|^q d\mu(Y) \right)^{1/q}, \qquad X \in \Sigma, \tag{4.3.5}$$

where $B_d(X, r) := \{Y \in \Sigma : \, d(X,Y) < r\}$ and we have set $f_{X,r} := ⨍_{B_d(X,r)} f \, d\mu$.

The following is a minor variation of Theorem 3.4 on p. 606 in [48].

Proposition 4.3.1 *Let (Σ, d, μ) be a measure metric space of finite diameter. Then for each $p \in (1, \infty)$ there exist constants with the following significance. First, the following statements are equivalent:*

(i) *$u \in W^{1,p}(\Sigma)$;*

(ii) *$u \in L^p(\Sigma, d\mu)$ and there exists a nonnegative function $g \in L^p(\Sigma, d\mu)$ such that the Poincaré inequality*

$$⨍_{B_d(X,r)} |u(Y) - u_{X,r}| \, d\mu(Y) \leq C\, r ⨍_{B_d(X,r)} g \, d\mu \tag{4.3.6}$$

holds for every $X \in \Sigma$ and $r > 0$;

(iii) *$u \in L^p(\Sigma, d\mu)$ and $\Lambda_{*,1} u \in L^p(\Sigma, d\mu)$.*

Second,

$$\begin{aligned}\|u\|_{W^{1,p}(\Sigma)} &\approx \|u\|_{L^p(\Sigma,d\mu)} + \inf\{\|g\|_{L^p(\Sigma,d\mu)} : g \text{ satisfies (4.3.6)}\}\\ &\approx \|u\|_{L^p(\Sigma,d\mu)} + \|\Lambda_{*,1}u\|_{L^p(\Sigma,d\mu)}. \end{aligned} \tag{4.3.7}$$

We shall also make use of the following Lusin-type approximation result from [46].

Proposition 4.3.2 *Assume that (Σ, d, μ) is a measure metric space of finite diameter and that $1 < p < \infty$. Then for each $f \in W^{1,p}(\Sigma)$ and $\varepsilon > 0$ there exists $h \in \mathrm{Lip}\,(\Sigma)$ for which*

$$\mu(\{X \in \Sigma :\, f(X) \neq h(X)\}) < \varepsilon \quad \textit{and} \quad \|f - h\|_{W^{1,p}(\Sigma)} < \varepsilon. \tag{4.3.8}$$

In particular, the collection of Lipschitz functions is dense in each $W^{1,p}(\Sigma)$, $1 < p < \infty$.

This completes our review of Sobolev spaces in abstract metric measure spaces and we return to the setting when Σ is the boundary of a sufficiently reasonable domain $\Omega \subset \mathbb{R}^{n+1}$. If $\Omega \subset \mathbb{R}^{n+1}$ is an open set with an Ahlfors regular boundary, then $\partial\Omega$ equipped with the Euclidean distance and the surface measure σ becomes a measure metric space. Consequently, we can consider the Sobolev space $W^{1,p}(\partial\Omega)$ as in (4.3.2)-(4.3.3), for each $p \in (1, \infty)$.

Proposition 4.3.3 *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with an Ahlfors regular boundary, which satisfies a two-sided local John condition (which is therefore a UR domain), and fix $p \in (1, \infty)$. Then the inclusion map*

$$L^p_1(\partial\Omega, d\sigma) \hookrightarrow W^{1,p}(\partial\Omega) \tag{4.3.9}$$

is well-defined and continuous.

Proof. Denote by $\mathcal{D}^{\pm}$ the harmonic double layers in $\Omega_{\pm}$. We shall show that there exists a finite constant $C = C(\Omega, p) > 0$ such that, given $f \in L^p_1(\partial\Omega, d\sigma)$,

$$g := C\Big(\mathcal{N}(\nabla\mathcal{D}^+ f) + \mathcal{N}(\nabla\mathcal{D}^- f) + |f|\Big) \tag{4.3.10}$$

is a generalized gradient for f (if $\partial\Omega$ is unbounded, $|f|$ can be dropped from the definition of g).

To prove this claim, fix $f \in L^p_1(\partial\Omega, d\sigma)$ and set $u^{\pm} := \mathcal{D}^{\pm} f$ in $\Omega_{\pm}$. Then for a.e. $X, Y \in \partial\Omega$ we can write $f(X) - f(Y) = (u^+(X) - u^+(Y)) - (u^-(X) - u^-(Y))$. If either $\partial\Omega$ is unbounded, or when $\partial\Omega$ is bounded and $R := |X - Y|$ is sufficiently small (say, $0 < R < R_o$, with R_o as in the definition of the local John condition) we let $A^{\pm}_R$ be the John centers of $B(X, 2R) \cap \Omega_{\pm}$ and estimate

$$\begin{aligned}|u^{\pm}(X) - u^{\pm}(Y)| &\leq |u^{\pm}(X) - u^{\pm}(A^{\pm}_R)| + |u^{\pm}(Y) - u^{\pm}(A^{\pm}_R)|\\ &\leq CR\Big(\mathcal{N}(\nabla u^{\pm})(X) + \mathcal{N}(\nabla u^{\pm})(X)\Big)\\ &\leq |X - Y|(g(X) + g(Y)). \end{aligned} \tag{4.3.11}$$

Here, to obtain the second inequality, we can write

$$u^{\pm}(X) - u^{\pm}(A_R^{\pm}) = \int_0^{C_X R} \frac{\partial}{\partial s} u\Big(z_X^{\pm}(s)\Big)\, ds, \tag{4.3.12}$$

where $z_X^{\pm}(s)$ is a unit speed parametrization of the non-tangential path γ_X, connecting X to $A_R^{\pm}$, of length $C_X R \leq CR$, whose existence is guaranteed by definition of the local John condition (and similarly with Y in place of X). Thus,

$$|f(X) - f(Y)| \leq |X - Y|(g(X) + g(Y)) \qquad \text{for a.e.} \quad X, Y \in \partial\Omega. \tag{4.3.13}$$

When $\partial\Omega$ is bounded and $R \geq R_o$, we trivially have $|f(X) - f(Y)| \leq R_o^{-1}|X - Y|(|f(X)| + |f(Y)|)$, so that (4.3.13) continues to hold in this case as well. Note that the function g given in (4.3.10) satisfies $\|g\|_{L^p(\partial\Omega, d\sigma)} \leq C\|f\|_{L_1^p(\partial\Omega, d\sigma)} < +\infty$, by (3.6.31).

Altogether, this shows that g is a generalized gradient for f. Hence, $f \in W^{1,p}(\partial\Omega)$ and $\|f\|_{W^{1,p}(\partial\Omega)} \leq \|f\|_{L^p(\partial\Omega, d\sigma)} + \|g\|_{L^p(\partial\Omega, d\sigma)} \leq C\|f\|_{L_1^p(\partial\Omega, d\sigma)}$. The desired conclusion follows. $\square$

We continue with another useful embedding result.

Lemma 4.3.4 *If Ω is a bounded UR domain then the inclusion*

$$\mathrm{Lip}\,(\partial\Omega) \hookrightarrow L_1^p(\partial\Omega, d\sigma) \tag{4.3.14}$$

is well-defined and continuous for each $p \in (1, \infty)$.

Proof. To fix ideas, assume that $p \in (1, \infty)$ and $f \in \mathrm{Lip}(\partial\Omega)$. Recall that Kirszbraun's Theorem asserts that any Lipschitz function defined on a subset of a metric space can be extended to a Lipschitz function on the entire space with the same Lipschitz constant (see, e.g., [117]). Thus, we can assume that $f = F|_{\partial\Omega}$, where F is a Lipschitz function in $\mathbb{R}^{n+1}$ with

$$\|\nabla F\|_{L^\infty(\mathbb{R}^{n+1})} \leq C_n \|f\|_{\mathrm{Lip}\,(\partial\Omega)}. \tag{4.3.15}$$

Pick a nice bump function η, set $\eta_\varepsilon(x) := \varepsilon^{-(n+1)}\eta(x/\varepsilon)$ and regularize $F^\varepsilon := F * \eta_\varepsilon$, $\varepsilon > 0$. Then $\|\nabla F^\varepsilon\|_{L^\infty(\mathbb{R}^{n+1})} \leq C_n\|f\|_{\mathrm{Lip}\,(\partial\Omega)}$ and for any $\psi \in C^\infty(\mathbb{R}^{n+1})$ we may estimate

$$\begin{aligned} \left|\int_{\partial\Omega} f \partial_{\tau_{jk}}\psi\, d\sigma\right| &= \left|\lim_{\varepsilon\to 0}\int_{\partial\Omega} F^\varepsilon \partial_{\tau_{jk}}\psi\, d\sigma\right| = \left|\lim_{\varepsilon\to 0}\int_{\partial\Omega} \partial_{\tau_{jk}} F^\varepsilon \psi\, d\sigma\right| \\ &\leq C\|\psi\|_{L^1(\partial\Omega, d\sigma)} \limsup_{\varepsilon\to 0} \|\nabla F^\varepsilon\|_{L^\infty(\mathbb{R}^{n+1})} \\ &\leq C\|\psi\|_{L^1(\partial\Omega, d\sigma)}\|f\|_{\mathrm{Lip}\,(\partial\Omega)}. \end{aligned} \tag{4.3.16}$$

Since $C^\infty(\mathbb{R}^{n+1})|_{\partial\Omega}$ is a dense subset of $L^1(\partial\Omega, d\sigma)$ (which is easily seen with the help of Lemma 2.4.9), it follows that for each $j, k \in \{1, ..., n+1\}$ the assignment $\psi \mapsto \int_{\partial\Omega} f\partial_{\tau_{jk}}\psi\, d\sigma$ extends to a functional in $(L^1(\partial\Omega, d\sigma))^* = L^\infty(\partial\Omega, d\sigma)$, of norm $\leq C\|f\|_{\mathrm{Lip}\,(\partial\Omega)}$. Hence, there exists unique $b_{jk} \in L^\infty(\partial\Omega, d\sigma)$, with $\|b_{jk}\|_{L^\infty(\partial\Omega, d\sigma)} \leq C\|f\|_{\mathrm{Lip}\,(\partial\Omega)}$ for which

$$\int_{\partial\Omega} f\partial_{\tau_{jk}}\psi\, d\sigma = -\int_{\partial\Omega} b_{jk}\psi\, d\sigma, \qquad \forall\, \psi \in C_0^\infty(\mathbb{R}^{n+1}). \tag{4.3.17}$$

This proves that $\partial_{\tau_{jk}} f = b_{jk} \in L^\infty(\partial\Omega, d\sigma)$ satisfies $\|\partial_{\tau_{jk}} f\|_{L^\infty(\partial\Omega,d\sigma)} \leq C\|f\|_{\mathrm{Lip}\,(\partial\Omega)}$. In particular, the inclusion (4.3.14) is well-defined and continuous. $\square$

Let Ω be an open set in $\mathbb{R}^{n+1}$ with an Ahlfors regular boundary. In analogy with (4.3.5), if $q \in [1, \infty)$ and $f \in L^q_{loc}(\partial\Omega, d\sigma)$, in the current setting we define Calderón's maximal operator as

$$\Lambda_{*,q} f(X) := \sup_{r>0} \left(\fint_{\Delta(X,r)} \left| \frac{f(Y) - f_{X,r}}{r} \right|^q d\sigma_Y \right)^{1/q}, \qquad X \in \partial\Omega, \tag{4.3.18}$$

where we have set $f_{X,r} := \fint_{\Delta(X,r)} f\, d\sigma$.

Proposition 4.3.5 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set with an Ahlfors regular boundary, and which satisfies a two-sided local John condition. Also, fix p, q with $1 \leq q < p$. Then*

$$\|f\|_{L^p_1(\partial\Omega,d\sigma)} \approx \|\Lambda_{*,q} f\|_{L^p(\partial\Omega,d\sigma)} + \|f\|_{L^p(\partial\Omega,d\sigma)}, \tag{4.3.19}$$

uniformly for $f \in L^p_1(\partial\Omega, d\sigma)$.

A few comments are in order. The idea of characterizing membership to classical Sobolev spaces, defined in the Euclidean setting, in terms of maximal operators (such as (4.3.18)) goes back to Calderón (see [12], [13]). In [105], Semmes has dealt with the issue of extending Calderón's theory when the flat Euclidean space is replaced by a more general manifold. Semmes' version of Calderón's theorem (closely related to the case $p = q = 2$ of (4.3.19)) is stated for a *smooth* surface albeit the comparability constants do not depend on smoothness (but only on the Ahlfors regularity constants and the NTA constants). In the same paper, Semmes also raises the issue of eliminating the a priori smoothness assumption on the surface. The latter is the main attribute of our result.

Proof of Proposition 4.3.5. Let Ω, p, q be as above and pick $q_o \in (1, p)$ with $q_o \geq q$. Also, select an arbitrary $f \in L^p_1(\partial\Omega, d\sigma)$. Let $\mathcal{M}$ denote the Hardy-Littlewood maximal operator on $\partial\Omega$. For each $X \in \partial\Omega$ and $r > 0$, the Poincaré inequality (4.2.1) then gives

$$\begin{aligned} \left(\fint_{\Delta(X,r)} \left| \frac{f(Y) - f_{X,r}}{r} \right|^q d\sigma_Y \right)^{1/q} &\leq \left(\fint_{\Delta(X,r)} \left| \frac{f(Y) - f_{X,r}}{r} \right|^{q_o} d\sigma_Y \right)^{1/q_o} \\ &\leq CR \Big[\fint_{\Delta(X,5r)} |\nabla_{\tan} f|^{q_o}\, d\sigma \Big]^{1/q_o} + CR \sum_{j=1}^{\infty} 2^{-j} \fint_{\Delta(X,2^{j+1}r)} |\nabla_{\tan} f|\, d\sigma \\ &\leq C \Big[\mathcal{M}(|\nabla_{\tan} f|^{q_o})(X) \Big]^{1/q_o}, \end{aligned} \tag{4.3.20}$$

provided that either $\partial\Omega$ is unbounded or $r \in (0, R_o)$, with R_o as in the statement of Proposition 4.2.1. If, on the other hand, $\partial\Omega$ is bounded and $r \geq R_o$, we simply estimate

$$\left(\fint_{\Delta(X,r)} \left| \frac{f(Y) - f_{X,r}}{r} \right|^q d\sigma_Y \right)^{1/q} \leq C \Big[\mathcal{M}(|f|^{q_o})(X) \Big]^{1/q_o}. \tag{4.3.21}$$

Either way, using (4.3.20), (4.3.21) and the boundedness of $\mathcal{M}$ on $L^{p/q_o}(\partial\Omega, d\sigma)$ we arrive at

$$\|\Lambda_{*,q} f\|_{L^p(\partial\Omega,d\sigma)} \le C\Big(\|\nabla_{\tan} f\|_{L^p(\partial\Omega,d\sigma)} + \|f\|_{L^p(\partial\Omega,d\sigma)}\Big). \tag{4.3.22}$$

This justifies the right-pointing inequality in (4.3.19).

Turning to the other direction, recall that

$$\partial_{\tau_{jk}} := \nu_j \partial_k - \nu_k \partial_j, \tag{4.3.23}$$

where ν is the outer unit normal to $\partial\Omega$, and ν_j is the j-th component of ν. Let $\mathcal{S}$, $\mathcal{D}$ denote, respectively, the single and double layer potentials associated with the Laplacian in $\mathbb{R}^{n+1}$ and set

$$u^{\pm} := \mathcal{D}f\Big|_{\Omega_{\pm}} \qquad \text{where } \ \Omega_+ := \Omega, \ \ \Omega_- := \mathbb{R}^{n+1} \setminus \bar{\Omega}. \tag{4.3.24}$$

In the current context, (3.6.31) then gives

$$\partial_j u^{\pm} = -\partial_r \mathcal{S}(\partial_{\tau_{jr}} f)|_{\Omega_{\pm}} \tag{4.3.25}$$

so that by Proposition 3.6.2, the second formula in (3.6.17) and (3.3.39),

$$\begin{aligned} \partial_{\tau_{jk}}(u^{\pm}|_{\partial\Omega}) &= \nu_j(\partial_k u^{\pm}|_{\partial\Omega}) - \nu_k(\partial_j u^{\pm}|_{\partial\Omega}) \\ &= \nu_j\Big[\pm\tfrac{1}{2}\nu_r\partial_{\tau_{kr}} f - \text{p.v.}\partial_r S(\partial_{\tau_{kr}} f)\Big] - \nu_k\Big[\pm\tfrac{1}{2}\nu_r\partial_{\tau_{jr}} f - \text{p.v.}\partial_r S(\partial_{\tau_{jr}} f)\Big], \end{aligned} \tag{4.3.26}$$

where p.v.$\partial_r S$ is the (convolution-like) principal value integral operator on $\partial\Omega$ whose kernel is $(\partial_r E)(X - Y)$ (with E as in (3.3.24)). Thus,

$$\begin{aligned} \partial_{\tau_{jk}}(u^+|_{\partial\Omega}) - \partial_{\tau_{jk}}(u^-|_{\partial\Omega}) &= \nu_j\nu_r\partial_{\tau_{kr}} f - \nu_k\nu_r\partial_{\tau_{jr}} f \\ &= \nu_j\nu_r(\nabla_{\tan} f)_j - \nu_k\nu_r(\nabla_{\tan} f)_k \\ &= \partial_{\tau_{jk}} f, \end{aligned} \tag{4.3.27}$$

on account of (4.3.26) and (3.6.43). The second formula in (3.6.17) and our earlier work on the nature of boundary traces allows for the following interpretation of the leftmost expression above. Given a point $X \in \partial\Omega$ that is weakly accessible from $\Omega_{\pm}$ (i.e., $X \in \overline{\Gamma_{\pm}(X)}$), and for which the respective non-tangential limits of $\nabla u^{\pm}$ exist (in particular, a.e. boundary point enjoys these properties), we form two non-tangential paths in $\Omega_{\pm}$, terminating at X, with arc-length parametrizations $\gamma^{\pm}(s)$, $0 \le s \le \delta_X$, $\gamma^{\pm}(0) = X$, $\text{dist}(\gamma^{\pm}(s), \partial\Omega) \approx s$ (cf. Definition 3.1.12), and we interpret the leftmost side of (4.3.27) evaluated at X as

$$\begin{aligned} \lim_{s\to 0}\Big(&\Big(\nu_j(X)(\partial_k u^+)(\gamma^+(s)) - \nu_k(X)(\partial_j u^+)(\gamma^+(s))\Big) \\ &- \Big(\nu_j(X)(\partial_k u^-)(\gamma^+(s)) - \nu_k(X)(\partial_j u^-)(\gamma^+(s))\Big)\Big). \end{aligned} \tag{4.3.28}$$

Fix a boundary point X along with two non-tangential paths $\gamma^{\pm}$ as above, and fix also $s \in (0, \delta_X)$. We claim that for each index $j \in \{1, ..., n+1\}$,

$$|(\partial_j u^+)(\gamma^+(s)) - (\partial_j u^-)(\gamma^-(s))| \leq C\Lambda_{*,1}f(X). \tag{4.3.29}$$

In turn, this and (3.6.42) entail the pointwise bound

$$|\nabla_{\tan} f(X)| \leq C\Lambda_{*,1}f(X) \qquad \text{for a.e.} \quad X \in \partial\Omega, \tag{4.3.30}$$

from which the left-pointing inequality in (4.3.19) follows immediately.

To prove the above claim, we first note that by (3.6.52), we may replace f by $f - f_{X,s}$, and we write

$$f - f_{X,s} = \sum_{i=0}^{\infty} f_i, \tag{4.3.31}$$

where

$$f_0 := (f - f_{X,s})\mathbf{1}_{\Delta(X,s)}, \qquad f_i := (f - f_{X,s})\mathbf{1}_{\Delta(X,2^i s)\setminus\Delta(X,2^{i-1}s)}, \quad i \geq 1. \tag{4.3.32}$$

(If $\partial\Omega$ is compact, the terms f_i may be vacuous for i large). Then

$$\begin{aligned} |(\partial_j u^+)(\gamma^+(s)) - (\partial_j u^-)(\gamma^-(s))| \quad &\leq \quad |(\partial_j \mathcal{D} f_0)(\gamma^+(s))| + |(\partial_j \mathcal{D} f_0)(\gamma^-(s))| \\ &\quad + \sum_{i=1}^{\infty} |(\partial_j \mathcal{D} f_i)(\gamma^+(s)) - (\partial_j \mathcal{D} f_i)(\gamma^-(s))| \; =: I + II + III. \end{aligned} \tag{4.3.33}$$

Standard estimates for derivatives of the fundamental solution for the Laplacian in $\mathbb{R}^{n+1}$ and the specific nature of the parametric paths $\gamma^{\pm}$ ensure that

$$\begin{aligned} I + II \quad &\leq \quad C \int_{\Delta(X,s)} \frac{1}{|\gamma^{\pm}(s) - Y|^{n+1}} |f(Y) - f_{X,s}| \, d\sigma_Y \\ &\leq \quad C \fint_{\Delta(X,s)} \left| \frac{f(Y) - f_{X,s}}{s} \right| d\sigma_Y \leq C\Lambda_{*,1}f(X). \end{aligned} \tag{4.3.34}$$

Similarly, using also the mean value theorem, we obtain that

$$\begin{aligned} &|(\partial_j \mathcal{D} f_i)(\gamma^+(s)) - (\partial_j \mathcal{D} f_i)(\gamma^-(s))| \\ &\quad \leq C \int\limits_{\Delta(X,2^i s)\setminus\Delta(X,2^{i-1}s)} |(\nabla^2 E)(\gamma^+(s) - Y) - (\nabla^2 E)(\gamma^-(s) - Y)| |f(Y) - f_{X,s}| \, d\sigma_Y \\ &\quad \leq C 2^{-i} \fint_{\Delta(X,2^i s)} \left| \frac{f(Y) - f_{X,s}}{2^i s} \right| d\sigma_Y, \end{aligned} \tag{4.3.35}$$

since $|\gamma^+(s)-\gamma^-(s)|\leq Cs$, and if $Y\in\Delta(X,2^is)\setminus\Delta(X,2^{i-1}s)$ and $Z\in\gamma^\pm(s)$ then for i large enough we may write $|Z-Y|\geq|X-Y|-|X-Z|\geq 2^{i-1}s-Cs\geq C2^is$. Going further, we estimate

$$\begin{aligned}
&⨍_{\Delta(X,2^is)}\left|\frac{f(Y)-f_{X,s}}{2^is}\right|\,d\sigma_Y\\
&\qquad\leq ⨍_{\Delta(X,2^is)}\left|\frac{f(Y)-f_{X,2^is}}{2^is}\right|\,d\sigma_Y\\
&\qquad+\sum_{j=1}^{i}⨍_{\Delta(X,2^is)}\left|\frac{f_{X,2^js}-f_{X,2^{j-1}s}}{2^is}\right|\,d\sigma_Y\\
&\qquad\leq\left|\frac{f(Y)-f_{X,2^is}}{2^is}\right|\,d\sigma_Y+C\sum_{j=1}^{i}⨍_{\Delta(X,2^js)}\left|\frac{f(Y)-f_{X,2^js}}{2^js}\right|\,d\sigma_Y\\
&\qquad\leq Ci\Lambda_{*,1}f(X).
\end{aligned}\tag{4.3.36}$$

In conjunction with (4.3.35), this yields the bound

$$III\leq C\sum_{i=1}^{\infty}i\,2^{-i}\Lambda_{*,1}f(X)\leq C\Lambda_{*,1}f(X).\tag{4.3.37}$$

This proves (4.3.29) and concludes the proof of Proposition 4.3.5. ☐

Recall the space $W^{1,p}(\partial\Omega)$ defined in the first part of this subsection, in (4.3.2).

Theorem 4.3.6 *Suppose that $\Omega\subset\mathbb{R}^{n+1}$ is an open set satisfying a two-sided local John condition and whose boundary is compact and Ahlfors regular. Then for each $p\in(1,\infty)$,*

$$L^p_1(\partial\Omega,d\sigma)=W^{1,p}(\partial\Omega)\tag{4.3.38}$$

with equivalence of norms.

Proof. Recall from Proposition 4.3.3 that $L^p_1(\partial\Omega,d\sigma)\hookrightarrow W^{1,p}(\partial\Omega)$ and note that, as seen from (4.3.7) and Proposition 4.3.5

$$\|f\|_{L^p_1(\partial\Omega,d\sigma)}\approx\|f\|_{W^{1,p}(\partial\Omega)},\qquad\text{uniformly for }\ f\in L^p_1(\partial\Omega,d\sigma).\tag{4.3.39}$$

We also know that $\mathrm{Lip}\,(\partial\Omega)$ is contained in $L^p_1(\partial\Omega,d\sigma)$ (cf. Lemma 4.3.4), and is dense in $W^{1,p}(\partial\Omega)$ (cf. Proposition 4.3.2). Since, by (4.3.39), the norms in these two Banach spaces are equivalent, the equality (4.3.38) follows. ☐

Corollary 4.3.7 *Let $\Omega\subset\mathbb{R}^{n+1}$ be an open set satisfying a two-sided local John condition and whose boundary is compact and Ahlfors regular. Then for each $p\in(1,\infty)$ the following hold:*

(i) The inclusion (4.3.14) has dense range, i.e., Lipschitz functions form a dense subset of the Sobolev space;

(ii) A function $f \in L^p(\partial\Omega, d\sigma)$ belongs to $L^p_1(\partial\Omega, d\sigma)$ if and only if $\Lambda_{*,1} f \in L^p(\partial\Omega, d\sigma)$.

Proof. Part (i) is a consequence of (4.3.38) and Proposition 4.3.2. The "if" direction of part (ii) follows from Proposition 4.3.1 and Theorem 4.3.6, whereas the "only if" direction is already contained in Proposition 4.3.5. □

Proposition 4.3.8 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set satisfying a two-sided local John condition and whose boundary is compact and Ahlfors regular. Then for each $p \in (1, \infty)$ the inclusion*

$$C^\infty(\mathbb{R}^{n+1})\Big|_{\partial\Omega} \hookrightarrow L^p_1(\partial\Omega, d\sigma) \tag{4.3.40}$$

is well-defined, with dense range.

Proof. To begin with, Lemma 4.3.4 shows that (4.3.40) is indeed well-defined. The main issue here is proving the denseness of the range, a task to which we now turn.

By the Hahn-Banach Theorem it suffices to show that if $\Lambda \in \Big(L^p_1(\partial\Omega, d\sigma)\Big)^*$ vanishes on $C^\infty(\mathbb{R}^{n+1})\Big|_{\partial\Omega}$ then it vanishes on $L^p_1(\partial\Omega, d\sigma)$. Invoking (3.6.39), this can be further rephrased as follows: If $f_0, f_{jk} \in L^{p'}(\partial\Omega, d\sigma)$, $1 \le j < k \le n+1$, $1/p + 1/p' = 1$, then

$$\begin{aligned} &\int_{\partial\Omega} \Big(f_0\varphi + \sum_{1\le j<k\le n+1} f_{jk}\partial_{\tau_{jk}}\varphi\Big)\, d\sigma = 0 \qquad \forall\, \varphi \in C^\infty(\mathbb{R}^{n+1}) \\ &\qquad \Longrightarrow \int_{\partial\Omega} \Big(f_0 f + \sum_{1\le j<k\le n+1} f_{jk}\partial_{\tau_{jk}} f\Big)\, d\sigma = 0 \qquad \forall\, f \in L^{p'}_1(\partial\Omega, d\sigma). \end{aligned} \tag{4.3.41}$$

To this end, assume that f_0, f_{jk} are as above and note that, by part (i) in Corollary 4.3.7, it suffices to show that the conclusion in (4.3.41) holds if $f \in \mathrm{Lip}\,(\partial\Omega)$.

Assuming that this is the case, consider $f_\varepsilon := F^\varepsilon|_{\partial\Omega}$, $\varepsilon > 0$, where F^ε are the functions constructed as in the proof of Lemma 4.3.4, in conjunction with this f. It is then clear that $f_\varepsilon \to f$ in $L^{p'}(\partial\Omega, d\sigma)$ as $\varepsilon \to 0$ and, since $\sup_{\varepsilon>0} \|\partial_{\tau_{jk}} f\|_{L^\infty(\partial\Omega,d\sigma)} \le C\|f\|_{\mathrm{Lip}\,(\partial\Omega)}$, Alaoglu's Theorem ensures that we can assume that $\partial_{\tau_{jk}} f_\varepsilon \to \partial_{\tau_{jk}} f$ weakly in $L^{p'}(\partial\Omega, d\sigma)$ as $\varepsilon \to 0$.

Based on these, we may then write

$$\int_{\partial\Omega} \Big(f_0 f + \sum_{1\le j<k\le n+1} f_{jk}\partial_{\tau_{jk}} f\Big)\, d\sigma = \lim_{\varepsilon\to 0} \int_{\partial\Omega} \Big(f_0 f_\varepsilon + \sum_{1\le j<k\le n+1} f_{jk}\partial_{\tau_{jk}} f_\varepsilon\Big)\, d\sigma = 0, \tag{4.3.42}$$

granted our hypotheses on f_0, f_{jk}, since f_ε is of the form $F^\varepsilon|_{\partial\Omega}$ with $F^\varepsilon \in C^\infty(\mathbb{R}^{n+1})$. This justifies (4.3.42) and finishes the proof of the proposition. □

Corollary 4.3.9 *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set satisfying a two-sided local John condition and whose boundary is compact and Ahlfors regular. Then the boundary integration by parts formula*

$$\int_{\partial\Omega} (\partial_{\tau_{jk}} f) g\, d\sigma = -\int_{\partial\Omega} f(\partial_{\tau_{jk}} g)\, d\sigma \tag{4.3.43}$$

holds for each $j,k\in\{1,...,n+1\}$, whenever $f\in L^p_1(\partial\Omega,d\sigma)$ and $g\in L^{p'}_1(\partial\Omega,d\sigma)$ with $p,p'\in(1,\infty)$, $1/p+1/p'=1$.

In particular, for each $j,k\in\{1,...,n+1\}$ and $1<p<\infty$, the tangential differential operator $\partial_{\tau_{jk}}:L^p_1(\partial\Omega,d\sigma)\to L^p(\partial\Omega,d\sigma)$ can be consistently extended as the bounded mapping

$$\partial_{\tau_{jk}}:L^p(\partial\Omega,d\sigma)\longrightarrow L^p_{-1}(\partial\Omega,d\sigma)=\Big(L^{p'}_1(\partial\Omega,d\sigma)\Big)^*,\quad \langle\partial_{\tau_{jk}}f,g\rangle:=-\int_{\partial\Omega}f(\partial_{\tau_{jk}}g)\,d\sigma. \tag{4.3.44}$$

Proof. By Proposition 4.3.8, it suffices to prove (4.3.43) when $f=u|_{\partial\Omega}$ and $g=v|_{\partial\Omega}$, for some $u,v\in C^1(\overline{\Omega})$. In this setting,

$$\int_{\partial\Omega}u\,(\partial_{\tau_{jk}}v)\,d\sigma=\int_\Omega\Big(\partial_ju(X)\partial_kv(X)-\partial_ku(X)\partial_jv(X)\Big)\,dX,\quad\forall\,j,k\in\{1,...,n+1\}. \tag{4.3.45}$$

Since the right-hand side of (4.3.45) is antisymmetric in u and v, we obtain that

$$\int_{\partial\Omega}u\,(\partial_{\tau_{jk}}v)\,d\sigma=\int_{\partial\Omega}(\partial_{\tau_{kj}}u)\,v\,d\sigma, \tag{4.3.46}$$

from which (4.3.43) follows. □

Having clarified the relationship between our Sobolev spaces and those defined on general measure metric spaces (cf. Proposition 4.3.3 and Theorem 4.3.6), the properties deduced in the abstract framework carry over to the current setting. As an example, we have:

Corollary 4.3.10 *Assume that $\Omega\subset\mathbb{R}^{n+1}$ is an open set satisfying a two-sided local John condition and whose boundary is compact and Ahlfors regular. Then*

$$L^p_1(\partial\Omega,d\sigma)\hookrightarrow L^{p^*}(\partial\Omega,d\sigma)\quad\textit{if } 1<p<n,\ \textit{where } p^*:=\tfrac{np}{n-p}, \tag{4.3.47}$$

$$L^p_1(\partial\Omega,d\sigma)\hookrightarrow C^\alpha(\partial\Omega)\quad\textit{if } n<p<\infty,\ \textit{where }\ \alpha:=1-\tfrac{n}{p}. \tag{4.3.48}$$

Corresponding to $p=n$, functions in $L^n_1(\partial\Omega,d\sigma)$ satisfy a global exponential integrability condition of John-Nirenberg type. If, in addition, $\partial\Omega$ is compact, then

$$L^p_1(\partial\Omega,d\sigma)\hookrightarrow L^p(\partial\Omega,d\sigma)\quad\textit{compactly, for each } p\in(1,\infty). \tag{4.3.49}$$

Proof. The embeddings (4.3.47)-(4.3.48) are consequences of Proposition 4.3.3 and Theorem 8.7 on p. 197 in [47] (cf. also Theorem 6 in [46]). The claim in (4.3.49) follows from Theorem 4.3.6 and Corollary 1 on p. 125 in [60]. □

Corollary 4.3.11 *Let $\Omega\subset\mathbb{R}^{n+1}$ be a bounded open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular. Also, recall the single layer potential operator $\mathcal{S}$ from (3.6.22). Then*

$$\|\mathcal{N}(\mathcal{S}f)\|_{L^q(\partial\Omega,d\sigma)}\leq C(\Omega,p,q)\,\|f\|_{L^p(\partial\Omega,d\sigma)}, \tag{4.3.50}$$

where $q:=np/(n-p)$ if $1<p<n$, and $q<\infty$ if $n\leq p<\infty$.

Proof. This is a direct consequence of (4.3.47)-(4.3.48), (3.6.14) and (3.6.36). $\square$

We conclude this subsection with the following global Poincaré inequality.

Proposition 4.3.12 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular, compact and connected. Then for any $p \in (1, \infty)$,*

$$\Big(⨍_{\partial\Omega} |f - f_{\partial\Omega}|^p \, d\sigma\Big)^{1/p} \leq C\Big(⨍_{\partial\Omega} |\nabla_{\tan} f|^p \, d\sigma\Big)^{1/p}, \tag{4.3.51}$$

uniformly for $f \in L^p_1(\partial\Omega, d\sigma)$, where $f_{\partial\Omega}$ denotes the integral average of f on $\partial\Omega$.

Proof. Seeking a contradiction, assume that there exists a sequence $f_j \in L^p_1(\partial\Omega, d\sigma)$, $j \in \mathbb{N}$, with $\|\nabla_{\tan} f\|_{L^p(\partial\Omega, d\sigma)} \to 0$ as $j \to \infty$ and yet $\|f_j - (f_j)_{\partial\Omega}\|_{L^p(\partial\Omega, d\sigma)} = 1$ for each $j \in \mathbb{N}$. Based on (4.3.49), there is no loss of generality in assuming that $f_j - (f_j)_{\partial\Omega} \to g$ in $L^p(\partial\Omega, d\sigma)$ as $j \to \infty$. In particular, this entails

$$\|g\|_{L^p(\partial\Omega, d\sigma)} = 1 \quad \text{and} \quad \int_{\partial\Omega} g \, d\sigma = 0. \tag{4.3.52}$$

We now claim that $g \in L^p_1(\partial\Omega, d\sigma)$ and, in fact, $\nabla_{\tan} g = 0$. Indeed, if $\varphi \in C^1(\mathbb{R}^{n+1})$, then for each $k, \ell \in \{1, ..., n+1\}$ we may write

$$\int_{\partial\Omega} g(\partial_{\tau_{k\ell}}\varphi) \, d\sigma = \lim_{j\to\infty} \int_{\partial\Omega} (f_j - (f_j)_{\partial\Omega})(\partial_{\tau_{k\ell}}\varphi) \, d\sigma = \lim_{j\to\infty} \int_{\partial\Omega} (\partial_{\tau_{\ell k}} f_j)\varphi \, d\sigma = 0, \tag{4.3.53}$$

justifying the claim. Going further, this, the connectivity assumption on $\partial\Omega$ and Proposition 3.6.6 further entail that g is constant on $\partial\Omega$. With this in hand, the desired contradiction is evident from (4.3.52). $\square$

4.4 Compactness of double layer-like operators on VMO_1 domains

Here we discuss the work in [50] and, when Ω is a VMO_1 domain, use it to establish the compactness on $L^p(\partial\Omega, d\sigma)$ of singular integral operators belonging to a distinguished class (which contains the principal value harmonic double layer K). Prior to presenting this in the form of a theorem, we isolate a key estimate. To state it, recall next that given $p \in (1, \infty)$, a positive, locally integrable function w defined in $\mathbb{R}^n$ is said to belong to the Muckenhoupt class A_p if

$$[w]_{A_p} := \sup_{Q \subset \mathbb{R}^n,\, cube} \Big(⨍_Q w(x)\, dx\Big)\Big(⨍_Q w(x)^{-1/(p-1)}\, dx\Big)^{p-1} < \infty. \tag{4.4.1}$$

For further reference, we recall that, corresponding to $p = 1$,

$$[w]_{A_1} := \sup_{Q \subset \mathbb{R}^n,\, cube} \Big(⨍_Q w(x)\, dx\Big)\Big(\operatorname*{ess\,inf}_Q w\Big)^{-1} < \infty, \tag{4.4.2}$$

and that, corresponding to $p = \infty$,

$$A_\infty := \bigcup_{p \geq 1} A_p. \tag{4.4.3}$$

The relevance of this concept in the current context stems from the following observation made in [50]:

$$\begin{aligned} & A \text{ as in (2.5.1)-(2.5.2)} \Longrightarrow w := \sqrt{1+|\nabla A|^2} \in \bigcap_{1<p<\infty} A_p \\ & \text{and} \quad [w]_{A_p} < C_{n,p}(1+\|\nabla A\|_*) \text{ for each } p \in (1,\infty). \end{aligned} \tag{4.4.4}$$

For $p \in (1,\infty)$ and $w \in A_p$ define $L^p_w(\mathbb{R}^n)$ as the weighted L^p space in $\mathbb{R}^n$ with respect to the measure $w\,dx$. We then have

Theorem 4.4.1 *For each $m,n \in \mathbb{N}$ there exists $N = N(n,m) \in \mathbb{N}$ with the following significance. Let A, B_j, $j = 1,...,m$, be functions in $\mathrm{BMO}_1(\mathbb{R}^n)$ and set $B := (B_1,...,B_m)$. Also, pick an even function $F : \mathbb{R}^m \to \mathbb{R}$ of class C^{N+2} with the property that $|F(w)| \leq C(1+|w|)^{-1}$ for $w \in \mathbb{R}^m$, and $\partial^\alpha F \in L^1(\mathbb{R}^m)$ whenever $0 \leq |\alpha| \leq N+2$. Finally, for each $x \in \mathbb{R}^n$ set*

$$T_*[A,B]f(x) := \sup_{\varepsilon>0}\Big| \int\limits_{\substack{y\in\mathbb{R}^n \\ |x-y|>\varepsilon}} \frac{A(x)-A(y)-\langle\nabla A(y), x-y\rangle}{|x-y|^{n+1}} F\Big(\frac{B(x)-B(y)}{|x-y|}\Big)\, f(y)\,dy\Big|. \tag{4.4.5}$$

Then for each $p \in (1,\infty)$ and $w \in A_p$, there holds

$$\begin{aligned} \|T_*[A,B]f\|_{L^p_w(\mathbb{R}^n)} \quad \leq \quad & C(n,p,[w]_{A_p})\Big(\sum_{|\alpha|\leq N+2} \|\partial^\alpha F\|_{L^1(\mathbb{R}^m)} + \sup_{w\in\mathbb{R}^m}[(1+|w|)|F(w)|]\Big) \times \\ & \times \|\nabla A\|_* \Big(1 + \sum_{j=1}^m \|\nabla B_j\|_*\Big)^N \|f\|_{L^p_w(\mathbb{R}^n)}. \end{aligned} \tag{4.4.6}$$

Proof. This is a mild extension of Theorem 1.10 in [50], where the case $m = 1$ has been treated. The current version can be proved along similar lines, the only notable difference being that the quantity denoted $m_B(\nabla A)$ at the bottom of page 490 in [50] now becomes $\Big(⨍_B \nabla B_j\,dx\Big)_{1\leq j\leq m}$ and the 'dot' products in Case 2 and Case 3 on pp. 491-492 in [50] should be interpreted accordingly. This allows for the same reduction (via good λ-inequalities) to the case when A, B are Lipschitz functions. The Lipschitz version of Theorem 4.4.1 is well-known; see, for instance, Theorem 11 on p.108 in [83] for the unweighted version of (4.4.6). The fact that a Muckenhoupt weight can be allowed is a well-known property of singular integral operators of Calderón-Zygmund type; cf. [19], [83]. When A is a Lipschitz function, the special algebraic expression $A(x) - A(y) - \langle\nabla A(y), x-y\rangle$ no longer plays a crucial role, in the sense that it can be decoupled into $A(x)-A(y)$ and $\langle\nabla A(y), y-x\rangle$, then both pieces can be treated separately, yielding (by linearity and rescaling) the factor $\|\nabla A\|_{L^\infty}$ in the right hand-side of the estimate (4.4.6). $\square$

Theorem 4.4.2 *Retain the same assumptions as in Theorem 4.4.1 and strengthen the hypothesis on A by requiring that this function belongs to $\mathrm{VMO}_1(\mathbb{R}^n)$. Also, for a fixed, arbitrary cube Q in $\mathbb{R}^n$, define*

$$T[A,B]f(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\mathbb{R}^n \\ |x-y|^2+(A(x)-A(y))^2>\varepsilon^2}} \frac{A(x)-A(y)-\langle\nabla A(y), x-y\rangle}{|x-y|^{n+1}} F\Big(\frac{B(x)-B(y)}{|x-y|}\Big)\, f(y)\,dy, \tag{4.4.7}$$

where $x \in Q$. Then for each $p \in (1,\infty)$, $w \in A_p$, and $f \in L^p_w(Q)$, the limit in (4.4.7) exists for a.e. $x \in Q$ and the operator

$$T[A,B] : L^p_w(Q) \longrightarrow L^p_w(Q) \tag{4.4.8}$$

is compact.

Proof. Once Theorem 4.4.1 has been established, the same proof as in [50] applies. $\square$

We conclude this subsection with several remarks.

Remark I. Let $\mathcal{L}(L^p(\partial\Omega, d\sigma))$ stand for the Banach space of linear, bounded operators on $L^p(\partial\Omega, d\sigma)$. Theorem 4.4.1 implies that, in the case when $\partial\Omega$ is the graph of a BMO_1 function A,

$$\|K\|_{\mathcal{L}(L^p(\partial\Omega,d\sigma))} \leq C\|\nabla A\|_*(1+\|\nabla A\|_*)^N, \tag{4.4.9}$$

for some $N = N(n) > 0$. In particular, for each $p \in (1,\infty)$, the operator $\frac{1}{2}I + K$ is invertible on $L^p(\partial\Omega, d\sigma)$ when $\|\nabla A\|_*$ is small enough. A closely related issue, namely whether

$$\Omega \text{ bounded VMO}_1 \text{ domain} \Longrightarrow K \text{ is compact on } L^p(\partial\Omega, d\sigma), \tag{4.4.10}$$

has been solved in the affirmative in [50] (cf. Theorem 1.17 there). On the other hand, if Ω is a bounded BMO_1 domain with the property that $\nu \in \mathrm{VMO}(\partial\Omega, d\sigma)$, then we may write

$$\tfrac{1}{2}I + K = \left[\tfrac{1}{2}I + \left(\frac{K-K^*}{2}\right)\right] + \left(\frac{K+K^*}{2}\right) =: K_1 + K_2 \tag{4.4.11}$$

and note that K_1 is accretive on $L^2(\partial\Omega, d\sigma)$ since $\int_{\partial\Omega} f\, K_1 f\, d\sigma = \frac{1}{2}\|f\|^2_{L^2(\partial\Omega,d\sigma)}$, whereas the integral kernel of K_2 is $\langle(\nabla E)(X-Y), \nu(X)-\nu(Y)\rangle$, i.e., of commutator type (see Theorem 2.4.2 and Theorem 2.4.5). All these can then be used to show that, in this case, $\frac{1}{2}I + K$ is Fredholm with index zero on $L^2(\partial\Omega, d\sigma)$. While we shall consider this point in greater detail later on, here we want to point out that the above observation invites the natural question whether the harmonic double layer K itself is actually compact on $L^2(\partial\Omega, d\sigma)$ (or, more generally, on $L^p(\partial\Omega, d\sigma)$ for each $p \in (1,\infty)$) whenever Ω is a bounded BMO_1 domain for which $\nu \in \mathrm{VMO}(\partial\Omega, d\sigma)$. Since BMO_1 domains have been shown in [55] to have the NTA property, the affirmative answer to this question follows from the results of the next subsection.

Remark II. Recall Zygmund's Λ_* class defined by (3.1.30), along with its counterpart λ_* defined by the requirement (3.1.32). There exist dimensional constants $C_n > 0$ and $\delta_n > 0$ with the property that if $\delta \in (0, \delta_n)$ and $\|\varphi\|_{\Lambda_*(\mathbb{R}^n)} \leq C_n\delta$ then $\Omega = \{(x, x_{n+1}) : x_{n+1} > \varphi(x)\}$ is δ-Reifenberg flat. Indeed, the separation property can be checked using the argument starting at the bottom of p. 95 in [55]. To also establish the estimate

$$\sup_{r>0} \sup_{Q\in\partial\Omega} \theta(Q,r) \leq C\|\varphi\|_{\Lambda_*(\mathbb{R}^n)}, \tag{4.4.12}$$

fix $\eta \in C^\infty_0(\mathbb{R}^n)$, nonnegative, even, with $\int \eta = 1$. If, for any $Q \in \partial\Omega$, $Q = (x, \varphi(x))$, and $r > 0$, we now introduce the hyperplane

$$L(Q,r) := \Big\{ \Big(z\,,\, \varphi(x) + \langle \nabla(\varphi * \eta_r)(x), z - x\rangle \Big) :\ z \in \mathbb{R}^n \Big\}, \tag{4.4.13}$$

then (4.4.12) follows from (3.1.36). A slight version of the above argument also shows that Ω is a Reifenberg flat domain with vanishing constant whenever $\varphi \in \lambda_*(\mathbb{R}^n)$. This is acknowledged at the beginning of §3 on p. 524 in [64]. The same remark is also made on p. 371 of [65]. As a consequence, one can deduce that each VMO_1 domain is a regular SKT domain. In particular, the material of the following subsection will extend the compactness result presented in Theorem 4.4.2.

4.5 Compactness of double layer-like operators on regular SKT domains

Here we shall extend the scope of work initiated in §4.4 by proving Theorem 4.5.1, which is the main result in this subsection. To state it, we find it convenient to introduce the following piece of notation. Given a Banach space $\mathcal{X}$, set $\mathcal{L}(\mathcal{X})$ for the Banach space of all bounded linear operators on $\mathcal{X}$ and define

$$\mathrm{Cp}\,(\mathcal{X}) := \text{ the space of all linear compact operators on } \mathcal{X}, \tag{4.5.1}$$

which, as is well-know, is a closed subspace of $\mathcal{L}(\mathcal{X})$. An extension of the compactness result below to more general, variable coefficient kernels appears in Theorem 4.5.4.

Theorem 4.5.1 *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain satisfying a two-sided local John condition and whose boundary is Ahlfors regular and compact. Also, fix an arbitrary $p \in (1,\infty)$. Then for every $\varepsilon > 0$ the following holds.*

Given a function k satisfying

$$k : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R} \text{ is smooth, even, and homogeneous of degree } -(n+1), \tag{4.5.2}$$

consider the operator

$$Tf(X) := \lim_{\eta\to 0} \int_{Y\in\partial\Omega, |X-Y|>\eta} \langle X - Y, \nu(Y)\rangle k(X-Y) f(Y)\, d\sigma(Y), \quad X \in \partial\Omega. \tag{4.5.3}$$

Then there exists $\delta > 0$, depending only on ε, the geometric characteristics $G(\Omega)$ of Ω, n, p and $\|k|_{S^n}\|_{C^N}$ (where the integer N is as in (3.2.5)) with the property that

$$\mathrm{dist}\,(\nu\,, \mathrm{VMO}\,(\partial\Omega, d\sigma)) < \delta \Longrightarrow \mathrm{dist}\,(T\,,\, \mathrm{Cp}\,(L^p(\partial\Omega, d\sigma)) < \varepsilon, \tag{4.5.4}$$

where the distance in the right-hand side is measured in $\mathcal{L}(L^p(\partial\Omega, d\sigma))$.

As a corollary, granted the initial geometrical assumptions on Ω and (4.5.2)-(4.5.3), then for every $p \in (1,\infty)$ there exists $\varepsilon' = \varepsilon'(G(\Omega), k, p, \varepsilon) > 0$ such that

$$\Omega \text{ is a } \varepsilon'\text{-regular SKT domain} \Longrightarrow \mathrm{dist}\,(T\,,\, \mathrm{Cp}\,(L^p(\partial\Omega, d\sigma)) < \varepsilon. \tag{4.5.5}$$

In particular, under the same background hypotheses, the following implication is valid for every $p \in (1,\infty)$:

$$\nu \in \mathrm{VMO}\,(\partial\Omega, d\sigma) \Longrightarrow T : L^p(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma) \text{ is a compact operator.} \tag{4.5.6}$$

Proof. To set the stage, we first note that, thanks to Proposition 4.1.12 and the current assumptions, the hypotheses of Theorem 3.5.2 are satisfied. In particular, the operator (4.5.6) is bounded whenever $1 < p < \infty$. Throughout the proof, $C = C(\Omega)$ will mean that the C depends only on n and the John constants of Ω (e.g., the corkscrew constants and the constants appearing in the Ahlfors regularity conditions for Ω).

Next, fix $p \in (1, \infty)$ along with an operator T as in (4.5.2)-(4.5.3). Fix also an arbitrary threshold $\varepsilon > 0$. For $\delta > 0$ to be specified later, assume that

$$\operatorname{dist}(\nu\,,\,\mathrm{VMO}\,(\partial\Omega, d\sigma)) < \delta. \tag{4.5.7}$$

Then it follows from (2.4.77) that it is possible to select $R_\delta > 0$ such that

$$\|\nu\|_*(\Delta(X,R)) < \delta, \qquad \forall\, X \in \partial\Omega, \quad \forall\, R \in (0, 100\, C R_\delta), \tag{4.5.8}$$

where $C > 0$ is a fixed, sufficiently large large constant, depending on Ω.

Our goal is to show that if δ is sufficiently small (relative to ε, the geometrical constants of Ω, n, p and $\|k|_{S^n}\|_{C^N}$, where the integer N is as in (3.2.5)) then the distance, measured in $\mathcal{L}(L^p(\partial\Omega, d\sigma))$, from T to the space of compact operators on $L^p(\partial\Omega, d\sigma)$ is $\leq \varepsilon$. To this end, we note that

$$\sup_{X\in\partial\Omega}\ \sup_{Y\in\Delta(X,2R)} R^{-1}|\langle X - Y, \nu_{\Delta(X,R)}\rangle| \leq C\delta, \qquad \text{provided} \ \ 0 < R < 10\, C R_\delta, \tag{4.5.9}$$

for some $C = C(\Omega) > 0$, by (4.5.7) and (4.2.23). Furthermore, we shall assume that R_δ is small enough so that the conclusions in Theorem 4.2.4 are valid as stated for the choice $R_* := 10CR_\delta$. Going further, cover

$$\partial\Omega \subset \bigcup_{j=1}^{N} B(X_j, R_\delta), \qquad X_j \in \partial\Omega,\ \ 1 \leq j \leq N, \tag{4.5.10}$$

and assume that this has been refined (using the Besicovitch covering theorem $-$cf. Theorem 2 on p. 30 in [36], or Lemma 11B.1 in [111]), so as to have bounded overlap, independent of δ and N. That is, there is a fixed constant dimensional $c(n)$ such that each each point lies in at most $c(n)$ of the balls $\{B(X_j, R_\delta)\}_{1\leq j\leq N}$. Pick now a family of smooth functions, $\{\varphi_j\}_{1\leq j\leq N}$, with the property that, for each j, $\varphi_j \in C_0^\infty(B(X_j, R_\delta))$, $0 \leq \varphi_j \leq 1$, and $\sum \varphi_j^2 = 1$ on a neighborhood of $\partial\Omega$. Also, for each $j = 1, ..., N$, select $\psi_j \in C_0^\infty(B(X_j, R_\delta))$ satisfying $0 \leq \psi_j \leq 1$ and which is identically one on the support of φ_j. Finally, generally speaking, denote by M_f the operator of multiplication by the function f. We may then write

$$T = \sum_{j=1}^{N} M_{1-\psi_j} T M_{\varphi_j^2} + \sum_{j=1}^{N} M_{\psi_j} T M_{\varphi_j^2} \tag{4.5.11}$$

and note that the first sum in the right hand-side of (4.5.11) is a compact operator on $L^p(\partial\Omega, d\sigma)$. Thus, for the purpose we have in mind, it suffices to show that

$$\Big\|\sum_{j=1}^{N} M_{\psi_j} T M_{\varphi_j^2}\Big\|_{\mathcal{L}(L^p(\partial\Omega,d\sigma))} \leq \varepsilon. \tag{4.5.12}$$

(We note that estimates on the gradients of φ_j and ψ_j are not used in subsequent calculations.) With this in mind, for an arbitrary $f \in L^p(\partial\Omega, d\sigma)$ we may write, using (twice) the bounded overlap property of the family $\{B(X_j, R_\delta)\}_{1\leq j\leq N}$,

$$\begin{aligned}
\Big(\int_{\partial\Omega}\Big|\sum_{j=1}^{N}\psi_j T(\varphi_j^2 f)\Big|^p\,d\sigma\Big)^{1/p} &\leq C_{p,n}\Big(\sum_{j=1}^{N}\int_{\partial\Omega}|\psi_j T(\varphi_j^2 f)|^p\,d\sigma\Big)^{1/p}\\
&\leq C_{p,n}\Big(\sum_{j=1}^{N}\|M_{\psi_j}TM_{\varphi_j}\|^p_{\mathcal{L}(L^p(\partial\Omega,d\sigma))}\int_{\partial\Omega}|\varphi_j f|^p\,d\sigma\Big)^{1/p}\\
&\leq C_{p,n}\max_{1\leq j\leq N}\|M_{\psi_j}TM_{\varphi_j}\|_{\mathcal{L}(L^p(\partial\Omega,d\sigma))}\Big(\int_{\partial\Omega}|f|^p\,d\sigma\Big)^{1/p}.
\end{aligned} \tag{4.5.13}$$

Hence (4.5.12) follows as soon as we show that δ can be chosen such that

$$\|M_{\psi_j}TM_{\varphi_j}\|_{\mathcal{L}(L^p(\partial\Omega,d\sigma))} \leq \varepsilon/C_{p,n}, \qquad \forall\, j \in \{1,...,N\}. \tag{4.5.14}$$

For the remaining of the proof we shall focus on establishing (4.5.14) for a fixed, arbitrary j. We therefore find it convenient to re-denote $X_o := X_j$, introduce $\Delta_o := \Delta(X_o, R_\delta)$ and drop the dependence on j for φ_j, ψ_j. To get started, fix $\gamma > 0$ and, for each locally integrable function f on $\partial\Omega$, define

$$\mathcal{M}_\gamma f(X) := \sup_{\Delta\ni X}\Big(⨍_{\Delta}|f|^{1+\gamma}\,d\sigma\Big)^{1/(1+\gamma)}, \qquad X \in \partial\Omega, \tag{4.5.15}$$

where the supremum is taken over all surface balls Δ containing X. Consider now a grid $\mathcal{Q}_o$ of dyadic cubes Q on $\partial\Omega$ (cf. Proposition 2.4.7) at a fixed scale, comparable to $\operatorname{diam}\Delta_o$. Also, set

$$I_o := \bigcup_{Q\in\mathcal{Q}_o,\, Q\cap 2\Delta_o\neq\emptyset} Q. \tag{4.5.16}$$

Let us also fix $f \in \mathrm{Lip}_c(\partial\Omega)$ and $\lambda > 0$ arbitrary. Also, assume that $A > 0$ is a fixed constant, whose actual size is to be specified later. The strategy of the proof is to deduce a good-λ inequality of the form

$$\begin{aligned}
\sigma\Big(\{X \in I_o : T_*(\varphi f)(X) > 3\lambda \text{ and } \mathcal{M}_\gamma f(X) \leq A\lambda\}\Big)&\\
\leq c(\delta)\sigma\Big(\{X \in I_o : T_*(\varphi f)(X) > \lambda\}\Big)&,
\end{aligned} \tag{4.5.17}$$

where

$$\lim_{\delta\to 0} c(\delta) = 0. \tag{4.5.18}$$

Here we wish to stress that, in contrast to the standard approach (cf. [19]), the constant A is taken to be *large*, rather than small. Such a choice is of utmost importance in the derivation of (4.5.14). The crucial ingredient allowing us to nonetheless implement the technology associated with good-λ inequalities in this context is the smallness of the local BMO norm of ν, which turns out to be an adequate counterbalance of the fact that A is large.

Of course, as far as (4.5.17) is concerned, it suffices to consider the case when

$$\mathcal{F}_\lambda := \{X \in I_o : T_*(\varphi f)(X) > 3\lambda \ \text{ and } \ \mathcal{M}_\gamma f(X) \leq A\lambda\} \neq \emptyset, \tag{4.5.19}$$

otherwise there is nothing to prove. Fix $X_\star \in \mathcal{F}_\lambda$ and bring in the decomposition $I_o = \mathcal{P}_\lambda \cup \mathcal{S}_\lambda$ where

$$\mathcal{P}_\lambda := \{X \in I_o : T_*(\varphi f)(X) \leq \lambda\}, \qquad \mathcal{S}_\lambda := \{X \in I_o : T_*(\varphi f)(X) > \lambda\}. \tag{4.5.20}$$

Let us first treat the case when $\mathcal{P}_\lambda = \emptyset$, i.e., when $\mathcal{S}_\lambda = I_o$.

We shall make use of Semmes's decomposition in the version presented in Theorem 4.2.4. In particular, (4.2.36)-(4.2.37) induce a splitting $I_o = G \cup E$ into two disjoint ('good' and 'evil') pieces satisfying the following two properties. First, in some suitable local co-ordinate system, G agrees with the graph $\mathcal{G}$ of a Lipschitz function h for which

$$\|\nabla h\|_{L^\infty} \leq \delta \tag{4.5.21}$$

and, second, there exist $C_1, C_2 > 0$, geometrical constants, such that

$$\sigma(E) \leq C_1 \exp(-C_2/\delta)\sigma(I_o). \tag{4.5.22}$$

Thanks to (4.5.22), in order to establish (4.5.17) in the current case, it is enough to look at

$$\begin{aligned} &\sigma\Big(\{X \in G : T_*(\varphi f \mathbf{1}_G)(X) > \tfrac{3}{2}\lambda \ \text{ and } \ \mathcal{M}_\gamma f(X) \leq A\lambda\}\Big) \\ &\qquad + \sigma\Big(\{X \in G : T_*(\varphi f \mathbf{1}_E)(X) > \tfrac{3}{2}\lambda \ \text{ and } \ \mathcal{M}_\gamma f(X) \leq A\lambda\}\Big). \end{aligned} \tag{4.5.23}$$

The first piece can be estimated by rewriting it in graph co-ordinates, based on (4.2.32)-(4.2.33). More specifically, denote by $\tilde{\sigma}$ the surface measure on $\mathcal{G}$, and by $\tilde{T}$ the operator associated with $\mathcal{G}$ much as T in (4.5.3) is associated with $\partial\Omega$. (It is well-known that the measures σ and $\tilde{\sigma}$ are compatible on $\partial\Omega \cap \mathcal{G}$. See, e.g., Proposition 12.9 in [111].) We can then apply Chebysheff's inequality and Theorem 4.4.1 in concert with the smallness condition (4.5.21) to estimate

$$
\begin{aligned}
\sigma\Big(\{X\in G:\, T_*(\varphi f\mathbf{1}_G)(X)>\tfrac{3}{2}\lambda\}\Big) &= \tilde\sigma\Big(\{X\in \mathcal{G}:\, \tilde T_*(\varphi f\mathbf{1}_G)(X)>\tfrac{3}{2}\lambda\}\Big)\\
&\le \frac{C}{\lambda^{1+\gamma}}\int_{\mathcal{G}}|T_*(\varphi f\mathbf{1}_G)|^{1+\gamma}\,d\tilde\sigma\\
&\le C\frac{\delta}{\lambda^{1+\gamma}}\int_{\mathcal{G}}|f\mathbf{1}_G|^{1+\gamma}\,d\tilde\sigma \le C\delta\,\frac{\sigma(I_o)}{\lambda^{1+\gamma}}⨍_{I_o}|f|^{1+\gamma}\,d\sigma\\
&\le C\delta\,\frac{\sigma(I_o)}{\lambda^{1+\gamma}}\Big[\mathcal{M}_\gamma f(X_\star)\Big]^{1+\gamma}\\
&\le C\,A^{1+\gamma}\delta\,\sigma(I_o).
\end{aligned}
\tag{4.5.24}
$$

As for the second term in (4.5.23), we let $q:=\sqrt{1+\gamma}\in(1,\infty)$ and write

$$
\begin{aligned}
&\sigma\Big(\{X\in G:\, T_*(\varphi f\mathbf{1}_E)(X)>\tfrac{3}{2}\lambda\}\Big)\\
&\qquad\le \frac{C}{\lambda^q}\int_{\partial\Omega}|T_*(\varphi f\mathbf{1}_E)|^q\,d\sigma \le \frac{C}{\lambda^q}\Big(\int_{I_o}|f|^q\mathbf{1}_E\,d\sigma\Big)\\
&\qquad\le C\frac{\sigma(E)^{(q-1)/q}}{\lambda^q}\Big(\int_{I_o}|f|^{q^2}\,d\sigma\Big)^{1/q}\\
&\qquad\le C\lambda^{-q}\sigma(I_o)\Big(\frac{\sigma(E)}{\sigma(I_o)}\Big)^{(q-1)/q}\Big(⨍_{I_o}|f|^{q^2}\,d\sigma\Big)^{1/q}\\
&\qquad\le C_1\lambda^{-q}\exp\Big(-\frac{C_2(q-1)}{\delta q}\Big)\sigma(I_o)\Big[\mathcal{M}_\gamma f(X_\star)\Big]^q\\
&\qquad\le C_1A^{\sqrt{1+\gamma}}\exp\Big(-\frac{C_2\,\gamma}{\delta\sqrt{1+\gamma}(\sqrt{1+\gamma}+1)}\Big)\sigma(I_o),
\end{aligned}
\tag{4.5.25}
$$

using Hölder's inequality and the estimate (4.5.22). Altogether, the above reasoning shows that (4.5.17) holds with a constant as in (4.5.18), when $\mathcal{P}_\lambda=\emptyset$.

Assume next that $\mathcal{P}_\lambda\neq\emptyset$. Perform a decomposition

$$
\mathcal{S}_\lambda=\bigcup_{k\in\mathcal{K}}Q_k \tag{4.5.26}
$$

into maximal dyadic subcubes of I_o. That is, using a stopping time argument which involves successive dyadic divisions of I_o (in the sense described in Proposition 2.4.7), we can produce a covering of $\mathcal{S}_\lambda$ with mutually disjoint dyadic cubes Q_k which are maximal in the sense that, if $\widetilde{Q}_k$ denotes the dyadic parent of Q_k, then

$$
\exists\, X_k^*\in\widetilde{Q}_k\cap\Big(I_o\setminus\mathcal{S}_\lambda\Big)=\widetilde{Q}_k\cap\mathcal{P}_\lambda. \tag{4.5.27}
$$

To each $k\in\mathcal{K}$ we associate a surface ball Δ_k with center in Q_k and of radius $\operatorname{diam}Q_k$. We also let $\widetilde{\Delta}_k$ be a surface ball with center at some point in $\widetilde{Q}_k$ and radius $2\operatorname{diam}\widetilde{Q}_k$.

Going further, we categorize the collection $\{\Delta_k\}_{k\in\mathcal{K}}$ into two classes. Specifically, denote by $\mathcal{K}_1$ the collection of all $k \in \mathcal{K}$ with the property that Δ_k contains a point X_k^{**} for which

$$\mathcal{M}_\gamma f(X_k^{**}) \leq A\lambda, \tag{4.5.28}$$

and set $\mathcal{K}_2 := \mathcal{K} \setminus \mathcal{K}_1$. It follows then that

$$\mathcal{F}_\lambda \cap \Delta_k = \emptyset, \qquad \forall\, k \in \mathcal{K}_2, \tag{4.5.29}$$

and, since $\mathcal{F}_\lambda \subseteq \mathcal{S}_\lambda$,

$$\sigma\Big(\{X \in I_o :\, T_*(\varphi f)(X) > 3\lambda \;\text{ and }\; \mathcal{M}_\gamma f(X) \leq A\lambda\}\Big) = \sum_{k\in\mathcal{K}_1} \sigma(\mathcal{F}_\lambda \cap \Delta_k). \tag{4.5.30}$$

If we now denote

$$F_k := \{X \in \Delta_k :\, T_*(\varphi f)(X) > 3\lambda\}, \qquad k \in \mathcal{K}_1, \tag{4.5.31}$$

it follows that $\mathcal{F}_\lambda \cap \Delta_k \subseteq F_k$ for each $k \in \mathcal{K}_1$, and our goal is to prove that

$$\sigma(F_k) \leq c(\delta)\,\sigma(\Delta_k), \qquad \forall\, k \in \mathcal{K}_1, \tag{4.5.32}$$

with $c(\delta) > 0$ as in (4.5.18). Granted this, we may then conclude that

$$\begin{aligned} \sigma(\mathcal{F}_\lambda) &= \sum_{k\in\mathcal{K}_1} \sigma(\mathcal{F}_\lambda \cap \Delta_k) \leq \sum_{k\in\mathcal{K}_1} \sigma(F_k) \\ &\leq c(\delta) \sum_{k\in\mathcal{K}_1} \sigma(\Delta_k) \leq c(\delta)\,\sigma(\mathcal{S}_\lambda), \end{aligned} \tag{4.5.33}$$

which justifies (4.5.17).

Turning to (4.5.32), fix $k \in \mathcal{K}_1$. In order to lighten notation, in the sequel we agree to suppress the dependence of Δ_k, $\widetilde{\Delta}_k$, F_k, X_k^* and X_k^{**} on k, and just simply write Δ, $\widetilde{\Delta}$, F, X^* and X^{**}, respectively. With this convention in mind, let R stand for the radius of Δ and denote by Δ^* the surface ball of center X^* and diameter $C\,\mathrm{diam}\,\Delta$, for a constant $C > 0$ depending only on Ω, chosen so that $2\widetilde{\Delta} \subset \Delta^* \subset C_o\Delta_o$, for a suitably large constant C_o, depending on Ω. We then decompose

$$\varphi f = g_1 + g_2, \qquad \text{where } \; g_1 := (\varphi f)\mathbf{1}_{2\Delta^*}, \quad g_2 := (\varphi f)\mathbf{1}_{\partial\Omega\setminus 2\Delta^*}, \tag{4.5.34}$$

so that

$$\sigma(F) \leq \sigma\Big(\{X \in \Delta :\, T_* g_1(X) > \tfrac{3}{2}\lambda\}\Big) + \sigma\Big(\{X \in \Delta :\, T_* g_2(X) > \tfrac{3}{2}\lambda\}\Big). \tag{4.5.35}$$

Now, the contribution from g_1 is handled as before, applying Semmes' decomposition to Δ^* and using the fact that there exists $X^{**} \in \Delta$ such that $\mathcal{M}_\gamma f(X^{**}) \leq A\lambda$. For this, we thus obtain

$$\sigma\Big(\{X\in\Delta:\,T_*g_1(X)>\tfrac{3}{2}\lambda\}\Big)\leq c(\delta)\,\sigma(\Delta) \tag{4.5.36}$$

with $c(\delta)>0$ as in (4.5.18), which is of the right order.

As for g_2, observe first that since the truncation in the definition of g_2 occurs at $X^*\in\widetilde{\Delta}\cap\mathcal{P}_\lambda$, we have

$$\Big|T_\varepsilon g_2(X^*)\Big|\leq T_*(\varphi f)(X^*)\leq\lambda,\qquad\forall\varepsilon>0. \tag{4.5.37}$$

With $\varepsilon>0$ momentarily fixed, consider now an arbitrary point $X\in\Delta$ and bound

$$\Big|T_\varepsilon g_2(X)-T_\varepsilon g_2(X^*)\Big|\leq I+II+II, \tag{4.5.38}$$

where

$$\begin{aligned}
I &:= \int\limits_{\partial\Omega\setminus 2\Delta^*}\Big|\langle X-Y,\nu(Y)\rangle k(X-Y)-\langle X^*-Y,\nu(Y)\rangle k(X^*-Y)\Big||(\varphi f)(Y)|\,d\sigma(Y),\\
II &:= \int\limits_{\substack{Y\in\partial\Omega\setminus 2\Delta^*\\ |X-Y|>\varepsilon,\,|X^*-Y|<\varepsilon}}|\langle X-Y,\nu(Y)\rangle|\cdot|k(X-Y)|\cdot|(\varphi f)(Y)|\,d\sigma(Y),\\
III &:= \int\limits_{\substack{Y\in\partial\Omega\setminus 2\Delta^*\\ |X^*-Y|>\varepsilon,\,|X-Y|<\varepsilon}}|\langle X^*-Y,\nu(Y)\rangle|\cdot|k(X^*-Y)|\cdot|(\varphi f)(Y)|\,d\sigma(Y).
\end{aligned} \tag{4.5.39}$$

In preparation for estimating I, we wish to analyze the difference between the original integrand and a similar expression in which $\nu(Y)$ has been replaced by $\nu_{\Delta^*}:=⨍_{\Delta^*}\nu\,d\sigma$. Let R denotes the radius of Δ^*. Keeping in mind that $X\in\Delta$, the error which occurs in this fashion can be estimate as follows

$$\begin{aligned}
&\Big|\int\limits_{\Delta_o\setminus 2\Delta^*}\Big(\langle X-Y,\nu(Y)-\nu_{\Delta^*}\rangle k(X-Y)\\
&\qquad\qquad -\langle X^*-Y,\nu(Y)-\nu_{\Delta^*}\rangle k(X^*-Y)\Big)f(Y)\,d\sigma(Y)\Big|\\
&\leq C\int\limits_{\Delta_o\setminus 2\Delta^*}\frac{R}{(R+|X^*-Y|)^{n+1}}|\nu(Y)-\nu_{\Delta^*}||f(Y)|\,d\sigma(Y)\\
&\leq C\sum_{\substack{j\in\mathbb{N}\\ 2^{j+1}\Delta^*\subseteq C_o\Delta_o}}2^{-j}⨍_{2^{j+1}\Delta^*\setminus 2^j\Delta^*}|\nu(Y)-\nu_{\Delta^*}||f(Y)|\,d\sigma(Y)\\
&\leq C\Big(\sum_{j=1}^\infty j\,2^{-j}\Big)\|\nu\|_*(C_o\Delta_o)\mathcal{M}_\gamma f(X^*)\\
&\leq CA\,\delta\lambda,
\end{aligned} \tag{4.5.40}$$

where we have used the well-known fact that $|\nu_{\Delta^*}-\nu_{2^j\Delta^*}|\leq C\,j\,\|\nu\|_*(2^{j+1}\Delta^*)$. On the other hand, based on the properties of $k(X)$ and the Mean-Value Theorem we have

$$
\begin{aligned}
&\Big|\int\limits_{\Delta_o\setminus 2\Delta^*}\Big(\langle X-Y,\nu_{\Delta^*}\rangle k(X-Y)-\langle X^*-Y,\nu_{\Delta^*}\rangle k(X^*-Y)\Big)f(Y)\,d\sigma(Y)\Big|\\
&\leq C\sum_{\substack{j\in\mathbb{N}\\ 2^{j+1}\Delta^*\subseteq C_o\Delta_o}}\int_{2^{j+1}\Delta^*\setminus 2^j\Delta^*}\Big(\frac{|\langle X-X^*,\nu_{\Delta^*}\rangle|}{|X^*-Y|^{n+1}}+R\frac{|\langle X-Y,\nu_{\Delta^*}\rangle|}{|X^*-Y|^{n+2}}\Big)|f(Y)|\,d\sigma(Y)\\
&\leq C\sum_{\substack{j\in\mathbb{N}\\ 2^{j+1}\Delta^*\subseteq C_o\Delta_o}}\int_{2^{j+1}\Delta^*\setminus 2^j\Delta^*}\frac{|\langle X-X^*,\nu_{\Delta^*}\rangle|}{|X^*-Y|^{n+1}}|f(Y)|\,d\sigma(Y)\\
&\quad +CR\sum_{\substack{j\in\mathbb{N}\\ 2^{j+1}\Delta^*\subseteq C_o\Delta_o}}\int_{2^{j+1}\Delta^*\setminus 2^j\Delta^*}\frac{|\langle X-Y,\nu_{\Delta^*}-\nu_{2^{j+1}\Delta^*}\rangle|}{|X^*-Y|^{n+2}}|f(Y)|\,d\sigma(Y)\\
&\quad +CR\sum_{\substack{j\in\mathbb{N}\\ 2^{j+1}\Delta^*\subseteq C_o\Delta_o}}\int_{2^{j+1}\Delta^*\setminus 2^j\Delta^*}\frac{|\langle X-Y,\nu_{2^{j+1}\Delta^*}\rangle|}{|X^*-Y|^{n+2}}|f(Y)|\,d\sigma(Y)\\
&=: I_1+I_2+I_3. \qquad (4.5.41)
\end{aligned}
$$

Now, much as before, with the help of (4.5.9) we obtain

$$
\begin{aligned}
|I_1|+|I_3| &\leq C\|\nu\|_*(C_o\Delta_o)\sum_{j=1}^{\infty}2^{-j}\fint_{2^{j+1}\Delta^*\setminus 2^j\Delta^*}|f(Y)|\,d\sigma(Y)\\
&\leq C\|\nu\|_*(C_o\Delta_o)\mathcal{M}_\gamma f(X^{**})\leq CA\,\delta\lambda, \qquad (4.5.42)
\end{aligned}
$$

and

$$
\begin{aligned}
|I_2| &\leq C\|\nu\|_*(C_o\Delta_o)\sum_{j=1}^{\infty}j\,2^{-j}\fint_{2^{j+1}\Delta^*\setminus 2^j\Delta^*}|f(Y)|\,d\sigma(Y)\\
&\leq C\|\nu\|_*(C_o\Delta_o)\,\mathcal{M}_\gamma f(X^{**})\leq CA\,\delta\lambda. \qquad (4.5.43)
\end{aligned}
$$

In summary, the estimates (4.5.40)-(4.5.43) prove that

$$I\leq CA\,\delta\lambda. \qquad (4.5.44)$$

Turning our attention to II and III in (4.5.39), let us first note that

$$
\begin{aligned}
&|X-Y|\approx|X^*-Y|\approx|X^{**}-Y|,\quad\text{uniformly for}\\
&X,X^{**}\in\Delta,\ X^*\in\widetilde{\Delta}\ \text{ and }\ Y\in\partial\Omega\setminus 2\Delta^*.
\end{aligned}
\qquad (4.5.45)
$$

In particular, $|X-Y|\approx|X^*-Y|\approx\varepsilon$ in the domain of integration in II and III. Let us also observe that it can be assumed that

$$0 < \varepsilon < C\,R_\delta \tag{4.5.46}$$

since otherwise both II and III vanish, out of simple support considerations. These considerations allow us to estimate

$$\begin{aligned} II &\leq C_1\varepsilon^{-n} \int\limits_{\substack{|X^{**}-Y|<C_2\varepsilon \\ |X-Y|<C_3\varepsilon}} \frac{|\langle X-Y, \nu(Y)\rangle|}{|X-Y|}|f(Y)|\,d\sigma(Y) \\ &\leq C_1\varepsilon^{-n} \int\limits_{\substack{|X^{**}-Y|<C_2\varepsilon \\ |X-Y|<C_3\varepsilon}} \frac{|\langle X-Y, \nu(Y)-\nu_{\Delta(X,\varepsilon)}\rangle|}{|X-Y|}|f(Y)|\,d\sigma(Y) \\ &\quad +C_1\varepsilon^{-n} \int\limits_{\substack{|X^{**}-Y|<C_2\varepsilon \\ |X-Y|<C_3\varepsilon}} \frac{|\langle X-Y, \nu_{\Delta(X,\varepsilon)}\rangle|}{|X-Y|}|f(Y)|\,d\sigma(Y) \\ &=: II_1 + II_2. \end{aligned} \tag{4.5.47}$$

Using Hölder and John-Nirenberg inequalities, (4.5.28), as well as (4.5.46) and the assumption (4.5.8), we may continue with

$$II_1 \leq C_4\|\nu\|_*(\Delta(X,C_5\varepsilon))\mathcal{M}_\gamma f(X^{**}) \leq CA\,\delta. \tag{4.5.48}$$

Proceeding in an analogous fashion and invoking (4.5.9) we also obtain

$$II_2 \leq C_6 \sup_{Y\in\Delta(X,C_3\varepsilon)} |\langle X-Y, \nu_{\Delta(X,\varepsilon)}\rangle|\mathcal{M}_\gamma f(X^{**}) \leq CA\,\delta. \tag{4.5.49}$$

In a similar manner,

$$III \leq CA\,\delta, \tag{4.5.50}$$

so that, altogether,

$$\Big|T_\varepsilon g_2(X) - T_\varepsilon g_2(X^*)\Big| \leq CA\,\delta\lambda, \qquad \forall\, X\in\Delta. \quad \forall\,\varepsilon>0, \tag{4.5.51}$$

by (4.5.44) and (4.5.49)-(4.5.50). In particular,

$$\Big|T_\varepsilon g_2(X) - T_\varepsilon g_2(X^*)\Big| \leq \tfrac{1}{2}\lambda, \qquad \forall\, X\in\Delta, \quad \forall\,\varepsilon>0, \tag{4.5.52}$$

provided

$$0 < \delta < \frac{1}{4C^2} \quad \text{and} \quad A = \frac{1}{\sqrt{\delta}}. \tag{4.5.53}$$

Note that, by virtue of (4.5.37), (4.5.52) entails

$$T_* g_2(X) \leq \tfrac{3}{2}\lambda, \qquad \forall\, X \in \Delta, \tag{4.5.54}$$

whenever (4.5.53) holds. For this choice, we then write

$$\begin{aligned} \sigma\Big(\{X \in \Delta : T_*(\varphi f)(X) > 3\lambda\}\Big) \;\; &\leq \;\; \sigma\Big(\{X \in \Delta : T_* g_1(X) > \tfrac{3}{2}\lambda\}\Big) \\ &\quad + \sigma\Big(\{X \in \Delta : T_* g_2(X) > \tfrac{3}{2}\lambda\}\Big) \end{aligned} \tag{4.5.55}$$

and observe that, by (4.5.54), the last term above drops out, as the corresponding set is empty. Thus, if A and δ are as in (4.5.53), estimate (4.5.32) is a consequence of this and (4.5.36). This finishes the proof of (4.5.17) in the case when A and δ are as in (4.5.53), which we shall assume henceforth.

Having proved (4.5.17), we make use of this inequality to deduce that

$$\begin{aligned} &\sigma\Big(\{X \in I_o : T_*(\varphi f)(X) > 3\lambda\}\Big) \\ &\qquad \leq \sigma\Big(\{X \in \partial\Omega : \mathcal{M}_\gamma f(X) \geq \delta^{-1/2}\lambda\}\Big) + c(\delta)\,\sigma\Big(\{X \in I_o : T_*(\varphi f)(X) > \lambda\}\Big). \end{aligned} \tag{4.5.56}$$

Multiplying (4.5.56) by λ^{p-1} and integrating in $\lambda \in (0, \infty)$ then yields

$$\begin{aligned} 3^{-p} \int_{I_o} |T_*(\varphi f)|^p \, d\sigma \;\; &\leq \;\; \delta^{p/2} \int_{\partial\Omega} |\mathcal{M}_\gamma f|^p \, d\sigma + c(\delta) \int_{I_o} |T_*(\varphi f)|^p \, d\sigma \\ &\leq \;\; C_{\partial\Omega,\gamma,p}\, \delta^{p/2} \int_{\partial\Omega} |f|^p \, d\sigma + c(\delta) \int_{I_o} |T_*(\varphi f)|^p \, d\sigma. \end{aligned} \tag{4.5.57}$$

Since $\int_{I_o} |T_*(\varphi f)|^p \, d\sigma < \infty$, we can absorb the last integral in (4.5.57) in the left hand-side and obtain

$$\int_{I_o} |T_*(\varphi f)|^p \, d\sigma \leq c(\delta) \int_{\partial\Omega} |f|^p \, d\sigma \tag{4.5.58}$$

with $c(\delta) > 0$ as in (4.5.18). Consequently,

$$\int_{\partial\Omega} |\psi T(\varphi f)|^p \, d\sigma \leq \int_{I_o} |T_*(\varphi f)|^p \, d\sigma \leq c(\delta) \int_{\partial\Omega} |f|^p \, d\sigma, \tag{4.5.59}$$

where $c(\delta) > 0$ is, once again, as in (4.5.18). Since f is arbitrary in $L^p(\partial\Omega, d\sigma)$, (4.5.14) follows by choosing $\delta > 0$ suitably small (relative to Ω, p, n, k and γ) to begin with. This finishes the proof of the theorem. $\square$

The following results illustrate the sharpness of the theorem just established. For $\theta \in (0, 2\pi)$, $p \in (1, \infty)$, consider the following ("bow" shaped) closed contour given by the parametric representation

$$\Sigma_\theta(p) := \left\{ \pm \frac{\sin\Big((\pi - \theta) z\Big)}{\sin(\pi z)} \, : \, z \in \tfrac{1}{p} + i\mathbb{R} \right\}. \tag{4.5.60}$$

Proposition 4.5.2 *Consider a bounded, simply connected curvilinear polygon $\Omega \subset \mathbb{R}^2$ with angles θ_j, $j = 1, ..., N$, and let $p \in (1,\infty)$. For each $j = 1, ..., N$, consider the bow-shaped curve $\Sigma_{\theta_j}(p)$ associated with the angle θ_j as in (4.5.60), and denote by $\widehat{\Sigma_{\theta_j}(p)}$ the (closed) two-dimensional region encompassed by $\Sigma_{\theta_j}(p)$. Let K stand for the harmonic double layer potential operator on $\partial\Omega$. Then the spectrum of K on $L^p(\partial\Omega, d\sigma)$ has the following structure:*

$$\mathrm{Spec}\Big(K, L^p(\partial\Omega, d\sigma)\Big) = \left(\bigcup_{1\leq j\leq N} \widehat{\Sigma_\theta(p)}\right) \bigcup \{\lambda_k\}_k, \tag{4.5.61}$$

where $\{\lambda_k\}_k$, the collection of eigenvalues of K on $L^p(\partial\Omega)$, is a finite subset of $(-1,1]$.

Moreover, for $z \in \cup_{j=1}^N \Sigma_{\theta_j}(p)$ the operator $zI - K$ is not Fredholm on $L^p(\partial\Omega, d\sigma)$, whereas for $z \in \mathbb{C} \setminus \Big(\cup_{j=1}^N \Sigma_{\theta_j}(p)\Big)$ the operator $zI - K$ is Fredholm on $L^p(\partial\Omega, d\sigma)$ and its index is given by

$$\mathrm{index}\Big(zI - K : L^p(\partial\Omega, d\sigma)\Big) = \sum_{j=1}^N W(z, \Sigma_{\theta_j}(p)), \tag{4.5.62}$$

where $W(z, \Sigma_{\theta_j}(p))$ denotes the winding number of z with respect to the curve $\Sigma_{\theta_j}(p)$.

Cf. [99]. Similar results hold, in fact, for double layer potential operators associated with the Lamé and Stokes systems; see [87].

It is clear from the result just stated that the presence of any angle $\theta \neq \pi$ prevents K from being compact on $L^p(\partial\Omega, d\sigma)$ when Ω is a curvilinear polygon in $\mathbb{R}^2$, for any $p \in (1,\infty)$. This failure of K to be compact can be quantified in a more precise fashion. Concretely, consider the case when Ω is a curvilinear polygon with precisely one angular point located at the origin $0 \in \mathbb{R}^2$. Furthermore, assume that, in a neighborhood of 0, $\partial\Omega$ agrees with a sector of aperture $\theta \in (0,\pi)$ with vertex at 0. In particular, the outward unit normal ν to Ω is smooth on $\partial\Omega \setminus \{0\}$ and is piecewise constant near 0, where it assumes two values, say, ν_+ and ν_-. Define

$$\{\nu\}_{\mathrm{Osc}(\partial\Omega)} := \limsup_{\varepsilon\to 0}\left(\sup_{B_\varepsilon} \fint_{B_\varepsilon\cap\partial\Omega}\fint_{B_\varepsilon\cap\partial\Omega}\Big|\,\nu(X) - \nu(Y)\,\Big|\,d\sigma_X d\sigma_Y\right), \tag{4.5.63}$$

where the supremum is taken over the collection $\{B_\varepsilon\}$ of disks with centers on $\partial\Omega$ and of radius ε. We obviously have $\{\nu\}_{\mathrm{Osc}(\partial\Omega)} \leq 2\,\mathrm{dist}\,(\nu, \mathrm{VMO}(\partial\Omega, d\sigma))$, where the distance is taken in $\mathrm{BMO}(\partial\Omega)$. Furthermore, as a consequence of a Proposition 2.4.8, there exists $C > 0$ such that $\mathrm{dist}\,(\nu, \mathrm{VMO}(\partial\Omega, d\sigma)) \leq C\{\nu\}_{\mathrm{Osc}(\partial\Omega)}$. Altogether,

$$\begin{aligned}\{\nu\}_{\mathrm{Osc}(\partial\Omega)} &\sim \limsup_{\varepsilon\to 0}\left(\sup_{B_\varepsilon} \fint_{B_\varepsilon\cap\partial\Omega}\fint_{B_\varepsilon\cap\partial\Omega}\Big|\,\nu(X) - \nu(Y)\,\Big|\,d\sigma_X d\sigma_Y\right)\\ &\sim \|\nu_+ - \nu_-\| \sim \sqrt{1+\cos\theta},\end{aligned} \tag{4.5.64}$$

which shows that there exists a family of domains $\Omega = \Omega_\theta$ as above for which

$$\mathrm{dist}\,(\nu, \mathrm{VMO}(\partial\Omega_\theta, d\sigma)) \longrightarrow 0, \quad \text{as } \theta \to \pi. \tag{4.5.65}$$

Based on this analysis, we may conclude the following.

Proposition 4.5.3 *For each $\varepsilon > 0$ there exists a bounded Lipschitz domain Ω (whose Lipschitz character is controlled by a universal constant) with the property that* $\operatorname{dist}(\nu, \mathrm{VMO}(\partial\Omega, d\sigma)) < \varepsilon$ *and yet for each $p \in (1, \infty)$ the operator K fails to be compact on $L^p(\partial\Omega, d\sigma)$.*

This shows that the hypotheses of Theorem 4.5.1 cannot be relaxed in a substantiative way.

We conclude this subsection by discussing a variable coefficient version of Theorem 4.5.1.

Theorem 4.5.4 *If Ω is a regular SKT domain, the compactness result (4.5.6) of Theorem 4.5.1 holds with the kernel $k(X-Y)$ replaced by $k(X, X-Y)$, where $k(X, Z)$ is even and homogeneous of degree $-(n+1)$ in Z, and $D_Z^\alpha k(X, Z)$ is continuous and bounded on $\mathbb{R}^{n+1} \times S^n$ for all $|\alpha| \le M(n)$. More generally, if Ω is an ε'-regular SKT domain, with $\varepsilon' = \varepsilon'(G(\Omega), \|k\|_{C^N}, p, \varepsilon)$, the result (4.5.5) holds in this setting.*

Proof. In the context of Proposition 3.2.1, we have

$$\|T\|_{\mathcal{L}(L^p)} \le C \big\| k\big|_{S^n} \big\|_{C^N}, \tag{4.5.66}$$

for some $C = C(p, G(\Omega)) > 0$ and $N = N(p, G(\Omega)) \in \mathbb{Z}_+$. An expansion

$$k(X, Z) = \sum_\ell a_\ell(X) \Psi_\ell\Big(\frac{Z}{|Z|}\Big) |Z|^{-(n+1)} \tag{4.5.67}$$

works just as in the proof of Theorem 3.5.1 in §3.5. □

4.6 Characterization of regular SKT domains via compactness

If $\Omega \subset \mathbb{R}^{n+1}$ is a UR domain with outward unit normal $\nu = (\nu_1, ..., \nu_{n+1})$ and surface measure $\sigma := \mathcal{H}^n \lfloor \partial\Omega$, then the Riesz transforms $(\mathcal{R}_k)_{1 \le k \le n+1}$ on $\partial\Omega$ are defined by

$$\mathcal{R}_k g(X) := \lim_{\varepsilon \to 0^+} \frac{2}{\omega_n} \int\limits_{\substack{Y \in \partial\Omega \\ |X-Y| > \varepsilon}} \frac{x_k - y_k}{|X - Y|^{n+1}} g(Y)\, d\sigma(Y), \qquad X \in \partial\Omega, \tag{4.6.1}$$

where $k \in \{1, ..., n+1\}$, and g is a real-valued function on $\partial\Omega$. Hence, formally,

$$\nabla S = \tfrac{1}{2} (\mathcal{R}_1, ..., \mathcal{R}_{n+1}) \tag{4.6.2}$$

where S is the harmonic single layer potential on $\partial\Omega$. In light of the identification (4.6.2), it is convenient to abbreviate

$$[M_\nu, \nabla S] := \tfrac{1}{2} \Big([M_{\nu_j}, \mathcal{R}_k] \Big)_{1 \le j,k \le n+1}. \tag{4.6.3}$$

We are now ready to state the main results in this subsection.

Theorem 4.6.1 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set that satisfies a two-sided local John condition, and for which $\partial\Omega$ is Ahlfors regular and compact. Then Ω is a regular SKT domain if and only if the harmonic double layer K along with the commutator $[M_\nu, \nabla S]$ are compact on $L^2(\partial\Omega, d\sigma)$.*

There is also a quantitative version of Theorem 4.6.1. To state it, recall the definition (4.5.1).

Theorem 4.6.2 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is a UR domain and that $p \in (1, \infty)$. Then there exists $C > 0$, depending only on n, p, as well as the UR and Ahlfors regularity constants of $\partial\Omega$, such that*

$$\begin{aligned} \operatorname{dist}\Big(\nu\,,\ \mathrm{VMO}\,(\partial\Omega, d\sigma)\Big) \leq C\Big(\operatorname{dist}\Big(K, \mathrm{Cp}\,(L^p(\partial\Omega, d\sigma))\Big)\Big)^{1/(n+1)} \\ +C\sum_{1\leq j,k\leq n+1}\Big(\operatorname{dist}\Big([M_{\nu_j}, \mathcal{R}_k], \mathrm{Cp}\,(L^p(\partial\Omega, d\sigma))\Big)\Big)^{1/(n+1)}. \end{aligned} \tag{4.6.4}$$

As a corollary, the following holds. Assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set that satisfies a two-sided local John condition, and for which $\partial\Omega$ is Ahlfors regular and compact. Then for every $\delta_o > 0$ and $p \in (1, \infty)$ there exists $\delta > 0$, depending only on δ_o, n, p and the geometry of Ω such that

$$\begin{aligned} \operatorname{dist}\Big(K, \mathrm{Cp}\,(L^p(\partial\Omega, d\sigma))\Big) + \sum_{1\leq j,k\leq n+1} \operatorname{dist}\Big([M_{\nu_j}, \mathcal{R}_k], \mathrm{Cp}\,(L^p(\partial\Omega, d\sigma))\Big) < \delta \\ \Longrightarrow\ \Omega \text{ is a } \delta_o\text{-SKT domain.} \end{aligned} \tag{4.6.5}$$

It is significant that similar results can be phrased purely in terms of the Riesz transforms $\mathcal{R}_k$, $1 \leq k \leq n+1$, from (4.6.1). In contrast to the harmonic double layer K, the latter are operators whose kernels are universal (i.e., independent of the underlying domain). Specifically, we have the following.

Theorem 4.6.3 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is a UR domain and that $p \in (1, \infty)$. Then there exists $C > 0$, depending only on n, p, as well as the UR and Ahlfors regularity constants of $\partial\Omega$, such that*

$$\begin{aligned} \operatorname{dist}\Big(\nu\,,\ \mathrm{VMO}\,(\partial\Omega, d\sigma)\Big) \leq C\Big(\operatorname{dist}\Big(I + \sum_{j=1}^{n+1}\mathcal{R}_j^2, \mathrm{Cp}\,(L^p(\partial\Omega, d\sigma))\Big)\Big)^{1/(n+1)} \\ +C\sum_{1\leq j,k\leq n+1}\Big(\operatorname{dist}\Big([\mathcal{R}_j, \mathcal{R}_k], \mathrm{Cp}\,(L^p(\partial\Omega, d\sigma))\Big)\Big)^{1/(n+1)}. \end{aligned} \tag{4.6.6}$$

As a consequence, the following is true. Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is an open set that satisfies a two-sided local John condition, and for which $\partial\Omega$ is Ahlfors regular and compact. Then for every $\delta_o > 0$ and $p \in (1, \infty)$ there exists $\delta > 0$, depending only on δ_o, n, p and the geometry of Ω such that

$$\begin{aligned} \operatorname{dist}\Big(I + \sum_{j=1}^{n+1}\mathcal{R}_j^2, \mathrm{Cp}\,(L^p(\partial\Omega, d\sigma))\Big) + \sum_{1\leq j,k\leq n+1} \operatorname{dist}\Big([\mathcal{R}_j, \mathcal{R}_k], \mathrm{Cp}\,(L^p(\partial\Omega, d\sigma))\Big) < \delta \\ \Longrightarrow\ \Omega \text{ is a } \delta_o\text{-SKT domain.} \end{aligned} \tag{4.6.7}$$

Theorem 4.6.4 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set that satisfies a two-sided local John condition, and for which $\partial\Omega$ is Ahlfors regular and compact. Then*

$$\Omega \textit{ is a regular SKT domain } \Longleftrightarrow \begin{cases} I+\displaystyle\sum_{j=1}^{n+1}\mathcal{R}_j^2 \in \mathrm{Cp}\Big(L^p(\partial\Omega, d\sigma)\Big) \quad \textit{and} \\ [\mathcal{R}_j,\mathcal{R}_k] \in \mathrm{Cp}\Big(L^p(\partial\Omega, d\sigma)\Big),\ 1\le j,k\le n+1, \end{cases} \tag{4.6.8}$$

for some (hence all) $p \in (1,\infty)$.

As the above theorem illustrates, there is significant geometric information encoded in the Riesz transforms associated with a given domain. In this vein, it is interesting to compare Theorem 4.6.4 with a recent result from [52] to the effect that, if $\Omega \subset \mathbb{R}^{n+1}$ is a two-sided NTA domain with an Ahlfors regular boundary, then

$$\partial\Omega \ \text{ is a sphere, or a } n\text{-plane } \Longleftrightarrow$$
$$I+\sum_{k=1}^{n+1}\mathcal{R}_k^2 = 0 \ \text{ and } \ [\mathcal{R}_j,\mathcal{R}_k]=0 \quad \forall\, j,k\in\{1,...,n+1\}. \tag{4.6.9}$$

The proofs of Theorems 4.6.1-4.6.4, presented at the end of the subsection, make use of the Clifford algebra formalism discussed in the first part of §3.4, which we now revisit. Recall that elements in $\mathcal{C}\!\ell_{n+1}$ can be uniquely written as $u=\sum_{l=0}^{n+1}\sum'_{|I|=l}u_I\, e_I$ with $u_I\in\mathbb{R}$, where e_I stands for the product $e_{i_1}\cdot e_{i_2}\dots e_{i_l}$ if $I=(i_1,i_2,\dots,i_l)$ with $1\le i_1<i_2<\cdots<i_l\le n+1$ and, as before, $e_0:=e_\emptyset:=1$. The Clifford conjugation on $\mathcal{C}\!\ell_{n+1}$, denoted by 'bar', is defined as the unique real-linear involution on $\mathcal{C}\!\ell_{n+1}$ for which $\overline{e_I}e_I=e_I\overline{e_I}=1$ for any multi-index I. We define the scalar part of $u=\sum_I u_Ie_I\in\mathcal{C}\!\ell_{n+1}$ as $u_0:=u_\emptyset$, and endow $\mathcal{C}\!\ell_{n+1}$ with the natural Hilbert space structure

$$\langle u,v\rangle := \sum_I u_Iv_I, \qquad \text{if } \ u=\sum_I u_Ie_I,\ v=\sum_I v_Ie_I\in\mathcal{C}\!\ell_{n+1}. \tag{4.6.10}$$

It follows that

$$|u|^2=(u\odot\bar u)_0=(\bar u\odot u)_0, \qquad \langle u,v\rangle=(u\odot\bar v)_0=(\bar u\odot v)_0, \quad \forall\, u,v\in\mathcal{C}\!\ell_{n+1}, \tag{4.6.11}$$
$$\bar X=-X \quad \text{for any } \ X\in\mathbb{R}^{n+1}, \tag{4.6.12}$$
$$u+\bar u=2\,u_0 \quad \text{for any } \ u\in\mathbb{R}^{n+1}\odot\mathbb{R}^{n+1}, \tag{4.6.13}$$
$$\bar{\bar u}=u \ \text{ and } \ \overline{u\odot v}=\bar v\odot\bar u, \quad \text{for any } \ u,v\in\mathcal{C}\!\ell_{n+1}, \tag{4.6.14}$$
$$|u\odot v|\le c_n|u||v|, \quad \text{for any } \ u,v\in\mathcal{C}\!\ell_{n+1}. \tag{4.6.15}$$

Lemma 4.6.5 *Let* M_b *denote the operator of multiplication by b (from the left). Then, in the above Clifford algebra setting,*

$$(M_b)^*=M_{\bar b}, \qquad \forall\, b\in\mathcal{C}\!\ell_{n+1}. \tag{4.6.16}$$

where star denotes adjunction with respect to the inner product in $\mathcal{C}\!\ell_{n+1}$.

Proof. Write $b = \sum_{\ell=0}^{n+1} \sum'_{|I|=\ell} b_I e_I$. By linearity, it suffices to show that for any multi-indices I, K and any $j \in \{1, ..., n\}$, there holds

$$\langle e_j e_I, e_K \rangle = -\langle e_I, e_j e_K \rangle. \tag{4.6.17}$$

In turn, this is seen by analyzing three cases. First, when $j \notin I$ and $j \notin K$, both sides in (4.6.17) vanish. Second, consider the case when $j \notin I$ and $K = K_1 \cup \{j\} \cup K_2$ (with $K_1 = \{k \in K : k < j\}$, $K_2 = \{k \in K : k > j\}$). On the one hand, if $I \neq K_1 \cup K_2$ then once again both sides of (4.6.17) vanish. If, on the other hand, $I = K_1 \cup K_2$ then both sides in (4.6.17) become $(-1)^{|K_1|+1}$. The third (and final) case, when $j \in I$ and $j \notin K$ is handled in a similar fashion. $\square$

Let now $\Omega \subset \mathbb{R}^{n+1}$ be a fixed UR domain. As usual, denote by $\nu = (\nu_1, ..., \nu_{n+1})$ the outward unit normal to $\partial\Omega$ and by $\sigma := \mathcal{H}^n \lfloor \partial\Omega$ the surface measure on $\partial\Omega$. The Cauchy-Clifford operator $\mathfrak{C}$ associated with $\partial\Omega$ is given by

$$\mathfrak{C} f(X) := \lim_{\varepsilon \to 0^+} \frac{1}{\omega_n} \int\limits_{\substack{Y \in \partial\Omega \\ |X-Y|>\varepsilon}} \frac{X - Y}{|X - Y|^{n+1}} \odot \nu(Y) \odot f(Y)\, d\sigma(Y), \qquad X \in \partial\Omega, \tag{4.6.18}$$

where f is a $\mathcal{C}\ell_{n+1}$-valued function defined on $\partial\Omega$. From Theorem 3.4.2, we know that

$$\mathfrak{C} : L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1} \longrightarrow L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1} \tag{4.6.19}$$

is well-defined and bounded for any $p \in (1, \infty)$. Similar considerations apply to the Riesz transforms $(\mathcal{R}_k)_{1 \leq k \leq n+1}$ on $\partial\Omega$, defined in (4.6.1).

Lemma 4.6.6 *Let Ω be a UR domain and recall that K stands for the harmonic double layer potential operator defined in (3.3.2) (with the understanding that K acts component-wise on $\mathcal{C}\ell_{n+1}$-valued functions). Also, recall that $[A, B] := AB - BA$ is the usual commutator bracket.*

If $\mathfrak{C}^*$ denotes the adjoint of $\mathfrak{C}$ in (4.6.18), then

$$(\mathfrak{C} - \mathfrak{C}^*) f = -2Kf - \sum_{l=0}^{n+1} \sum_{|I|=l}{}' \sum_{j,k=1}^{n+1} \Big([M_{\nu_j}, \mathcal{R}_k] f_I \Big) e_j \odot e_k \odot e_I, \tag{4.6.20}$$

for each sufficiently nice $\mathcal{C}\ell_{n+1}$-valued function $f = \sum_{l=0}^{n+1} \sum'_{|I|=l} f_I$ defined on $\partial\Omega$.

Proof. With the help of Lemma 4.6.5 it is straightforward to compute

$$\mathfrak{C}^* f(X) = -\lim_{\varepsilon \to 0^+} \frac{1}{\omega_n} \int\limits_{\substack{Y \in \partial\Omega \\ |X-Y|>\varepsilon}} \nu(X) \odot \frac{X - Y}{|X - Y|^{n+1}} \odot f(Y)\, d\sigma(Y), \qquad X \in \partial\Omega, \tag{4.6.21}$$

i.e.,

$$\mathfrak{C}^* = M_\nu \mathfrak{C} M_\nu. \tag{4.6.22}$$

Hence, at a.e. $X \in \partial\Omega$,

$$(\mathfrak{C} - \mathfrak{C}^*)f(X) \qquad (4.6.23)$$
$$= \lim_{\varepsilon \to 0^+} \frac{1}{\omega_n} \int\limits_{\substack{Y \in \partial\Omega \\ |X-Y| > \varepsilon}} \Big[(X - Y) \odot \nu(Y) + \nu(X) \odot (X - Y)\Big] \odot \frac{f(Y)}{|X - Y|^{n+1}}\, d\sigma(Y).$$

Based on (4.6.13)-(4.6.14) and (4.6.11), we may rewrite the expression in the brackets as

$$\begin{aligned}
&(X - Y) \odot \nu(Y) + \nu(X) \odot (X - Y) \\
&\quad = \Big[(X - Y) \odot \nu(Y) + \overline{(X - Y) \odot \nu(Y)}\Big] + (\nu(X) - \nu(Y)) \odot (X - Y) \\
&\quad = 2\Big((X - Y) \odot \nu(Y)\Big)_0 + \sum_{j,k=1}^{n+1} (\nu_j(X) - \nu_j(Y))(x_k - y_k) e_j \odot e_k \\
&\quad = 2\langle X - Y, \nu(Y)\rangle + \sum_{j,k=1}^{n+1} (\nu_j(X) - \nu_j(Y))(x_k - y_k) e_j \odot e_k. \qquad (4.6.24)
\end{aligned}$$

Now the identity (4.6.20) readily follows from (4.6.24) and definitions. □

Assume that $\Omega \subset \mathbb{R}^{n+1}$ is a domain satisfying a two-sided local John condition and whose boundary is Ahlfors regular and compact. Also, fix $p \in (1, \infty)$. It follows from Corollary 2.4.5, Theorem 4.5.1 and Lemma 4.6.6 that for every $\varepsilon > 0$ there exists $\delta > 0$, depending only on ε, the geometry of Ω, n and p, such that

$$\operatorname{dist}(\nu\,, \mathrm{VMO}\,(\partial\Omega, d\sigma)) < \delta \Longrightarrow \operatorname{dist}\Big(\mathfrak{C} - \mathfrak{C}^*\,,\ \mathrm{Cp}\,(L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\!\ell_{n+1})\Big) < \varepsilon, \qquad (4.6.25)$$

where the distance in the right-hand side is measured in $\mathcal{L}(L^p(\partial\Omega, d\sigma))$. Thus, informally, ν close to being in $\mathrm{VMO}\,(\partial\Omega, d\sigma)$ implies $\mathfrak{C}$ close to being self-adjoint, modulo compact operators. Remarkably, the opposite implication in this statement is also true, and this is made precise in the theorem below.

Theorem 4.6.7 *Let $\Omega \subset \mathbb{R}^{n+1}$ be a UR domain and assume that $p \in (1, \infty)$. Then there exists $C > 0$, depending only on n, p, as well as the UR and Ahlfors regularity constants of $\partial\Omega$, such that*

$$\operatorname{dist}\Big(\nu\,,\ \mathrm{VMO}\,(\partial\Omega, d\sigma)\Big) \leq C\Big[\operatorname{dist}\Big(\mathfrak{C} - \mathfrak{C}^*\,,\ \mathrm{Cp}\,(L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\!\ell_{n+1})\Big)\Big]^{1/(n+1)}. \qquad (4.6.26)$$

As a consequence, if $\mathfrak{C} - \mathfrak{C}^*$ is a compact operator on $L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\!\ell_{n+1}$ for some $p \in (1, \infty)$ then $\nu \in \mathrm{VMO}\,(\partial\Omega, d\sigma)$.

Proof. As a preliminary step, we establish the following result of general nature: Assume that $T : L^p(\partial\Omega, d\sigma) \to L^p(\partial\Omega, d\sigma)$ is a compact operator for some $p \in (1, \infty)$. Then for every $\eta > 0$ there exists $R_\eta > 0$ such that

$$\|\mathbf{1}_{\Delta_r} T\|_{\mathcal{L}\left(L^p(\partial\Omega,d\sigma)\right)} \leq \eta, \qquad \forall\, r \in (0, R_\eta), \tag{4.6.27}$$

where Δ_r stands for an arbitrary surface ball of radius r.

To justify this, we reason by contradiction and assume that that there exist a threshold $\eta_o > 0$ along with a sequence of surface balls $\{\Delta_{r_j}\}_{j\in\mathbb{N}}$ in $\partial\Omega$, with $r_j \to 0^+$ as $j \to \infty$, for which $\|\mathbf{1}_{\Delta_{r_j}} T\|_{\mathcal{L}\left(L^p(\partial\Omega,d\sigma)\right)} \geq \eta_o$. In particular, there exist $f_j \in L^p(\partial\Omega, d\sigma)$, $j \in \mathbb{N}$, with $\|f_j\|_{L^p(\partial\Omega,d\sigma)} = 1$ for each j, and such that

$$\|\mathbf{1}_{\Delta_{r_j}} T f_j\|_{L^p(\partial\Omega,d\sigma)} \geq \eta_o/2, \qquad \forall\, j \in \mathbb{N}. \tag{4.6.28}$$

Since T is compact, by eventually passing to a subsequence, we may assume that there exists $g \in L^p(\partial\Omega, d\sigma)$ such that $Tf_j \to g$ in $L^p(\partial\Omega, d\sigma)$ as $j \to \infty$. Next, write

$$\mathbf{1}_{\Delta_{r_j}} T f_j = \mathbf{1}_{\Delta_{r_j}}(Tf_j - g) + \mathbf{1}_{\Delta_{r_j}} g \tag{4.6.29}$$

and observe that $\|\mathbf{1}_{\Delta_{r_j}}(Tf_j - g)\|_{L^p(\partial\Omega,d\sigma)} \leq \|Tf_j - g\|_{L^p(\partial\Omega,d\sigma)} \to 0$ as $j \to \infty$, whereas $\mathbf{1}_{\Delta_{r_j}} g \to 0$ in $L^p(\partial\Omega, d\sigma)$ as $j \to \infty$, by Lebesgue's Dominated Convergence Theorem. Hence, $\mathbf{1}_{\Delta_{r_j}} T f_j \to 0$ in $L^p(\partial\Omega, d\sigma)$ as $j \to \infty$, contradicting (4.6.28).

After this preamble, assume that $T \in \mathrm{Cp}\left(L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\!\ell_{n+1}\right)$ is such that

$$\|(\mathfrak{C} - \mathfrak{C}^*) - T\|_{\mathcal{L}\left(L^p(\partial\Omega,d\sigma)\otimes\mathcal{C}\!\ell_{n+1}\right)} \leq \eta/2 \tag{4.6.30}$$

for some $\eta \in (0, 1/10)$. If we now select $R_\eta > 0$ such that $\|\mathbf{1}_{\Delta_r} T\|_{\mathcal{L}\left(L^p(\partial\Omega,d\sigma)\otimes\mathcal{C}\!\ell_{n+1}\right)} \leq \eta/2$ whenever $r \in (0, R_\eta)$, it follows that

$$\|\mathbf{1}_{\Delta_r}(\mathfrak{C} - \mathfrak{C}^*)\|_{\mathcal{L}\left(L^p(\partial\Omega,d\sigma)\otimes\mathcal{C}\!\ell_{n+1}\right)} \leq \eta, \qquad \forall\, r \in (0, R_\eta). \tag{4.6.31}$$

By further decreasing R_η if necessary, it can be assumed that $0 < R_\eta < \eta^{1/(n+1)}(\operatorname{diam}\Omega)/100$. We now claim that there exists $C > 0$, which depends only on the Ahlfors regularity constants of $\partial\Omega$ and p, such that

$$\sup_{Q\in\partial\Omega} \inf_{A\in\mathbb{R}^{n+1}} \Big(⨍_{\Delta(Q,r)} |\nu(X) - A|^p \, d\sigma(X) \Big)^{1/p} \leq C\eta^{1/(n+1)}, \qquad \forall\, r \in (0, R_\eta). \tag{4.6.32}$$

From this and Proposition 2.4.8, we may then conclude that $\operatorname{dist}(\nu, \mathrm{VMO}(\partial\Omega, d\sigma)) < C\eta^{1/(n+1)}$ which readily proves (4.6.26).

To prove this claim, fix $Q \in \partial\Omega$ and $r \in (0, R_\eta)$ arbitrary. Then

$$\begin{aligned}
&\int_{\Delta(Q,r)} \Big| \int_{\Delta(P,r)} \frac{\nu(X) \odot (X - Y) + (X - Y) \odot \nu(Y)}{|X - Y|^{n+1}} \, d\sigma(Y) \Big|^p d\sigma(X) \\
&\qquad = \|\mathbf{1}_{\Delta(Q,r)}(\mathfrak{C} - \mathfrak{C}^*)\mathbf{1}_{\Delta(P,r)}\|^p_{L^p(\partial\Omega,d\sigma)\otimes\mathcal{C}\!\ell_{n+1}} \\
&\qquad \leq \|\mathbf{1}_{\Delta(Q,r)}(\mathfrak{C} - \mathfrak{C}^*)\|^p_{\mathcal{L}\left(L^p(\partial\Omega,d\sigma)\otimes\mathcal{C}\!\ell_{n+1}\right)} \|\mathbf{1}_{\Delta(P,r)}\|^p_{L^p(\partial\Omega,d\sigma)} \\
&\qquad \leq C\eta^p r^n.
\end{aligned} \tag{4.6.33}$$

Next, pick a point $P \in \partial\Omega$ with $|Q-P| = \eta^{-1/(n+1)} r$ and observe that

$$\int_{\Delta(Q,r)} \Big| \int_{\Delta(P,r)} \frac{\nu(X)\odot(Q-Y)+(Q-Y)\odot\nu(Y)}{|Q-Y|^{n+1}}\, d\sigma(Y)\Big|^p d\sigma(X) \leq C\eta^p r^n \tag{4.6.34}$$

since the difference between the left-most integrands in (4.6.33) and (4.6.34) can be pointwise bounded by

$$C|X-Q| \sup_{Z\in[X,Q]} |Z-Y|^{-(n+1)} \leq C\eta r^{-n}, \tag{4.6.35}$$

on the domain of integration. Using once more the same type of argument shows that

$$\int_{\Delta(Q,r)} \Big| \int_{\Delta(P,r)} \Big(\frac{\nu(X)\odot(Q-P)}{|Q-P|^{n+1}} + \frac{(Q-Y)\odot\nu(Y)}{|Q-Y|^{n+1}}\Big)\, d\sigma(Y)\Big|^p d\sigma(X) \leq C\eta^p r^n. \tag{4.6.36}$$

Selecting

$$A := |Q-P|^{n-1}\Big(⨍_{\Delta(P,r)} \frac{(Q-Y)\odot\nu(Y)}{|Q-Y|^{n+1}}\, d\sigma(Y)\Big)\odot(Q-P) \tag{4.6.37}$$

and using (4.6.14)-(4.6.15), allows us to estimate

$$\begin{aligned}
&\int_{\Delta(Q,r)} |\nu(X)-A|^p\, d\sigma(X) \\
&= \big(\eta^{-1/(n+1)}r\big)^{p(n-1)} \int_{\Delta(Q,r)} \Big|(\nu(X)-A)\odot\frac{Q-P}{|Q-P|^{n+1}}\odot(Q-P)\Big|^p d\sigma(X) \\
&\leq C\big(\eta^{-1/(n+1)}r\big)^{pn} \int_{\Delta(Q,r)} \Big|\frac{\nu(X)\odot(Q-P)}{|Q-P|^{n+1}} - A\odot\frac{Q-P}{|Q-P|^{n+1}}\Big|^p d\sigma(X) \\
&\leq C\big(\eta^{-1/(n+1)}r\big)^{pn} \int_{\Delta(Q,r)} \Big|\frac{\nu(X)\odot(Q-P)}{|Q-P|^{n+1}} + ⨍_{\Delta(P,r)} \frac{(Q-Y)\odot\nu(Y)}{|Q-Y|^{n+1}}\, d\sigma(Y)\Big|^p d\sigma(X) \\
&\leq C\big(\eta^{-1/(n+1)}r\big)^{pn} \int_{\Delta(Q,r)} \Big|⨍_{\Delta(P,r)} \Big(\frac{\nu(X)\odot(Q-P)}{|Q-P|^{n+1}} + \frac{(Q-Y)\odot\nu(Y)}{|Q-Y|^{n+1}}\Big)\, d\sigma(Y)\Big|^p d\sigma(X) \\
&\leq C\big(\eta^{-1/(n+1)}r\big)^{pn}(\eta^p r^n) r^{-pn} \leq C\sigma(\Delta(Q,r))\eta^{p/(n+1)}.
\end{aligned}$$

From this (4.6.32) follows, finishing the proof of the theorem. $\square$

To state our next result, recall the Clifford-Cauchy operator from (4.6.18) and the Riesz transforms from (4.6.1).

Theorem 4.6.8 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set satisfying a two-sided local John condition and such that $\partial\Omega$ is Ahlfors regular and compact. Also, denote by $\nu = (\nu_1, ..., \nu_{n+1})$ the outward unit normal to Ω. Then the following statements are equivalent:*

(i) $\mathfrak{C} - \mathfrak{C}^*$ is a compact operator on $L^p(\partial\Omega, d\sigma)\otimes \mathcal{C}\ell_{n+1}$ for some (hence all) $p \in (1,\infty)$;

(ii) $[\mathfrak{C}, \mathfrak{C}^*]$ *is a compact operator on* $L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}$ *for some (hence all)* $p \in (1, \infty)$*;*

(iii) the harmonic double layer K *and the commutators* $[M_{\nu_j}, \mathcal{R}_k]$, $1 \leq j, k \leq n+1$, *between the Riesz transforms and multiplication by the components of the unit normal, are compact operators on* $L^p(\partial\Omega, d\sigma)$ *for some (and, hence, all)* $p \in (1, \infty)$*;*

(iv) $I + \sum_{j=1}^{n+1} \mathcal{R}_j^2$ *and the commutators* $[\mathcal{R}_j, \mathcal{R}_k]$, $1 \leq j, k \leq n+1$, *are compact operators on* $L^p(\partial\Omega, d\sigma)$ *for some (hence, all)* $p \in (1, \infty)$*;*

(v) the commutators $[M_{\nu_j}, \mathcal{R}_k]$ *and* $[\mathcal{R}_j, \mathcal{R}_k]$, $1 \leq j, k \leq n+1$, *are are compact operators on* $L^p(\partial\Omega, d\sigma)$ *for some (and, hence, all)* $p \in (1, \infty)$*;*

(vi) Ω *is a regular SKT domain.*

Proof. We shall need the fact that, in the context of Theorem 4.6.8, the Cauchy-Clifford operator satisfies

$$\mathfrak{C}^2 = \tfrac{1}{4} I \quad \text{on} \quad L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}, \tag{4.6.38}$$

for every $p \in (1, \infty)$. As to do not disrupt the flow of the presentation, the proof of this identity is postponed for the next subsection. Granted (4.6.38), we obtain via duality $(\mathfrak{C}^*)^2 = \frac{1}{4} I$ on $L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}$, $1 < p < \infty$. Hence,

$$[\mathfrak{C}, \mathfrak{C}^*] = (\mathfrak{C} - \mathfrak{C}^*)(\mathfrak{C} + \mathfrak{C}^*) \quad \text{on} \quad L^2(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}. \tag{4.6.39}$$

We now make the observation that

$$\mathfrak{C} + \mathfrak{C}^* \quad \text{is an invertible operator on} \quad L^2(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}. \tag{4.6.40}$$

Indeed, on the one hand, $\mathfrak{C} + \mathfrak{C}^* = \mathfrak{C}(I + 4\mathfrak{C}\mathfrak{C}^*)$ thanks to (4.6.38). On the other hand, $\mathfrak{C}\mathfrak{C}^*$ is a nonnegative, self-adjoint operator so $I + 4\mathfrak{C}\mathfrak{C}^*$ is invertible on $L^2(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}$. Now, (4.6.40) readily follows from these considerations.

In concert, (4.6.39) and (4.6.40) prove that

$$\begin{aligned} &[\mathfrak{C}, \mathfrak{C}^*] \ \text{compact on} \ L^2(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1} \\ &\qquad \Longleftrightarrow \ \mathfrak{C} - \mathfrak{C}^* \ \text{compact on} \ L^2(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}, \end{aligned} \tag{4.6.41}$$

which, in turn, justifies the equivalence $(i) \Leftrightarrow (ii)$.

Moving on, the implication $(iii) \Rightarrow (i)$ is a consequence of Lemma 4.6.6. If (i) holds, then Theorem 4.6.7 gives that $\nu \in \mathrm{VMO}\,(\partial\Omega, d\sigma)$. Consequently, Ω is a regular SKT domain by virtue of Theorem 4.2.9. This proves $(i) \Rightarrow (vi)$. Assume next that (vi) holds. From Theorem 4.5.1 we know that the harmonic double layer K is a compact operator on $L^p(\partial\Omega, d\sigma)$. Also, since $\nu \in \mathrm{VMO}\,(\partial\Omega, d\sigma)$, Theorem 2.4.5 and Theorem 3.4.2 give that $[M_{\nu_j}, \mathcal{R}_k]$, the commutator between the operator of multiplication by the j-th component of ν and the k-th Riesz transform on $\partial\Omega$ is also compact on $L^p(\partial\Omega, d\sigma)$, for each $j, k \in \{1, ..., n+1\}$. Hence, $(vi) \Rightarrow (iii)$.

In summary, $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (vi)$. To consider (iv), we first re-write (4.6.18) in the form

$$\mathfrak{C}M_\nu = -\tfrac{1}{2}\sum_{k=1}^{n}\mathcal{R}_k M_{e_k}, \tag{4.6.42}$$

so that

$$\begin{aligned}\mathfrak{C}M_\nu\mathfrak{C}M_\nu &= \tfrac{1}{4}\sum_{j,k=1}^{n+1}\mathcal{R}_j\mathcal{R}_k M_{e_j}M_{e_k} = -\tfrac{1}{4}\sum_{k=1}^{n+1}\mathcal{R}_k^2 + \tfrac{1}{4}\sum_{1\leq j\neq k\leq n+1}\mathcal{R}_j\mathcal{R}_k M_{e_j}M_{e_k} \\ &= -\tfrac{1}{4}\sum_{k=1}^{n+1}\mathcal{R}_k^2 + \tfrac{1}{4}\sum_{1\leq j<k\leq n+1}(\mathcal{R}_j\mathcal{R}_k - \mathcal{R}_k\mathcal{R}_j)M_{e_j\odot e_k}.\end{aligned} \tag{4.6.43}$$

Thus, based on (4.6.43) and (4.6.22), we may conclude that

$$\mathfrak{C}\mathfrak{C}^* = -\tfrac{1}{4}\sum_{k=1}^{n+1}\mathcal{R}_k^2 + \tfrac{1}{4}\sum_{1\leq j<k\leq n+1}(\mathcal{R}_j\mathcal{R}_k - \mathcal{R}_k\mathcal{R}_j)M_{e_j\odot e_k}. \tag{4.6.44}$$

In turn, this and (4.6.38) imply

$$\mathfrak{C}(\mathfrak{C}^* - \mathfrak{C}) = -\tfrac{1}{4}\Big(I + \sum_{k=1}^{n+1}\mathcal{R}_k^2\Big) + \tfrac{1}{4}\sum_{1\leq j<k\leq n+1}[\mathcal{R}_j,\mathcal{R}_k]M_{e_j\odot e_k}, \tag{4.6.45}$$

$$\mathfrak{C} - \mathfrak{C}^* = \mathfrak{C}\Big(I + \sum_{k=1}^{n+1}\mathcal{R}_k^2\Big) + \sum_{1\leq j<k\leq n+1}\mathfrak{C}[\mathcal{R}_j,\mathcal{R}_k]M_{e_j\odot e_k}. \tag{4.6.46}$$

It is then clear from (4.6.45) that $(i) \Rightarrow (iv)$, and from (4.6.46) that $(iv) \Rightarrow (i)$. Hence $(i) \Leftrightarrow (iv)$.

Finally, consider (v). To this end, we use (4.6.22) and (4.6.43) to write

$$\begin{aligned}\mathfrak{C}^*\mathfrak{C} &= -M_\nu\mathfrak{C}M_\nu\mathfrak{C}M_\nu M_\nu = -M_\nu\mathfrak{C}\mathfrak{C}^*M_\nu \\ &= -\tfrac{1}{4}\sum_{k=1}^{n+1}\mathcal{R}_k^2 + \tfrac{1}{4}\sum_{1\leq j\neq k\leq n+1}M_\nu\mathcal{R}_j\mathcal{R}_k M_{e_j}M_{e_k}M_\nu.\end{aligned} \tag{4.6.47}$$

Note that $M_\nu\mathcal{R}_j\mathcal{R}_k M_{e_j}M_{e_k}M_\nu$ and $\mathcal{R}_j\mathcal{R}_k M_\nu M_{e_j}M_{e_k}M_\nu$ differ by expressions containing commutators between the Riesz transforms and multiplication by the components of ν, and that

$$\sum_{1\leq j\neq k\leq n+1}\mathcal{R}_j\mathcal{R}_k M_\nu M_{e_j}M_{e_k}M_\nu = \sum_{1\leq j<k\leq n+1}[\mathcal{R}_j,\mathcal{R}_k]M_\nu M_{e_j}M_{e_k}M_\nu. \tag{4.6.48}$$

Furthermore

$$\Big[M_\nu\Big(\sum_{k=1}^{n+1}\mathcal{R}_k^2\Big)M_\nu\Big]^* = M_\nu\Big(\sum_{k=1}^{n+1}\mathcal{R}_k^2\Big)M_\nu. \tag{4.6.49}$$

These observations readily give that $(v) \Rightarrow (ii)$. On the other hand, it is clear that $(iii)-(iv) \Rightarrow (v)$, completing the proof of the theorem (modulo the justification of (4.6.38), which is postponed for the next subsection). $\square$

We are now in a position to complete the proofs of our main results in this subsection:

Proof of Theorem 4.6.1. This is a direct consequence of Theorem 4.6.8. $\square$

Proof of Theorem 4.6.2. This is an immediate consequence of (4.6.26) and (4.6.20). $\square$

Proof of Theorem 4.6.3. This readily follows from (4.6.46) and (4.6.26). $\square$

Proof of Theorem 4.6.4. This is implied by Theorem 4.6.8. $\square$

4.7 Clifford-Szegö projections and regular SKT domains

Let $\Omega \subset \mathbb{R}^{n+1}$ be an NTA domain, with an Ahlfors regular boundary. Retain the same Clifford algebra formalism as in §4.6, and recall the Dirac operator

$$D := \sum_{j=1}^{n+1} M_{e_j}\partial_j. \tag{4.7.1}$$

In particular,

$$-D^2 = \Delta, \tag{4.7.2}$$

the Laplacian in $\mathbb{R}^{n+1}$. In analogy with the classical setting of functions of one complex variable, for each $p \in (1,\infty)$ define the Hardy spaces

$$\mathcal{H}^p(\Omega) := \{u : \Omega \to \mathcal{C}\ell_{n+1} : \mathcal{N}(u) \in L^p(\partial\Omega, d\sigma),\ Du = 0 \text{ in } \Omega\}, \tag{4.7.3}$$

with the convention that that if Ω is unbounded and $\partial\Omega$ is bounded, the decay condition

$$u(x) = O(|x|^{-n}) \quad \text{as} \quad |x| \to \infty \tag{4.7.4}$$

is also included. Next, consider the boundary version of the Hardy spaces (4.7.3)

$$\mathcal{H}^p(\partial\Omega) := \{u|_{\partial\Omega} : u \in \mathcal{H}^p(\Omega)\}. \tag{4.7.5}$$

(The existence of the boundary trace will be established in (4.7.13).) Furthermore, define the *Clifford-Szegö projection*

$$\mathfrak{P} : L^2(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1} \longrightarrow \mathcal{H}^2(\partial\Omega) \hookrightarrow L^2(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1} \tag{4.7.6}$$

as the orthogonal projection of $L^2(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}$ onto the closed subspace $\mathcal{H}^2(\partial\Omega)$. The issue we wish to study first is whether (4.7.6) extends to

$$\mathfrak{P} : L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1} \longrightarrow \mathcal{H}^p(\partial\Omega) \tag{4.7.7}$$

in a continuous and onto fashion for other values of $p \in (1,\infty)$. In the classical setting, when Ω is the unit disk in the complex plane (and D is the Cauchy-Riemann operator), this holds for $1 < p < \infty$

according to a famous theorem of M. Riesz which, in fact, is equivalent to the L^p-boundedness of the Hilbert transform on the unit circle. See pp. 151–152 in [50] for more details. Our main result in this respect, which can be viewed as a higher-dimensional generalization of Theorem 2.1(1)(4) on p. 67 in [72], is Theorem 4.7.2, given below. Before stating it, we mention the following extension of a theorem of Calderón, established in [52], Theorem 2.6.

Theorem 4.7.1 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is a two-sided NTA domain with an Ahlfors regular boundary (making it a UR domain). Then the following decomposition is valid for each $p \in (1, \infty)$:*

$$L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1} = \mathcal{H}^p_+(\partial\Omega) \oplus \mathcal{H}^p_-(\partial\Omega), \tag{4.7.8}$$

where $\mathcal{H}^p_\pm(\partial\Omega)$ are the boundary Hardy spaces corresponding to $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^{n+1} \setminus \overline{\Omega}$.

(In particular, this implies $\mathcal{H}^2(\partial\Omega)$ in (4.7.6) is closed.) Here is our result.

Theorem 4.7.2 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is an NTA domain with an Ahlfors regular boundary. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ with the property that $\mathfrak{P}$ extends to a bounded mapping of $L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}$ onto $\mathcal{H}^p(\partial\Omega)$ for each $p \in (2-\varepsilon, 2+\varepsilon)$. In particular, with p as above, the following decomposition is valid:*

$$L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1} = \mathcal{H}^p(\partial\Omega) \oplus \Big(\nu \odot \mathcal{H}^p(\partial\Omega)\Big), \tag{4.7.9}$$

where the direct sum is topological (when $p = 2$ this is an orthogonal decomposition).

Moreover, if $\partial\Omega$ is compact then given any $p \in (1, \infty)$ there exists $\varepsilon > 0$ depending only on p, n and the geometrical constants of of Ω with the property that if Ω is an ε-regular SKT domain then the Clifford-Szegö projection can be extended to a bounded operator from $L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}$ onto $\mathcal{H}^p(\partial\Omega)$. Furthermore, (4.7.9) holds.

As a corollary, when Ω is a regular SKT domain with compact boundary, the Clifford-Szegö projection is extendible to a bounded operator from $L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}$ onto $\mathcal{H}^p(\partial\Omega)$ for each $p \in (1, \infty)$ and (4.7.9) is valid in this range.

Proof. Consider the Cauchy-Clifford operator

$$\mathcal{C}f(X) := \frac{1}{\omega_n} \int_{\partial\Omega} \frac{X - Y}{|X - Y|^{n+1}} \odot \nu(Y) \odot f(Y)\, d\sigma(Y), \qquad X \in \Omega, \tag{4.7.10}$$

where f is a $\mathcal{C}\ell_{n+1}$-valued function defined on $\partial\Omega$. It follows from Theorem 3.4.2 and Proposition 3.2.3 that for each $p \in (1, \infty)$

$$\|\mathcal{N}(\mathcal{C}f)\|_{L^p(\partial\Omega, d\sigma)} \leq C\|f\|_{L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}}, \tag{4.7.11}$$

$$\mathcal{C}f\Big|_{\partial\Omega} = (\tfrac{1}{2}I + \mathfrak{C})f, \qquad D(\mathcal{C}f) = 0 \quad \text{in } \ \Omega, \tag{4.7.12}$$

for every $f \in L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}$.

We now claim that if $p \in (1, \infty)$, the following Fatou type theorem and Cauchy reproducing formula hold:

$$u\Big|_{\partial\Omega} \text{ exists, } \quad \text{and} \quad u = \mathcal{C}(u|_{\partial\Omega}) \ \text{ in } \ \Omega, \qquad \forall\, u \in \mathcal{H}^p(\Omega). \tag{4.7.13}$$

Indeed, according to Theorem 6.4 on p. 112 of [55],

> any function u which is harmonic in an NTA domain Ω and nontangentially bounded from below on $E \subset \partial\Omega$ has a nontangential limit ω^{X_o}-a.e. on E (where ω^{X_o} is the harmonic measure with pole at $X_o \in \Omega$).

Now, the fact that $u|_{\partial\Omega}$ exists for every $u \in \mathcal{H}^p(\Omega)$, $p \in (1,\infty)$, is a consequence of (4.7.2), the above local Fatou theorem applied to $E_k := \{X \in \partial\Omega : \mathcal{N}u(X) < 2^k\}$, $k = 1, 2...$, and the mutual absolute continuity between the surface and harmonic measures proved in [31]; cf. Proposition 3.1.16.

We shall now establish Cauchy's reproducing formula in (4.7.13) in the case when the domain Ω has an unbounded boundary (the argument when $\partial\Omega$ is compact is similar and simpler). Thus, assume that $\partial\Omega$ is unbounded, fix $X^* \in \Omega$ and pick $X_o \in \partial\Omega$ such that $|X^* - X_o| = \eta$, where $\eta := \operatorname{dist}(X^*, \partial\Omega)$.

Given $u \in \mathcal{H}^p(\Omega_+)$ and $0 < \varepsilon < \eta$, the idea now is to employ Theorem 3.2.8 for the vector field

$$v(X) := \Big(\frac{X^* - X}{|X^* - X|^{n+1}} \odot e_j \odot u(X)\Big)_{1 \leq j \leq n+1} \tag{4.7.14}$$

in the domain $\Omega \setminus \overline{B(X^*, \varepsilon)}$. Since $Du = 0$ in Ω (cf. (4.7.3)), one may readily check that the vector field v is divergence free. In more detail, a calculation gives

$$\begin{aligned}(\operatorname{div} v)(X) &= \sum_{j=1}^{n+1} \partial_j \Big(\frac{X^* - X}{|X^* - X|^{n+1}} \odot e_j \odot u(X)\Big) \\ &= \Big(\sum_{j=1}^{n+1} \partial_j \Big[\frac{X^* - X}{|X^* - X|^{n+1}}\Big] \odot e_j\Big) \odot u(X) + \frac{X^* - X}{|X^* - X|^{n+1}} \odot (Du)(X).\end{aligned} \tag{4.7.15}$$

The last term vanishes, since $Du = 0$ in Ω. Also, for any scalar-valued function f one has the readily verified identity

$$\sum_j \partial_j (Df) \odot e_j = -\Delta f, \tag{4.7.16}$$

which, in concert with the observation that $(X^* - X)/|X^* - X|^{n+1}$ is a constant times $(Df)(X)$ if $f(X) := |X^* - X|^{1-n}$, shows that the penultimate term in (4.7.15) also vanishes. Hence v is divergence free in $\Omega \setminus \overline{B(X^*, \varepsilon)}$. Consequently,

$$\begin{aligned}\frac{1}{\omega_n} \int_{\partial\Omega} & \frac{X^* - X}{|X^* - X|^{n+1}} \odot \nu(X) \odot u(X)\, d\sigma(X) \\ &= \frac{1}{\omega_n} \int_{\partial B(X^*, \varepsilon)} \frac{X^* - X}{|X^* - X|^{n+1}} \odot \frac{X^* - X}{\varepsilon} \odot u(X)\, d\sigma(X) \\ &= \fint_{\partial B(X^*, \varepsilon)} u(X)\, d\sigma(X) \\ &= u(X^*),\end{aligned} \tag{4.7.17}$$

by the Mean Value Formula for harmonic functions. Thus, $u(X^*) = \mathcal{C}(u|_{\partial\Omega})(X^*)$ follows, as soon as we show that $\mathcal{N}_\varepsilon v \in L^1(\partial\Omega, d\sigma)$ where $\mathcal{N}_\varepsilon$ is the nontangential maximal operator relative to

$\Omega \setminus \overline{B(X^*, \varepsilon)}$. (That $\mathcal{N}_\varepsilon v \in L^p_{loc}(\partial\Omega, d\sigma)$ is easily checked using $\mathcal{N} v \in L^p(\partial\Omega, d\sigma)$ and the Ahlfors regularity of $\partial\Omega$.)

To this end, it suffices to note that $\mathcal{N}_\varepsilon v(X) \leq \frac{C}{(\eta+|X_o-X|)^n}(\mathcal{N}u)(X)$ for σ-a.e. $X \in \partial\Omega$ and that, as a function of X, $(\eta+|X_o-X|)^{-n}$ belongs to $L^{p'}(\partial\Omega, d\sigma)$, where $1/p+1/p'=1$. The latter claim is easily checked by decomposing $\partial\Omega$ in dyadic annuli of the form $\Delta(X_o, 2^{j+1}\eta) \setminus \Delta(X_o, 2^j\eta)$ and using the Ahlfors regularity of $\partial\Omega$. This concludes the proof of (4.7.13).

Proceeding further, we note from (4.7.11), (4.7.12), and (4.7.13) that the operator

$$\mathcal{C} : L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1} \longrightarrow \mathcal{H}^p(\Omega) \tag{4.7.18}$$

is well-defined, bounded and onto for each $p \in (1, \infty)$. From this and (4.7.11)-(4.7.13) we then obtain

$$\mathfrak{C}^2 = \tfrac{1}{4} I \quad \text{ on } \quad L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}, \tag{4.7.19}$$

which has been already employed in the proof of Theorem 4.6.8.

A careful inspection of the argument reveals that *(4.7.19) is valid if Ω is merely a UR domain.* We elaborate on this. Strictly speaking, (4.7.19) has been obtained using (4.7.10)–(4.7.13). While (4.7.10)–(4.7.11) work if Ω is only UR, the Fatou type theorem contained in (4.7.13) requires that Ω is NTA. However, an inspection of the arguments leading up to (4.7.19) shows that we only need to know that functions of the type $u = \mathcal{C}f$, for $f \in L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}$, have pointwise nontangential trace σ-a.e. on the boundary, which follows from our previously established jump formulas, valid for UR domains.

Next, formula (4.7.19) further entails

$$\operatorname{Im}\Big(\tfrac{1}{2}I + \mathfrak{C} :\ L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}\Big) = \mathcal{H}^p(\partial\Omega) = \operatorname{Ker}\Big(-\tfrac{1}{2}I + \mathfrak{C} :\ L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}\Big). \tag{4.7.20}$$

We now establish a version of Kerzman-Stein's formula (cf. [68]) in the Clifford algebra setting. To get started, note that the identities

$$\mathfrak{P}(\tfrac{1}{2}I + \mathfrak{C}) = \tfrac{1}{2}I + \mathfrak{C}, \qquad (-\tfrac{1}{2}I + \mathfrak{C})\mathfrak{P} = 0 \text{ in } L^2(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}, \tag{4.7.21}$$

can be easily justified in light of (4.7.20). Taking the adjoint of the second equality and subtracting it from the first yields $\mathfrak{P}(I + \mathfrak{C} - \mathfrak{C}^*) = \frac{1}{2}I + \mathfrak{C}$. Next, introduce the bounded operator

$$\mathfrak{A} := \mathfrak{C} - \mathfrak{C}^* : L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1} \longrightarrow L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}, \quad 1 < p < \infty. \tag{4.7.22}$$

On $L^2(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1}$, since $\mathfrak{A}$ is a bounded, skew-adjoint operator, the operator $I + \mathfrak{A}$ is accretive and, hence, invertible. From general stability results it follows then that there exists $\varepsilon > 0$ so that

$$I + \mathfrak{A} : L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1} \longrightarrow L^p(\partial\Omega, d\sigma) \otimes \mathcal{C}\ell_{n+1} \quad \text{is invertible for } 2 - \varepsilon < p < 2 + \varepsilon. \tag{4.7.23}$$

In this fashion, we arrive at the Kerzman-Stein type formula

$$\mathfrak{P} = (\tfrac{1}{2}I + \mathfrak{C}) \circ (I + \mathfrak{A})^{-1} \tag{4.7.24}$$

valid in $L^p(\partial\Omega, d\sigma)\otimes \mathcal{C}\ell_{n+1}$, $2-\varepsilon<p<2+\varepsilon$. This, (4.7.23) and Theorem 3.4.2 then show that $\mathfrak{P}$ extends as a bounded operator in $L^p(\partial\Omega, d\sigma)\otimes \mathcal{C}\ell_{n+1}$ for $2-\varepsilon<p<2+\varepsilon$ and, in fact,

$$\mathfrak{P}: L^p(\partial\Omega, d\sigma)\otimes \mathcal{C}\ell_{n+1} \longrightarrow \mathcal{H}^p(\partial\Omega) \text{ is onto for } 2-\varepsilon<p<2+\varepsilon. \tag{4.7.25}$$

Consider next $\mathfrak{Q}$, the complementary orthogonal projection of (4.7.6). Hence, by (4.6.22),

$$\begin{aligned}\mathfrak{Q}f &= (I-\mathfrak{P})f = \Big[I-(\tfrac{1}{2}I+\mathfrak{C})(I+\mathfrak{A})^{-1}\Big]f = (\tfrac{1}{2}I-\mathfrak{C}^*)(I+\mathfrak{A})^{-1}f & (4.7.26)\\ &= -M_\nu(\tfrac{1}{2}I+\mathfrak{C})M_\nu(I+\mathfrak{A})^{-1}f, & (4.7.27)\end{aligned}$$

since $M_\nu^2=-I$. We therefore arrive at the conclusion that $\mathfrak{Q}$ extends as a bounded, surjective operator

$$\mathfrak{Q}: L^p(\partial\Omega, d\sigma)\otimes \mathcal{C}\ell_{n+1} \longrightarrow \nu\odot\mathcal{H}^p(\partial\Omega) \tag{4.7.28}$$

whenever $2-\varepsilon<p<2+\varepsilon$. From (4.7.25) and (4.7.28), the conclusion regarding (4.7.9) follows. This finishes the proof of the first part of the theorem.

Consider next the claim made in the second part of the statement of the theorem. Fix $p\in(1,\infty)$ and denote by p' its Hölder conjugate exponent. Then Theorem 4.6.8 implies that

$$\operatorname{dist}\Big(\mathfrak{A}\,,\ \mathrm{Cp}\,(L^q(\partial\Omega, d\sigma)\otimes \mathcal{C}\ell_{n+1})\Big)<1, \qquad \text{for every } q \text{ in between } p \text{ and } p', \tag{4.7.29}$$

if Ω is an ε-regular SKT domain for a sufficiently small ε. Assuming that this is the case, we may then conclude that the operator $I+\mathfrak{A}$ is Fredholm with index zero on $L^q(\partial\Omega, d\sigma)\otimes \mathcal{C}\ell_{n+1}$ for every q in between p and p'. Also, from what we have proved already, this operator is also invertible when $q=2$. It is then easy to show that the operator in question is, in fact, invertible for every q in between p and p'. Background material on Fredholm operators can be found in the Functional Analysis appendix in Vol. 1 of [112].

With this in hand and proceeding as in the first part of the proof, we arrive at the conclusion that (4.7.25) and (4.7.28) actually hold for the given p. As a consequence, (4.7.9) is also valid for the given p.

Finally, when Ω is a regular SKT domain then, by Proposition 4.1.11, Ω is an ε-regular SKT domain for every $\varepsilon>$ and, hence, (4.7.9) is valid for the full range $1<p<\infty$ in this case. $\square$

Denote by $\mathfrak{P}_\pm$ the Clifford-Szegö projections corresponding to $\Omega_+:=\Omega$ and $\Omega_-:=\mathbb{R}^{n+1}\setminus\bar{\Omega}$, respectively.

Theorem 4.7.3 *Let $\Omega\subset\mathbb{R}^{n+1}$ be a two-sided NTA domain, with a compact, Ahlfors regular boundary. Then the following statements are equivalent:*

(i) $\frac{1}{2}I+\mathfrak{C}-\mathfrak{P}_+$ *and* $-\frac{1}{2}I+\mathfrak{C}+\mathfrak{P}_-$ *are compact operators on* $L^2(\partial\Omega, d\sigma)\otimes \mathcal{C}\ell_{n+1}$;

(ii) $\mathfrak{P}_+\mathfrak{P}_-$ *(or, equivalently,* $\mathfrak{P}_-\mathfrak{P}_+$*) is a compact operator on* $L^2(\partial\Omega, d\sigma)\otimes \mathcal{C}\ell_{n+1}$;

(iii) $I-\mathfrak{P}_+-\mathfrak{P}_-$ *is a compact operator on* $L^2(\partial\Omega, d\sigma)\otimes \mathcal{C}\ell_{n+1}$;

(iv) Ω *is a regular SKT domain.*

Proof. This will follow from Theorem 4.6.8 as soon as we establish the following operator identities:

$$\tfrac{1}{2}I + \mathfrak{C} - \mathfrak{P}_+ = (\tfrac{1}{2}I + \mathfrak{C})(\mathfrak{C} - \mathfrak{C}^*)(I + \mathfrak{A})^{-1}, \tag{4.7.30}$$

$$-\tfrac{1}{2}I + \mathfrak{C} + \mathfrak{P}_- = (-\tfrac{1}{2}I + \mathfrak{C})(\mathfrak{C} - \mathfrak{C}^*)(-I + \mathfrak{A})^{-1}, \tag{4.7.31}$$

$$\begin{aligned} I - \mathfrak{P}_+ - \mathfrak{P}_- = {} & (\tfrac{1}{2}I + \mathfrak{C})(\mathfrak{C} - \mathfrak{C}^*)(I + \mathfrak{A})^{-1} \\ & -(-\tfrac{1}{2}I + \mathfrak{C})(\mathfrak{C} - \mathfrak{C}^*)(-I + \mathfrak{A})^{-1}, \end{aligned} \tag{4.7.32}$$

$$-\mathfrak{P}_+(I - \mathfrak{P}_+ - \mathfrak{P}_-) = \mathfrak{P}_+\mathfrak{P}_-, \quad -\mathfrak{P}_-(I - \mathfrak{P}_+ - \mathfrak{P}_-) = \mathfrak{P}_-\mathfrak{P}_+, \tag{4.7.33}$$

$$\begin{aligned} \mathfrak{C} - \mathfrak{C}^* = {} & (\tfrac{1}{2}I + \mathfrak{C} - \mathfrak{P}_+)(I + \mathfrak{A}) \\ & -(-\tfrac{1}{2}I + \mathfrak{C} + \mathfrak{P}_-)(-I + \mathfrak{A}), \end{aligned} \tag{4.7.34}$$

$$\mathfrak{C} - \mathfrak{C}^* = \mathfrak{P}_+\mathfrak{P}_-(\mathfrak{C} + \mathfrak{C}^*) - (\mathfrak{C} + \mathfrak{C}^*)\mathfrak{P}_-\mathfrak{P}_+. \tag{4.7.35}$$

To this end, note that (4.7.30) is a direct consequence of (4.7.24), and that (4.7.31) is proved similarly. Next, (4.7.32) is obtained by subtracting (4.7.31) from (4.7.30). The two identities in (4.7.33) are easily verified by multiplying through and using the fact that $\mathfrak{P}_\pm$ are projections. Also, (4.7.34) follows easily from (4.7.30) and (4.7.31). As for (4.7.35), we start with

$$\mathfrak{P}_+(-\tfrac{1}{2}I + \mathfrak{C}) = \mathfrak{P}_+\Big((\tfrac{1}{2}I + \mathfrak{C}) - I\Big) = \tfrac{1}{2}I + \mathfrak{C} - \mathfrak{P}_+, \tag{4.7.36}$$

and use it to obtain

$$\mathfrak{P}_+\mathfrak{P}_-(-\tfrac{1}{2}I + \mathfrak{C}) = \tfrac{1}{2}I + \mathfrak{C} - \mathfrak{P}_+. \tag{4.7.37}$$

Dualizing $(\frac{1}{2}I + \mathfrak{C})\mathfrak{P}_- = 0$ we also obtain $\mathfrak{P}_-(\frac{1}{2}I + \mathfrak{C}^*) = 0$, so that

$$\mathfrak{P}_+\mathfrak{P}_-(\tfrac{1}{2}I + \mathfrak{C}^*) = 0. \tag{4.7.38}$$

Adding (4.7.38) and (4.7.37) then yields

$$\tfrac{1}{2}I + \mathfrak{C} - \mathfrak{P}_+ = \mathfrak{P}_+\mathfrak{P}_-(\mathfrak{C} + \mathfrak{C}^*), \tag{4.7.39}$$

so by subtracting (4.7.39) from its dual version we arrive at (4.7.35). □

5 Laplace-Beltrami layer potentials and the Dirichlet and Neumann problems

In this section we study the Dirichlet and Neumann problems for the Laplace operator Δ on a regular SKT domain Ω in a Riemannian manifold M. We consider more generally operators of the form $L = \Delta - V$ where $V \in L^\infty(M)$ is ≥ 0. We set up the double layer operator $\mathcal{D}$ and the single layer operator $\mathcal{S}$ in §5.1 and investigate their basic properties of L^p-boundedness and nontangential boundary limit behavior, when Ω is a UR domain. Compactness results are derived in §5.2 when Ω is a regular SKT domain. Results of §§3–4 are crucial for the work here. In §5.3 we extend Green formulas of §2.3 to the manifold setting. This extension is used in §5.4 to establish injectivity of certain boundary layer operators, which in combination with the compactness results of §5.2 then yields invertibility. Then in §5.5 the Dirichlet and Neumann problems are solved, in terms of $\mathcal{D}$ and of $\mathcal{S}$ acting on inverses of appropriate boundary layer operators, applied to the boundary data. Results here will be complemented in §6 by results on various second order elliptic systems. In §5.6 we extend the results of §§5.4–5.5 to the setting of ε-regular SKT domains.

5.1 Boundedness and jump-relations

To set the stage, we let M be a compact, connected manifold, of dimension $n+1$, endowed with a Riemannian metric tensor $g = \sum_{j,k} g_{jk} dx_j \otimes dx_k$, perhaps of limited smoothness, as we will discuss further below. As is customary, we also let g denote $\det(g_{jk})$ and set $d\mathcal{V} := \sqrt{g}\, dX$ for the volume element on M. Finally, let Δ denote the Laplace-Beltrami operator associated to the metric tensor and fix a nonnegative potential $V \in L^\infty(M)$ which is not identically zero. Then

$$L := \Delta - V, \tag{5.1.1}$$

is a second-order elliptic operator which maps the classical Sobolev space $H^{1,2}(M)$ isomorphically onto its dual, $H^{-1,2}(M)$, and whose inverse can be represented as an integral operator, say

$$L^{-1}u(X) = \int_M E(X,Y)u(Y)\, d\mathcal{V}(Y), \qquad X \in M. \tag{5.1.2}$$

The invariance of various classes of rough domains under C^1-diffeomorphisms has been studied in [53]. In the context of Riemannian manifolds, this allows one to define, in a coordinate invariant fashion, classes of rough domains much as in the Euclidean setting.

If $\Omega \subset M$ is a UR domain, then the double layer potential associated with Ω in the compact $(n+1)$-dimensional Riemannian manifold M has the form

$$\mathcal{D}f(X) := \int_{\partial\Omega} \frac{\partial E}{\partial \nu_Y}(X,Y)\, f(Y)\, d\sigma_g(Y), \tag{5.1.3}$$

with $E(X,Y)$ as in (5.1.2), and where σ_g stands for the surface measure induced by the metric on $\partial\Omega$. It is important to point out that, in any local coordinate system, if σ is the surface measure induced by the Euclidean metric (i.e. when $g_{jk} = \delta_{jk}$) on $\partial\Omega$, then

$$d\sigma_g = \rho\, d\sigma, \quad \text{ where } \quad \rho,\ \rho^{-1} \in L^\infty(\partial\Omega, d\sigma). \tag{5.1.4}$$

The proof of (5.1.4) can be seen by noting that both surface measures σ and σ_g are equal to n-dimensional Hausdorff measure, for $\partial\Omega \cap U \subset M$, where U is a coordinate patch with either the Euclidean metric or the Riemannian metric of M, together with the fact that subsets of U have diameters in these two metrics varying by a bounded factor. In fact, one has, in local coordinates,

$$\rho(X) = \sqrt{g(X)}\, G(X, n(X))^{1/2}, \tag{5.1.5}$$

where $g(X) = \det(g_{jk}(X))$, $G(X,\xi) = g^{jk}(X)\xi_j\xi_k$, and $n = (n_j)_j$ is the unit conormal to $\partial\Omega$ in the Euclidean metric. See §5.3 for more on this.

Concerning the nature of the singularity in $E(x,y)$, let us recall here that a parametrix construction, detailed for progressively rougher metric tensors in [91]–[93], gives

$$\sqrt{g(X)}\, E(X,Y) = e_0(X-Y, X) + e_1(Y,X), \tag{5.1.6}$$

in local coordinates, where the leading term has the form

$$e_0(Z,X) := C_n\Big(\sum g_{jk}(X)z_jz_k\Big)^{-(n-1)/2}, \qquad Z=(z_i)_i \in \mathbb{R}^{n+1}, \tag{5.1.7}$$

where C_n is a purely dimensional constant and the residue $e_1(Y,X)$ satisfies the following estimates if the metric tensor is Hölder continuous, say $g_{jk} \in C^\alpha$ for some $\alpha \in (0,1)$,

$$|e_1(Y,X)| \le C|X-Y|^{-(n-1-\alpha)}, \quad |\nabla_Y e_1(Y,X)| \le C|X-Y|^{-(n-\alpha)}. \tag{5.1.8}$$

Cf. Proposition 2.4 of [93], which improves (2.70)–(2.71) of [92]. (Here the dimension of M is $n+1$ rather than n.) There are related estimates in [93] when g_{jk} has modulus of continuity $\omega(h)$, satisfying

$$\int_0^1 \frac{\sqrt{\omega(h)}}{h}\, dh < \infty. \tag{5.1.9}$$

We will return to this matter later in this subsection.

Theorem 3.5.1 treats the contribution to $\mathcal{D}f(X)$ due to $g(X)^{-1/2}\partial_{\nu(Y)}e_0(X-Y,X)$. Now we examine the behavior of

$$\mathcal{K}_1 f(X) := \int\limits_{\partial\Omega} k_1(X,Y)f(Y)\,d\sigma_g(Y), \quad k_1(X,Y) := g(X)^{-1/2}\,\partial_{\nu(Y)}e_1(Y,X), \quad X\in\Omega, \tag{5.1.10}$$

and

$$K_1 f(X) := \int\limits_{\partial\Omega} k_1(X,Y)f(Y)\,d\sigma_g(Y), \quad X\in\partial\Omega. \tag{5.1.11}$$

We will estimate $\mathcal{N}\mathcal{K}_1 f$, examine the behavior of (5.1.10) as X approaches the boundary, and show that $K_1 : L^p(\partial\Omega, d\sigma) \to L^p(\partial\Omega, d\sigma)$, compactly, for $p \in (1,\infty)$.

Behind this analysis is the following rather general result. Let $\mathcal{X}$ be a compact metric space, with distance function $d(X,Y)$ and positive measure μ having the property that for some $0 < A_1, A_2 < \infty$ and $n \in \mathbb{N}$,

$$A_1 r^n \le \mu(B_r(X)) \le A_2 r^n, \tag{5.1.12}$$

for all $X \in \mathcal{X}$, $r \in (0, \operatorname{diam}\mathcal{X}]$. As it follows from (5.1.4) and the discussion in §3, $\mathcal{X} = \partial\Omega$ has this property when Ω is a bounded, $(n+1)$-dimensional UR domain and μ is n-dimensional surface area.

Proposition 5.1.1 *Assume $\mathcal{X}$ is as above and let $k_1(X,Y)$ be a measurable function on $\mathcal{X}\times\mathcal{X}$ satisfying*

$$|k_1(X,Y)| \le \psi(d(X,Y))\,d(X,Y)^{-n}, \tag{5.1.13}$$

where $\psi(t)$ is monotone increasing, slowly varying, with

$$\int_0^1 \frac{\psi(t)}{t}\, dt < \infty. \tag{5.1.14}$$

Consider

$$K_1 f(X) := \int_{\mathcal{X}} k_1(X,Y) f(Y)\, d\mu(Y). \tag{5.1.15}$$

Then $K_1 : L^p(\mathcal{X}, d\mu) \to L^p(\mathcal{X}, d\mu)$ *is compact, for each* $p \in (1,\infty)$.

Proof. Under the hypothesis (5.1.12), this result is a special case of Lemma 2.4.6. □

To apply Proposition 5.1.1 to $\mathcal{K}_1$ and K_1, first note that if the metric tensor is Hölder continuous, so (5.1.8) holds, then (5.1.13) holds with $\psi(t) = t^\alpha$, $\alpha \in (0,1)$. Thus K_1 is compact on $L^p(\partial\Omega, d\sigma_g)$ for each $p \in (1,\infty)$. Also

$$\mathcal{N}(\mathcal{K}_1 f)(X) \le \int_{\partial\Omega} k_1^M(X,Y)|f(Y)|\, d\sigma_g(Y), \qquad X \in \partial\Omega, \tag{5.1.16}$$

where

$$k_1^M(X,Y) := \sup_{Z\in\Gamma(X)} |k_1(Z,Y)|, \quad X,Y \in \partial\Omega. \tag{5.1.17}$$

If (5.1.8) holds, then, since $\operatorname{dist}(Z,X) \le C\operatorname{dist}(Z,\partial\Omega)$ for $Z \in \Gamma(X)$, we have the same sort of estimate (5.1.13) on $k_1^M(X,Y)$. Hence

$$\|\mathcal{N}(\mathcal{K}_1 f)\|_{L^p(\partial\Omega,d\sigma_g)} \le C_p \|f\|_{L^p(\partial\Omega,d\sigma_g)}, \qquad 1 < p < \infty. \tag{5.1.18}$$

We are ready to establish a result on boundary behavior.

Lemma 5.1.2 *Given* $p \in (1,\infty)$, $f \in L^p(\partial\Omega, d\sigma_g)$, *we have*

$$\lim_{\substack{Z\to X\\ Z\in\Gamma(X)}} \mathcal{K}_1 f(Z) = K_1 f(X), \qquad \sigma_g - a.e.\ \ X \in \partial\Omega. \tag{5.1.19}$$

Proof. Given the bound (5.1.18) on $\|\mathcal{N}(\mathcal{K}f)\|_{L^p(\partial\Omega,d\sigma_g)}$, it suffices to show that convergence in (5.1.19) holds for each $f \in C^0(\partial\Omega)$. To see this, write

$$\begin{aligned}
\lim_{\substack{Z\to X\\ Z\in\Gamma(X)}} \mathcal{K}_1 f(Z) &= \lim_{\varepsilon\to 0} \lim_{\substack{Z\to X\\ Z\in\Gamma(X)}} \int_{|X-Y|>\varepsilon} k_1(Z,Y) f(Y)\, d\sigma_g(Y) \\
&\quad + \lim_{\varepsilon\to 0} \lim_{\substack{Z\to X\\ Z\in\Gamma(X)}} \int_{|X-Y|<\varepsilon} k_1(Z,Y) f(Y)\, d\sigma_g(Y) \qquad (5.1.20) \\
&=: I_1 + I_2.
\end{aligned}$$

For each $\varepsilon > 0$, the Lebesgue Dominated Convergence Theorem applies to I_1 twice, giving

$$I_1 = \lim_{\varepsilon\to 0} \int_{|X-Y|>\varepsilon} k_1(X,Y) f(Y)\, d\sigma_g(Y) = K_1 f(X). \tag{5.1.21}$$

Meanwhile, with $k_1^M(X,Y)$ as in (5.1.17), we have

$$I_2 \le \|f\|_{L^\infty(\partial\Omega,d\sigma)} \lim_{\varepsilon\to 0} \int_{|X-Y|<\varepsilon} k_1^M(X,Y)\, d\sigma_g(Y) = 0, \tag{5.1.22}$$

by estimates parallel to (2.4.51). $\square$

A useful notational convention is as follows. Fix a smooth background metric g_o and denote by $d_o(X,Y)$ the geodesic distance between $X, Y \in M$, taken with respect to g_o. We then set

$$\text{P.V.} \int_{\partial\Omega} F(X,Y)\, d\sigma_g(Y) := \lim_{\varepsilon \to 0^+} \int\limits_{\substack{Y \in \partial\Omega \\ d_o(X,Y) > \varepsilon}} F(X,Y)\, d\sigma_g(Y), \qquad X \in \partial\Omega. \tag{5.1.23}$$

Theorem 5.1.3 *Let Ω be a UR domain in a compact, connected Riemannian manifold whose metric tensor is Hölder continuous, let L be given by (5.1.1) with $V \geq 0$ and $V > 0$ on a set of positive measure, and let $\mathcal{D}$ be given by (5.1.3). Then, for $p \in (1,\infty)$, $f \in L^p(\partial\Omega, d\sigma)$,*

$$\|\mathcal{N}(\mathcal{D}f)\|_{L^p(\partial\Omega, d\sigma_g)} \leq C_p \|f\|_{L^p(\partial\Omega, d\sigma_g)}, \tag{5.1.24}$$

and

$$\lim_{\substack{Z \to X \\ Z \in \Gamma(X)}} \mathcal{D}f(Z) = (\tfrac{1}{2}I + K)f(X), \quad \textit{for} \ \ \sigma_g - \textit{a.e.} \ \ X \in \partial\Omega, \tag{5.1.25}$$

where

$$Kf(X) := \text{P.V.} \int\limits_{\partial\Omega} \frac{\partial E}{\partial \nu_Y}(X,Y) f(Y)\, d\sigma_g(Y) \tag{5.1.26}$$

has the property

$$\|Kf\|_{L^p(\partial\Omega, d\sigma_g)} \leq C_p \|f\|_{L^p(\partial\Omega, d\sigma_g)}. \tag{5.1.27}$$

Proof. Most of the claims follow in a straightforward manner by combining Lemma 5.1.2 with the results of §§3–4. In fact, the only remaining task is to verify the limit (5.1.25). Note that

$$\begin{aligned} \mathcal{D}f(Z) &= \int\limits_{\partial\Omega} \langle d_Y E(Z,Y), \nu(Y) \rangle f(Y)\, d\sigma_g(Y) \\ &= \mathcal{E}(\nu f)(z), \end{aligned} \tag{5.1.28}$$

where we have set

$$\mathcal{E}h(Z) := \int\limits_{\partial\Omega} d_Y E(Z,Y) h(Y)\, d\sigma_g(Y). \tag{5.1.29}$$

The same arguments as above bound $\mathcal{N}(\mathcal{E}h)$ and appeal to results of §4 gives

$$\lim_{\substack{Z \to X \\ Z \in \Gamma(X)}} \mathcal{D}f(Z) = A(X)f(X) + Kf(X), \tag{5.1.30}$$

with K as in (5.1.26) and $A(X)$ of the form

$$A(X) = -\frac{1}{2\sqrt{-1}} \langle p(X, n(X)), \nu(X) \rangle, \tag{5.1.31}$$

where $n(X)$ is the unit conormal to $\partial\Omega$ at the point X (in the Euclidean metric, in a local coordinate system) and $p(X,\xi)$ is the principal symbol of the operator on functions on $\mathbb{R}^{n+1}$ with integral kernel $d_Y e_0(X-Y,X)$. This is an $(n+1)$-covector with components

$$p_j(X,\xi) = -\sqrt{-1}\,\frac{\rho(X)}{\sqrt{g(X)}}\,G(X,\xi)^{-1}\xi_j, \quad G(X,\xi) = g^{jk}(X)\xi_j\xi_k, \tag{5.1.32}$$

where $(g^{jk})_{jk}$ is the inverse matrix of $(g_{jk})_{jk}$. Now the unit conormal to $\partial\Omega$ with respect to the metric g_{jk} is given by

$$\nu_j(X) = G(X,n(X))^{-1/2}n_j(X), \qquad \text{for a.e. } X\in\partial\Omega, \tag{5.1.33}$$

and the unit normal to $\partial\Omega$ with respect to this metric is given by

$$\nu^j(X) = g^{jk}(X)\nu_k(X), \qquad \text{for a.e. } X\in\partial\Omega. \tag{5.1.34}$$

Thus, we have

$$\begin{aligned} A(X) &= \frac{1}{2}\frac{\rho(X)}{\sqrt{g(X)}}g^{jk}(X)n_k(X)n_j(X)G(X,n(X))^{-3/2} \\ &= \frac{1}{2}\frac{\rho(X)}{\sqrt{g(X)}}G(X,n(X))^{-1/2} \\ &= \frac{1}{2}, \end{aligned} \tag{5.1.35}$$

the last identity by (5.1.5). $\square$

We now discuss how results of [93] allow one to extend Theorem 5.1.3 to a class of Riemannian manifolds whose metric tensors have a weaker modulus of continuity, namely one satisfying (5.1.9). Indeed, according to Proposition 2.4 of [93], if ω satisfies the Dini condition $\int_0^1 \omega(t)t^{-1}\,dt < \infty$, while $t^{-\alpha}\omega(t) \searrow$ for some $\alpha\in(0,1)$, then one has these estimates on $e_1(Y,X)$:

$$|e_1(Y,X)| \le C\frac{\sigma(|X-Y|)}{|X-Y|^{n-1}}, \quad |\nabla_Y e_1(Y,X)| \le C\frac{\beta(|X-Y|)}{|X-Y|^n} \tag{5.1.36}$$

(given that dim $M = n+1$), where

$$\sigma(h) := \int_0^h \frac{\omega(t)}{t}\,dt, \tag{5.1.37}$$

and $\beta(h)$ is required to satisfy two conditions. The first is $\beta(h)\ge\sigma(h)$. The second is that there exists $\omega_1(h)$ satisfying the Dini condition

$$\int_0^1 \frac{\omega_1(t)}{t}\,dt < \infty, \tag{5.1.38}$$

such that

$$\omega(\rho h) \le \beta(\rho)\omega_1(h), \quad \forall\,\rho,h\in(0,1]. \tag{5.1.39}$$

In such a case, the hypothesis (5.1.13) of Proposition 5.1.1 holds with $\psi(t) = \beta(t)$, so Proposition 5.1.1 applies provided $\int_0^1 \beta(t)t^{-1}\,dt < \infty$.

Now if $\omega(t) = t^\alpha$ with $\alpha \in (0,1)$, then one can take $\sigma(t) = \beta(t) = \omega_1(t) = t^\alpha$. On the other hand, if $\omega(t) = (\log 1/t)^{-\alpha}$ for $t << 1$, one needs $\alpha > 2$. For example, if $0 < a < b$, one can take

$$\begin{aligned} \omega(h) &= \Bigl(\log \frac{1}{h}\Bigr)^{-2-b}, \quad \sigma(h) = \Bigl(\log \frac{1}{h}\Bigr)^{-1-b}, \\ \beta(h) &= \Bigl(\log \frac{1}{h}\Bigr)^{-1-a}, \quad \omega_1(h) = \Bigl(\log \frac{1}{h}\Bigr)^{-1-(b-a)}. \end{aligned} \tag{5.1.40}$$

More generally, if ω satisfies (5.1.9), then (5.1.37)–(5.1.39) hold with $\omega_1(t) = \sqrt{\omega(t)}$, $\sigma(t) \le C\omega_1(t)$, and $\beta(t) = \omega_1(t)$. Thus Proposition 5.1.1 is applicable with $\psi(t) = \beta(t) = \omega_1(t)$, and all the steps in the proof of Theorem 5.1.3 extend to yield:

Theorem 5.1.4 *The results of Theorem 5.1.3 hold whenever the metric tensor on M has a modulus of continuity satisfying (5.1.9).*

We now turn to the single-layer potential

$$\mathcal{S}f(X) := \int\limits_{\partial\Omega} E(X,Y) f(Y)\, d\sigma_g(Y), \quad X \in \Omega, \tag{5.1.41}$$

and its gradient

$$\nabla \mathcal{S}f(X) = \int\limits_{\partial\Omega} \nabla_X E(X,Y) f(Y)\, d\sigma_g(Y), \quad X \in \Omega. \tag{5.1.42}$$

In this case it is convenient to exploit the symmetry $E(X,Y) = E(Y,X)$ and replace (5.1.6) by

$$\sqrt{g(Y)} E(X,Y) = e_0(X-Y,Y) + e_1(X,Y). \tag{5.1.43}$$

In this scenario, the second estimate in (5.1.36) becomes

$$|\nabla_X e_1(X,Y)| \le C \frac{\beta(|X-Y|)}{|X-Y|^n}. \tag{5.1.44}$$

Also, in this case, Theorem 3.5.2 applies to the contribution of $g(Y)^{-1/2} \nabla_X e_0(X-Y,Y)$ to $\nabla \mathcal{S}f(X)$, and the same arguments as used above apply to the analysis of

$$\widetilde{\mathcal{K}}_1 f(X) := \int\limits_{\partial\Omega} \tilde{k}_1(X,Y) f(Y)\, d\sigma_g(Y), \quad \tilde{k}_1(X,Y) := g(Y)^{-1/2} \nabla_X e_1(X,Y). \tag{5.1.45}$$

We obtain the following extension of (3.3.36)–(3.3.39):

Theorem 5.1.5 *In the setting of Theorem 5.1.4, we have, for $p \in (1,\infty)$,*

$$\|\mathcal{N}(\nabla \mathcal{S}f)\|_{L^p(\partial\Omega, d\sigma_g)} \le C_p \|f\|_{L^p(\partial\Omega, d\sigma_g)}. \tag{5.1.46}$$

Also, given $p \in (1,\infty)$ and $f \in L^p(\partial\Omega, d\sigma_g)$, one has for almost all $X \in \partial\Omega$,

$$\lim_{\substack{Z \to X \\ Z \in \Gamma(X)}} \Bigl(\partial_{\nu(X)} \mathcal{S}f\Bigr)(Z) = (-\tfrac{1}{2} I + K^*) f(X), \tag{5.1.47}$$

where K^* is the adjoint of K, given by (5.1.26).

5.2 Compactness of K

We now establish compactness of K on $L^p(\partial\Omega, d\sigma)$ for $p \in (1, \infty)$, when Ω is a regular SKT domain.

Theorem 5.2.1 *Retain the hypotheses of Theorem 5.1.3 (or 5.1.4), but this time assume that Ω is a regular SKT domain. Then the operator K defined in (5.1.26) is compact when acting from $L^p(\partial\Omega, d\sigma_g)$ into itself, whenever $1 < p < \infty$.*

Proof. To get started, we peel off some pieces that are always compact on $L^p(\partial\Omega, d\sigma_g)$ and look at what remains. So far we have

$$K = K^b + K_1, \tag{5.2.1}$$

where K_1, given by (5.1.11), is known to be compact, and (in a local coordinate patch)

$$K^b f(X) = \text{P.V.} \int\limits_{\partial\Omega} \partial_{\nu(Y)} e_0(X - Y, X) f(Y)\, d\sigma_g(Y). \tag{5.2.2}$$

Now, by (5.1.7), setting

$$\Gamma(X, X - Y) := g_{jk}(X)(x_j - y_j)(x_k - y_k), \quad X = (x_i)_i,\ Y = (y_i)_i \in \mathbb{R}^{n+1}, \tag{5.2.3}$$

it follows that, if C_n is as in (5.1.7), we have

$$\partial_{\nu(Y)} e_0(X - Y, X) = -\frac{C_n}{2}(n-1)\Big[\partial_{\nu(Y)} \Gamma(X, X - Y)\Big] \Gamma(X, X - Y)^{-(n+1)/2}, \tag{5.2.4}$$

and, via (5.1.33)–(5.1.34),

$$\begin{aligned} \partial_{\nu(Y)} \Gamma(X, X - Y) &= G(Y, n(Y))^{-1/2} g^{\ell m}(Y) n_\ell(Y) \frac{\partial}{\partial y_m}\big[g_{jk}(X)(x_j - y_j)(x_k - y_k)\big] \\ &= -2G(Y, n(Y))^{-1/2} n_\ell(Y) g^{\ell m}(Y) g_{mk}(X)(x_k - y_k) \\ &= -2G(Y, n(Y))^{-1/2} n^j(Y) g_{jk}(X)(x_k - y_k), \end{aligned} \tag{5.2.5}$$

with $n^j := n_\ell g^{\ell j}$. Hence

$$K^b f(X) = (n-1) C_n\, \text{P.V.} \int\limits_{\partial\Omega} \frac{n^j(Y) g_{jk}(X)(x_k - y_k)}{\Gamma(X, X - Y)^{(n+1)/2}} G(Y, n(Y))^{-1/2} f(Y)\, d\sigma_g(Y). \tag{5.2.6}$$

Next we set

$$K^b = K^\# + K_2, \tag{5.2.7}$$

where K_2 is defined by substituting $g_{jk}(X) - g_{jk}(Y)$ for $g_{jk}(X)$ in (5.2.6). Thus K_2 is compact, by Proposition 5.1.1, and $K^\#$ has the form (5.2.6) with $n^j(Y) g_{jk}(X)$ replaced by $n^j(Y) g_{jk}(Y) = n_k(Y)$, so

$$K^\# f(X) = (n-1) C_n\, \text{P.V.} \int\limits_{\partial\Omega} \frac{n_k(Y)(x_k - y_k)}{\Gamma(X, X - Y)^{(n+1)/2}} G(Y, n(Y))^{-1/2} f(Y)\, d\sigma_g(Y). \tag{5.2.8}$$

Compare this formula with (3.3.1). Now $K^{\#}$ has the form

$$K^{\#} f = TMf, \quad Mf(Y) = G(Y, n(Y))^{-1/2} f(Y), \tag{5.2.9}$$

and Theorem 4.5.4 is directly applicable to T. Since $G(Y, n(Y))^{-1/2}$ is bounded, this yields compactness of $K^{\#}$ and completes the proof of Theorem 5.2.1. $\square$

5.3 Green formulas on Riemannian manifolds

Let $\mathbb{R}^m$ carry a continuous metric tensor (g_{jk}), in addition to the Euclidean metric tensor (δ_{jk}). A vector field $v = v^j \partial_j$ has divergence $\operatorname{div} v \in \mathcal{D}'(\mathbb{R}^m)$ given by

$$\langle \varphi, \operatorname{div} v \rangle = -\langle \partial_j \varphi, g^{1/2} v^j \rangle, \tag{5.3.1}$$

where $g = \det(g_{jk})$, and we use the summation convention. If $\operatorname{div} v$ is a locally integrable multiple of Lebesgue measure, or equivalently of $d\mathcal{V} = g^{1/2}\, dx$, we identify $\operatorname{div} v$ with the density $\operatorname{div} v \, d\mathcal{V}$. We denote by $\operatorname{div}_0 v$ this quantity associated to (δ_{jk}) rather than (g_{jk}), so $\operatorname{div} v = g^{-1/2} \operatorname{div}_0(g^{1/2} v)$, in the locally integrable case.

Let $\Omega \subset \mathbb{R}^m$ be an open set with locally finite perimeter. Assume v belongs to

$$\begin{aligned} \mathcal{D} &= \{ v \in C_0^0(\mathbb{R}^m, \mathbb{R}^m) : \operatorname{div} v \in L^1(\mathbb{R}^m) \} \\ &= \{ v \in C_0^0(\mathbb{R}^m, \mathbb{R}^m) : \operatorname{div}_0(g^{1/2} v) \in L^1(\mathbb{R}^m) \}. \end{aligned} \tag{5.3.2}$$

Then

$$\int_\Omega \operatorname{div} v \, d\mathcal{V} = \int_\Omega \operatorname{div}_0(g^{1/2} v)\, dx. \tag{5.3.3}$$

Hence Green's theorem from §2.2 gives

$$\int_\Omega \operatorname{div} v \, d\mathcal{V} = \int_{\partial^* \Omega} \langle n, v \rangle \, g^{1/2} \, d\sigma, \tag{5.3.4}$$

where n is the outward-pointing unit normal with respect to the metric (δ_{jk}) and σ is the $(m-1)$-dimensional Hausdorff measure defined by (δ_{jk}). We claim that

$$\int_{\partial^* \Omega} \langle n, v \rangle \, g^{1/2} \, d\sigma = \int_{\partial^* \Omega} \langle \nu, v \rangle_g \, d\sigma_g, \tag{5.3.5}$$

where ν is the unit outward-pointing normal determined by (g_{jk}), $\langle \, , \, \rangle_g$ is the inner product determined by (g_{jk}), and σ_g is $(m-1)$-dimensional Hausdorff measure determined by (g_{jk}). The vectors $\nu = \nu^j \partial_j$ and $n = n^j \partial_j$ have associated covectors

$$\nu^b = \sum \nu_j \, dx_j, \quad n^b = \sum n_j \, dx_j, \tag{5.3.6}$$

where

$$\nu_j = g_{jk}\nu^k, \quad n_j = \delta_{jk}n^k. \tag{5.3.7}$$

The covectors ν^b and n^b are parallel and both have unit length, with respect to their associated metric tensors, so

$$n_j = a\nu_j, \quad a^2 = \langle n^b, n^b\rangle_g = g^{jk}n_j n_k. \tag{5.3.8}$$

Hence

$$\begin{aligned} \langle n, v\rangle\, g^{1/2} &= n_j v^j g^{1/2} \\ &= g^{1/2}\langle n^b, n^b\rangle_g^{1/2}\, \nu_j v^j \\ &= g^{1/2}\langle n^b, n^b\rangle_g^{1/2}\, \langle \nu, v\rangle_g. \end{aligned} \tag{5.3.9}$$

Thus (5.3.5) is equivalent to the assertion that

$$\sigma_g = g^{1/2}\langle n^b, n^b\rangle_g^{1/2}\, \sigma \tag{5.3.10}$$

on measurable subsets of $\partial^*\Omega$. The following result establishes (5.3.10).

Proposition 5.3.1 *Let $S \subset \mathbb{R}^m$ be a countably rectifiable $(m-1)$-dimensional set in $\mathbb{R}^m$, with measure-theoretic unit normal n determined by the Euclidean structure. If σ is $(m-1)$-dimensional Hausdorff measure determined by (δ_{jk}) and σ_g is $(m-1)$-dimensional Hausdorff measure determined by (g_{jk}), then (5.3.10) holds on S.*

Proof. It is clear from the definitions that for each compact $K \subset \mathbb{R}^m$ there exists $C_K \in (1, \infty)$ such that

$$C_K^{-1}\sigma(A) \le \sigma_g(A) \le C_K\sigma(A), \quad A \subset K. \tag{5.3.11}$$

The hypothesis of countable rectifiability implies there is a disjoint union

$$S = \bigcup_{k\ge 1} M_k \cup N, \tag{5.3.12}$$

where each M_k is a Borel subset of some $(m-1)$-dimensional C^1 submanifold of $\mathbb{R}^m$, while $\sigma(N) = 0$, hence $\sigma_g(N) = 0$. It is elementary that (5.3.10) holds on each C^1 submanifold of $\mathbb{R}^m$, of dimension $m-1$, so by (5.3.12) it holds on S. $\square$

Since $\partial^*\Omega$ is countably rectifiable, we have the following variant of Proposition 2.2.4.

Proposition 5.3.2 *Let $\mathbb{R}^m$ have a continuous metric tensor (g_{jk}). Let $\Omega \subset \mathbb{R}^m$ be an open set with locally finite perimeter. Then*

$$\int\limits_{\Omega} \operatorname{div} v \, d\mathcal{V} = \int\limits_{\partial^*\Omega} \langle \nu, v \rangle_g \, d\sigma_g, \tag{5.3.13}$$

for all $v \in \mathcal{D}$, defined by (5.3.2).

Similarly we can apply Proposition 2.2.5 to deduce that

$$\int\limits_{\Omega} \operatorname{div}_0(g^{1/2} v) \, dx = \int\limits_{\partial^*\Omega} \langle n, v \rangle \, g^{1/2} \, d\sigma, \tag{5.3.14}$$

whenever Ω has a tame interior approximation and v belongs to

$$\begin{aligned} \widetilde{\mathcal{D}} &= \{v \in C_0^0(\overline{\Omega}, \mathbb{R}^m) : \operatorname{div}_0(g^{1/2} v) \in L^1(\Omega)\} \\ &= \{v \in C_0^0(\overline{\Omega}, \mathbb{R}^m) : \operatorname{div} v \in L^1(\Omega)\}. \end{aligned} \tag{5.3.15}$$

We obtain

Proposition 5.3.3 *If $\mathbb{R}^m$ has a continuous metric tensor (g_{jk}) and $\Omega \subset \mathbb{R}^m$ has a tame interior approximation, then (5.3.13) holds for all $v \in \widetilde{\mathcal{D}}$.*

Also the results of §2.3 together with Proposition 5.3.1 yield the following.

Proposition 5.3.4 *If $g = (g_{jk})$ is a continuous metric tensor on $\mathbb{R}^m$ and $\Omega \subset \mathbb{R}^m$ is a bounded domain with Ahlfors regular boundary, then (5.3.13) holds whenever*

$$\operatorname{div} v \in L^1(\Omega) \ \ \textit{and} \ \ v \in \mathfrak{L}^p, \tag{5.3.16}$$

for some $p > 1$, where

$$\mathfrak{L}^p := \{v \in C(\Omega) : \mathcal{N} v \in L^p(\partial\Omega, d\sigma_g) \ \textit{and} \ \exists \, \text{nontangential limit } v_b, \, \sigma_g\text{–a.e.}\}. \tag{5.3.17}$$

Remark. Using partitions of unity, we can extend the scope of these results to $\Omega \subset M$, where M is a smooth manifold with a continuous metric tensor.

We can apply Proposition 5.3.4 in the following setting. Let M be a compact manifold with a Riemannian metric whose components are continuous with a modulus of continuity ω satisfying

$$\int_0^1 \frac{\sqrt{\omega(t)}}{t} \, dt < \infty. \tag{5.3.18}$$

Let $V \in L^\infty(M)$ satisfy $V \geq 0$ on M and $V > 0$ on a set of positive measure. Then let $E(x, y)$ be the integral kernel of $(\Delta - V)^{-1}$ on $L^2(M)$. Let $\Omega \subset M$ be a connected UR domain. For $f \in L^p(\partial\Omega, d\sigma_g)$, set

$$\mathcal{S}f(x) := \int_{\partial\Omega} E(x,y)f(y)\,d\sigma_g(y), \quad x \in \Omega. \tag{5.3.19}$$

A fundamental result is

$$\|\mathcal{N}(\nabla\mathcal{S}f)\|_{L^p(\partial\Omega,d\sigma_g)} \le C_p\|f\|_{L^p(\partial\Omega,d\sigma_g)}, \quad 1 < p < \infty, \tag{5.3.20}$$

and that nontangential limits of $\nabla\mathcal{S}f$ exist σ_g-a.e. on $\partial\Omega$. In addition,

$$\lim_{y\to x \text{ in } \Gamma_x} \langle \nu(x), \nabla\mathcal{S}f(y)\rangle = \Big(-\tfrac{1}{2} + K^*\Big)f(x), \quad \sigma_g\text{-a.e.}, \tag{5.3.21}$$

where $K^* : L^p(\partial\Omega, d\sigma_g) \to L^p(\partial\Omega, d\sigma_g)$, for $1 < p < \infty$. Also, by Proposition 3.2.5,

$$\|\nabla\mathcal{S}f\|_{L^p(\Omega)} + \|\mathcal{S}f\|_{L^q(\Omega)} \le C\|f\|_{L^2(\partial\Omega,d\sigma_g)} \tag{5.3.22}$$

for some $p, q > 2$, and elementary estimates give $\|\mathcal{N}\mathcal{S}f\|_{L^r(\partial\Omega,d\sigma_g)} \le C\|f\|_{L^2(\partial\Omega,d\sigma_g)}$ for some $r > 2$. Now if we take $f \in L^2(\partial\Omega, d\sigma_g)$ and set

$$u = \mathcal{S}f, \quad v = u\nabla u, \tag{5.3.23}$$

we have

$$\operatorname{div} v = |\nabla u|^2 + u\Delta u = |\nabla u|^2 + Vu^2, \quad \text{on } \Omega. \tag{5.3.24}$$

Thus Proposition 5.3.4 applies to give

$$\int_\Omega (|\nabla u|^2 + Vu^2)\,d\mathcal{V} = \int_{\partial\Omega} u\left(-\tfrac{1}{2}I + K^*\right)f\,d\sigma_g. \tag{5.3.25}$$

5.4 Invertibility of boundary layer potentials

As in §5.1, let Ω be an open subset of a compact, connected manifold M, endowed with a Riemannian metric tensor g whose components have a modulus of continuity satisfying the Dini-type condition (5.1.9). We set $L = \Delta - V$ with bounded $V \ge 0$ on M, $V > 0$ on a set of positive measure, and define the double layer potential $\mathcal{D}$ by (5.1.3) and the single layer potential $\mathcal{S}$ by (5.1.41). By Theorem 5.1.3, if Ω is a UR domain, then, for a.e. $X \in \partial\Omega$,

$$\lim_{Z\to X, Z\in\Gamma(X)} \mathcal{D}f(Z) = \Big(\tfrac{1}{2}I + K\Big)f(X), \tag{5.4.1}$$

and

$$K : L^p(\partial\Omega, d\sigma_g) \longrightarrow L^p(\partial\Omega, d\sigma_g), \quad 1 < p < \infty. \tag{5.4.2}$$

Similarly, if $Z \to X \in \partial\Omega$ from a nontangential region in $\Omega^- := M \setminus \overline{\Omega}$, we have

$$\mathcal{D}f\big|_{\partial\Omega^-} = \Big(-\tfrac{1}{2}I + K\Big)f. \tag{5.4.3}$$

Also, by Theorem 5.1.5 and its counterpart for Ω^-,

$$\partial_\nu \mathcal{S}f\big|_{\partial\Omega^\pm} = \Big(\mp\tfrac{1}{2}I + K^*\Big)f. \tag{5.4.4}$$

We establish invertibility of these operators on $L^p(\partial\Omega, d\sigma_g)$, under certain conditions. Here is our starting point.

Proposition 5.4.1 *Let $\Omega \subset M$ be a connected UR domain. Assume $V > 0$ on a set of positive measure in each connected component of $\Omega^- = M \setminus \overline{\Omega}$. Then*

$$\tfrac{1}{2}I + K^* \text{ is injective on } L^2(\partial\Omega, d\sigma_g). \tag{5.4.5}$$

If $V > 0$ on a set of positive measure in Ω, then also $-\frac{1}{2}I + K^*$ is injective on $L^2(\partial\Omega, d\sigma)$, while

$$V \equiv 0 \text{ on } \Omega \Longrightarrow -\tfrac{1}{2}I + K^* : L^2_0(\partial\Omega, d\sigma_g) \to L^2_0(\partial\Omega, d\sigma_g), \text{ injectively.} \tag{5.4.6}$$

Here and elsewhere, $L^2_0(\partial\Omega, d\sigma_g)$ consists of functions in $L^2(\partial\Omega, d\sigma_g)$ that integrate to 0 on $\partial\Omega$ with respect to σ_g.

Proof. We have finally assembled all the tools needed to make the standard argument work. Here it is. Assume $f \in L^2(\partial\Omega, d\sigma_g)$ is annihilated by $\frac{1}{2}I + K^*$, and set

$$u := \mathcal{S}f. \tag{5.4.7}$$

Then we can apply the Green formula (5.3.25), with Ω replaced by $\Omega^- = M \setminus \overline{\Omega}$, and hence with $-\frac{1}{2}I + K^*$ replaced by $\frac{1}{2}I + K^*$. We have

$$\int_{\Omega^-} (|\nabla u|^2 + Vu^2)\, d\mathcal{V} = \int_{\partial\Omega} u\Big(\tfrac{1}{2}I + K^*\Big)f\, d\sigma_g = 0. \tag{5.4.8}$$

Hence $|\nabla u| \equiv 0$ on Ω^-, so u is constant on each connected component of Ω^-, and the hypothesis on V implies that each such constant must be 0. Thus $u \equiv 0$ on $M \setminus \overline{\Omega}$. Hence u has zero boundary trace on $\partial\Omega$ from inside Ω, so

$$\int_{\Omega} (|\nabla u|^2 + Vu^2)\, d\mathcal{V} = \int_{\partial\Omega} u\Big(-\tfrac{1}{2}I + K^*\Big)f\, d\sigma_g = 0. \tag{5.4.9}$$

Hence u is constant on Ω and since u does not jump across $\partial\Omega$, because its trace from both sides is Sf, the constant must be 0, so $u \equiv 0$ on M. Thus, by (5.4.4), $(\frac{1}{2}I + K^*)f = 0 = (-\frac{1}{2}I + K^*)f$, so $f = 0$. This proves (5.4.5).

The proof that $-\frac{1}{2}I + K^*$ is injective on $L^2(\partial\Omega, d\sigma_g)$ if $V > 0$ on a set of positive measure in Ω is the same. As for (5.4.6), given $f \in L^2_0(\partial\Omega, \sigma_g)$, again define u by (5.4.7). Applying the Green formula to $v = \nabla u$ gives, via (5.3.21),

$$\int_{\partial\Omega} \Big(-\tfrac{1}{2}I + K^*\Big) f \, d\sigma_g = \int_{\Omega} \Delta u \, d\mathcal{V} = 0, \tag{5.4.10}$$

so indeed $-\frac{1}{2}I + K^*$ maps $L^2_0(\partial\Omega, \sigma_g)$ to itself. Next, if $f \in L^2_0(\partial\Omega, \sigma_g)$ is annihilated by $-\frac{1}{2}I + K^*$, then (5.4.9) applies, and we deduce that u is constant on Ω; say $u \equiv a$ on Ω. Now we have $(\frac{1}{2}I + K^*)f = f$, so

$$\begin{aligned} \int_{\Omega^-} (|\nabla u|^2 + Vu^2) \, d\mathcal{V} &= \int_{\partial\Omega} u \Big(\tfrac{1}{2}I + K^*\Big) f \, d\sigma_g \\ &= a \int_{\partial\Omega} f \, d\sigma_g = 0. \end{aligned} \tag{5.4.11}$$

This forces $u \equiv 0$ on Ω^-. Since u does not jump across $\partial\Omega$, this forces $a = 0$, so $u \equiv 0$ on M. As before, this forces $f = 0$. $\square$

Now for the invertibility result.

Proposition 5.4.2 *Let $\Omega \subset M$ be a connected, regular SKT domain. Assume as usual that $V \geq 0$ is bounded on M, and $V > 0$ on a set of positive measure on each connected component of $\Omega^- = M \setminus \overline{\Omega}$. Then*

$$\tfrac{1}{2}I + K, \ \tfrac{1}{2}I + K^* : L^p(\partial\Omega, d\sigma_g) \to L^p(\partial\Omega, d\sigma_g), \ \textit{isomorphically}, \ \forall\, p \in (1, \infty). \tag{5.4.12}$$

If $V > 0$ on a set of positive measure in Ω, then $-\frac{1}{2}I + K$ and $-\frac{1}{2} + K^*$ are isomorphisms on $L^p(\partial\Omega, d\sigma_g)$ for each $p \in (1, \infty)$, while

$$V \equiv 0 \ \textit{on} \ \Omega \Longrightarrow -\tfrac{1}{2}I + K^* : L^p_0(\partial\Omega, \sigma_g) \to L^p_0(\partial\Omega, \sigma_g), \ \textit{isomorphically}, \tag{5.4.13}$$

for each $p \in (1, \infty)$.

Proof. In this setting, Proposition 5.4.1 applies, and we also know that K and K^* are compact on $L^p(\partial\Omega, d\sigma_g)$ for each $p \in (1, \infty)$, so all the operators in (5.4.12) and (5.4.13) are Fredholm of index zero. By Proposition 5.4.1, $\frac{1}{2}I + K^*$ is injective on $L^2(\partial\Omega, d\sigma_g)$, hence on $L^p(\partial\Omega, d\sigma_g)$ for each $p \in [2, \infty)$. Thus $\frac{1}{2}I + K^*$ is invertible on $L^p(\partial\Omega, d\sigma_g)$ for each $p \in [2, \infty)$, so $\frac{1}{2}I + K$ is invertible on $L^p(\partial\Omega, d\sigma_g)$ for each $p \in (1, 2]$, hence injective on $L^p(\partial\Omega, d\sigma_g)$ for each $p \in (1, \infty)$, and hence invertible on $L^p(\partial\Omega, d\sigma_g)$ for each $p \in (1, \infty)$, and we have (5.4.12). Similar arguments yield the rest of Proposition 5.4.2. $\square$

5.5 The Dirichlet and Neumann problems

We can now apply the results of §5.4 to the Dirichlet and Neumann problems. We make the standing assumption that Ω is a connected regular SKT domain in M, a compact manifold with a Riemannian metric tensor whose components satisfy the Dini-type condition (5.1.9). We set $L = \Delta - V$ and assume $V \in L^\infty(M)$ is ≥ 0 and that $V > 0$ on a set of positive measure on each connected component of $M \setminus \overline{\Omega}$. Since we are studying $Lu = 0$ on Ω, we can alter V at will off Ω, so there is no loss of generality in making this last assumption. Here is our result.

Theorem 5.5.1 *Given $p \in (1,\infty)$, take*

$$f, g \in L^p(\partial\Omega, d\sigma_g). \tag{5.5.1}$$

Then the Dirichlet problem

$$Lu = 0 \text{ on } \Omega, \quad u\big|_{\partial\Omega} = f \tag{5.5.2}$$

has a unique solution $u \in C^1(\Omega)$ satisfying

$$\|\mathcal{N}u\|_{L^p(\partial\Omega,d\sigma_g)} \leq C\|f\|_{L^p(\partial\Omega,d\sigma_g)}. \tag{5.5.3}$$

If $V > 0$ on a set of positive measure on Ω, the Neumann problem

$$Lu = 0 \text{ on } \Omega, \quad \partial_\nu u\big|_{\partial\Omega} = g \tag{5.5.4}$$

has a unique solution $u \in C^1(\Omega)$ satisfying

$$\|\mathcal{N}\nabla u\|_{L^p(\partial\Omega,d\sigma_g)} \leq C\|g\|_{L^p(\partial\Omega,d\sigma_g)}. \tag{5.5.5}$$

If $V \equiv 0$ on Ω, then, provided $g \in L^p_0(\partial\Omega, \sigma_g)$, (5.5.4) has a solution satisfying (5.5.5), unique up to an additive constant.

Proof. For existence in (5.5.2), we take

$$u = \mathcal{D}\Big(\tfrac{1}{2}I + K\Big)^{-1} f, \tag{5.5.6}$$

and for existence in (5.5.4) we take

$$u = \mathcal{S}\Big(-\tfrac{1}{2}I + K^*\Big)^{-1} g. \tag{5.5.7}$$

The respective properties (5.5.3) and (5.5.5) then follow from the results of §5.1.

As for uniqueness, the argument given in §7.1 applies here with only minor modifications, so to avoid redundancy we refer the reader to §7.1. □

5.6 Extensions to ε-regular SKT domains

We extend the results of §5.2 and §§5.4–5.5 from the setting of regular SKT domains to the setting of ε-regular SKT domains. Take M to be a compact manifold, with a Riemannian metric tensor g, satisfying a Dini-type condition as in §5.1. Take $L = \Delta - V$ and K as before. The following replaces Theorem 5.2.1.

Proposition 5.6.1 *Let $\Omega \subset M$ be a domain, satisfying a two-sided John condition, with Ahlfors regular boundary. Fix $p \in (1, \infty)$. For each $\delta > 0$, the following holds. Let $G(\Omega)$ denote the geometrical characteristics of Ω (consisting in this case of the Ahlfors regularity, UR, and John constants, measured with respect to some coordinate chart). There exists $\varepsilon = \varepsilon\big(G(\Omega), M, g, p, \delta\big) > 0$ such that*

$$\Omega \textit{ is an } \varepsilon\textit{-regular SKT domain} \Longrightarrow \operatorname{dist}(K, \operatorname{Cp}(L^p(\partial\Omega, d\sigma_g)) < \delta. \tag{5.6.1}$$

Proof. As in the proof of Theorem 5.2.1, $K = K^\# + K_1 + K_2$ where, by Proposition 5.1.1, K_1 and K_2 are compact. Furthermore, $K^\# = TM$ as in (5.2.9), and Theorem 4.5.4 applies to T, to yield (5.6.1). □

The injectivity results of Proposition 5.4.1 remain at our disposal. We can now extend Proposition 5.4.2.

Proposition 5.6.2 *Let $\Omega \subset M$ be as in Proposition 5.6.1 and assume also that Ω is connected. Assume $V > 0$ on a set of positive measure on each connected component of $\Omega^- = M \setminus \overline{\Omega}$. Take $q \in (1, 2]$. Assume (cf. notation from Proposition 5.6.1)*

$$\varepsilon' \leq \inf_{q \leq p \leq q'} \varepsilon\Big(G(\Omega), M, g, p, \tfrac{1}{2}\Big) \tag{5.6.2}$$

and that Ω is an ε'-regular SKT domain. Then, for each $p \in [q, q']$,

$$\tfrac{1}{2}I + K : L^p(\partial\Omega, d\sigma_g) \to L^p(\partial\Omega, d\sigma_g), \quad \tfrac{1}{2}I + K^* : L^p(\partial\Omega, d\sigma_g) \to L^p(\partial\Omega, d\sigma_g), \tag{5.6.3}$$

isomorphically. If $V > 0$ on a set of positive measure in Ω, then $-\frac{1}{2}I + K$ and $-\frac{1}{2}I + K^*$ are isomorphisms on $L^p(\partial\Omega, d\sigma_g)$, while

$$V \equiv 0 \text{ on } \Omega \Longrightarrow -\tfrac{1}{2}I + K^* : L^p_0(\partial\Omega, d\sigma_g) \to L^p_0(\partial\Omega, d\sigma_g), \text{ isomorphically.} \tag{5.6.4}$$

Proof. The hypotheses yield

$$\operatorname{dist}\big(K, \operatorname{Cp}\left(L^p(\partial\Omega), d\sigma_g\right)\big) < \tfrac{1}{2}. \tag{5.6.5}$$

Hence the operators $\pm\frac{1}{2}I + K$ and $\pm\frac{1}{2}I + K^*$ are Fredholm on $L^p(\partial\Omega, d\sigma_g)$, of index zero. The arguments used in Proposition 5.4.2 finish the proof, except that now we work on $p \in [q, q']$ rather than $p \in (1, \infty)$. □

From here, we can extend Theorem 5.5.1, as follows.

Theorem 5.6.3 *Take Ω as in Proposition 5.6.2, $q \in (1, 2]$, $p \in [q, q']$, and ε as in (5.6.2). Take $f, g \in L^p(\partial\Omega, d\sigma_g)$. Then the Dirichlet problem (5.5.2) has a unique solution $u \in C^1(\Omega)$ satisfying (5.5.3). If $V > 0$ on a set of positive measure on Ω, the Neumann problem (5.5.4) has a unique solution $u \in C^1(\Omega)$ satisfying (5.5.5). If $V \equiv 0$ on Ω, then, provided $g \in L^p_0(\partial\Omega, d\sigma_g)$, (5.5.4) has a solution satisfying (5.5.5), unique up to an additive constant.*

Proof. We can appeal to Proposition 5.6.2 to write solutions in the form (5.5.6) and (5.5.7). Uniqueness again follows from arguments that will be given in §7. □

6 Second order elliptic systems on regular SKT domains: set-up

In this section we apply the methods of §§2–4 to a variety of second order elliptic systems: the Lamé system of linear elasticity, the Stokes system for steady fluid flows, and the Maxwell system for time-harmonic electromagnetic fields. We also consider various boundary problems for the scalar Laplace operator, complementing and supplementing results of §5. Unlike in §5, we restrict attention to constant-coefficient equations in Euclidean space in this section.

Section 6.1 is devoted to a more detailed description of the various boundary problems to be studied for the Lamé system, Stokes system, and Maxwell system, and the various layer potential operators that arise to solve these boundary problems. Some of these layer potentials conform to the form (4.5.3), yielding compactness for regular SKT domains and small norm modulo compacts for ε-regular SKT domains, and some do not; further techniques will be brought to bear on these.

Section 6.2 derives compactness results on L^p-Sobolev spaces $L^p_1(\partial\Omega, d\sigma)$, for regular SKT domains, which complement our compactness results on $L^p(\partial\Omega, d\sigma)$. In addition, there are Fredholm results on $L^p_1(\partial\Omega, d\sigma)$, obtained when $\mathrm{dist}(\nu, \mathrm{VMO}(\partial\Omega, d\sigma))$ is small.

Section 6.3 studies the invertibility of various double layer potential operators. When they have the form $\lambda I + K$ with K either compact or of norm modulo compacts $< |\lambda|$, the crux is to establish injectivity, and separate techniques are involved in the various cases. This yields results for regular SKT domains, of the form that $\lambda I + K$ is invertible on $L^p(\partial\Omega, d\sigma)$ for each $p \in (1, \infty)$. For the more general class of ε-regular SKT domains, we obtain invertibility for a range of p, depending on how small ε is.

Section 6.4 studies the invertibility of single layer potentials, typically from $L^p(\partial\Omega, d\sigma) \to L^p_1(\partial\Omega, d\sigma)$ or $L^p_{-1}(\partial\Omega, d\sigma) \to L^p(\partial\Omega, d\sigma)$, or some variant. Section 6.5 studies the invertibility of the magnetostatic layer potential. This is a double layer potential, but the invertibility results have a significantly different flavor from those of §6.3.

6.1 Examples

To illustrate the scope of the analysis pertaining to operators of the form (4.5.2)-(4.5.3), which was carried out in §4.5, here we shall discuss in detail three examples, namely integral operators arising in the study of linear elasticity, in the study of the Stokes system for hydrostatics, and in the study of the time-harmonic Maxwell system.

Our first example comes from linear elasticity problems on domains in $\mathbb{R}^{n+1}$. Specifically, let

$$\mu > 0 \quad \text{and} \quad \lambda > -\frac{2\mu}{n+1}, \tag{6.1.1}$$

and, for a fixed, arbitrary parameter $r \in \mathbb{R}$, set (using the standard δ-Kronecker formalism)

$$a_{jk}^{\alpha\beta}(r) := \mu\delta_{jk}\delta_{\alpha\beta} + (\mu+\lambda-r)\,\delta_{j\alpha}\delta_{k\beta} + r\,\delta_{j\beta}\delta_{k\alpha}. \tag{6.1.2}$$

Then for any vector field $\vec{u} = (u_\alpha)_{1\leq\alpha\leq n+1}$ and any $\alpha = 1, ..., n+1$ we have (using the repeated index summation convention)

$$\partial_j\left(a_{jk}^{\alpha\beta}(r)\partial_k u_\beta\right) = \mu\,\Delta u_\alpha + (\mu+\lambda)\partial_\alpha(\operatorname{div}\vec{u}), \tag{6.1.3}$$

i.e., the α-component of the Lamé operator $\mu\,\Delta+(\mu+\lambda)\nabla\operatorname{div}$ acting on $\vec{u}$. The conormal derivative associated with the above choice of coefficients in the writing of the Lamé operator is then given by

$$\partial_\nu^r\vec{u} := \left(\nu_j a_{jk}^{\alpha\beta}(r)\partial_k u_\beta\right)_\alpha = [\mu(\nabla\vec{u})^\top + r(\nabla\vec{u})]\Big|_{\partial\Omega}\nu + (\mu+\lambda-r)(\operatorname{div}\vec{u})\Big|_{\partial\Omega}\nu, \tag{6.1.4}$$

where the superscript $\top$ denotes transposition.

The approach to solving the Dirichlet problem

$$\begin{cases} \mu\,\Delta\vec{u} + (\mu+\lambda)\nabla\operatorname{div}\vec{u} = 0 \quad \text{in } \Omega, \\ \vec{u}\Big|_{\partial\Omega} = \vec{f} \in \left[L^p(\partial\Omega, d\sigma)\right]^{n+1}, \\ \mathcal{N}(\vec{u}) \in L^p(\partial\Omega, d\sigma), \end{cases} \tag{6.1.5}$$

via the method of boundary integral operators proceeds as follows. Recall that ω_n denotes the surface measure of the unit sphere in $\mathbb{R}^{n+1}$, and let $E(X) = (E_{jk}(X))_{1\leq j,k\leq n+1}$ be the standard fundamental solution for the Lamé system, defined at each $X = (x_j)_j \in \mathbb{R}^{n+1}\setminus\{0\}$ by

$$E_{jk}(X) := \begin{cases} \dfrac{-1}{2\mu(2\mu+\lambda)\omega_n}\left[\dfrac{3\mu+\lambda}{n-1}\dfrac{\delta_{jk}}{|X|^{n-1}} + \dfrac{(\mu+\lambda)x_jx_k}{|X|^{n+1}}\right], & \text{if } n\geq 2, \\ \dfrac{1}{2\pi\mu(2\mu+\lambda)}\left[(3\mu+\lambda)\delta_{jk}\log|X| - \dfrac{(\mu+\lambda)x_jx_k}{|X|^2}\right], & \text{if } n=1. \end{cases} \tag{6.1.6}$$

See, e.g., [71] and (9.2) in Chapter 9 of [70]. Assuming that $-\mu\leq r\leq\mu$, we then define the elastic double layer potential operator $\mathcal{D}_r$ by setting

$$\mathcal{D}_r\vec{g}(X) := \int_{\partial\Omega}[\partial_{\nu(Y)}^r E(Y-X)]^\top\vec{g}(Y)\,d\sigma(Y), \qquad X\in\Omega, \tag{6.1.7}$$

for each $\vec{g}\in\left[L^p(\partial\Omega, d\sigma)\right]^{n+1}$. Assuming that the domain Ω is reasonable, we seek a solution to (6.1.5) in the form $\vec{u} = \mathcal{D}_r\vec{g}$ for a suitable $\vec{g}\in\left[L^p(\partial\Omega, d\sigma)\right]^{n+1}$, in which case it is useful to know that

$$\mathcal{D}_r\vec{g}\Big|_{\partial\Omega} = (\tfrac{1}{2}I + K_r)\vec{g}, \tag{6.1.8}$$

where I denotes the identity operator, and

$$K_r\vec{g}(X) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} [\partial^r_{\nu(Y)} E(Y-X)]^\top \vec{g}(Y)\, d\sigma(Y), \qquad X \in \partial\Omega. \tag{6.1.9}$$

Explicitly, the integral kernel of the operator (6.1.9) is a $(n+1)\times(n+1)$ matrix whose (j,k) entry is given by

$$\begin{aligned} -L_1(r)\frac{\delta_{jk}}{\omega_n}\frac{\langle X-Y,\nu(Y)\rangle}{|X-Y|^{n+1}} \quad &- \quad (1-L_1(r))\frac{n+1}{\omega_n}\frac{\langle X-Y,\nu(Y)\rangle(x_j-y_j)(x_k-y_k)}{|X-Y|^{n+3}} \\ &- \quad L_2(r)\frac{1}{\omega_n}\frac{(x_j-y_j)\nu_k(Y)-(x_k-y_k)\nu_j(Y)}{|X-Y|^{n+1}}, \end{aligned} \tag{6.1.10}$$

where

$$L_1(r) := \frac{\mu(3\mu+\lambda)-r(\mu+\lambda)}{2\mu(2\mu+\lambda)}, \qquad L_2(r) := \frac{\mu(\mu+\lambda)-r(3\mu+\lambda)}{2\mu(2\mu+\lambda)}. \tag{6.1.11}$$

It is here that the usefulness of making a judicious choice for the parameter r is most apparent. Specifically, for

$$r := \frac{\mu(\mu+\lambda)}{3\mu+\lambda} \tag{6.1.12}$$

we have $L_2(r) = 0$ and, hence, the last term in (6.1.10) drops out. Consequently, the operator (6.1.10) corresponding to the choice (6.1.12), referred to in the literature as the *pseudo-stress elastic double layer* (cf., e.g., [70]), takes the form (4.5.3). We shall denote this operator by K_ψ.

Another particular conormal derivative which has received a lot of attention is the so-called *traction* which corresponds to (6.1.4) written for $r = \mu$, i.e.,

$$\partial^\mu_\nu \vec{u} = \mu[\nabla\vec{u} + \nabla\vec{u}^\top]\Big|_{\partial\Omega}\nu + \lambda(\operatorname{div}\vec{u})\Big|_{\partial\Omega}\nu. \tag{6.1.13}$$

The operator (6.1.9)–(6.1.10) written for $r = \mu$ is called the *traction elastic double layer* and is denoted by K_{trac}. As is apparent from (6.1.10), K_{trac} *fails* to be of the form (4.5.3).

Our second example pertains to the Stokes system of hydrostatics. In this case, for a given, sufficiently nice domain $\Omega \subset \mathbb{R}^{n+1}$ and $1 < p < \infty$, the Dirichlet problem for the velocity field $\vec{u}$ and the scalar pressure π reads

$$\begin{cases} \Delta\vec{u} - \nabla\pi = 0 \ \text{ in } \ \Omega, \\ \operatorname{div}\vec{u} = 0 \ \text{ in } \ \Omega, \\ \vec{u}\Big|_{\partial\Omega} = \vec{f} \in L^p_\nu(\partial\Omega, d\sigma), \\ \mathcal{N}(\vec{u}) \in L^p(\partial\Omega, d\sigma), \end{cases} \tag{6.1.14}$$

where, for each $1 < p < \infty$, we have set

$$L^p_\nu(\partial\Omega, d\sigma) := \Big\{\vec{f} \in \Big[L^p(\partial\Omega, d\sigma)\Big]^{n+1} : \int_{\partial\Omega} \langle \nu, \vec{f}\rangle \, d\sigma = 0\Big\}. \tag{6.1.15}$$

In order to implement the method of layer potentials for this problem, for each fixed parameter $\gamma \in \mathbb{R}$, consider the coefficients

$$a^{\alpha\beta}_{jk}(\gamma) := \delta_{jk}\delta_{\alpha\beta} + \gamma\, \delta_{j\beta}\delta_{k\alpha}, \tag{6.1.16}$$

and note that for every $\alpha = 1, ..., n+1$,

$$\partial_j \Big(a^{\alpha\beta}_{jk}(\gamma)\partial_k u_\beta \Big) = \Delta u_\alpha + \gamma \partial_\alpha (\operatorname{div} \vec{u}). \tag{6.1.17}$$

In particular,

$$\operatorname{div} \vec{u} = 0 \Longrightarrow \partial_j \Big(a^{\alpha\beta}_{jk}(\gamma)\partial_k u_\beta \Big) = (\Delta \vec{u})_\alpha, \qquad \alpha = 1, ..., n+1. \tag{6.1.18}$$

For an arbitrary, given pair $(\vec{u}, \pi)$, where $\vec{u}$ is a divergence-free field and π is a scalar function, define the conormal derivative associated with (6.1.16), (6.1.17) by

$$\partial^\gamma_\nu(\vec{u}, \pi) := [(\nabla \vec{u})^\top + \gamma(\nabla \vec{u})]\Big|_{\partial\Omega} \nu - \pi\Big|_{\partial\Omega} \nu. \tag{6.1.19}$$

Going further, let $E(X) = (E_{jk}(X))_{1 \le j,k \le n+1}$ be the canonical matrix-valued fundamental solution for the Stokes system, where

$$E_{jk}(X) := \begin{cases} -\dfrac{1}{2\omega_n}\left(\dfrac{1}{n-1}\dfrac{\delta_{jk}}{|X|^{n-1}} + \dfrac{x_j x_k}{|X|^{n+1}}\right), & \text{if } \ n \ge 2, \\ -\dfrac{1}{4\pi}\left(\delta_{jk}\log|X| + \dfrac{x_j x_k}{|X|^2}\right), & \text{if } \ n = 1, \end{cases} \qquad X = (x_j)_j \in \mathbb{R}^{n+1} \setminus \{0\}, \tag{6.1.20}$$

and the corresponding pressure vector

$$\vec{q}(X) := -\frac{1}{\omega_n}\frac{X}{|X|^{n+1}}, \qquad X \in \mathbb{R}^{n+1} \setminus \{0\}. \tag{6.1.21}$$

For each $X \in \mathbb{R}^{n+1} \setminus \{0\}$, these functions satisfy

$$\partial_k E_{jk}(X) = 0 \ \text{ for } \ 1 \le j \le n+1 \ \text{ and } \ \partial_j E_{jk}(X) = 0 \ \text{ for } \ 1 \le k \le n+1, \tag{6.1.22}$$

$$\Delta E_{jk}(X) = \Delta E_{kj}(X) = \partial_k q_j(X) = \partial_j q_k(X) \quad \text{for} \quad 1 \le j,k \le n+1. \tag{6.1.23}$$

Now, assume that $-1 < \gamma \le 1$ and, for each $\vec{g} \in L^p_\nu(\partial\Omega, d\sigma)$ define the hydrostatic double layer potential operator $\mathcal{D}_\gamma$ by

$$\mathcal{D}_\gamma \vec{g}(X) \;\; := \;\; \int\limits_{\partial\Omega} [\partial^\gamma_{\nu(Y)}(E,\vec{q})(Y-X)]^\top \vec{g}(Y)\,d\sigma(Y), \qquad X\in\Omega, \tag{6.1.24}$$

where, in this context, $\partial^\gamma_{\nu(Y)}$ is applied to each pair consisting of the j-th column in $E(Y-X)$ and the j-th component of $\vec{q}(Y-X)$, i.e.

$$(\partial^\gamma_\nu(E,\vec{q}))_{jk} = \nu_\alpha\partial_\alpha E_{kj} + \gamma\nu_\alpha\partial_k E_{\alpha j} - q_j\nu_k. \tag{6.1.25}$$

Let us also define the corresponding potential for the pressure by setting

$$\mathcal{P}\vec{g}(X) \;\; := \;\; \int\limits_{\partial\Omega} \langle \partial_{\nu(Y)}\vec{q}(Y-X)\,,\,\vec{g}(Y)\rangle\,d\sigma(Y), \qquad X\in\Omega, \tag{6.1.26}$$

where the normal derivative is applied component-wise. We seek a solution for (6.1.14) in the form $\vec{u} = \mathcal{D}_\gamma\vec{g}$, $\pi = \mathcal{P}\vec{g}$, for a suitable $\vec{g}\in L^p_\nu(\partial\Omega, d\sigma)$. It is then of interest to know that

$$\mathcal{D}_\gamma\vec{g}\Big|_{\partial\Omega} = (\tfrac{1}{2}I + K_\gamma)\vec{g}, \tag{6.1.27}$$

where

$$K_\gamma\vec{g}(X) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}} [\partial^\gamma_{\nu(Y)}(E,\vec{q})(Y-X)]^\top \vec{g}(Y)\,d\sigma(Y), \qquad X\in\partial\Omega. \tag{6.1.28}$$

The integral kernel of the operator (6.1.28) is a $(n+1)\times(n+1)$ matrix whose (j,k) entry is

$$\begin{aligned} -(1-\gamma)\frac{\delta_{jk}}{\omega_n}\frac{\langle X-Y,\nu(Y)\rangle}{|X-Y|^{n+1}} \quad &- \quad (1+\gamma)\frac{n+1}{\omega_n}\frac{\langle X-Y,\nu(Y)\rangle(x_j-y_j)(x_k-y_k)}{|X-Y|^{n+3}} \\ &- \quad (1-\gamma)\frac{1}{\omega_n}\frac{(x_j-y_j)\nu_k(Y)-(x_k-y_k)\nu_j(Y)}{|X-Y|^{n+1}}. \end{aligned} \tag{6.1.29}$$

For $\gamma = 1$, in which case the operator (6.1.28) is known as the *slip hydrostatic double layer* (cf., e.g., [71]), the last term in (6.1.29) vanishes. Thus, for this particular choice of the parameter γ, the operator (6.1.28) becomes of the type (4.5.3).

Our last example concerns the time-harmonic Maxwell's equations with wave number $k\in\mathbb{C}$, $\operatorname{Im} k \geq 0$, in a domain $\Omega\subset\mathbb{R}^3$:

$$\operatorname{curl}\vec{E} + ik\vec{H} = 0 \quad \text{and} \quad \operatorname{curl}\vec{H} - ik\vec{E} = 0 \quad \text{in } \Omega, \qquad \nu\times\vec{E}\Big|_{\partial\Omega} = \vec{f} \text{ on } \partial\Omega. \tag{6.1.30}$$

Eliminating $\vec{H}$ then leads us to considering

$$\begin{cases} (\Delta + k^2)\vec{E} = 0 \text{ in } \Omega, \\ \operatorname{div} \vec{E} = 0 \text{ in } \Omega, \\ \mathcal{N}(\vec{E}) \in L^p(\partial\Omega, d\sigma), \\ \nu \times \vec{E}\Big|_{\partial\Omega} = \vec{f} \in L^p_{\tan}(\partial\Omega, d\sigma), \end{cases} \tag{6.1.31}$$

where $1 < p < \infty$ and

$$L^p_{\tan}(\partial\Omega, d\sigma) := \Big\{ \vec{f} = (f_1, f_2, f_3) : \ f_j \in L^p(\partial\Omega, d\sigma), \ j = 1, 2, 3 \text{ and } \langle \nu, \vec{f} \rangle = 0 \text{ a.e. on } \partial\Omega \Big\} \tag{6.1.32}$$

stands for the space of vector fields which are tangential to $\partial\Omega$, with p-th power integrable components. In this setting, the method of layer potentials consists of looking for a solution $\vec{E}$ for (6.1.31) in the form

$$\vec{E}(X) := \operatorname{curl} \int_{\partial\Omega} \Phi_k(X - Y)\vec{g}(Y)\, d\sigma(Y), \qquad X \in \Omega, \tag{6.1.33}$$

where $\vec{g} \in L^p_{\tan}(\partial\Omega, d\sigma)$ is yet to be determined and Φ_k is the canonical radial fundamental solution for the Helmholtz operator $\Delta + k^2$ in $\mathbb{R}^3$, i.e.

$$\Phi_k(X) := -\frac{e^{\sqrt{-1}\,k|X|}}{4\pi|X|}, \qquad X \in \mathbb{R}^3 \setminus \{0\}. \tag{6.1.34}$$

In particular, $\Phi_0(X) = -\frac{1}{4\pi|X|}$ is the usual fundamental solution for the Laplacian in $\mathbb{R}^3$. Then, if Ω is sufficiently nice, we have the following trace formula

$$\nu \times \vec{E}\Big|_{\partial\Omega} = (-\tfrac{1}{2}I + M_k)\vec{g}, \tag{6.1.35}$$

where

$$M_k\vec{g}(X) := \lim_{\varepsilon \to 0^+} \int\limits_{\substack{Y \in \partial\Omega \\ |X-Y|>\varepsilon}} \nu(X) \times \operatorname{curl}_X \{\Phi_k(X - Y)\vec{g}(Y)\}\, d\sigma(Y), \qquad X \in \partial\Omega, \tag{6.1.36}$$

is the so-called *magneto-static layer potential* (cf., e.g., [24]). The elementary algebraic identity

$$\vec{a} \times (\vec{b} \times \vec{c}) = -\langle \vec{a}, \vec{b} \rangle\, \vec{c} + \langle \vec{a}, \vec{c} \rangle\, \vec{b}, \qquad \forall\, \vec{a},\, \vec{b},\, \vec{c} \in \mathbb{R}^3, \tag{6.1.37}$$

plus the fact that $\langle \nu(Y), \vec{g}(Y) \rangle = 0$ for a.e. $Y \in \partial\Omega$, allow us to express the integrand in $M_k\vec{g}(X)$ in the form

$$\begin{aligned} \nu(X) \times \Big((\nabla\Phi_k)(X - Y) \times \vec{g}(Y) \Big) \quad &= \quad -\partial_{\nu(X)}\Phi_k(X - Y)\vec{g}(Y) \\ &\qquad +\langle \nu(X) - \nu(Y), \vec{g}(Y) \rangle (\nabla\Phi_k)(X - Y), \\ &=: \quad k_1(X, Y)\vec{g}(Y) + k_2(X, Y)\vec{g}(Y). \end{aligned} \tag{6.1.38}$$

Now, $k_1(X,Y)$ can be decomposed further as

$$k_1(X,Y) = -\partial_{\nu(X)}\Phi_0(X-Y) + \partial_{\nu(X)}\Big[\Phi_0(X-Y) - \Phi_k(X-Y)\Big], \tag{6.1.39}$$

where the first term in the right hand-side of (6.1.39) is of the type (4.5.3) (in fact, up to a sign, this is the kernel of the adjoint harmonic double layer), and the second one is a bounded function on $\partial\Omega \times \partial\Omega$, thus giving rise to a compact operator on $L^p(\partial\Omega, d\sigma)$. Finally, $k_2(X,Y)\vec{g}(Y)$ can be written as

$$\begin{aligned} k_2(X,Y)\vec{g}(Y) &= \langle \nu(X) - \nu(Y), \vec{g}(Y)\rangle (\nabla\Phi_0)(X-Y) \\ &\quad + \langle \nu(X) - \nu(Y), \vec{g}(Y)\rangle \Big[(\nabla\Phi_0)(X-Y) - (\nabla\Phi_k)(X-Y)\Big] \end{aligned} \tag{6.1.40}$$

where the structure of the first term is that of a commutator between a nice singular integral and the operator of multiplication by ν, whereas the expression in the brackets is a bounded function on $\partial\Omega \times \partial\Omega$, hence once again giving rise to a compact operator on $L^p(\partial\Omega, d\sigma)$. In particular, for this first term in the right hand-side of (6.1.40) the homogeneous space version of the commutator theorem of Coifman-Rochberg-Weiss applies (see Theorem 2.4.2 and Theorem 2.4.5).

6.2 Compactness of layer potential operators on Sobolev spaces

Consider a differential operator L as in (3.6.20)-(3.6.21) and, as before, let $E = (E_{\beta\gamma})_{\beta,\gamma}$ be a fundamental solution for L in $\mathbb{R}^{n+1}$ which decays at infinity. Given a bounded domain $\Omega \subset \mathbb{R}^{n+1}$, of finite perimeter, whose boundary is Ahlfors regular and satisfies (2.3.1), introduce the double layer potential operator and its boundary version as in (3.6.24), (3.3.3).

The goal is to study the compactness of K on the Sobolev space $L^p_1(\partial\Omega, d\sigma)$, for $1 < p < \infty$. In the case of the harmonic double layer potential such a compactness result was established for bounded C^1 domains in [37], via methods which make essential use of the local graph structure of the boundary of the domain in question. Of course, the domains we consider in this paper typically lack this key feature, so a new approach is required. We nonetheless have:

Theorem 6.2.1 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded regular SKT domain, and let L, E, K be defined as before. Furthermore, assume that the double layer operator K has the form (4.5.3). Then the operator K is compact on $L^p_1(\partial\Omega, d\sigma)$ for every $p \in (1,\infty)$.*

As a preamble, we first record the following jump formula.

Lemma 6.2.2 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded UR domain, and let L, E, be defined as before. Recall the single layer from (3.6.22) and, for each multi-indices α, β, set*

$$\mathcal{S}_{\alpha\beta}\, g(X) := \int_{\partial\Omega} E_{\alpha\beta}(X-Y) g(Y)\, d\sigma(Y), \qquad X \in \mathbb{R}^{n+1} \setminus \partial\Omega. \tag{6.2.1}$$

Also, consider

$$B := \Big[\Big(a^{\alpha\beta}_{jk}\nu_j\nu_k\Big)_{\alpha,\beta}\Big]^{-1}, \qquad B = \Big(b_{\alpha\beta}\Big)_{\alpha,\beta}. \tag{6.2.2}$$

Then for every $r \in \{1, ..., n+1\}$

$$\partial_r \mathcal{S}_{\alpha\beta}\, g\Big|_{\partial\Omega_\pm}(X) = \mp\tfrac{1}{2}\nu_r(X) b_{\alpha\beta}\, g(X) + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} (\partial_r E_{\alpha\beta})(X-Y) g(Y)\, d\sigma(Y), \tag{6.2.3}$$

at a.e. $X \in \partial\Omega$, whenever $g \in L^p(\partial\Omega, d\sigma)$, $1 < p < \infty$.

Proof. This is a direct consequence of Theorem 3.5.2. □

We are ready to present the

Proof of Theorem 6.2.1. Recall (6.2.1). By relying on (3.6.43), we may then further transform formula (3.6.31) into

$$\begin{aligned}
\partial_j \Big(\mathcal{D} f\Big)_\gamma(X) &= \int_{\partial\Omega} \nu_j(Y) a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(X-Y)(\nabla_{\tan} f_\alpha)_s(Y)\, d\sigma(Y) \\
&\quad - \int_{\partial\Omega} \nu_s(Y) a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(X-Y)(\nabla_{\tan} f_\alpha)_j(Y)\, d\sigma(Y). \\
&= a^{\beta\alpha}_{rs} \partial_r \mathcal{S}_{\gamma\beta}\Big(\nu_j (\nabla_{\tan} f_\alpha)_s\Big)(X) + \Big(\mathcal{D}\big((\nabla_{\tan} f)^j\big)\Big)_\gamma(X), \qquad (6.2.4)
\end{aligned}$$

where $(\nabla_{\tan} f)^j$ is a vector whose component of order α is $(\nabla_{\tan} f_\alpha)_j$. As a consequence of Lemma 3.6.1, (6.2.4) and jump relations, we then have

$$
\begin{aligned}
\partial_{\tau_{jk}}(Kf)_\gamma(X) &= \partial_{\tau_{jk}}(\tfrac{1}{2}f + Kf)_\gamma(X) - \tfrac{1}{2}\partial_{\tau_{jk}} f_\gamma(X) \\
&= \nu_j(\partial_k \mathcal{D} f)_\gamma\Big|_{\partial\Omega}(X) - \nu_k(\partial_j \mathcal{D} f)_\gamma\Big|_{\partial\Omega}(X) - \tfrac{1}{2}\partial_{\tau_{jk}} f_\gamma(X) \\
&= \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} \nu_j(X)\nu_k(Y) a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(X-Y)(\nabla_{\tan} f_\alpha)_s(Y)\, d\sigma(Y) \\
&\quad + \nu_j(X)\Big(K\big(\nabla_{\tan} f\big)^k\Big)_\gamma(X) \\
&\quad + \tfrac{1}{2}\nu_j(X)(\nabla_{\tan} f_\gamma)_k(X) - \tfrac{1}{2}\nu_j(X)\nu_k(X)\nu_r(X) a_{rs}^{\beta\alpha} b_{\gamma\beta}(\nabla_{\tan} f_\alpha)_s(X) \\
&\quad - \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} \nu_k(X)\nu_j(Y) a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(X-Y)(\nabla_{\tan} f_\alpha)_s(Y)\, d\sigma(Y) \\
&\quad - \nu_k(X)\Big(K\big(\nabla_{\tan} f\big)^j\Big)_\gamma(X) \\
&\quad - \tfrac{1}{2}\nu_j(X)(\nabla_{\tan} f_\gamma)_k(X) + \tfrac{1}{2}\nu_j(X)\nu_k(X)\nu_r(X) a_{rs}^{\beta\alpha} b_{\gamma\beta}(\nabla_{\tan} f_\alpha)_s(X) \\
&\quad - \tfrac{1}{2}\partial_{\tau_{jk}} f_\gamma(X) \\
&= - \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} \nu_k(X)\nu_j(Y) a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(X-Y)(\nabla_{\tan} f_\alpha)_s(Y)\, d\sigma(Y) \\
&\quad + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} \nu_j(X)\nu_k(Y) a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(X-Y)(\nabla_{\tan} f_\alpha)_s(Y)\, d\sigma(Y) \\
&\quad + \Big(\Big[K, M_{\nu_k}\Big]\big(\nabla_{\tan} f\big)^j\Big)_\gamma(X) - \Big(\Big[K, M_{\nu_j}\Big]\big(\nabla_{\tan} f\big)^k\Big)_\gamma(X) \\
&\quad + \Big(K\big(\nu_j(\nabla_{\tan} f)^k - \nu_k(\nabla_{\tan} f)^j\big)\Big)_\gamma(X), \qquad (6.2.5)
\end{aligned}
$$

where, generally speaking, $[A,B] := AB - BA$ and M_h is the operator of multiplication by the function h. Note that the terms in the 5th, 6th and 8th line above cancel. Also, by (3.6.43), in the last line of (6.2.5) we may write $\nu_j(\nabla_{\tan} f)^k - \nu_k(\nabla_{\tan} f)^j = \partial_{\tau_{jk}} f$. Thus, if $\nabla_{\tan} f$ is regarded as a matrix-valued function whose (α, s) entry is the s-th component of $\nabla_{\tan} f_\alpha$, then the above identity can be summarized as

$$
\partial_{\tau_{jk}}(Kf) = K(\partial_{\tau_{jk}} f) + T_{jk}(\nabla_{\tan} f) \qquad (6.2.6)
$$

where, if $g = (g_{\alpha s})_{\alpha,s}$ is an arbitrary matrix valued function with components in $L^p(\partial\Omega, d\sigma)$,

$$
\begin{aligned}
(T_{jk}g)_\gamma(X) := &- \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}} [\nu_k(X)-\nu_k(Y)]\nu_j(Y) a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(X-Y) g_{\alpha s}(Y)\, d\sigma(Y)\\
&+ \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}} [\nu_j(X)-\nu_j(Y)]\nu_k(Y) a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(X-Y) g_{\alpha s}(Y)\, d\sigma(Y)\\
&+ \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}} [\nu_k(Y)-\nu_k(X)]\nu_s(Y) a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(X-Y) g_{\alpha j}(Y)\, d\sigma(Y)\\
&- \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}} [\nu_j(Y)-\nu_j(X)]\nu_s(Y) a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(X-Y) g_{\alpha k}(Y)\, d\sigma(Y) \quad (6.2.7)
\end{aligned}
$$

for every index γ and σ-a.e. point $X \in \partial\Omega$. Upon noticing that, by Theorem 2.4.5, T_{jk} is a compact operator on $L^p(\partial\Omega, d\sigma)$, the desired conclusion now follows from (6.2.6) and Theorem 4.5.1. $\Box$

Compared to the case of Lebesgue spaces, for Sobolev spaces it is not clear whether ν close to $\mathrm{VMO}\,(\partial\Omega, d\sigma)$ implies that K is close to $\mathrm{Cp}\,(L^p_1(\partial\Omega, d\sigma))$. Nonetheless, the following holds.

Theorem 6.2.3 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is an open set satisfying a two-sided local John condition and such that $\partial\Omega$ is Ahlfors regular and compact. Let L, E, K be defined as before and suppose that the double layer operator K has the form (4.5.3). Also, fix $p \in (1,\infty)$ and $\lambda \in \mathbb{R}\setminus\{0\}$. Then there exists a small $\delta > 0$, depending only on L, n, p, λ and the geometry of Ω, with the property that*

$$
\mathrm{dist}\,(\nu\,, \mathrm{VMO}\,(\partial\Omega, d\sigma)) < \delta \Longrightarrow \lambda I + K \textit{ is Fredholm with index zero on } L^p_1(\partial\Omega, d\sigma), \quad (6.2.8)
$$

where the distance is measured in $\mathrm{BMO}(\partial\Omega, d\sigma)$.

Proof. Fix $\lambda \in \mathbb{R}$ with $\lambda \neq 0$ along with $p \in (1,\infty)$. From Theorem 4.5.1 we know that there exists $\delta > 0$ such that

$$
\begin{aligned}
&\mathrm{dist}\,(\nu\,, \mathrm{VMO}\,(\partial\Omega, d\sigma)) < \delta \Longrightarrow K = K_0 + K_1\\
&\quad \text{with } \|K_0\|_{\mathcal{L}(L^p(\partial\Omega,d\sigma))} < |\lambda|/2 \text{ and } K_1 \in \mathrm{Cp}\,(L^p(\partial\Omega, d\sigma)).
\end{aligned} \quad (6.2.9)
$$

In particular, $\|(\lambda I + K_0)^{-1}\|_{\mathcal{L}(L^p(\partial\Omega,d\sigma))} < 2/|\lambda|$ which further entails

$$
|\lambda|\|f\|_{L^p(\partial\Omega,d\sigma)} \leq 2\|(\lambda I + K)f\|_{L^p(\partial\Omega,d\sigma)} + 2\|K_1 f\|_{L^p(\partial\Omega,d\sigma)}, \qquad \forall\, f \in L^p(\partial\Omega, d\sigma). \quad (6.2.10)
$$

Next, introduce $T := (T_{jk})_{1\leq j,k\leq n+1}$, where $T_{jk} : L^p(\partial\Omega, d\sigma) \to L^p(\partial\Omega, d\sigma)$ is defined in (6.2.7). For each such operator, Theorem 2.4.5 ensures that there exists $K_{jk} \in \mathrm{Cp}\,(L^p(\partial\Omega, d\sigma))$ with the property that

$$
\|T_{jk} - K_{jk}\|_{\mathcal{L}(L^p(\partial\Omega,d\sigma))} \leq C\,\mathrm{dist}\,(\nu, \mathrm{VMO}(\partial\Omega, d\sigma)), \quad (6.2.11)
$$

where C depends only on Ω, L, p and n. Also, the identity (6.2.6) gives

$$\partial_{\tau_{jk}}\Big[(\lambda I+K)f\Big]=(\lambda I+K)(\partial_{\tau_{jk}}f)+T_{jk}(\nabla_{\tan}f),\qquad\forall\, f\in L^p(\partial\Omega,d\sigma).\tag{6.2.12}$$

Set $K_2:=(K_{jk})_{1\leq j,k\leq n+1}$. Together, (6.2.10) and (6.2.12) then prove that

$$\|f\|_{L^p_1(\partial\Omega,d\sigma)}\leq C\|(\lambda I+K)f\|_{L^p_1(\partial\Omega,d\sigma)}+\|K_2(\nabla_{\tan}f)\|_{L^p(\partial\Omega,d\sigma)},\quad\forall\, f\in L^p_1(\partial\Omega,d\sigma),\tag{6.2.13}$$

where $C=C(\Omega,L,\lambda,n,p)>0$, granted that $\operatorname{dist}(\nu,\mathrm{VMO}(\partial\Omega,d\sigma))<\delta$, with $\delta>0$ sufficiently small. The fact that $\lambda I+K$ is bounded from below, modulo compact operators, on $L^p_1(\partial\Omega,d\sigma)$ can then be used, in conjunction with the homotopic invariance of the index, to show that in fact $\lambda I+K$ is Fredholm with index zero on $L^p_1(\partial\Omega,d\sigma)$. □

Our next result can be viewed as a quantitative version of Theorem 6.2.3.

Proposition 6.2.4 *Retain the same hypotheses as in Theorem 6.2.3. Then for every $\varepsilon>0$ there exist a small $\delta>0$ and a large $N\in\mathbb{N}$ such that*

$$\operatorname{dist}(\nu\,,\mathrm{VMO}\,(\partial\Omega,d\sigma))<\delta\Longrightarrow\Big[\operatorname{dist}\Big(K^m,\mathrm{Cp}\,(L^p_1(\partial\Omega,d\sigma))\Big)\Big]^{1/m}<\varepsilon\tag{6.2.14}$$

for each $m\in\mathbb{N}$ satisfying $m\geq N$ (above, K^m denotes the m-fold composition of K with itself).

Proof. The same type of argument as in the proof of Theorem 6.2.3 shows that for every $\varepsilon>0$ there exists $\delta>0$ with the property that the essential spectrum of K on $L^p_1(\partial\Omega,d\sigma)$ (i.e., the set of complex numbers λ for which $\lambda I-K$ is not Fredholm) is included in $B(0,\varepsilon)$ if $\operatorname{dist}(\nu\,,\mathrm{VMO}\,(\partial\Omega,d\sigma))<\delta$. As is well-known, the essential spectrum of an operator T acting on a Banach space $\mathcal{X}$ is just the ordinary spectrum of $[T]$, the class of T in the Calkin Algebra $\mathcal{L}(\mathcal{X})/\mathrm{Cp}\,(\mathcal{X})$. Then the the spectral radius formula gives

$$\limsup_{m\to\infty}\sqrt[m]{\|[K]^m\|_{\mathcal{L}(L^p_1(\partial\Omega,d\sigma))/\mathrm{Cp}\,(L^p_1(\partial\Omega,d\sigma))}}<\varepsilon,\tag{6.2.15}$$

from which the desired conclusion follows. □

Let us now turn our attention to the double layers associated with the Stokes system (cf. the discussion in §6.1). First, from (6.1.24)-(6.1.25), for each $\gamma\in\mathbb{R}$, $j\in\{1,...,n+1\}$ and $\vec{g}\in\Big[L^p(\partial\Omega,d\sigma)\Big]^{n+1}$, $1<p<\infty$, we have

$$\begin{aligned}\Big(\mathcal{D}_\gamma\vec{g}\Big)_j(X)\;&=\;\int_{\partial\Omega}\Big(\nu_\alpha(Y)(\partial_\alpha E_{jk})(X-Y)+\gamma\,\nu_\alpha(Y)(\partial_j E_{\alpha k})(X-Y)\\&\qquad-\nu_j(Y)q_k(X-Y)\Big)g_k(Y)\,d\sigma(Y),\qquad X\in\mathbb{R}^{n+1}\setminus\partial\Omega.\end{aligned}\tag{6.2.16}$$

Then for each $\vec{g}\in\Big[L^p_1(\partial\Omega,d\sigma)\Big]^{n+1}$, $1<p<\infty$, $r,j\in\{1,...,n+1\}$, and $X\in\Omega$ we may write

$$
\begin{aligned}
\partial_r\Big(\mathcal{D}_\gamma\vec{g}\Big)_j(X) &= \int_{\partial\Omega}\Big(\nu_\alpha(Y)(\partial_r\partial_\alpha E_{jk})(X-Y)+\gamma\,\nu_\alpha(Y)(\partial_r\partial_j E_{\alpha k})(X-Y) \\
&\qquad -\nu_j(Y)(\partial_r q_k)(X-Y)\Big)g_k(Y)\,d\sigma(Y) \\
&= \int_{\partial\Omega}\Big(-\partial_{\tau_{\alpha r}(Y)}[(\partial_\alpha E_{jk})(X-Y)]-\gamma\,\partial_{\tau_{\alpha r}(Y)}[(\partial_j E_{\alpha k})(X-Y)] \\
&\qquad +\partial_{\tau_{jr}(Y)}[q_k(X-Y)]\Big)g_k(Y)\,d\sigma(Y) \\
&\quad +\int_{\partial\Omega}\Big(\nu_r(Y)(\Delta E_{jk})(X-Y)+\gamma\,\nu_r(Y)(\partial_j\partial_\alpha E_{\alpha k})(X-Y) \\
&\qquad -\nu_r(Y)(\partial_j q_k)(X-Y)\Big)g_k(Y)\,d\sigma(Y) \\
&= \int_{\partial\Omega}\Big(-\partial_{\tau_{\alpha r}(Y)}[(\partial_\alpha E_{jk})(X-Y)]-\gamma\,\partial_{\tau_{\alpha r}(Y)}[(\partial_j E_{\alpha k})(X-Y)] \\
&\qquad +\partial_{\tau_{jr}(Y)}[q_k(X-Y)]\Big)g_k(Y)\,d\sigma(Y),
\end{aligned}
\tag{6.2.17}
$$

where we have used the fact that, by (6.1.22)-(6.1.23), the integrands in the 5-th and 6-th lines of (6.2.17) vanish. By further integrating by parts (cf. (3.6.4) the tangential derivatives in (6.2.17) we arrive at the identity

$$
\begin{aligned}
\partial_r\Big(\mathcal{D}_\gamma\vec{g}\Big)_j(X) &= \int_{\partial\Omega}\Big\{\Big[(\partial_\alpha E_{jk})(X-Y)+\gamma\,(\partial_j E_{\alpha k})(X-Y)\Big](\partial_{\tau_{\alpha r}}g_k)(Y) \\
&\qquad +q_k(X-Y)(\partial_{\tau_{rj}}g_k)(Y)\Big\}\,d\sigma(Y).
\end{aligned}
\tag{6.2.18}
$$

Proposition 6.2.5 *Let $\Omega\subset\mathbb{R}^{n+1}$ be a bounded* UR *domain. Also, recall the hydrostatic double layer potential operator (6.1.24) and the pressure potential (6.1.26). Then for every $\gamma\in\mathbb{R}$ and $p\in(1,\infty)$, there exists a finite constant $C=C(\Omega,\gamma,p)>0$ such that*

$$
\|\mathcal{N}(\nabla\mathcal{D}_\gamma\vec{g})\|_{L^p(\partial\Omega,d\sigma)}\leq C\|\vec{g}\|_{\big[L^p_1(\partial\Omega,d\sigma)\big]^{n+1}},
\tag{6.2.19}
$$

and

$$
\|\mathcal{N}(\mathcal{P}\vec{g})\|_{L^p(\partial\Omega,d\sigma)}\leq C\|\vec{g}\|_{\big[L^p_1(\partial\Omega,d\sigma)\big]^{n+1}},
\tag{6.2.20}
$$

for every $\vec{g}\in\Big[L^p_1(\partial\Omega,d\sigma)\Big]^{n+1}$.

Furthermore, for each $\vec{g}\in\Big[L^p_1(\partial\Omega,d\sigma)\Big]^{n+1}$, the nontangential boundary traces $\mathcal{P}\vec{g}\Big|_{\partial\Omega}$, $\partial_j\mathcal{D}\vec{g}\Big|_{\partial\Omega}$, $1\leq j\leq n+1$, exist at σ-a.e. point on $\partial\Omega$.

Proof. The estimate (6.2.19) and the well-definiteness of $\partial_j\mathcal{D}\vec{g}\Big|_{\partial\Omega}$, $1\leq j\leq n+1$, are consequences of the identity (6.2.18) and Theorem 3.5.2. The same type of reasoning applies to (6.2.20) once we notice that, if E is as in (3.3.24),

$$
\begin{aligned}
\mathcal{P}\vec{g}(X) &= \int\limits_{\partial\Omega} \nu_j(Y)(\partial_j\partial_k E)(X-Y)g_k(Y)\,d\sigma(Y) \\
&= \int\limits_{\partial\Omega} \partial_{\tau_{kj}}(Y)[(\partial_j E)(X-Y)]g_k(Y)\,d\sigma(Y) \\
&= \int\limits_{\partial\Omega} (\partial_j E)(X-Y)(\partial_{\tau_{jk}} g_k)(Y)\,d\sigma(Y) \qquad (6.2.21)
\end{aligned}
$$

for each $\vec{g} \in \Big[L_1^p(\partial\Omega, d\sigma)\Big]^{n+1}$ and each $X \in \Omega$. □

Corollary 6.2.6 *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded* UR *domain, and recall the principal value hydrostatic double layer potential operator K_γ from (6.1.28). Then*

$$
K_\gamma : \Big[L_1^p(\partial\Omega, d\sigma)\Big]^{n+1} \longrightarrow \Big[L_1^p(\partial\Omega, d\sigma)\Big]^{n+1} \qquad (6.2.22)
$$

is well-defined and bounded each $\gamma \in \mathbb{R}$ and $p \in (1, \infty)$.

Proof. This is an immediate consequence of Proposition 6.2.5, Lemma 3.6.1 and Theorem 3.5.2. □

Theorem 6.2.7 *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded regular SKT domain, and let K_{slip} be slip hydrostatic double layer defined as in (6.1.28) for $\gamma = 1$. Then*

$$
K_{slip} : \Big[L_1^p(\partial\Omega, d\sigma)\Big]^{n+1} \longrightarrow \Big[L_1^p(\partial\Omega, d\sigma)\Big]^{n+1} \qquad (6.2.23)
$$

is compact for every $p \in (1, \infty)$.

Proof. To begin with, for each $\gamma \in \mathbb{R}$, we may use (3.6.43) to further transform (6.2.18) into

$$
\begin{aligned}
\partial_r\Big(\mathcal{D}_\gamma\vec{g}\Big)_j(X) &= \int_{\partial\Omega}\Big\{\Big[(\partial_\alpha E_{jk})(X-Y) + \gamma\,(\partial_j E_{\alpha k})(X-Y)\Big] \\
&\qquad\qquad \times\Big[\nu_\alpha(Y)(\nabla_{\tan} g_k)_r(Y) - \nu_r(Y)(\nabla_{\tan} g_k)_\alpha(Y)\Big] \\
&\qquad + q_k(X-Y)\Big[\nu_r(Y)(\nabla_{\tan} g_k)_j(Y) - \nu_j(Y)(\nabla_{\tan} g_k)_r(Y)\Big]\Big\}\,d\sigma(Y) \\
&= \Big(\mathcal{D}_\gamma(\nabla_{\tan}\vec{g})^r\Big)_j(X) \\
&\quad + \int_{\partial\Omega} \nu_r(Y)\Big[(\partial_\alpha E_{jk})(X-Y) + \gamma\,(\partial_j E_{\alpha k})(X-Y)\Big](\nabla_{\tan} g_k)_\alpha(Y)\,d\sigma(Y) \\
&\quad + \int_{\partial\Omega} \nu_r(Y) q_k(X-Y)(\nabla_{\tan} g_k)_j(Y)\,d\sigma(Y), \qquad (6.2.24)
\end{aligned}
$$

where $(\nabla_{\tan}\vec{g})^r$ is the vector whose component of order k is $(\nabla_{\tan} g_k)_r$. By relying on Lemma 3.6.1, (6.2.24) and Theorem 3.5.2, for any $r, s, j \in \{1, ..., n+1\}$ we may then write

$$
\begin{aligned}
\partial_{\tau_{rs}}(K_\gamma\vec{g})_j(X) &= \partial_{\tau_{rs}}(\tfrac{1}{2}\vec{g}+K_\gamma\vec{g})_j(X)-\tfrac{1}{2}\partial_{\tau_{rs}}g_j(X)\\
&= \nu_r(\partial_s\mathcal{D}_\gamma\vec{g})_j\Big|_{\partial\Omega}(X)-\nu_s(\partial_r\mathcal{D}_\gamma\vec{g})_j\Big|_{\partial\Omega}(X)-\tfrac{1}{2}\partial_{\tau_{rs}}g_j(X)\\
&= \lim_{\varepsilon\to 0^+}\int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}}\Big[\nu_r(Y)\nu_s(X)-\nu_s(Y)\nu_r(X)\Big]\\
&\qquad\qquad\times\Big[(\partial_\alpha E_{jk})(X-Y)+\gamma\,(\partial_j E_{\alpha k})(X-Y)\Big](\nabla_{\tan}g_k)_\alpha(Y)\,d\sigma(Y)\\
&\quad+\lim_{\varepsilon\to 0^+}\int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}}\Big[\nu_r(X)\nu_s(Y)-\nu_s(X)\nu_r(Y)\Big]q_k(X-Y)(\nabla_{\tan}g_k)_j(Y)\,d\sigma(Y)\\
&\quad+\nu_r(X)\Big(K_\gamma\big((\nabla_{\tan}\vec{g})^s\big)\Big)_j(X)-\nu_s(X)\Big(K_\gamma\big((\nabla_{\tan}\vec{g})^r\big)\Big)_j(X)\\
&= \lim_{\varepsilon\to 0^+}\int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}}\Big[\nu_r(Y)\nu_s(X)-\nu_s(Y)\nu_r(X)\Big]\\
&\qquad\qquad\times\Big[(\partial_\alpha E_{jk})(X-Y)+\gamma\,(\partial_j E_{\alpha k})(X-Y)\Big](\nabla_{\tan}g_k)_\alpha(Y)\,d\sigma(Y)\\
&\quad+\lim_{\varepsilon\to 0^+}\int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}}\Big[\nu_r(X)\nu_s(Y)-\nu_s(X)\nu_r(Y)\Big]q_k(X-Y)(\nabla_{\tan}g_k)_j(Y)\,d\sigma(Y)\\
&\quad+\nu_r(X)\Big(K_\gamma\big(\nabla_{\tan}\vec{g}\big)^s\Big)_j(X)-\nu_s(X)\Big(K_\gamma\big(\nabla_{\tan}\vec{g}\big)^r\Big)_j(X),
\end{aligned}
\tag{6.2.25}
$$

where we regard $\nabla_{\tan}\vec{g}$ as a matrix-valued function whose (j,k) entry is the k-th component of the vector $\nabla_{\tan}g_j$. The last line in (6.2.25) can be further transformed into

$$
\begin{aligned}
&\Big(K_\gamma\big(\nu_r(\nabla_{\tan}\vec{g})^s-\nu_s(\nabla_{\tan}\vec{g})^r\big)\Big)_j(X)\\
&\qquad-\Big(\Big[K_\gamma,M_{\nu_r}\Big]\big(\nabla_{\tan}\vec{g}\big)^r\Big)_j(X)+\Big(\Big[K_\gamma,M_{\nu_s}\Big]\big(\nabla_{\tan}\vec{g}\big)^s\Big)_j(X)\\
&=\Big(K_\gamma\big(\partial_{\tau_{rs}}\vec{g}\big)\Big)_j(X)-\Big(\Big[K_\gamma,M_{\nu_r}\Big]\big(\nabla_{\tan}\vec{g}\big)^r\Big)_j(X)\\
&\qquad+\Big(\Big[K_\gamma,M_{\nu_s}\Big]\big(\nabla_{\tan}\vec{g}\big)^s\Big)_j(X)
\end{aligned}
\tag{6.2.26}
$$

since, by (3.6.43), we have $\nu_r(\nabla_{\tan}\vec{g})^s-\nu_s(\nabla_{\tan}\vec{g})^r=\partial_{\tau_{rs}}\vec{g}$. Thus, altogether, the identity (6.2.25) can be summarized in the form

$$
\partial_{\tau_{rs}}\big(K_\gamma\vec{g}\big)=K_\gamma\big(\partial_{\tau_{rs}}\vec{g}\big)+R^{rs}_\gamma(\nabla_{\tan}\vec{g}),
\tag{6.2.27}
$$

where, for $f=(f_{\alpha\beta})_{\alpha,\beta}$, we have set

$$
\begin{aligned}
(R^{rs}_\gamma)_j f(X) := \lim_{\varepsilon\to 0^+} & \int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}} \Big[(\nu_r(Y)-\nu_r(X))\nu_s(X)-(\nu_s(Y)-\nu_s(X))\nu_r(X)\Big] \\
& \qquad \times\Big[(\partial_\alpha E_{jk})(X-Y)+\gamma\,(\partial_j E_{\alpha k})(X-Y)\Big] f_{k\alpha}(Y)\,d\sigma(Y) \\
& + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega\\ |X-Y|>\varepsilon}} \Big[(\nu_r(X)-\nu_r(Y))\nu_s(Y)-(\nu_s(X)-\nu_s(Y))\nu_r(Y)\Big] \\
& \qquad \times q_k(X-Y) f_{kj}(Y)\,d\sigma(Y) \qquad (6.2.28) \\
& -\Big(\Big[K_\gamma, M_{\nu_r}\Big](f_{\alpha r})_{1\le\alpha\le n+1}\Big)_j(X)+\Big(\Big[K_\gamma, M_{\nu_s}\Big](f_{\alpha s})_{1\le\alpha\le n+1}\Big)_j(X).
\end{aligned}
$$

With (6.2.27) in hand, the fact that the operator (6.2.23) is compact follows from Theorem 4.5.1, and the observation that, due the commutator structure of the operators R^{rs}_γ in (6.2.28), Theorem 2.4.5 shows that these are compact on $L^p(\partial\Omega, d\sigma)$, for each $\gamma\in\mathbb{R}$ and $1<p<\infty$. □

Theorem 6.2.8 *Suppose that $\Omega\subset\mathbb{R}^{n+1}$ is a domain satisfying a two-sided local John condition and whose boundary is Ahlfors regular and compact. Also, fix $p\in(1,\infty)$ and $\lambda\in\mathbb{R}\setminus\{0\}$. Then there exists a small $\delta>0$, depending only on L, n, p, λ and the geometry of Ω, with the property that*

$$
\left.\begin{array}{c}\mathrm{dist}\,(\nu\,,\mathrm{VMO}\,(\partial\Omega, d\sigma))<\delta\\ \text{and } 1-\delta<\gamma\le 1\end{array}\right\}\Longrightarrow \begin{array}{l}\lambda I+K_\gamma \text{ is Fredholm with}\\ \text{index zero on } L^p_1(\partial\Omega, d\sigma),\end{array} \qquad (6.2.29)
$$

where, as usual, the distance is measured in $\mathrm{BMO}(\partial\Omega, d\sigma)$.

Proof. This is justified much as in the proof of Theorem 6.2.3, with the help of (6.2.27). □

6.3 The invertibility of boundary double layer potentials

One important consequence of Theorem 2.3.1 is the fact that Green's formula continues to hold for functions representable in the form of layer potentials. To state this in a proper form, assume that L is a constant (real) coefficient, second order operator as in (3.6.20), for which Legendre-Hadamard condition (3.6.21) holds. As before, denote by $E\in C^\infty(\mathbb{R}^{n+1}\setminus\{0\})$ a (matrix-valued) fundamental solution for L which is even and homogeneous of degree $-(n-1)$. Let us also set

$$
A := \Big(a^{\alpha\beta}_{rs}\Big)_{\alpha,\beta,r,s} \qquad (6.3.1)
$$

and introduce

$$
\langle A\xi,\zeta\rangle := a^{\alpha\beta}_{rs}\xi^\alpha_r\zeta^\beta_s, \qquad \forall\,\xi:=(\xi^\alpha_r)_{\alpha,r},\ \forall\,\zeta:=(\zeta^\beta_s)_{\beta,s}. \qquad (6.3.2)
$$

In particular, $\langle A\nabla u,\nabla v\rangle = a^{\alpha\beta}_{rs}\partial_r u_\alpha\partial_s v_\beta$. We shall then call A *symmetric* if $\langle A\xi,\zeta\rangle=\langle A\zeta,\xi\rangle$ for every ξ, ζ, i.e., if

$$a_{rs}^{\alpha\beta} = a_{sr}^{\beta\alpha}, \qquad \forall\, \alpha, \beta, r, s. \tag{6.3.3}$$

Moreover, call A *semi-positive definite* if

$$\langle A\xi, \xi\rangle \geq 0 \qquad \forall\, \xi, \tag{6.3.4}$$

and *positive definite* if

$$\langle A\xi, \xi\rangle \geq \kappa|\xi|^2 \qquad \forall\, \xi, \tag{6.3.5}$$

for some $\kappa > 0$. Finally, given a UR domain $\Omega \subset \mathbb{R}^{n+1}$ with outward unit normal ν, call

$$\left(\partial_\nu^A u\right)_\alpha := \nu_r a_{rs}^{\alpha\beta} \partial_s u_\beta \tag{6.3.6}$$

the *conormal derivative* associated with the writing of the operator L as in (3.6.20). Recall next the single and double layers, as well as their boundary versions from (3.6.22), and (3.6.24)-(3.6.25). Also, denote by K^* the adjoint of K and recall the convention (3.6.26).

Proposition 6.3.1 *In the above context,*

$$\partial_\nu^A \mathcal{S} f\Big|_{\partial\Omega_\pm} = (\mp\tfrac{1}{2}I + K^*)f, \tag{6.3.7}$$

for each $f \in L^p(\partial\Omega, d\sigma)$, $p \in (1, \infty)$.

Proof. This is a consequence of (6.2.3) and (6.3.6). □

Proposition 6.3.2 *Assume that* $\Omega \subset \mathbb{R}^{n+1}$ *is a bounded* UR *domain. As usual, set* $\sigma := \mathcal{H}^n \lfloor \partial\Omega$ *and denote by* ν *the (measure theoretic) outward unit normal to* $\partial\Omega$*. Let* L, E, $\mathcal{S}$, S, K, K^* *be as above. Next, let* f *be an arbitrary vector-valued function (with components) in* $L^2(\partial\Omega, d\sigma)$ *and consider*

$$u^\pm(X) := \mathcal{S}f(X), \qquad X \in \Omega_\pm. \tag{6.3.8}$$

Then

$$\int_{\Omega_+} \langle A\nabla u^+, \nabla u^+\rangle \, dX = \int_{\partial\Omega} \Big\langle Sf\,,\, \Big(-\tfrac{1}{2}I + K^*\Big)f\Big\rangle \, d\sigma, \tag{6.3.9}$$

and, if either $\int_{\partial\Omega} f \, d\sigma = 0$ *or* $n \geq 2$,

$$\int_{\Omega_-} \langle A\nabla u^-, \nabla u^-\rangle \, dX = \int_{\partial\Omega} \Big\langle Sf\,,\, \Big(-\tfrac{1}{2}I - K^*\Big)f\Big\rangle \, d\sigma. \tag{6.3.10}$$

Proof. Consider the vector field

$$v := \Big(u^+_\alpha a^{\alpha\beta}_{rs}\partial_s u^+_\beta\Big)_{1\leq r\leq n+1} \in C^0(\Omega) \tag{6.3.11}$$

which, thanks to Theorem 3.4.2, Proposition 3.2.3 Corollary 4.3.11, and our hypotheses, satisfies the conditions listed in (2.3.3). Theorem 2.3.1 applied to v gives

$$\int_\Omega \langle A\nabla u^+, \nabla u^+\rangle\, dX = \int_{\partial\Omega} u^+_\alpha(X)\,\nu_r(X) a^{\alpha\beta}_{rs}\partial_s u^+_\beta(X)\, d\sigma(X). \tag{6.3.12}$$

Then (6.3.9) follows from this and the jump-relations (3.6.27), (6.3.7). Formula (6.3.10) when $n\geq 2$ is proved in a similar manner by working in the domain $B_R\setminus\bar\Omega$, where B_R is the ball centered at the origin and having a sufficiently large radius R, then passing to the limit $R\to\infty$. Given that the outward unit normal for Ω_- is $-\nu$ and that, from (6.3.8),

$$|u^-(X)| + |X||\nabla u^-(X)| = O(|X|^{1-n}) \ \text{ as } \ |X|\to\infty, \tag{6.3.13}$$

the desired conclusion follows. In the case when $n=2$ and $\int_{\partial\Omega} f\, d\sigma = 0$, the proof follows the same pattern since, this time, we have the improved decay condition

$$|u^-(X)| + |X||\nabla u^-(X)| = O(|X|^{-n}) \ \text{ as } \ |X|\to\infty, \tag{6.3.14}$$

in place of (6.3.13). $\square$

Let $W^{1,p}_{loc}(\mathbb{R}^{n+1})$, $p\in(1,\infty)$, denote the local version of the usual scale of L^p-based Sobolev spaces of order one in $\mathbb{R}^{n+1}$.

Proposition 6.3.3 *Assume that $\Omega\subset\mathbb{R}^{n+1}$ is a* UR *domain. Then for every $p\in(1,\infty)$, the operator*

$$\mathcal{S} : L^p(\partial\Omega, d\sigma) \longrightarrow W^{1,p(n+1)/n}_{loc}(\mathbb{R}^{n+1}), \tag{6.3.15}$$

is well-defined and bounded.

Proof. This follows from Proposition 3.2.7, Theorem 2.3.1, and the fact that the single layer 'does not jump' across $\partial\Omega$ (cf. (3.6.27)) which gives that, in the distributional sense, $[\partial_j(\mathcal{S}f)]\Big|_{\Omega_\pm} = u^\pm_j$ for each $f\in L^p(\partial\Omega, d\sigma)$ and $1\leq j\leq n+1$, where $u^\pm_j(X) := (\partial_j\mathcal{S}f)(X)$ for $X\in\Omega_\pm$. $\square$

To state our next result, define

$$L^p_0(\partial\Omega, d\sigma) := \Big\{f\in L^p(\partial\Omega, d\sigma) : \int_{\partial\Omega} f\, d\sigma = 0\Big\}, \qquad 1<p<\infty. \tag{6.3.16}$$

Proposition 6.3.4 *Retain the same notation and hypotheses as in Proposition 6.3.2 and, in addition, assume that the coefficient tensor A (introduced in (6.3.1)) is semi-positive definite. Then the operator $\lambda I + K^*$ is injective on $L^2(\partial\Omega, d\sigma)$ for any $\lambda\in\mathbb{R}\setminus[-\frac12,\frac12]$. On the other hand, assuming that A is positive definite, it follows that $\frac12 I + K^*$ is injective on $L^2(\partial\Omega, d\sigma)$ if $\mathbb{R}^{n+1}\setminus\bar\Omega$ is connected, and that $-\frac12 I + K^*$ is injective on $L^2_0(\partial\Omega, d\sigma)$ when Ω is connected, and on $L^2(\partial\Omega, d\sigma)$ when Ω is connected and $n\geq 2$.*

Proof. Let $\lambda \in \mathbb{R} \setminus [-\frac{1}{2}, \frac{1}{2}]$ and $f \in L^2(\partial\Omega, d\sigma)$ be such that $(\lambda I + K^*)f = 0$. Based on this and Green's formula we may then write

$$\int_{\partial\Omega} f \, d\sigma = -\tfrac{2}{1+2\lambda} \int_{\partial\Omega} (-\tfrac{1}{2}I + K^*)f \, d\sigma = -\tfrac{2}{1+2\lambda} \int_{\partial\Omega} \partial_\nu^A \mathcal{S} f \, d\sigma = 0 \tag{6.3.17}$$

which shows that $f \in L_0^2(\partial\Omega, d\sigma)$. Thus, if $u^\pm := \mathcal{S}f$ in $\Omega_\pm$ it follows that (6.3.14) holds. Also, if we set

$$u := \begin{cases} u^+ & \text{in } \Omega_+, \\ u^- & \text{in } \Omega_-, \end{cases} \tag{6.3.18}$$

then from Proposition 6.3.3 we may deduce that for all $\varepsilon > 0$

$$u \in W^{1,2(n+1)/n-\varepsilon}_{loc}(\mathbb{R}^{n+1}). \tag{6.3.19}$$

On the other hand, our current hypotheses and Proposition 6.3.2, allow us to write

$$\begin{aligned} 0 &= \int_{\partial\Omega} \langle (\lambda I + K^*)f, Sf \rangle \, d\sigma \\ &= \int_{\partial\Omega} \Big\langle (-\lambda + \tfrac{1}{2})(-\tfrac{1}{2}I + K^*)f + (\lambda + \tfrac{1}{2})(\tfrac{1}{2}I + K^*)f, Sf \Big\rangle \, d\sigma \\ &= (-\lambda + \tfrac{1}{2}) \int_{\Omega_+} \langle A\nabla u, \nabla u \rangle \, dX + (-\lambda - \tfrac{1}{2}) \int_{\Omega_-} \langle A\nabla u, \nabla u \rangle \, dX. \end{aligned} \tag{6.3.20}$$

Consequently,

$$\int_{\mathbb{R}^{n+1}} \langle A\nabla u, \nabla u \rangle \, dX = 0, \tag{6.3.21}$$

since $-\lambda - \frac{1}{2}$ and $-\lambda + \frac{1}{2}$ have the same sign and the integrands in the last line in (6.3.20) are nonnegative.

Next, pick a function $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$ which is identically one in a neighborhood of the origin and set $\varphi_j(X) := \varphi(X/j)$, $j \in \mathbb{N}$. We may then write

$$\begin{aligned} \lim_{j\to\infty} \int_{\mathbb{R}^{n+1}} \langle A\nabla(\varphi_j u), \nabla(\varphi_j u) \rangle \, dX &= \lim_{j\to\infty} \int_{\mathbb{R}^{n+1}} \varphi_j^2 \langle A\nabla u, \nabla u \rangle \, dX \\ &\quad + \lim_{j\to\infty} \int_{\mathbb{R}^{n+1}} \mathcal{O}\Big(|\varphi_j||\nabla\varphi_j||u||\nabla u| + |\nabla\varphi_j|^2|u|^2 \Big) \, dX \\ &= 0, \end{aligned} \tag{6.3.22}$$

thanks to (6.3.21) and the decay of u at infinity (cf. (6.3.13)). Since, by (6.3.19), $\varphi_j u \in W^{1,2}(\mathbb{R}^{n+1})$, Plancherel's formula (used twice) and the Legendre-Hadamard condition (3.6.21) then give

$$
\begin{aligned}
0 &= \lim_{j\to\infty} \int_{\mathbb{R}^{n+1}} \langle A\nabla(\varphi_j u), \nabla(\varphi_j u)\rangle \, dX \geq \kappa \lim_{j\to\infty} \int_{\mathbb{R}^{n+1}} |\nabla(\varphi_j u)|^2 \, dX \\
&= \kappa \int_{\mathbb{R}^{n+1}} |\nabla u|^2 \, dX. \qquad (6.3.23)
\end{aligned}
$$

Thus, u is a constant in $\mathbb{R}^{n+1}$ and, ultimately,

$$
u = 0 \ \text{ in } \ \mathbb{R}^{n+1} \qquad (6.3.24)
$$

by (6.3.14). Recall now (6.3.6) and observe that

$$
\partial_\nu^A u^\pm = (\mp\tfrac{1}{2}I + K^*)f. \qquad (6.3.25)
$$

In concert with (6.3.24), this gives $f = \partial_\nu^A u^- - \partial_\nu^A u^+ = 0$, proving that $\lambda I + K^*$ is injective on $L^2(\partial\Omega, d\sigma)$.

Consider now the case when the coefficient tensor A is positive definite and $f \in L^2(\partial\Omega, d\sigma)$ is such that $(\frac{1}{2}I + K^*)f = 0$. Now, as before, $f = -(-\frac{1}{2}I + K^*)f \in L^2_0(\partial\Omega, d\sigma)$, so that (6.3.14) holds. Thus, we may write

$$
0 = -\int_{\partial\Omega} \langle (\tfrac{1}{2}I + K^*)f, Sf\rangle \, d\sigma = \int_{\Omega_-} \langle A\nabla u, \nabla u\rangle \, dX \geq \kappa \int_{\Omega_-} |\nabla u|^2 \, dX, \qquad (6.3.26)
$$

so that $u = 0$ in Ω_-, by (6.3.14) and the fact that Ω_- is connected. In turn, this entails $Sf = u^-\Big|_{\partial\Omega} = 0$ and, further,

$$
0 = \int_{\partial\Omega} \langle (-\tfrac{1}{2}I + K^*)f, Sf\rangle \, d\sigma = \int_{\Omega_+} \langle A\nabla u, \nabla u\rangle \, dX \geq \kappa \int_{\Omega_+} |\nabla u|^2 \, dX. \qquad (6.3.27)
$$

It follows that $\nabla u = 0$ in Ω_+, so once again $f = \partial_\nu^A u^- - \partial_\nu^A u^+ = 0$. Hence the operator $\frac{1}{2}I + K^*$ is, as claimed, injective when acting on the space $L^2(\partial\Omega, d\sigma)$.

Finally, assume that Ω is connected, A is positive definite, that the function $f \in L^2(\partial\Omega, d\sigma)$ satisfies $(-\frac{1}{2}I + K^*)f = 0$ and that either $n \geq 2$ or f has vanishing moment. Then

$$
0 = \int_{\partial\Omega} \langle (-\tfrac{1}{2}I + K^*)f, Sf\rangle \, d\sigma = \int_{\Omega_+} \langle A\nabla u, \nabla u\rangle \, dX \geq \kappa \int_{\Omega_+} |\nabla u|^2 \, dX, \qquad (6.3.28)
$$

hence, since Ω_+ is connected, there exists a constant c such that $u \equiv c$ in Ω. We may thus conclude that $Sf = u^+\Big|_{\partial\Omega} = c$ and, hence, $(\frac{1}{2}I + K^*)f = (-\frac{1}{2}I + K^*)f + f = f$. Consequently, since when either $n \geq 2$, or f has vanishing moment, the exterior Green's formula holds, we may write

$$
0 = -\Big\langle c, \int_{\partial\Omega} f \, d\sigma \Big\rangle = -\int_{\partial\Omega} \langle (\tfrac{1}{2}I + K^*)f, Sf\rangle \, d\sigma = \int_{\Omega_-} \langle A\nabla u, \nabla u\rangle \, dX \geq \kappa \int_{\Omega_-} |\nabla u|^2 \, dX \qquad (6.3.29)
$$

Thus, $\nabla u = 0$ in Ω_- which then gives $\partial_\nu^A u^- = 0$. The same argument based on jump relations now shows that $f = 0$, and the desired conclusion follows. □

Theorem 6.3.5 *Let $p \in (1,\infty)$ be given and assume that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. Also, assume that L is as in (3.6.20)-(3.6.21) and recall the double layer potential operator K from (3.6.25).*

Then there exists $\varepsilon > 0$ which depends only on p, n and the John and Ahlfors regularity constants of Ω, such that if Ω is an ε-regular SKT domain then the following operators are invertible:

$$\lambda I + K^* : L^p(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma) \quad \forall\, \lambda \in \mathbb{R} \setminus [-\tfrac{1}{2}, \tfrac{1}{2}], \quad \text{if } \ A \geq 0, \tag{6.3.30}$$

$$-\tfrac{1}{2} I + K^* : L^p_0(\partial\Omega, d\sigma) \longrightarrow L^p_0(\partial\Omega, d\sigma) \ \text{ if } A > 0 \text{ and } \Omega \text{ is connected}, \tag{6.3.31}$$

$$-\tfrac{1}{2} I + K^* : L^p(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma) \ \text{ if } A > 0,\ \Omega \text{ is connected, } n \geq 2, \tag{6.3.32}$$

$$\tfrac{1}{2} I + K^* : L^p(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma) \ \text{ if } A > 0 \text{ and } \mathbb{R}^{n+1} \setminus \bar{\Omega} \text{ is connected.} \tag{6.3.33}$$

If, in addition, the double layer potential operator K (originally defined in (3.6.25)) can be represented as in (4.5.3) then the following operators are also invertible:

$$\lambda I + K : L^p_1(\partial\Omega, d\sigma) \longrightarrow L^p_1(\partial\Omega, d\sigma) \quad \forall\, \lambda \in \mathbb{R} \setminus [-\tfrac{1}{2}, \tfrac{1}{2}], \quad \text{if } \ A \geq 0, \tag{6.3.34}$$

$$-\tfrac{1}{2} I + K : L^p_1(\partial\Omega, d\sigma)\Big/\mathbb{R} \longrightarrow L^p_1(\partial\Omega, d\sigma)\Big/\mathbb{R} \ \text{ if } A > 0 \text{ and } \Omega \text{ is connected}, \tag{6.3.35}$$

$$-\tfrac{1}{2} I + K : L^p_1(\partial\Omega, d\sigma) \longrightarrow L^p_1(\partial\Omega, d\sigma) \ \text{ if } A > 0,\ \Omega \text{ is connected, } n \geq 2, \tag{6.3.36}$$

$$\tfrac{1}{2} I + K : L^p_1(\partial\Omega, d\sigma) \longrightarrow L^p_1(\partial\Omega, d\sigma) \ \text{ if } A > 0 \text{ and } \mathbb{R}^{n+1} \setminus \bar{\Omega} \text{ is connected.} \tag{6.3.37}$$

Proof. The claims in the first part of the theorem are immediate consequences of Theorem 4.5.1, Proposition 6.3.4 and classical Fredholm theory (compare the proof of Proposition 5.4.2). The claims in the second part of the theorem then follow from this, duality, the extra assumption on K and Theorem 6.2.3. □

Theorem 6.3.5 applies directly to the case of the Laplacian. This time, however, given the explicit nature of the differential operator in question, the results become more specific. To state them, for an open set $\Omega \subset \mathbb{R}^{n+1}$ and $k \in \mathbb{N}$ introduce

$$\mathbb{R}^k_{\partial\Omega} := \Big\{ \sum_j c_j \mathbf{1}_{\Sigma_j} \, : \, c_j \in \mathbb{R}^k \text{ and } \Sigma_j \text{ connected component of } \partial\Omega \Big\}, \tag{6.3.38}$$

$$\mathbb{R}^k_{\partial\Omega_\pm} := \Big\{ \sum_j c_j \mathbf{1}_{\partial\mathcal{O}_j} \, : \, c_j \in \mathbb{R}^k \text{ and } \mathcal{O}_j \text{ bounded connected component of } \Omega_\pm \Big\}, \tag{6.3.39}$$

$$\mathbb{R}^k_{\Omega_\pm} := \Big\{ \sum_j c_j \mathbf{1}_{\mathcal{O}_j} \, : \, c_j \in \mathbb{R}^k \text{ and } \mathcal{O}_j \text{ bounded connected component of } \Omega_\pm \Big\}, \tag{6.3.40}$$

with the convention that, when $k = 1$, we agree to drop it as a superscript. In particular, we have

$$\mathbb{R}^k_{\partial\Omega_\pm} = (\mathbb{R}^k_{\Omega_\pm})\Big|_{\partial\Omega} \quad \text{and} \quad \mathbb{R}^k_{\partial\Omega} = \mathbb{R}^k_{\partial\Omega_+} \oplus \mathbb{R}^k_{\partial\Omega_-}, \tag{6.3.41}$$

where the sum is direct (but not orthogonal). Let us also point out here that

$$\dim \mathbb{R}^k_{\Omega_+} = \dim \mathbb{R}^k_{\partial\Omega_+} = k \cdot b_0, \qquad \dim \mathbb{R}^k_{\Omega_-} = \dim \mathbb{R}^k_{\partial\Omega_-} = k \cdot b_n,$$
$$\dim \mathbb{R}^k_{\partial\Omega} = k \cdot (b_0 + b_n), \tag{6.3.42}$$

where the Betti numbers b_0, b_n represent the number of bounded connected components of Ω_+ and Ω_-, respectively. Therefore, the intuitive interpretation of b_n is the number of $(n+1)$-dimensional "holes" of Ω_+. Granted Theorem 4.5.1, by arguing as in [84] and the proof of Proposition 6.3.4 we then obtain:

Theorem 6.3.6 *Let $p \in (1, \infty)$ be given and assume that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. Also, let K be as in (3.3.2)–(3.3.3). Then there exists $\varepsilon > 0$ which depends only on p, n and the John and Ahlfors regularity constants of Ω, such that if Ω is an ε-regular SKT domain then the following operators are invertible:*

$$\lambda I + K^* : L^p(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma) \quad \textit{if } \lambda \in \mathbb{R} \setminus [-\tfrac{1}{2}, \tfrac{1}{2}], \tag{6.3.43}$$

$$\pm\tfrac{1}{2} I + K^* \ \textit{mapping} \ \big\{f \in L^p(\partial\Omega, d\sigma) : \textstyle\int_{\partial\Omega} f\psi \, d\sigma = 0, \ \forall\, \psi \in \mathbb{R}_{\partial\Omega}\big\} \ \textit{onto itself}, \tag{6.3.44}$$

$$\pm\tfrac{1}{2} I + K^* \ \textit{mapping} \ \big\{f \in L^p(\partial\Omega, d\sigma) : \textstyle\int_{\partial\Omega} f\psi \, d\sigma = 0, \ \forall\, \psi \in \mathbb{R}_{\partial\Omega_\mp}\big\} \ \textit{onto itself}, \tag{6.3.45}$$

$$\pm\tfrac{1}{2} I + K : L^p(\partial\Omega, d\sigma)\Big/\mathbb{R}_{\partial\Omega} \longrightarrow L^p(\partial\Omega, d\sigma)\Big/\mathbb{R}_{\partial\Omega}, \tag{6.3.46}$$

$$\pm\tfrac{1}{2} I + K : L^p(\partial\Omega, d\sigma)\Big/\mathbb{R}_{\partial\Omega_\mp} \longrightarrow L^p(\partial\Omega, d\sigma)\Big/\mathbb{R}_{\partial\Omega_\mp}, \tag{6.3.47}$$

$$\lambda I + K : L^p_1(\partial\Omega, d\sigma) \longrightarrow L^p_1(\partial\Omega, d\sigma) \quad \textit{if } \lambda \in \mathbb{R} \setminus [-\tfrac{1}{2}, \tfrac{1}{2}], \tag{6.3.48}$$

$$\pm\tfrac{1}{2} I + K : L^p_1(\partial\Omega, d\sigma)\Big/\mathbb{R}_{\partial\Omega} \longrightarrow L^p(\partial\Omega, d\sigma)\Big/\mathbb{R}_{\partial\Omega}, \tag{6.3.49}$$

$$\pm\tfrac{1}{2} I + K : L^p_1(\partial\Omega, d\sigma)\Big/\mathbb{R}_{\partial\Omega_\mp} \longrightarrow L^p_1(\partial\Omega, d\sigma)\Big/\mathbb{R}_{\partial\Omega_\mp}. \tag{6.3.50}$$

Below, we record a suitable version of Theorem 6.3.6 which holds in a slightly different geometrical measure theoretic setting.

Proposition 6.3.7 *Assume that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded* UR *domain, for which the outward unit normal ν belongs to* $\mathrm{VMO}(\partial\Omega, d\sigma)$*. Also, recall the harmonic double layer K introduced in (3.3.2)–(3.3.3). Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that the operators (6.3.43)–(6.3.50) are invertible for each $p \in (2 - \varepsilon, 2 + \varepsilon)$.*

Proof. Consider first the operators (6.3.43)-(6.3.47). That the operators in question are Fredholm with index zero when $p = 2$ can be seen as in (4.4.11) and the subsequent analysis. Then the extension to $2 - \varepsilon < p < 2 + \varepsilon$ follows from this and well-known stability results (cf. [106] and the discussion in [61]).

Next, we treat the operator in (6.4.2) for some fixed $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$. From what we have proved already we know that that there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that for each $2 - \varepsilon < p < 2 + \varepsilon$ one can find $C = C(\Omega, p) > 0$ with the property that

$$\|f\|_{L^p(\partial\Omega, d\sigma)} \leq C \|(\lambda I + K) f\|_{L^p(\partial\Omega, d\sigma)}, \qquad \forall\, f \in L^p(\partial\Omega, d\sigma). \tag{6.3.51}$$

Next, introduce $T := (T_{jk})_{1\leq j,k\leq n+1}$, where $T_{jk} : L^p(\partial\Omega, d\sigma) \to L^p(\partial\Omega, d\sigma)$ is defined in (6.2.7). Given that we are assuming $\nu \in \mathrm{VMO}(\partial\Omega, d\sigma)$, these operators are compact for every $p \in (1,\infty)$. Also, the identity (6.2.6) gives

$$\partial_{\tau_{jk}}\Big[(\lambda I + K)f\Big] = (\lambda I + K)(\partial_{\tau_{jk}} f) + T_{jk}(\nabla_{\tan} f), \qquad \forall\, f \in L^p(\partial\Omega, d\sigma). \tag{6.3.52}$$

Together, (6.3.51) and (6.3.52) prove that

$$\|f\|_{L^p_1(\partial\Omega,d\sigma)} \leq C\|(\lambda I + K)f\|_{L^p_1(\partial\Omega,d\sigma)} + \|T(\nabla_{\tan} f)\|_{L^p(\partial\Omega,d\sigma)}, \quad \forall\, f \in L^p_1(\partial\Omega, d\sigma). \tag{6.3.53}$$

The above analysis shows that there exists $\varepsilon = \varepsilon(\Omega) > 0$ with the property that if $2-\varepsilon < p < 2+\varepsilon$ then the operator $\lambda I + K$ is injective and semi-Fredholm on the space $L^p_1(\partial\Omega, d\sigma)$ for each $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{1}{2}$. That this operator is in fact invertible, is now an easy consequence of the homotopic invariance of the index, along with the simple observation that, if $|\lambda|$ is large, $\lambda I + K$ can be inverted on $L^p_1(\partial\Omega, d\sigma)$ (via a Neumann series). This completes the proof of the claim made in the statement of the proposition about the operator (6.4.2).

In a similar fashion, the operators (6.3.49), (6.3.50), can be shown to be injective and semi-Fredholm if $2-\varepsilon < p < 2+\varepsilon$ for some small $\varepsilon = \varepsilon(\Omega) > 0$. When used in concert with the fact that, for the same range of p's, the operator (6.3.48) is, as a trivial consequence of we have just proved, semi-Fredholm when acting on either $L^p_1(\partial\Omega, d\sigma)$ or $L^p_1(\partial\Omega, d\sigma)/\mathbb{R}$, the same argument based on index theory can be used to conclude that (6.3.49)-(6.3.50) are invertible in the present context if p is near 2. $\square$

We wish to prove results analogous to Theorem 6.3.6 for the Stokes system (6.1.14). To set the stage, we momentarily digress for the purpose of introducing notation which will facilitate stating this result. Let Ψ be the $(n+1)(n+2)/2$-dimensional linear space of $\mathbb{R}^{n+1}$-valued functions $\psi = (\psi_j)_{1\leq j\leq n+1}$ defined in $\mathbb{R}^{n+1}$ and satisfying

$$\partial_j\psi_k + \partial_k\psi_j = 0, \qquad 1 \leq j,k \leq n+1, \tag{6.3.54}$$

and note that

$$\Psi = \Big\{\psi(X) = AX + \vec{a} :\ A,\ (n+1)\times(n+1) \text{ antisymmetric matrix, and } \vec{a} \in \mathbb{R}^{n+1}\Big\}. \tag{6.3.55}$$

Given an open set $\Omega \subset \mathbb{R}^{n+1}$ define, as usual, $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^{n+1} \setminus \bar{\Omega}$, and introduce

$$\Psi(\Omega_\pm) := \Big\{\sum_j \psi_j \mathbf{1}_{\mathcal{O}_j} :\ \psi_j \in \Psi,\ \mathcal{O}_j \text{ bounded component of } \Omega_\pm\Big\}. \tag{6.3.56}$$

Then for $\gamma \in (-1,1]$, we can define

$$\Psi^\gamma(\Omega_\pm) := \begin{cases} \mathbb{R}^{n+1}_{\Omega_\pm}, & |\gamma| < 1, \\ \Psi(\Omega_\pm), & \gamma = 1, \end{cases} \tag{6.3.57}$$

and

$$\Psi^\gamma(\partial\Omega_\pm) := \{\psi|_{\partial\Omega} : \psi \in \Psi^\gamma(\Omega_\pm)\}, \tag{6.3.58}$$

so that

$$\dim \Psi^\gamma(\partial\Omega_+) = \begin{cases} (n+1)\cdot b_0 & \text{if } \ |\gamma| < 1, \\ \frac{(n+1)(n+2)}{2}\cdot b_0 & \text{if } \ \gamma = 1, \end{cases} \tag{6.3.59}$$

and

$$\dim \Psi^\gamma(\partial\Omega_-) = \begin{cases} (n+1)\cdot b_n & \text{if } \ |\gamma| < 1, \\ \frac{(n+1)(n+2)}{2}\cdot b_n & \text{if } \ \gamma = 1. \end{cases} \tag{6.3.60}$$

Finally, assuming that $\Omega \subset \mathbb{R}^{n+1}$ is bounded domain of finite perimeter and $p \in (1,\infty)$, set

$$L^p_{\Psi^\gamma_\pm}(\partial\Omega, d\sigma) := \Big\{\vec{f} \in \Big[L^p(\partial\Omega, d\sigma)\Big]^{n+1} : \int_{\partial\Omega}\langle\psi, \vec{f}\rangle\, d\sigma = 0,\ \forall\,\psi \in \Psi^\gamma(\partial\Omega_\pm)\Big\}, \tag{6.3.61}$$

$$L^p_{1,\nu_\pm}(\partial\Omega, d\sigma) := \Big\{\vec{f} \in \Big[L^p_1(\partial\Omega, d\sigma)\Big]^{n+1} : \int_{\partial\Omega}\langle\psi, \vec{f}\rangle\, d\sigma = 0,\ \forall\,\psi \in \nu\,\mathbb{R}_{\partial\Omega_\pm}\Big\}, \tag{6.3.62}$$

$$L^p_{1,\nu}(\partial\Omega, d\sigma) := \Big\{\vec{f} \in \Big[L^p_1(\partial\Omega, d\sigma)\Big]^{n+1} : \int_{\partial\Omega}\langle\psi, \vec{f}\rangle\, d\sigma = 0,\ \forall\,\psi \in \nu\,\mathbb{R}_{\partial\Omega}\Big\}, \tag{6.3.63}$$

$$L^p_{\nu_\pm}(\partial\Omega, d\sigma) := \Big\{\vec{f} \in \Big[L^p(\partial\Omega, d\sigma)\Big]^{n+1} : \int_{\partial\Omega}\langle\psi, \vec{f}\rangle\, d\sigma = 0,\ \forall\,\psi \in \nu\,\mathbb{R}_{\partial\Omega_\pm}\Big\}, \tag{6.3.64}$$

$$L^p_{\nu}(\partial\Omega, d\sigma) := \Big\{\vec{f} \in \Big[L^p(\partial\Omega, d\sigma)\Big]^{n+1} : \int_{\partial\Omega}\langle\psi, \vec{f}\rangle\, d\sigma = 0,\ \forall\,\psi \in \nu\,\mathbb{R}_{\partial\Omega}\Big\}. \tag{6.3.65}$$

Recall the hydrostatic double layer potential K_γ from (6.1.28). For this, we have:

Theorem 6.3.8 *Let $p \in (1,\infty)$ be given and assume that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. Then there exists $\varepsilon > 0$ which depends only on p, n and the John and Ahlfors regularity constants of Ω, such that if Ω is an ε-regular SKT domain and $1-\varepsilon < \gamma \leq 1$ then the following operators are invertible:*

$$\pm\tfrac{1}{2}I + K^*_\gamma : L^p_{\Psi^\gamma_\mp}(\partial\Omega)\Big/\nu\mathbb{R}_{\partial\Omega_\pm} \longrightarrow L^p_{\Psi^\gamma_\mp}(\partial\Omega)\Big/\nu\mathbb{R}_{\partial\Omega_\pm}, \tag{6.3.66}$$

$$\pm\tfrac{1}{2}I + K_\gamma : L^p_{1,\nu_\pm}(\partial\Omega)\Big/\Psi^\gamma(\partial\Omega_\mp) \longrightarrow L^p_{1,\nu_\pm}(\partial\Omega)\Big/\Psi^\gamma(\partial\Omega_\mp), \tag{6.3.67}$$

$$\pm\tfrac{1}{2}I + K_\gamma : L^p_{\nu_\pm}(\partial\Omega)\Big/\Psi^\gamma(\partial\Omega_\mp) \longrightarrow L^p_{\nu_\pm}(\partial\Omega)\Big/\Psi^\gamma(\partial\Omega_\mp), \tag{6.3.68}$$

along with

$$\lambda I + K_\gamma : \Big[L^p(\partial\Omega, d\sigma)\Big]^{n+1} \longrightarrow \Big[L^p(\partial\Omega, d\sigma)\Big]^{n+1}, \tag{6.3.69}$$

$$\lambda I + K_\gamma : \Big[L^p_1(\partial\Omega, d\sigma)\Big]^{n+1} \longrightarrow \Big[L^p_1(\partial\Omega, d\sigma)\Big]^{n+1}, \tag{6.3.70}$$

if $\lambda \in \mathbb{R}\setminus[-\frac{1}{2},\frac{1}{2}]$.

Proof. In the case of Lipschitz domains (and p near 2) this has been proved in [94] and, given Theorem 6.2.8 and the techniques used in the proof of Proposition 6.3.4, the same type of invertibility results can then be established in the current more general setting. $\square$

We also have the following analogue of Theorem 6.3.8 for the case of the Lamé system (6.1.3). Below, K_r refers to the elastostatic double layer defined in (6.1.9) and we set

$$\Psi^r(\Omega_\pm) := \begin{cases} \mathbb{R}^{n+1}_{\Omega_\pm}, & |r| < \mu, \\ \Psi(\Omega_\pm), & r = \mu, \end{cases} \tag{6.3.71}$$

and

$$\Psi^r(\partial\Omega_\pm) := \{\psi|_{\partial\Omega} : \psi \in \Psi^r(\Omega_\pm)\}, \tag{6.3.72}$$

where λ, μ are as in (6.1.1).

Theorem 6.3.9 *Let $p \in (1,\infty)$ be given and assume that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. Then there exists $\varepsilon > 0$ which depends only on p, n and the John and Ahlfors regularity constants of Ω, such that if Ω is an ε-regular SKT domain and*

$$-\mu < r \leq \mu \quad \text{is such that} \quad \left| r - \frac{\mu(\mu+\lambda)}{3\mu+\lambda} \right| < \varepsilon, \tag{6.3.73}$$

then the following operators are invertible:

$$\pm\tfrac{1}{2}I + K_r^* : L^p_{\Psi^r_\mp}(\partial\Omega) \longrightarrow L^p_{\Psi^r_\mp}(\partial\Omega), \tag{6.3.74}$$

$$\pm\tfrac{1}{2}I + K_r : \Big[L^p(\partial\Omega, d\sigma)\Big]^{n+1} \Big/ \Psi^r(\partial\Omega_\mp) \longrightarrow \Big[L^p(\partial\Omega, d\sigma)\Big]^{n+1} \Big/ \Psi^r(\partial\Omega_\mp), \tag{6.3.75}$$

$$\pm\tfrac{1}{2}I + K_r : \Big[L^p_1(\partial\Omega, d\sigma)\Big]^{n+1} \Big/ \Psi^r(\partial\Omega_\mp) \longrightarrow \Big[L^p_1(\partial\Omega, d\sigma)\Big]^{n+1} \Big/ \Psi^r(\partial\Omega_\mp), \tag{6.3.76}$$

along with

$$\eta I + K_r : \Big[L^p(\partial\Omega, d\sigma)\Big]^{n+1} \longrightarrow \Big[L^p(\partial\Omega, d\sigma)\Big]^{n+1} \tag{6.3.77}$$

$$\eta I + K_r : \Big[L^p_1(\partial\Omega, d\sigma)\Big]^{n+1} \longrightarrow L^p_1(\partial\Omega, d\sigma)\Big]^{n+1}, \tag{6.3.78}$$

if $\eta \in \mathbb{R} \setminus [-\frac{1}{2}, \frac{1}{2}]$.

Proof. This follows along the lines of known results for Lipschitz domains, with the help of Theorem 6.2.3 and the remarks made in § 6.1. $\square$

6.4 The invertibility of boundary single layer potentials

Recall the definitions of the Sobolev spaces of order ± 1, i.e., $L^p_1(\partial\Omega, d\sigma)$ and $L^p_{-1}(\partial\Omega, d\sigma)$ from (3.6.10)-(3.6.11) and (3.6.14), respectively. Also, for each $p \in (1,\infty)$, set

$$L^p_{-1,0}(\partial\Omega, d\sigma) := \Big\{ f \in L^p_{-1}(\partial\Omega, d\sigma) : \langle f, 1\rangle = 0 \Big\}. \tag{6.4.1}$$

Theorem 6.4.1 *Let $p \in (1,\infty)$ be given and assume that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. Also, recall the (boundary version) harmonic single layer S introduced in (3.3.35).*

Then there exists $\varepsilon > 0$ which depends only on p, n and the John and Ahlfors regularity constants of Ω, such that if Ω is an ε-regular SKT domain then the following operators are invertible:

$$S : L^p(\partial\Omega, d\sigma) \longrightarrow L^p_1(\partial\Omega, d\sigma) \quad \text{if } n \geq 2, \tag{6.4.2}$$

$$S : L^p_{-1}(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma) \quad \text{if } n \geq 2, \tag{6.4.3}$$

$$S : L^p_0(\partial\Omega, d\sigma) \longrightarrow L^p_1(\partial\Omega, d\sigma)\Big/\mathbb{R}, \tag{6.4.4}$$

$$S : L^p_{-1,0}(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma)\Big/\mathbb{R}. \tag{6.4.5}$$

As opposed to the operators discussed in § 6.3 which are of the form "identity +small perturbation of a compact operator" the case of the (boundary version of the) single layer is different in nature and main difficulty is establishing that $S : L^p(\partial\Omega, d\sigma) \to L^p_1(\partial\Omega, d\sigma)$ is Fredholm with index zero. To circumvent this problem, we find it convenient to work with the acoustic layer potentials associated with Ω, which we now proceed to define. In order to facilitate the subsequent exposition, we choose to discuss here a few definitions and preliminary results. To get started, denote by $E_{n+1}(X;k)$ the fundamental solution of the Helmholtz operator $\Delta + k^2$ in $\mathbb{R}^{n+1}$, $n \geq 1$, explicitly given by

$$E_{n+1}(X;k) = \begin{cases} -\frac{\sqrt{-1}}{4}\left(\frac{2\pi|X|}{k}\right)^{(1-n)/2} H^{(1)}_{(n-1)/2}\big(k|X|\big), & n \geq 1,\ k \in \mathbb{C}\backslash\{0\}, \\ \frac{1}{2\pi}\ln|X|, & n = 1,\ k = 0, \\ -\frac{1}{(n-1)\omega_n}|X|^{1-n}, & n \geq 2,\ k = 0, \end{cases} \tag{6.4.6}$$

for $\operatorname{Im} k \geq 0$ and $X \in \mathbb{R}^{n+1}\backslash\{0\}$, where $H^{(1)}_\nu(\cdot)$ denotes the Hankel function of the first kind with index $\nu \geq 0$ and ω_n represents the area of the unit sphere S^n in $\mathbb{R}^{n+1}$. In particular,

$$E_3(X;k) = -\frac{e^{\sqrt{-1}\,k|X|}}{4\pi|X|}, \qquad X \in \mathbb{R}^3 \setminus \{0\}, \tag{6.4.7}$$

is precisely the function $\Phi_k(X)$ introduced in (6.1.34). Next, we record the following useful lemma which appears in [45].

Lemma 6.4.2 *For each fixed $k \in \mathbb{C}\backslash\{0\}$ and $R > 0$, the function $E_{n+1}(\cdot;k)$ satisfies the following estimates uniformly for $0 < |X| < R$:*

$$|E_{n+1}(X;k) - E_{n+1}(X;0)| \leq \begin{cases} C, & n = 1, 2, \\ C[1 + |\ln |X||], & n = 3, \\ C\big[1 + |X|^{3-n}\big], & n \geq 4, \end{cases} \tag{6.4.8}$$

$$|\partial_j E_{n+1}(X;k) - \partial_j E_{n+1}(X;0)| \leq \begin{cases} C, & n = 1, 2, \\ C\big[1 + |X|^{2-n}\big], & n \geq 3, \end{cases} \tag{6.4.9}$$

$$|(\partial_i \partial_j E_{n+1})(X;k) - (\partial_i \partial_j E_{n+1})(X;0)| \leq \begin{cases} C[1 + |\ln |X||], & n = 1, \\ C\big[1 + |X|^{2-n}\big], & n \geq 2. \end{cases} \tag{6.4.10}$$

Above $C = C(R, n, k) > 0$, and $\partial_j = \partial/\partial x_j$ for $1 \leq j \leq n+1$.

After these preparations we are ready to discuss the

Proof of Theorem 6.4.1. For each $k \in \sqrt{-1}\mathbb{R}_+$ define the operators $\mathcal{S}_k$, $\mathcal{D}_k$, S_k, K_k as well as its adjoint K_k^*, in a similar fashion to (3.3.34), (3.3.1), (3.3.35) and (3.3.2), respectively, by replacing the fundamental solution for the Laplacian (3.3.24) with (6.4.6). From (6.4.8)-(6.4.10), we may then conclude that these operators enjoy the same estimates, trace formulas and compactness properties as their counterparts for $k = 0$ do. A significant difference is that, for $n \geq 2$, Green's formulas (6.3.9)-(6.3.10) now read

$$\int_{\Omega_\pm} \Big[|\nabla u^\pm|^2 - k^2 |u^\pm|^2\Big]\, dX = \pm \int_{\partial\Omega} (S_k f)^c \left(\pm\tfrac{1}{2}I + K_k^*\right) f\, d\sigma, \tag{6.4.11}$$

for every $f \in L^2(\partial\Omega, d\sigma)$ and $u^\pm := \mathcal{S}_k f$ in $\Omega_\pm$ (where the superscript c indicates complex conjugation). When we run the same type of argument as in the proof of Proposition 6.3.4, the presence of the positive factor $-k^2$ ensures that the operators

$$\pm\tfrac{1}{2}I + K_k^*,\ \pm\tfrac{1}{2}I + K_k : L^p(\partial\Omega, d\sigma) \longrightarrow L^p(\partial\Omega, d\sigma), \tag{6.4.12}$$

$$\pm\tfrac{1}{2}I + K_k : L_1^p(\partial\Omega, d\sigma) \longrightarrow L_1^p(\partial\Omega, d\sigma), \tag{6.4.13}$$

are injective when $p = 2$, without any additional topological restrictions imposed on the domain. Thus, if Ω is as in the statement of the theorem, these operators in fact are invertible.

Next, for an arbitrary function $f \in L_1^p(\partial\Omega, d\sigma)$, set $u := \mathcal{D}_k f$ in Ω. Then $(\Delta + k^2)u = 0$ in Ω, and $\mathcal{N}u$, $\mathcal{N}(\nabla u) \in L^p(\partial\Omega, d\sigma)$. Also, in the nontangential limit sense, $u|_{\partial\Omega}$, $\nabla u|_{\partial\Omega}$ exist pointwise σ-a.e. on $\partial\Omega$. These conditions ensure that the following Green's representation formula holds

$$u(X) = \mathcal{D}_k(u|_{\partial\Omega})(X) - \mathcal{S}_k(\partial_\nu u)(X), \qquad X \in \Omega. \tag{6.4.14}$$

Since $u|_{\partial\Omega} = (\frac{1}{2}I + K_k)f$, letting the point X in (6.4.14) approach nontangentially the boundary of Ω yields (after some trivial algebra)

$$S_k(\partial_\nu \mathcal{D}_k f) = \Big(\tfrac{1}{2}I + K_k\Big)\Big(-\tfrac{1}{2}I + K_k\Big)f, \qquad \forall\, f \in L^p_1(\partial\Omega, d\sigma). \tag{6.4.15}$$

From (6.4.15) and the fact that the operators (6.4.13) are invertible, it follows that

$$S_k : L^p(\partial\Omega, d\sigma) \longrightarrow L^p_1(\partial\Omega, d\sigma) \tag{6.4.16}$$

is surjective. In a similar fashion (more specifically, by using (6.4.14) with $u = \mathcal{S}_k f$, $f \in L^p(\partial\Omega, d\sigma)$, and then taking the normal derivatives of both sides), we obtain

$$\partial_\nu \mathcal{D}_k(S_k f) = \Big(-\tfrac{1}{2}I + K_k^*\Big)\Big(\tfrac{1}{2}I + K_k^*\Big)f, \qquad \forall\, f \in L^p(\partial\Omega, d\sigma). \tag{6.4.17}$$

In turn, this and the fact that the operators (6.4.12) are invertible ensure that (6.4.16) is injective. Altogether, S_k in (6.4.16) is both surjective and injective, hence invertible.

Going further, we observe that the estimates in Lemma 6.4.2 imply that

$$S - S_k : L^p(\partial\Omega, d\sigma) \longrightarrow L^p_1(\partial\Omega, d\sigma) \quad \text{ is compact.} \tag{6.4.18}$$

Writing $S = S_k + (S - S_k)$ we see that (6.4.2) is a Fredholm operator, with index zero if $n \geq 1$. Since the same type of argument as for S_k proves that S in (6.4.2) is injective when $p = 2$ and $n \geq 2$, we may finally conclude that this operator is invertible.

The fact that (6.4.3) is invertible is now an immediate corollary of what we have just proved and duality. Since the operator (6.4.5) is the dual of (6.4.4), we are finally left with proving that the latter is an isomorphism. From what we have shown above, it follows that (6.4.3) is Fredholm with index zero, so it suffices to prove that (6.4.3) is injective when $p = 2$. To this end, assume that $f \in L^2(\partial\Omega, d\sigma)$ is such that $Sf = c$, a constant, on $\partial\Omega$. Set $u^\pm := \mathcal{S}f$ in $\Omega_\pm$ and note that $\int_{\Omega_+} |\nabla u^+|^2\, dX = c\int_{\partial\Omega} \partial_\nu u^+\, d\sigma = 0$, so that $\nabla u^+ = 0$ in Ω_+. In particular, $f = \partial_\nu u^- - \partial_\nu u^+ = \partial_\nu u^-$. Also, since f has mean value zero, Green's formula continues to work in the complement of the domain and gives $\int_{\Omega_-} |\nabla u^-|^2\, dX = -c\int_{\partial\Omega} \partial_\nu u^-\, d\sigma = -c\int_{\partial\Omega} f\, d\sigma = 0$. Thus $\nabla u^- = 0$ in Ω_-, which forces $f = 0$. $\square$

Similar invertibility results to those contained in Theorem 6.4.1 are valid for the single layer associated with the Lamé and Stokes systems. More specifically, define

$$S_{Lame}\vec{f}(X) := \int_{\partial\Omega} E(X - Y)\vec{f}(Y)\, d\sigma(Y), \quad X \in \partial\Omega, \tag{6.4.19}$$

if $E(X) = (E_{jk}(X))_{1 \leq j,k \leq n+1}$ with $E_{jk}(X)$ as in (6.1.6), and

$$S_{Stokes}\vec{f}(X) := \int_{\partial\Omega} E(X - Y)\vec{f}(Y)\, d\sigma(Y), \quad X \in \partial\Omega, \tag{6.4.20}$$

if $E(X) = (E_{jk}(X))_{1 \leq j,k \leq n+1}$ with $E_{jk}(X)$ as in (6.1.20). Also, recall the space $L^p_{1,\nu}(\partial\Omega, d\sigma)$ from (6.3.63). We then have:

Theorem 6.4.3 *Let $p \in (1, \infty)$ be given and assume that $\Omega \subset \mathbb{R}^{n+1}$, with $n \geq 2$, is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. Then there exists $\varepsilon > 0$ which depends only on p, n and the John and Ahlfors regularity constants of Ω, such that if Ω is an ε-regular SKT domain then*

$$S_{Lame} : \Big[L^p(\partial\Omega, d\sigma)\Big]^{n+1} \longrightarrow \Big[L^p_1(\partial\Omega, d\sigma)\Big]^{n+1}, \tag{6.4.21}$$

$$S_{Lame} : \Big[L^p_{-1}(\partial\Omega, d\sigma)\Big]^{n+1} \longrightarrow \Big[L^p(\partial\Omega, d\sigma)\Big]^{n+1}, \tag{6.4.22}$$

as well as

$$S_{Stokes} : \Big[L^p(\partial\Omega, d\sigma)\Big]^{n+1} \Big/ \nu\,\mathbb{R}_{\partial\Omega} \longrightarrow L^p_{1,\nu}(\partial\Omega, d\sigma), \tag{6.4.23}$$

$$S_{Stokes} : \Big[L^p_{-1}(\partial\Omega, d\sigma)\Big]^{n+1} \Big/ \nu\,\mathbb{R}_{\partial\Omega} \longrightarrow L^p_{\nu}(\partial\Omega, d\sigma), \tag{6.4.24}$$

are isomorphisms.

The proof of this theorem relies on the results developed in § 6.3 and proceeds along the same lines as the proof of Theorem 6.4.1. In the process, we are free to select any type of conormal derivative and we choose one corresponding to a coefficient tensor which is positive definite (a slight adaptation is needed in the case of the Stokes system). We leave the details to the interested reader.

6.5 The invertibility of the magnetostatic layer potential

Here we shall work in the three-dimensional setting. Given $k \in \mathbb{C}$, playing the role of the wave number, recall the standard fundamental solution (6.1.34) for the Helmholtz operator $\Delta + k^2$ in $\mathbb{R}^3$. Let the scatterer occupy a region $\Omega \subset \mathbb{R}^3$ which, for now, we assume to be a bounded domain of finite perimeter whose boundary is Ahlfors regular (of dimension 2) and for which (2.3.1) holds. The single layer acoustic potential operator and its boundary versions are then defined by

$$\begin{aligned} \mathcal{S}_k f(X) &:= \int_{\partial\Omega} \Phi_k(X - Y) f(Y)\, d\sigma(Y), \quad X \in \mathbb{R}^3 \setminus \partial\Omega, \\ S_k f(X) &:= \int_{\partial\Omega} \Phi_k(X - Y) f(Y)\, d\sigma(Y), \quad X \in \partial\Omega. \end{aligned} \tag{6.5.1}$$

The action of the operators $\mathcal{S}_k$, S_k on vector fields is then defined component-wise. It follows that for every $f \in L^p(\partial\Omega, d\sigma)$, $1 < p < \infty$,

$$\mathcal{S}_k f\Big|_{\partial\Omega}(X) = S_k f(X), \qquad X \in \partial\Omega, \tag{6.5.2}$$

and

$$\partial_\nu \mathcal{S}_k f\Big|_{\partial\Omega}(X) = \left(-\tfrac{1}{2} I + K^*_k\right) f(X) \tag{6.5.3}$$

at σ-almost any $X \in \partial\Omega$. Above, K^*_k is the formal transpose of the principal-value integral operator

$$K_k f(X) := \frac{1}{4\pi} \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} \frac{\langle \nu(Y), Y-X\rangle}{|X-Y|^3} e^{\sqrt{-1}k|X-Y|}(1-\sqrt{-1}k|X-Y|) f(Y)\, d\sigma(Y), \quad (6.5.4)$$

where $X \in \partial\Omega$, i.e., the so-called (boundary-version) double layer acoustic potential operator. As a consequence of the jump relations of §3.4, we have the following.

Proposition 6.5.1 *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded* UR *domain. Then for each vector field $\vec{f}$ in $\Big[L^p(\partial\Omega, d\sigma)\Big]^3$, $1 < p < \infty$, we have*

$$\operatorname{div}(\mathcal{S}_k \vec{f})\Big|_{\partial\Omega_\pm}(X) = \mp\tfrac{1}{2}\langle \nu(X), \vec{f}(X)\rangle + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} \langle (\nabla\Phi_k)(X-Y), \vec{f}(Y)\rangle\, d\sigma(Y), \qquad (6.5.5)$$

and

$$(\operatorname{curl}\mathcal{S}_k \vec{f})\Big|_{\partial\Omega_\pm}(X) = \mp\tfrac{1}{2}\nu(X)\times\vec{f}(X) + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{Y\in\partial\Omega \\ |X-Y|>\varepsilon}} (\nabla\Phi_k)(X-Y)\times\vec{f}(Y)\, d\sigma(Y), \qquad (6.5.6)$$

at σ-almost any $X \in \partial\Omega$. In particular, if M_k is the magnetostatics (or, magnetic dipole) layer potential defined in (6.1.36), then

$$\nu\times(\operatorname{curl}\mathcal{S}_k\vec{f})\Big|_{\partial\Omega_\pm} = (\pm\tfrac{1}{2}+M_k)\vec{f}, \qquad \forall\, \vec{f}\in L^p_{\tan}(\partial\Omega, d\sigma), \qquad (6.5.7)$$

whenever $1 < p < \infty$.

We shall now define function spaces which are well suited to the Maxwell system. To this end, recall the space $L^p_{\tan}(\partial\Omega, d\sigma)$ introduced in (6.1.32) and then set

$$L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma) := \{\vec{f}\in L^p_{\tan}(\partial\Omega, d\sigma) :\ \operatorname{Div}\vec{f}\in L^p(\partial\Omega, d\sigma)\}. \qquad (6.5.8)$$

For each $p \in (1,\infty)$, this becomes a Banach space when equipped with the natural norm,

$$\|\vec{f}\|_{L^{p,\mathrm{Div}}_{\tan}(\partial\Omega,d\sigma)} := \|\vec{f}\|_{\big[L^p(\partial\Omega,d\sigma)\big]^3} + \|\operatorname{Div}\vec{f}\|_{L^p(\partial\Omega,d\sigma)}. \qquad (6.5.9)$$

Above, Div is the *surface divergence operator* which we now proceed to define. Specifically, if $\vec{f}\in L^p_{\tan}(\partial\Omega, d\sigma)$, consider the functional $\operatorname{Div}\vec{f}$ acting on $\{\varphi|_{\partial\Omega} :\ \varphi\in C^1 \text{ near } \partial\Omega\}$ according to

$$\langle \operatorname{Div}\vec{f}, (\varphi|_{\partial\Omega})\rangle := -\int_{\partial\Omega} \langle \vec{f}, \nabla_{\tan}(\varphi|_{\partial\Omega})\rangle\, d\sigma. \qquad (6.5.10)$$

In particular, whenever $\vec{f}\in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$ and φ is a scalar function, of class C^1 near $\partial\Omega$,

$$\int_{\partial\Omega} (\operatorname{Div}\vec{f})\,\varphi\, d\sigma = -\int_{\partial\Omega} \langle \vec{f}, \nabla_{\tan}\varphi\rangle\, d\sigma. \qquad (6.5.11)$$

Lemma 6.5.2 *If $\vec{u} \in C^1(\Omega, \mathbb{R}^3)$ is a vector field satisfying*

$$\mathcal{N}(\vec{u}),\, \mathcal{N}(\operatorname{curl}\vec{u}) \in L^p(\partial\Omega, d\sigma), \quad \exists\, \vec{u}\Big|_{\partial\Omega},\;\; \exists\, (\operatorname{curl}\vec{u})\Big|_{\partial\Omega}, \tag{6.5.12}$$

for some $p \in (1, \infty)$, then $\nu \times (\vec{u}|_{\partial\Omega}) \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$ and

$$\operatorname{Div}\Big[\nu \times (\vec{u}|_{\partial\Omega})\Big] = -\langle \nu, (\operatorname{curl}\vec{u})|_{\partial\Omega}\rangle. \tag{6.5.13}$$

Proof. For each scalar-valued function φ which is of class C^1 near $\bar{\Omega}$, we may use Proposition 3.2.5 and Theorem 2.3.1 in order to write

$$\begin{aligned}
\int_{\partial\Omega} \langle \nu \times (\vec{u}|_{\partial\Omega}), \nabla_{\tan}\varphi\rangle\, d\sigma &= \int_{\partial\Omega} \langle \nu \times (\vec{u}|_{\partial\Omega}), \nabla\varphi\rangle\, d\sigma \\
&= \int_{\Omega} \langle \operatorname{curl}\vec{u}, \nabla\varphi\rangle\, dX \\
&= \int_{\partial\Omega} \langle \nu, (\operatorname{curl}\vec{u})|_{\partial\Omega}\rangle \varphi\, d\sigma.
\end{aligned} \tag{6.5.14}$$

This proves (6.5.13). □

Next we study the action of the surface divergence operator in connection with the boundary integral operators introduced above.

Proposition 6.5.3 *Let Ω, k, p, be as before. Then for each $\vec{f} \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$ we have*

$$\operatorname{div}(\mathcal{S}_k\vec{f}) = \mathcal{S}_k(\operatorname{Div}\vec{f}) \qquad \text{in } \mathbb{R}^3 \setminus \partial\Omega, \tag{6.5.15}$$

and

$$\operatorname{Div}(M_k\vec{f}) = -k^2\langle \nu, S_k\vec{f}\rangle - K_k^*(\operatorname{Div}\vec{f}) \qquad \text{on} \qquad \partial\Omega. \tag{6.5.16}$$

In particular, for every $k \in \mathbb{C}$, the operator M_k is well-defined and bounded on the space (6.5.8).

Proof. For an arbitrary point $X \in \mathbb{R}^3 \setminus \partial\Omega$ we may write, based on the tangentiality of $\vec{f}$ and (6.5.10),

$$\begin{aligned}
(\operatorname{div}\mathcal{S}_k\vec{f})(X) &= -\int_{\partial\Omega} \langle (\nabla\Phi_k)(X-Y), \vec{f}(Y)\rangle\, d\sigma(Y) \\
&= -\int_{\partial\Omega} \langle (\nabla_{\tan}\Phi_k)(X-Y), \vec{f}(Y)\rangle\, d\sigma(Y) \\
&= \int_{\partial\Omega} \Phi_k(X-Y)(\operatorname{Div}\vec{f})(Y)\, d\sigma(Y) \\
&= \mathcal{S}_k(\operatorname{Div}\vec{f})(X),
\end{aligned} \tag{6.5.17}$$

which proves (6.5.15). As for (6.5.16), if $\vec{u} := \operatorname{curl}(\mathcal{S}_k\vec{f})$ in Ω, then

$$\mathcal{N}(\vec{u}) \in L^p(\partial\Omega, d\sigma) \text{ and } \vec{u}\Big|_{\partial\Omega} \text{ exists,} \tag{6.5.18}$$

by the jump relations of §3. In fact, by Proposition 6.5.1,

$$\nu \times \vec{u}\Big|_{\partial\Omega} = (\tfrac{1}{2}I + M_k)\vec{f}. \tag{6.5.19}$$

Next, making use of the identity

$$\operatorname{curl}\operatorname{curl} = -\Delta + \nabla \operatorname{div} \tag{6.5.20}$$

and (6.5.15), we compute

$$\operatorname{curl}\vec{u} = -\Delta\,\mathcal{S}_k\vec{f} + \nabla\operatorname{div}\mathcal{S}_k\vec{f} = k^2\,\mathcal{S}_k\vec{f} + \nabla\,\mathcal{S}_k(\operatorname{Div}\vec{f}). \tag{6.5.21}$$

Thus, in addition to (6.5.18), $\vec{u}$ also satisfies

$$\mathcal{N}(\operatorname{curl}\vec{u}) \in L^p(\partial\Omega, d\sigma) \text{ and } (\operatorname{curl}\vec{u})\Big|_{\partial\Omega} \text{ exists.} \tag{6.5.22}$$

Let us also note here that, as a consequence of (6.5.21) and (6.5.2)-(6.5.3),

$$\langle \nu, (\operatorname{curl}\vec{u})|_{\partial\Omega}\rangle = k^2\,\langle \nu, S_k\vec{f}\rangle + (-\tfrac{1}{2}I + K_k^*)(\operatorname{Div}\vec{f}). \tag{6.5.23}$$

In turn, properties (6.5.18), (6.5.22) allow us to employ Lemma 6.5.2. In concert with (6.5.19) and (6.5.23), this implies that $M_k\vec{f} \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$ and the identity (6.5.16) holds. □

Lemma 6.5.4 *Let Ω and k be as before. Then for each $f \in L^p_1(\partial\Omega, d\sigma)$, $1 < p < \infty$,*

$$\operatorname{curl}\mathcal{S}_k(f\nu) = \mathcal{S}_k(\nu \times \nabla_{\tan} f) \quad \text{in } \mathbb{R}^3 \setminus \partial\Omega. \tag{6.5.24}$$

Furthermore, for any $f \in L^p_1(\partial\Omega, d\sigma)$, $1 < p < \infty$, we have that

$$\nu \times \nabla_{\tan} K_k f = k^2\,\nu \times S_k(f\nu) + M_k(\nu \times \nabla_{\tan} f). \tag{6.5.25}$$

Proof. If $f \in L^p_1(\partial\Omega, d\sigma)$ then

$$\begin{aligned}
\nu \times \nabla_{\tan} f &= \Big[\nu_2(\nabla_{\tan} f)_3 - \nu_3(\nabla_{\tan} f)_2\Big]\,e_1 + \Big[\nu_3(\nabla_{\tan} f)_1 - \nu_1(\nabla_{\tan} f)_3\Big]\,e_2 \\
&\quad + \Big[\nu_1(\nabla_{\tan} f)_2 - \nu_2(\nabla_{\tan} f)_1\Big]\,e_3 \\
&= \Big(\partial_{\tau_{23}} f\,,\ \partial_{\tau_{31}} f\,,\ \partial_{\tau_{12}} f\Big),
\end{aligned} \tag{6.5.26}$$

where the first equality is the definition of the cross-product in $\mathbb{R}^3$ (with e_j, $1 \leq j \leq 3$, the canonical orthonormal basis there), while the second one follows from (3.6.43). Consequently, for each $X \in \mathbb{R}^3 \setminus \partial\Omega$,

$$\begin{aligned}
\Big(\mathcal{S}_k(\nu\times\nabla_{\tan}f)\Big)_1(X) &= \int_{\partial\Omega}\Phi_k(X-Y)(\partial_{\tau_{23}}f)(Y)\,d\sigma_Y \\
&= \int_{\partial\Omega}[\nu_2(Y)(\partial_3\Phi_k)(X-Y)-\nu_3(Y)(\partial_2\Phi_k)(X-Y)]f(Y)\,d\sigma_Y, \\
\Big(\mathcal{S}_k(\nu\times\nabla_{\tan}f)\Big)_2(X) &= \int_{\partial\Omega}\Phi_k(X-Y)(\partial_{\tau_{31}}f)(Y)\,d\sigma_Y \qquad (6.5.27)\\
&= \int_{\partial\Omega}[\nu_3(Y)(\partial_1\Phi_k)(X-Y)-\nu_1(Y)(\partial_3\Phi_k)(X-Y)]f(Y)\,d\sigma_Y, \\
\Big(\mathcal{S}_k(\nu\times\nabla_{\tan}f)\Big)_3(X) &= \int_{\partial\Omega}\Phi_k(X-Y)(\partial_{\tau_{12}}f)(Y)\,d\sigma_Y \\
&= \int_{\partial\Omega}[\nu_1(Y)(\partial_2\Phi_k)(X-Y)-\nu_2(Y)(\partial_1\Phi_k)(X-Y)]f(Y)\,d\sigma_Y,
\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{S}_k(\nu\times\nabla_{\tan}f)(X) &= \int_{\partial\Omega}\Big[(\nabla\Phi_k)(X-Y)\times\nu(Y)\Big]f(Y)\,d\sigma_Y \\
&= \operatorname{curl}\mathcal{S}_k(f\nu)(X), \qquad (6.5.28)
\end{aligned}$$

proving (6.5.24). With this in hand, we may then compute

$$\begin{aligned}
\nabla\mathcal{D}_k f &= -\nabla\operatorname{div}\mathcal{S}_k(f\nu) = -(\Delta+\operatorname{curl}\operatorname{curl})\mathcal{S}_k(f\nu) \\
&= k^2\,\mathcal{S}_k(f\nu)+\operatorname{curl}\mathcal{S}_k(\nu\times\nabla_{\tan}f), \qquad (6.5.29)
\end{aligned}$$

where the last equality uses (6.5.24). Going to the boundary and taking $\nu\times$ of both sides of (6.5.29) yields (6.5.25). $\square$

Proposition 6.5.5 *Assume that $\Omega\subset\mathbb{R}^3$ is a bounded* UR *domain. Then the operator $\lambda I+M_k$ is injective on $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega,d\sigma)$ granted that either*

$$k\in\mathbb{C}\ \ \textit{has}\ \ \operatorname{Im}k>0\ \ \textit{and}\ \ \lambda\in\mathbb{R}\ \ \textit{has}\ \ |\lambda|\geq\tfrac{1}{2}, \qquad (6.5.30)$$

or

$$k\in\mathbb{R}\setminus\{0\}\ \ \textit{and}\ \ \lambda\in(-\infty,-\tfrac{1}{2})\cup(\tfrac{1}{2},\infty). \qquad (6.5.31)$$

Proof. Let $\vec{f}\in L^{2,\mathrm{Div}}_{\tan}(\partial\Omega,d\sigma)$ be such that $(\lambda I+M_k)\vec{f}=0$. Our goal is to show that $\vec{f}=0$. We shall prove this by analyzing several cases. Assume first that $k\in\mathbb{C}$ has $\operatorname{Im}k>0$ and that $\lambda\in(\frac{1}{2},\infty)$. Set

$$E^{\pm}:=\operatorname{curl}\mathcal{S}_k\vec{f},\quad H^{\pm}:=\tfrac{1}{\sqrt{-1}k}\operatorname{curl}E^{\pm}\quad\text{in}\ \ \Omega_{\pm}, \qquad (6.5.32)$$

so that

$$(\Delta + k^2)E^{\pm} = (\Delta + k^2)H^{\pm} = 0 \quad \text{in} \ \ \Omega_{\pm}, \tag{6.5.33}$$

$$H^{\pm} = \tfrac{1}{\sqrt{-1}k}\Big(k^2\mathcal{S}_k\vec{f} + \nabla\mathcal{S}_k(\mathrm{Div}\,\vec{f})\Big) \quad \text{in} \ \ \Omega_{\pm}, \tag{6.5.34}$$

$$\mathcal{N}(E^{\pm}),\, \mathcal{N}(H^{\pm}) \in L^2(\partial\Omega, d\sigma) \ \text{ and } \ \exists\, E^{\pm}\Big|_{\partial\Omega}, \quad \exists\, H^{\pm}\Big|_{\partial\Omega}, \tag{6.5.35}$$

$$\mathrm{curl}\, E^{\pm} = \sqrt{-1}kH^{\pm} \quad \text{and} \quad \mathrm{curl}\, H^{\pm} = -\sqrt{-1}kE^{\pm} \quad \text{in} \ \ \Omega_{\pm}. \tag{6.5.36}$$

In particular, from (6.5.32), (6.5.34), and the fact that $\mathrm{Im}\, k > 0$, the fields E^-, H^- decay exponentially at infinity, so that Green's formula works both in Ω_+ and Ω_-. Thus Theorem 2.3.1, whose applicability is ensured by (6.5.33), (6.5.35) and (6.5.36), then allows us to write

$$\pm\int_{\partial\Omega}\langle \nu \times (E^{\pm})^c, \mathrm{curl}\, E^{\pm}\rangle\, d\sigma = \int_{\Omega_{\pm}}\Big[|\mathrm{curl}\, E^{\pm}|^2 - k^2|E^{\pm}|^2\Big]\, dX, \tag{6.5.37}$$

$$\pm\int_{\partial\Omega}\langle \nu \times (H^{\pm})^c, \mathrm{curl}\, H^{\pm}\rangle\, d\sigma = \int_{\Omega_{\pm}}\Big[|\mathrm{curl}\, H^{\pm}|^2 - k^2|H^{\pm}|^2\Big]\, dX, \tag{6.5.38}$$

where the superscript c indicates complex conjugation. We continue by noting that (6.5.34) implies

$$\nu \times H^+\Big|_{\partial\Omega} = \nu \times H^-\Big|_{\partial\Omega} \quad \text{on} \ \ \partial\Omega. \tag{6.5.39}$$

Furthermore, if

$$\mu := \frac{2\lambda - 1}{2\lambda + 1} \in (0,1) \tag{6.5.40}$$

then

$$\begin{aligned} \nu \times E^+\Big|_{\partial\Omega} - \mu\,\nu \times E^-\Big|_{\partial\Omega} &= (\tfrac{1}{2}I + M_k)\vec{f} - \mu(-\tfrac{1}{2}I + M_k)\vec{f} \\ &= (1-\mu)(\lambda I + M_k)\vec{f} = 0 \quad \text{on} \ \ \partial\Omega. \end{aligned} \tag{6.5.41}$$

Hence, for the choice (6.5.40),

$$\nu \times E^+\Big|_{\partial\Omega} = \mu\,\nu \times E^-\Big|_{\partial\Omega} \quad \text{on} \ \ \partial\Omega. \tag{6.5.42}$$

Using (6.5.37), (6.5.39), (6.5.42), we may now write

$$\begin{aligned} \int_{\Omega_+}\Big[|\mathrm{curl}\, E^+|^2 - k^2|E^+|^2\Big]\, dX &= \int_{\partial\Omega}\langle \nu \times (E^+)^c, \mathrm{curl}\, E^+\rangle\, d\sigma \\ &= \mu\int_{\partial\Omega}\langle \nu \times (E^-)^c, \mathrm{curl}\, E^-\rangle\, d\sigma \\ &= -\mu\int_{\Omega_-}\Big[|\mathrm{curl}\, E^-|^2 - k^2|E^-|^2\Big]\, dX. \end{aligned} \tag{6.5.43}$$

Thus,

$$\int_{\Omega_+}\Big[|\mathrm{curl}\,E^+|^2-k^2|E^+|^2\Big]\,dX=-\mu\int_{\Omega_-}\Big[|\mathrm{curl}\,E^-|^2-k^2|E^-|^2\Big]\,dX, \tag{6.5.44}$$

and, similarly,

$$\int_{\Omega_+}\Big[|\mathrm{curl}\,H^+|^2-k^2|H^+|^2\Big]\,dX=-\mu\int_{\Omega_-}\Big[|\mathrm{curl}\,H^-|^2-k^2|H^-|^2\Big]\,dX. \tag{6.5.45}$$

Use now (6.5.36) to re-write (6.5.45) in terms of the electric fields $E^\pm$, obtaining

$$\int_{\Omega_+}\Big[|k^2||E^+|^2-\tfrac{k^2}{|k|^2}|\mathrm{curl}\,E^+|^2\Big]\,dX=-\mu\int_{\Omega_-}\Big[|k|^2|E^-|^2-\tfrac{k^2}{|k|^2}|\mathrm{curl}\,E^-|^2\Big]\,dX. \tag{6.5.46}$$

Finally, combining (6.5.44) with (6.5.46) we arrive at

$$\begin{aligned} k^2\Big[\int_{\Omega_+}|E^+|^2\,dX+\mu\int_{\Omega_-}|E^-|^2\,dX\Big] &= \int_{\Omega_+}|\mathrm{curl}\,E^+|^2\,dX+\mu\int_{\Omega_-}|\mathrm{curl}\,E^-|^2\,dX \\ &= \frac{|k|^2}{k^2}\Big[\int_{\Omega_+}|E^+|^2\,dX+\mu\int_{\Omega_-}|E^-|^2\,dX\Big] \end{aligned} \tag{6.5.47}$$

or, equivalently,

$$(k^4-|k|^4)\Big[\int_{\Omega_+}|E^+|^2\,dX+\mu\int_{\Omega_-}|E^-|^2\,dX\Big]=0. \tag{6.5.48}$$

If, on the one hand, the expression in the brackets vanishes then $E^\pm=0$ in $\Omega_\pm$ and, hence,

$$0=\nu\times E^+\Big|_{\partial\Omega}-\nu\times E^-\Big|_{\partial\Omega}=\vec f, \tag{6.5.49}$$

as desired. If, on the other hand, $k^4-|k|^4=0$, then since $\mathrm{Im}\,k>0$ it follows that $k\in\sqrt{-1}\mathbb{R}_+$. In this scenario, we return to (6.5.37) and denote by $A_\pm$ the right-hand side of this identity. The assumption $k\in\sqrt{-1}\mathbb{R}_+$ entails $A_\pm\geq 0$ and we may also assume that $A_++A_->0$ (otherwise (6.5.49) holds). Since $\nu\times(E^\pm|_{\partial\Omega})=(\pm\frac{1}{2}I+M_k)\vec f=(-\lambda\pm\frac{1}{2})\vec f$, this becomes

$$(\mp\lambda+\tfrac{1}{2})\int_{\partial\Omega}\langle\vec f,\mathrm{curl}\,E^\pm\rangle\,d\sigma=A_\pm. \tag{6.5.50}$$

Going further, we note that by (6.5.36) and (6.5.39) the vector fields $(\mathrm{curl}\,E^+)|_{\partial\Omega}$ and $(\mathrm{curl}\,E^-)|_{\partial\Omega}$ have the same tangential components. Thus, $\int_{\partial\Omega}\langle\vec f,\mathrm{curl}\,E^+\rangle\,d\sigma=\int_{\partial\Omega}\langle\vec f,\mathrm{curl}\,E^-\rangle\,d\sigma$, since $\vec f$ is tangential. If these integrals vanish, then $A_\pm=0$ and we may once again conclude that $\vec f=0$ as in (6.5.49). If they are different from zero, taking the quotient of the two versions of (6.5.50) gives, after some elementary algebra,

$$\lambda = \frac{1}{2}\frac{A_- - A_+}{A_+ + A_-}. \tag{6.5.51}$$

Thus, necessarily, $\lambda \le \frac{1}{2}$, contradicting our original assumption.

In summary, the above reasoning shows that the operator $\lambda I + M_k$ is injective on $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$ whenever $\lambda \in (\frac{1}{2}, \infty)$. The case $\lambda \in (-\infty, -\frac{1}{2})$ can be treated similarly, and we now turn to the task of proving that $\pm\frac{1}{2}I + M_k$ are injective on $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$ while retaining the assumption that $\mathrm{Im}\, k > 0$. We shall consider in detail the case of the operator $\frac{1}{2}I + M_k$ since, again, $-\frac{1}{2}I + M_k$ is handled analogously. To this end, if $\vec{f} \in L^{2,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$ is such that $(\frac{1}{2}I + M_k)\vec{f} = 0$, define $E^{\pm}$, $H^{\pm}$ as in (6.5.32) and notice that, this time, $\nu \times E^+ = 0$. Then (6.5.37) gives $\int_\Omega \Big[|k|^2|H^+| - k^2|E^+|^2\Big]\, dX = 0$ and by inspecting the real and imaginary parts, it follows that either $H^+ = 0$ or $E^+ = 0$ in Ω_+. Based on this and (6.5.36), we then have $E^+ = H^+ = 0$ in Ω_+. Next, (6.5.39) gives $\nu \times H^- = 0$ in Ω_- and by performing the same type of analysis and before we arrive at the conclusion that $E^- = H^- = 0$ in Ω_-. Hence, ultimately, $\vec{f} = 0$ from (6.5.42). This finishes the proof of the proposition under the hypotheses listed in (6.5.30).

Next, assume that the conditions in (6.5.31) hold. In this scenario, the expression $\exp(\sqrt{-1}k|X|)$ intervening in (6.1.34) is only oscillatory in nature. Hence the decay of E^-, H^- at infinity, far from being exponential, is more subtle and this prevents us from writing the versions of the identities (6.5.37)-(6.5.38) for the unbounded domain Ω_-. Indeed, if $k \in \mathbb{R}\setminus\{0\}$, then (6.5.32), (6.5.34) only give

$$\begin{aligned} &H^-(X) \times (X/|X|) - |X|E^-(X) = o(1), \\ &E^-(X) \times (X/|X|) + |X|H^-(X) = o(1), \end{aligned} \qquad \text{as } |X| \to \infty, \tag{6.5.52}$$

uniformly in all directions in $\mathbb{R}^3$. These are the so-called Silver-Müller radiation conditions and are known to be equivalent (cf. the discussion on pp. 154-156 of [25]) to

$$\int_{|X|=R} |E^- \times \nu + H^-|^2\, d\sigma = o(1), \quad \int_{|X|=R} |H^- \times \nu - E^-|^2\, d\sigma = o(1) \;\text{ as }\; R \to \infty, \tag{6.5.53}$$

and even to the seemingly much weaker conditions

$$\int_{|X|=R} |E^-|^2\, d\sigma = O(1), \quad \int_{|X|=R} |H^-|^2\, d\sigma = O(1), \;\text{ as }\; R \to \infty. \tag{6.5.54}$$

To illustrate the delicate balance between various radiation-type conditions, it is worth comparing (6.5.54) with the celebrated Rellich lemma (cf., e.g., Lemma 2.11 on p. 31 in [25]) according to which

$$\left.\begin{array}{l} (\Delta + k^2)u = 0 \text{ in a connected, open neighborhood of infinity} \\ \text{where } k \in \mathbb{R}\setminus\{0\} \text{ and } \displaystyle\int_{|X|=R} |u|^2\, d\sigma = o(1) \;\text{ as }\; R \to \infty, \end{array}\right\} \Longrightarrow u \equiv 0. \tag{6.5.55}$$

Returning to the mainstream discussion, we write

$$
\begin{aligned}
\lim_{R\to\infty}\int_{|X|=R}&\big[|H^-\times\nu|^2+|E^-|^2\big]\,d\sigma\\
&=\lim_{R\to\infty}\Big(\int_{|X|=R}|H^-\times\nu-E^-|^2\,d\sigma+2\,\mathrm{Re}\int_{|X|=R}\langle\nu\times(E^-)^c,H^-\rangle\,d\sigma\Big)\\
&=\lim_{R\to\infty}2\,\mathrm{Re}\int_{|X|=R}\langle\nu\times(E^-)^c,H^-\rangle\,d\sigma\\
&=\lim_{R\to\infty}2\,\mathrm{Re}\Big(\int_{\partial\Omega}\langle\nu\times(E^-)^c,H^-\rangle\,d\sigma+\sqrt{-1}k\int_{B(0,R)\setminus\bar\Omega}\big[|H^-|^2-|E^-|^2\big]\,dX\Big)\\
&=2\,\mathrm{Re}\int_{\partial\Omega}\langle\nu\times(E^-)^c,H^-\rangle\,d\sigma\\
&=2\,\mu\,\mathrm{Re}\int_{\partial\Omega}\langle\nu\times(E^+)^c,H^+\rangle\,d\sigma\\
&=2\,\mu\,\mathrm{Re}\Big(\sqrt{-1}k^{-1}\int_{\Omega}\big[|\mathrm{curl}\,E^+|^2-k^2|E^+|^2\big]\,dX\Big)\\
&=0.
\end{aligned}
\tag{6.5.56}
$$

The first equality above is obtained by expanding $|H^-\times\nu-E^-|^2$, while the second one is a consequence of (6.5.53). The third equality is a consequence of (6.5.36) and the version of (6.5.37) written for the bounded domain $B(0,R)\setminus\bar\Omega$ (here, Theorem 2.3.1 and Proposition 3.2.5 are also implicitly used). The forth equality rests on the observation that the solid integral, as well as the wave number k are real. The fifth equality employs the identities (6.5.39) and (6.5.42). The sixth equality is the version of (6.5.37) corresponding to the sign plus. Finally, the last equality uses once again the fact that the solid integral, along with the wave number k, are real numbers.

Having justified (6.5.56), we may now deduce from this and (6.5.55) that E^- vanishes in Ω_-, at least if Ω_- is connected. The general case (i.e., when no topological assumption is made on Ω_-) is proved in a similar manner by working with the connected components of Ω_- and using the unique continuation property for the Helmholtz operator $\Delta+k^2$. See the proof of Theorem 2.1 in [85] for details in somewhat similar circumstances.

Altogether, the above reasoning shows that $E^-=0$ in Ω_-. This, (6.5.42) and (6.5.49) now prove that $\vec f=0$, as wanted. This concludes the proof of the proposition. □

Theorem 6.5.6 *Assume that $\Omega\subset\mathbb{R}^3$ is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. Then for every $k\in\mathbb{C}$ with $\mathrm{Im}\,k\geq 0$, $p\in(1,\infty)$ and $\lambda\in\mathbb{R}\setminus\{0\}$ there exists a small $\delta>0$, depending only on λ, k, p and the geometry of Ω, with the property that*

$$
\begin{aligned}
&\mathrm{dist}\,(\nu\,,\mathrm{VMO}\,(\partial\Omega,d\sigma))<\delta\Longrightarrow\\
&\qquad\lambda I+M_k \text{ is Fredholm with index zero on } L^{p,\mathrm{Div}}_{\tan}(\partial\Omega,d\sigma).
\end{aligned}
\tag{6.5.57}
$$

Furthermore, under the same background hypotheses, the following implication is valid for every $p\in(1,\infty)$:

$$
\Omega \text{ is a regular SKT domain} \Longrightarrow
$$

$$
M_k : L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma) \longrightarrow L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma) \text{ is a compact operator.} \tag{6.5.58}
$$

Proof. The claim in (6.5.57) is proved much as in Theorem 6.2.3, with the help of the identity (6.5.16) and Theorem 4.5.1. The claim in (6.5.58) is a direct consequence of the identity (6.5.16) and Theorem 4.5.1. □

Theorem 6.5.7 *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. Also, fix $p \in (1,\infty)$ along with some $\lambda \in (-\infty, -\frac{1}{2}] \cup [\frac{1}{2}, \infty)$. Then there exist $\varepsilon > 0$ and a sequence of complex numbers $\{\zeta_j\}_j$ (depending only on p, λ and the geometry of Ω) with $\mathrm{Im}\,\zeta_j \leq 0$ and no finite accumulation points, and such that for each $k \in \mathbb{C}\backslash\{\zeta_j\}_j$ the operator*

$$
\lambda I + M_k : L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma) \longrightarrow L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma) \tag{6.5.59}
$$

is invertible if Ω is an ε-regular SKT domain. Moreover, when $\lambda \in (-\infty, -\frac{1}{2}) \cup (\frac{1}{2}, \infty)$, then all ζ_j's satisfy $\mathrm{Im}\,\zeta_j < 0$.

Proof. To begin with, Theorem 6.5.6, Proposition 6.5.5 and standard Fredholm theory give that the operator (6.5.59) is an isomorphism when $p = 2$ for every $k \in \mathbb{C}$ with $\mathrm{Im}\,k > 0$. Next, fix $k_o \in \mathbb{C}$ with $\mathrm{Im}\,k_o > 0$, $\lambda \in \mathbb{R}$ with $|\lambda| \geq \frac{1}{2}$ and, for each $k \in \mathbb{C}$ write

$$
\lambda I + M_k = \lambda I + M_{k_o} + (M_k - M_{k_o}) = (\lambda I + M_{k_o})^{-1}\Big[I + (\lambda I + M_{k_o})(M_k - M_{k_o})\Big], \tag{6.5.60}
$$

regarded as operators on $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$. A useful observation is that $\lambda I + M_k$ is invertible on this space if and only if $I + (\lambda I + M_{k_o})(M_k - M_{k_o})$ is. Now, $M_k - M_{k_o}$ has a weakly singular kernel (since $\Phi_k - \Phi_{k_o}$ is, in fact, bounded). Also, from (6.5.16),

$$
\mathrm{Div}\,(M_k - M_{k_o})\vec{f} = -k^2\langle \nu, S_k\vec{f}\rangle + k_o^2\langle \nu, S_{k_o}\vec{f}\rangle - (K^*_k - K^*_{k_o})(\mathrm{Div}\,\vec{f}) \tag{6.5.61}
$$

for every $\vec{f} \in L^{2,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$. Since S_k, S_{k_o} and $K^*_k - K^*_{k_o}$ are also weakly singular integral operators, it follows that $M_k - M_{k_o}$ is compact on $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$. Hence, so is $(\lambda I + M_{k_o})(M_k - M_{k_o})$. Consider next the operator-valued holomorphic function

$$
\mathbb{C} \ni k \mapsto \mathcal{A}(k) := (\lambda I + M_{k_o})(M_k - M_{k_o}) \in \mathcal{L}\Big(L^{2,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)\Big) \tag{6.5.62}
$$

and note that $\mathcal{A}(k_o) = 0$. From the Analytic Fredholm Theorem (cf., e.g., [62]), it follows that $I + \mathcal{A}(k)$ has a bounded inverse on $L^{2,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$ except at isolated points $k \in \mathbb{C}$ which, in fact, are poles for the meromorphic function $(I + \mathcal{A}(k))^{-1}$. By Proposition 6.5.5 (and an observation made earlier in the proof), these poles belong to the closed lower-half complex plane. Thus, the conclusion in the theorem corresponding to the case when $p = 2$ follows.

Finally, having established the theorem in the case $p = 2$, the general case $p \in (1,\infty)$ follows from this, Theorem 6.5.6 and standard Fredholm theory. □

7 Second order elliptic systems: specific cases

With the material of §6 in place, we are ready to tackle the various elliptic boundary problems introduced there. Section 7.1 deals with boundary problems for the scalar Laplace operator, including the Dirichlet problem and the Neumann problem. In these cases, existence arguments are special cases of those in §5.5–§5.6. The major effort here is devoted to uniqueness proofs. Also the regularity problem, given Dirichlet data in $L^p_1(\partial\Omega, d\sigma)$, is analyzed. In addition, §7.1 treats transmission problems.

Section 7.2 is devoted to natural boundary problems for Stokes systems, §7.3 to boundary problems for Lamé systems, and §7.4 to boundary problems for Maxwell's equations. These subsections make heavy use of the material from §6.

7.1 Boundary value problems for the Laplacian

For a chord-arc domain Ω in the plane $\mathbb{R}^2 \equiv \mathbb{C}$, Laurentiev [73] has proved, in effect, that the harmonic measure and the arc-length measure on the boundary are A_∞ equivalent. This fact, as is well-known, is equivalent to the well-posedness of the Dirichlet problem

$$\left\{\begin{array}{l} \Delta u = 0 \text{ in } \Omega, \\ \mathcal{N}(u) \in L^p(\partial\Omega, d\sigma), \\ u\Big|_{\partial\Omega} = f \in L^p(\partial\Omega, d\sigma), \text{ given,} \end{array}\right. \tag{7.1.1}$$

whenever p is sufficiently large. By further combining this with certain conformal mapping techniques developed in [63], it has been observed in [54] that one may also obtain the solvability of the Neumann and Regularity problems for the dual range of indices.

In the higher dimensional setting, if Ω is a bounded NTA domain whose boundary is Ahlfors regular, by combining the A_∞ equivalence of the surface and harmonic measures on $\partial\Omega$ (cf. Proposition 3.1.16) with Theorem 5.8 on p. 105 in [55], it follows that (7.1.1) is well-posed if p is sufficiently large.

The goal of this subsection is to explore the extent to which such results hold in the higher dimensional setting. In this context, as a substitute for the conformal mapping and harmonic measure techniques alluded to above, we shall rely on the method of boundary layer potentials. Our first result in this regard is as follows.

Theorem 7.1.1 *Let $p_o \in (1,\infty)$ be given and assume that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. Then there exists $\varepsilon > 0$ which depends only on p_o, n and the John and Ahlfors regularity constants of Ω, such that if Ω is an ε-regular SKT domain then the interior Dirichlet boundary value problem (7.1.1) is well-posed for every $p > p_o$.*

Proof. This is a consequence of Theorem 7.1.2 below and interpolation with the case $p = \infty$, when the Maximum Principle applies. □

Theorem 7.1.2, which is our main result here, treats the interior and exterior Dirichlet problems with data from $L^p(\partial\Omega, d\sigma)$ and $L^p_1(\partial\Omega, d\sigma)$, $1 < p < \infty$, in the case when Ω is either a bounded ε-regular SKT domain, or the complement of the closure of such a domain.

Theorem 7.1.2 *Assume that $p \in (1, \infty)$ is given and that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded domain satisfying a two-sided local John condition and whose boundary is Ahlfors regular. Then there exists $\varepsilon > 0$ which depends only on p, n and the John and Ahlfors regularity constants of Ω with the following significance.*

If Ω is an ε-regular SKT domain then the interior Dirichlet boundary value problem (7.1.1) has a unique solution. In addition, there exists a finite constant $C > 0$ such that, for each $f \in L^p(\partial\Omega, d\sigma)$, the solution u of (7.1.1) obeys the natural estimate

$$\|\mathcal{N}(u)\|_{L^p(\partial\Omega,d\sigma)} \leq C\|f\|_{L^p(\partial\Omega,d\sigma)}, \tag{7.1.2}$$

and has the following regularity property:

$$\begin{aligned} &f \in L^p_1(\partial\Omega, d\sigma) \Longrightarrow \mathcal{N}(\nabla u) \in L^p(\partial\Omega, d\sigma) \\ &\textit{and} \quad \|\mathcal{N}(\nabla u)\|_{L^p(\partial\Omega,d\sigma)} \leq C\|f\|_{L^p_1(\partial\Omega,d\sigma)}. \end{aligned} \tag{7.1.3}$$

Similar results are valid for the exterior Dirichlet problem. When $n \geq 2$, this reads as follows. Given a function $f \in L^p(\partial\Omega, d\sigma)$ find $u \in C^0(\mathbb{R}^{n+1} \setminus \bar{\Omega})$ such that

$$\begin{cases} \Delta u = 0 \ \textit{in}\ \mathbb{R}^{n+1} \setminus \bar{\Omega}, \\ \mathcal{N}(u) \in L^p(\partial\Omega, d\sigma), \quad u\Big|_{\partial\Omega} = f, \\ u(X) = O(|X|^{1-n}) \ \textit{as}\ |X| \to \infty. \end{cases} \tag{7.1.4}$$

In the case when $n = 1$, the above decay condition should be changed to

$$u(X) = a \log|X| + O(1) \ \textit{as}\ |X| \to \infty, \tag{7.1.5}$$

for some a priori given constant $a \in \mathbb{R}$. Also, the standard nontangential maximal operator $\mathcal{N}$ in (7.1.2) should be replaced by its truncated version $\mathcal{N}^\delta$ (defined as in (2.3.26) but for the domain Ω_- in place of Ω), for some arbitrary, fixed $\delta > 0$, in which case the constant C in (7.1.2) depends on δ as well.

Proof. We divide the proof into several steps, starting with

`Step 1.` *The above interior and exterior Dirichlet problems are formulated in a meaningful fashion.* By combining Theorem 6.4 on p. 81 in [55] with Theorem 2 on p. 842 in [31], it follows that the nontangential boundary trace $u|_{\partial\Omega}$ exists for every harmonic function u in a domain Ω, as in the statement of the theorem, for which $\mathcal{N}u \in L^p(\partial\Omega, d\sigma)$. This shows that (7.1.1) is meaningful as stated. The argument for (7.1.4) is similar.

`Step 2.` *Existence of a solution for the interior Dirichlet problem obeying the natural estimate (7.1.2) and the regularity property (7.1.3), in the case when $n \geq 2$.* Relying on the fact that the operator (6.4.3) is invertible, we may take

$$u(X) := \mathcal{S}\Big(S^{-1}f\Big)(X), \qquad X \in \Omega. \tag{7.1.6}$$

The fact that this is a solution of (7.1.1) which satisfies the estimate (7.1.2) is now a consequence of Proposition 3.6.4. When $f \in L^p_1(\partial\Omega, d\sigma)$, we may invoke (6.4.2) and Theorem 3.5.2 in order to conclude that, for the solution (7.1.6), the estimate (7.1.3) holds.

`Step 3.` *Uniqueness for the interior Dirichlet problem when $n \geq 2$.* To start the proof, fix $X \in \Omega$ and consider the following Green function

$$G(X,Y) := E(X-Y) - \mathcal{S}\Big(S^{-1}(E(X-\cdot)|_{\partial\Omega})\Big)(Y), \qquad Y \in \Omega. \tag{7.1.7}$$

Then

$$G(X,\cdot) \in C^\infty(\Omega \setminus \{X\}), \quad \Delta_Y G(X,Y) = \delta_X(Y) \text{ in } \Omega, \qquad G(X,\cdot)\Big|_{\partial\Omega} = 0, \tag{7.1.8}$$

$$\mathcal{N}^\delta(\nabla_Y G(X,\cdot)), \mathcal{N}^\delta(G(X,\cdot)) \in L^{p'}(\partial\Omega, d\sigma), \quad \text{if } \ 0 < \delta < \tfrac{1}{4}\,\text{dist}\,(X, \partial\Omega), \tag{7.1.9}$$

where the truncated nontangential maximal function $\mathcal{N}^\delta$ has been introduced in (2.3.26), and $1/p + 1/p' = 1$. Also, as in (2.3.6), for each $0 < \delta < \text{diam}\,(\Omega)$, consider the (one-sided) δ-collar of the boundary, i.e., $\mathcal{O}_\delta := \{Z \in \Omega :\, \text{dist}\,(Z, \partial\Omega) \leq \delta\}$, and pick a family of functions ψ_δ, indexed by $0 < \delta < \frac{1}{4}\,\text{dist}\,(X, \partial\Omega)$, with the following properties

$$\psi_\delta \in C^\infty_0\,(\Omega), \quad 0 \leq \psi_\delta \leq 1, \quad |\partial^\alpha \psi_\delta| \leq C_\alpha \delta^{-|\alpha|} \quad \forall\, \alpha, \tag{7.1.10}$$

$$\psi_\delta \equiv 1 \text{ on } \Omega \setminus \mathcal{O}_\delta \quad \text{and} \quad \psi_\delta \equiv 0 \text{ on } \mathcal{O}_{\delta/2}. \tag{7.1.11}$$

Such a family can be constructed by, e.g., further regularizing the functions χ_δ introduced in (2.3.37). Let now u be a solution for the homogeneous version of (7.1.1). Then, if $0 < \delta < \frac{1}{4}\,\text{dist}\,(X, \partial\Omega)$, $\psi_\delta u \in C^\infty_0(\Omega)$ and successive integrations by parts give, for $X \in \Omega \setminus \mathcal{O}_\delta$,

$$\begin{aligned} u(X) &= (\psi_\delta u)(X) = \int_\Omega G(X,Y)\Delta_Y(\psi_\delta u)(Y)\,dY \\ &= 2\int_\Omega G(X,Y)\langle \nabla\psi_\delta(Y), \nabla u(Y)\rangle\,dY + \int_\Omega G(X,Y)(\Delta\psi_\delta)(Y)u(Y)\,dY \\ &= -2\int_\Omega \langle \nabla_Y G(X,Y), (\nabla\psi_\delta)(Y)\rangle u(Y)\,dY - \int_\Omega G(X,Y)(\Delta\psi_\delta)(Y)u(Y)\,dY \\ &=: I + II, \end{aligned} \tag{7.1.12}$$

since $\Delta u = 0$ in Ω and $\psi_\delta \equiv 0$ near $\partial\Omega$. Next, if $1 < p' < \infty$ is such that $1/p + 1/p' = 1$, based on (7.1.10)-(7.1.11) and (2.3.25), we may estimate

$$\begin{aligned} |I| &\leq \frac{C}{\delta}\int_{\mathcal{O}_\delta} |\nabla_Y G(X,Y)||u(Y)|\,dY \leq C\int_{\partial\Omega} \mathcal{N}^\delta(\nabla_Y G(X,\cdot)u)\,d\sigma \\ &\leq C\int_{\partial\Omega} \mathcal{N}^\delta(\nabla_Y G(X,\cdot))\mathcal{N}^\delta(u)\,d\sigma \\ &\leq C\|\mathcal{N}^\delta(\nabla_Y G(X,\cdot))\|_{L^{p'}(\partial\Omega,d\sigma)}\|\mathcal{N}^\delta(u)\|_{L^p(\partial\Omega,d\sigma)} \end{aligned} \tag{7.1.13}$$

and note that, by (7.1.9) and the same type of reasoning as in the justification of (2.3.61),

$$\|\mathcal{N}^\delta(\nabla_Y G(X,\cdot))\|_{L^{p'}(\partial\Omega,d\sigma)} = O(1) \quad \text{and} \quad \|\mathcal{N}^\delta(u)\|_{L^p(\partial\Omega,d\sigma)} = o(1) \quad \text{as} \ \ \delta \to 0^+. \qquad (7.1.14)$$

This proves that, for each fixed $X \in \Omega$,

$$\lim_{\delta\to 0^+} |I| = 0. \qquad (7.1.15)$$

Next, we turn our attention to II in (7.1.12). To set the stage, let $\{I_k\}_k$ be a decomposition of Ω into nonoverlapping Whitney cubes and, for each fixed $\delta > 0$, set

$$\mathcal{J}_\delta := \{k : \, I_k^\delta := I_k \cap \mathcal{O}_\delta \neq \emptyset\}. \qquad (7.1.16)$$

It follows that the side-length of each I_k^δ is comparable with δ. Going further, since $\partial\Omega$ (equipped with the measure σ and the Euclidean distance) is a space of homogeneous type, there exists a decomposition of $\partial\Omega$ into a grid of dyadic boundary "cubes" Q^δ, of side-length comparable with δ. For each $k \in \mathcal{J}_\delta$, select one such boundary dyadic cube Q_k^δ with the property that

$$\operatorname{dist}(I_k^\delta, \partial\Omega) = \operatorname{dist}(I_k^\delta, Q_k^\delta). \qquad (7.1.17)$$

Matters can be arranged so that the concentric dilates of these boundary dyadic cubes have bounded overlap. That is, for every $c \geq 1$ there exists a finite constant $C > 0$ such that

$$\sum_{k\in\mathcal{J}_\delta} \mathbf{1}_{cQ_k^\delta} \leq C \quad \text{on} \ \ \partial\Omega. \qquad (7.1.18)$$

Next, fix $X \in \Omega$, and assume that $\delta > 0$ is much smaller than $\operatorname{dist}(X, \partial\Omega)$. We may then write

$$\begin{aligned} \frac{1}{\delta^2} \int_{\substack{\frac{\delta}{2}\leq \operatorname{dist}(Y,\partial\Omega)\leq\delta \\ Y\in\Omega}} |G(X,Y)|\,|u(Y)|\,dY &\leq \sum_{k\in\mathcal{J}_\delta} \frac{1}{\delta^2} \int_{I_k^\delta} |G(X,Y)|\,|u(Y)|\,dY \\ &\leq \sum_{k\in\mathcal{J}_\delta} \left(\frac{1}{\delta}\int_{I_k^\delta} \left(\frac{|G(X,Y)|}{\delta}\right)^{p'} dY\right)^{1/p'} \left(\frac{1}{\delta}\int_{\mathcal{O}_\delta} |(\mathbf{1}_{I_k^\delta} u)(Y)|^p\,dY\right)^{1/p}. \end{aligned} \qquad (7.1.19)$$

Using $G(X,\cdot)|_{\partial\Omega} = 0$, the fact that there exists $c \geq 1$ such that

$$\operatorname{supp} \mathcal{N}^\delta(\mathbf{1}_{I_k^\delta} u) \subset c\, Q_k^\delta \qquad (7.1.20)$$

and (2.3.25), we then obtain

$$\frac{1}{\delta^2}\int\limits_{\substack{\frac{\delta}{2}\leq\mathrm{dist}(Y,\partial\Omega)\leq\delta\\ Y\in\Omega}}|G(X,Y)|\,|u(Y)|\,dY\leq\sum_{k\in\mathcal{J}_\delta}\frac{1}{\delta^2}\int_{I_k^\delta}|G(X,Y)|\,|u(Y)|\,dY$$

$$\leq\sum_{k\in\mathcal{J}_\delta}\frac{1}{\delta}\left(\frac{1}{\delta}\int_{I_k^\delta}\Big|G(X,Y)-\fint_{Q_k^\delta}G(X,\cdot)\,d\sigma\Big|^{p'}dY\right)^{1/p'}\left(\int_{cQ_k^\delta}|\mathcal{N}^\delta u|^p\,d\sigma\right)^{1/p}.$$

Note that every point $Y\in I_k^\delta$ is a corkscrew point, relative to any point $Z\in Q_k^\delta$. Then Lemma 3.1.13 guarantees the existence of a rectifiable path $\gamma(Z,Y)$ from Z to Y, of length bounded by $C\delta$ and such that for each point $P\in\gamma(Z,Y)$ we have $\mathrm{dist}\,(P,Z)\approx\mathrm{dist}\,(P,\partial\Omega)$. In particular, there exists $\beta>0$ such that

$$\gamma(Z,Y)\subset\Gamma_\beta(Z). \tag{7.1.21}$$

Fix now $Y\in I_k^\delta$ and, based on the Fundamental Theorem of Calculus and (7.1.21), write

$$\begin{aligned}\frac{1}{\delta}\left|G(X,Y)-\fint_{Q_k^\delta}G(X,Z)\,d\sigma(Z)\right|&\leq\frac{1}{\delta}\fint_{Q_k^\delta}|G(X,Y)-G(X,Z)|\,d\sigma(Z)\\&\leq\left|\fint_{Q_k^\delta}\left(\frac{1}{\delta}\int_{\gamma(Z,Y)}\partial_s G(X,P)\,ds(P)\right)d\sigma(Z)\right|\\&\leq\fint_{Q_k^\delta}|\mathcal{N}_\beta(\nabla G(X,\cdot))(Z)|\,d\sigma(Z),\end{aligned}\tag{7.1.22}$$

where ds and ∂_s denote, respectively, the arc-length measure and tangential derivative along $\gamma(Z,Y)$ (considered in the second variable of the function G). Returning with this back in (7.1.21) allows us to estimate

$$\begin{aligned}&\frac{1}{\delta^2}\int\limits_{\substack{\frac{\delta}{2}\leq\mathrm{dist}(Y,\partial\Omega)\leq\delta\\ Y\in\Omega}}|G(X,Y)|\,|u(Y)|\,dY\\&\qquad\leq\sum_{k\in\mathcal{J}_\delta}\left(\int_{Q_k^\delta}|\mathcal{N}_\beta(\nabla G(X,\cdot))|^{p'}\,d\sigma\right)^{1/p'}\left(\int_{cQ_k^\delta}|\mathcal{N}^\delta u|^p\,d\sigma\right)^{1/p}\\&\qquad\leq C\left(\int_{\partial\Omega}|\mathcal{N}_\beta(\nabla G(X,\cdot))|^{p'}d\sigma\right)^{1/p'}\left(\int_{\partial\Omega}|\mathcal{N}^\delta u|^p\,d\sigma\right)^{1/p},\end{aligned}\tag{7.1.23}$$

by Hölder's inequality and (7.1.18). Replacing δ by $2^{-j}\delta$, with $j\geq0$, in (7.1.23) then yields

$$\begin{aligned}&\frac{1}{\delta^2}\int\limits_{\substack{2^{-j-1}\delta\leq\mathrm{dist}(Y,\partial\Omega)\leq 2^{-j}\delta\\ Y\in\Omega}}|G(X,Y)|\,|u(Y)|\,dY\\&\qquad\leq C4^{-j}\left(\int_{\partial\Omega}|\mathcal{N}_\beta(\nabla G(X,\cdot))|^{p'}d\sigma\right)^{1/p'}\left(\int_{\partial\Omega}|\mathcal{N}^\delta u|^p\,d\sigma\right)^{1/p}.\end{aligned}\tag{7.1.24}$$

Thus, summing up (7.1.24) for $j = 0, 1, ...$ gives

$$\frac{1}{\delta^2}\int\limits_{\mathcal{O}_\delta} |G(X,Y)|\,|u(Y)|\,dY \leq C\left(\int_{\partial\Omega} |\mathcal{N}_\beta(\nabla G(X,\cdot))|^{p'}\,d\sigma\right)^{1/p'}\left(\int_{\partial\Omega} |\mathcal{N}^\delta u|^p\,d\sigma\right)^{1/p}, \quad (7.1.25)$$

which further entails

$$\begin{aligned} |II| &\leq \frac{C}{\delta^2}\int_{\mathcal{O}_\delta} |G(X,Y)||u(Y)|\,dY, \\ &\leq C\left(\int_{\partial\Omega} |\mathcal{N}_\beta(\nabla G(X,\cdot))|^{p'}\,d\sigma\right)^{1/p'}\left(\int_{\partial\Omega} |\mathcal{N}^\delta u|^p\,d\sigma\right)^{1/p}. \end{aligned} \quad (7.1.26)$$

Hence, thanks to (7.1.14), we also have

$$\lim_{\delta\to 0^+} |II| = 0. \quad (7.1.27)$$

Altogether, (7.1.12) and (7.1.14), (7.1.27), prove that $u(X) = 0$. Since $X \in \Omega$ was arbitrary, this shows that the problem (7.1.1) has a unique solution.

`Step 4.` *Uniqueness for the exterior Dirichlet problem when $n \geq 2$.* Fix a point $X^* \in \Omega$ and, for each $R > 2\,\mathrm{diam}\,(\Omega)$, set $\Omega_-^R := B(X^*, 2R) \setminus \bar{\Omega}$. Also, much as in (2.3.26), for an arbitrary function $v \in C^0(\Omega_-)$, we define the (exterior) truncated maximal function by

$$\mathcal{N}^R v(Q) := \sup\Big\{|v(X)| : \ X \in \Omega_- \text{ has } |X - Q| < \min\{R\,,\,2\,\mathrm{dist}\,(X, \partial\Omega)\}\Big\}, \quad Q \in \partial\Omega. \quad (7.1.28)$$

Then, if u solves (7.1.4) written for $f = 0$, it follows that for each R as above

$$\begin{cases} u \in C^0(\Omega_-^R) \text{ with } \Delta u = 0 \text{ in } \Omega_-^R, \\ \mathcal{N}^R(u) \in L^p(\partial\Omega_-^R, d\sigma), \\ u\Big|_{\partial\Omega_-^R} \in \bigcap_{1<q<\infty} L_1^q(\partial\Omega_-^R, d\sigma). \end{cases} \quad (7.1.29)$$

Now, the existence and uniqueness result for bounded domains in $\mathbb{R}^{n+1}$, $n \geq 2$, the regularity statement (7.1.3) and the integral representation (7.1.6), it follows that

$$\mathcal{N}^R(\nabla u) \in L^2(\partial\Omega_-^R, d\sigma). \quad (7.1.30)$$

In particular, this suffices to justify Green's formula

$$\int_{\Omega_-^R} |\nabla u|^2\,dX = -\int_{\partial\Omega_-^R} u\,\partial_\nu u\,d\sigma = -\int_{\partial\Omega} u\,\partial_\nu u\,d\sigma + \int_{\partial B(X^*,2R)} u\,\partial_\nu u\,d\sigma. \quad (7.1.31)$$

From assumptions, $u|_{\partial\Omega} = 0$ and $u|_{\partial B(X^*,2R)} = O(R^{1-n})$ as $R \to \infty$. Furthermore, as is well-known, if a function u, defined in the complement of a compact set in $\mathbb{R}^{n+1}$, is harmonic at infinity then

$$(\partial_r u)(r\omega) = \begin{cases} O(r^{-n}) \text{ if } n \geq 2, \\ O(r^{-2}) \text{ if } n = 1, \end{cases} \qquad \text{as } r \to \infty, \text{ uniformly for } \omega \in S^n. \tag{7.1.32}$$

Consequently, $\partial_\nu u\Big|_{\partial B(X^*,2R)} = O(R^{-n})$ as $R \to \infty$. Thus, from this and (7.1.31), we have that $\int_{\partial B(X^*,2R)} u\partial_\nu u\, d\sigma = O(R^{1-n}) = o(1)$ as $R \to \infty$, since $n \geq 2$. In turn, this implies

$$\int_{\Omega_-} |\nabla u|^2\, dX = \lim_{R\to\infty} \int_{\Omega_-^R} |\nabla u|^2\, dX = 0, \tag{7.1.33}$$

hence, $u \equiv 0$ in the unbounded component of Ω_-. To finish the proof of uniqueness in the current case, we therefore need to show that u also vanishes in any other bounded component of Ω_-. This, however, is a direct consequence of our uniqueness result for bounded domains, already proved above.

`Step 5.` *Uniqueness for the interior Dirichlet problem when $n = 1$.* To get started, we make the claim that, for each $p \in (1, \infty)$, the operator

$$L^p(\partial\Omega, d\sigma) \oplus \mathbb{R} \ni (g, c) \mapsto \Big(Sg + c, \int_{\partial\Omega} g\, d\sigma\Big) \in L^p_1(\partial\Omega, d\sigma) \oplus \mathbb{R} \tag{7.1.34}$$

is an isomorphism. Indeed, this is can be written as the sum of two operators, $(g, c) \mapsto (Sg, 0)$ and $(g, c) \mapsto \Big(c, \int_{\partial\Omega} g\, d\sigma\Big)$ which are, respectively, Fredholm with index zero and compact (in the context of (7.1.34)). Thus, on the one hand, (7.1.34) is Fredholm with index zero. On the other hand, the fact that (6.4.4) is invertible ensures that the operator (7.1.34) is injective. Altogether, this proves the claim made about (7.1.34). As a consequence, given an arbitrary, fixed point $X \in \Omega$, there exist (unique) $g_X \in \bigcap_{1<q<\infty} L^q_0(\partial\Omega, d\sigma)$ and $c(X) \in \mathbb{R}$ such that $Sg_X + c(X) = E(X - \cdot)|_{\partial\Omega}$. If, in place of (7.1.7), we now take

$$G(X, Y) := E(X - Y) - c(X) - \Big(\mathcal{S}g_X\Big)(Y), \qquad Y \in \Omega, \tag{7.1.35}$$

then this function continues to satisfy all the properties listed in (7.1.8)-(7.1.9). Once such a Green function has been constructed, the proof of uniqueness for the problem (7.1.1) when $n = 1$ proceeds as in the higher dimensional case (cf. Step 3).

`Step 6.` *Existence of a solution for the exterior Dirichlet problem which satisfying a natural estimate and regularity property.* In the case when $n \geq 2$, the function $u(X) := \mathcal{S}(S^{-1}f)$, $X \in \Omega_-$, does the job. Assume next that $n = 1$. The same type of reasoning as for (7.1.34) shows that the operator

$$L^p_{-1}(\partial\Omega, d\sigma) \oplus \mathbb{R} \ni (g, c) \mapsto \Big(Sg + c, \langle g, 1\rangle\Big) \in L^p(\partial\Omega, d\sigma) \oplus \mathbb{R} \tag{7.1.36}$$

is an isomorphism whenever $1 < p < \infty$. Thus, there exists $C > 0$ such that for any given function $f \in L^p(\partial\Omega, d\sigma)$ and constant $a \in \mathbb{R}$ one can find $g \in L^p_{-1}(\partial\Omega, d\sigma)$ and $c \in \mathbb{R}$ with

$$\|g\|_{L^p_{-1}(\partial\Omega,d\sigma)} + |c| \leq C\|f\|_{L^p(\partial\Omega,d\sigma)} \quad \text{and} \quad Sg + c = f, \ \ \langle g, 1\rangle = 2\pi\, a. \tag{7.1.37}$$

If we now set $u(X) := (\mathcal{S}g)(X) + c$ for $X \in \Omega_-$, it follows that u is a harmonic function for which $\|\mathcal{N}^\delta u\|_{L^p(\partial\Omega,d\sigma)} \leq C(\Omega, \delta)\|f\|_{L^p(\partial\Omega,d\sigma)}$, for every $\delta > 0$, and

$$u(X) = \frac{1}{2\pi}\langle g, 1\rangle \log|X| + c = a\log|X| + O(1) \quad \text{ as } \ |X| \to \infty. \tag{7.1.38}$$

Since, by virtue of (7.1.34) being invertible and compatible with (7.1.36), $f \in L^p_1(\partial\Omega, d\sigma)$ implies $g \in L^p(\partial\Omega, d\sigma)$, this function also satisfies (7.1.3).

`Step 7.` *Uniqueness for the exterior Dirichlet problem when $n = 1$.* Let u solve (7.1.4) for $f = 0$ and $a = 0$. In particular, $u(X) = O(1)$ as $|X| \to \infty$ so u is harmonic at infinity. Thus, from (7.1.32), $\partial_\nu u(X)\Big|_{\partial B(X^*,2R)} = O(R^{-2})$ as $R \to \infty$ and, hence, $\int_{\partial B(X^*,2R)} u\partial_\nu u\, d\sigma = O(R^{-1}) = o(1)$ as $R \to \infty$. These are the only alterations needed for the reasoning in Step 4 to go through in the current setting.

This concludes the proof of Theorem 7.1.2. $\square$

Theorem 7.1.3 *Let $p \in (1,\infty)$ be given and assume that $\Omega \subset \mathbb{R}^{n+1}$ is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. Then there exists $\varepsilon > 0$ which depends only on p, n and the John and Ahlfors regularity constants of Ω having the following role.*

If Ω is an ε-regular SKT domain then the Neumann boundary value problem, which asks to find a function $u \in C^0(\Omega)$ satisfying

$$\begin{cases} \Delta u = 0 \ \textit{in}\ \Omega, \\ \mathcal{N}(\nabla u) \in L^p(\partial\Omega, d\sigma), \\ \partial_\nu u = f \in L^p(\partial\Omega, d\sigma), \end{cases} \tag{7.1.39}$$

has a solution if and only if the datum f satisfies the necessary compatibility conditions

$$\int_{\partial D} f\, d\sigma = 0, \qquad \forall\, D \ \textit{connected component of}\ \Omega. \tag{7.1.40}$$

Moreover, this solution is unique modulo functions which are locally constant in Ω, and there exists a finite constant $C > 0$ such that if $f \in L^p(\partial\Omega, d\sigma)$ is as in (7.1.40) then any solution u of (7.1.39) satisfies

$$\|\mathcal{N}(\nabla u)\|_{L^p(\partial\Omega,d\sigma)} \leq C\|f\|_{L^p(\partial\Omega,d\sigma)}. \tag{7.1.41}$$

Finally, similar results are valid for the exterior Neumann problem, i.e., when Ω is replaced by $\Omega_- := \mathbb{R}^{n+1}\setminus\bar{\Omega}$ in (7.1.40)-(7.1.39), provided this time D in (7.1.40) stands for arbitrary bounded connected components of Ω_-, and a decay condition of the form

$$u(X) = \begin{cases} a + O(|X|^{1-n}) \ \textit{as } |X| \to \infty, \quad \textit{for some } a \in \mathbb{R}, \textit{ if } n \geq 2, \\ \frac{1}{2\pi}\Big(\int_{\partial D_\infty} f\, d\sigma\Big) \log|X| + O(1) \ \textit{as } |X| \to \infty, \quad \textit{if } n = 1, \end{cases} \tag{7.1.42}$$

where D_∞ is the unbounded component of Ω_-, is incorporated into the statement of the problem.

Proof. For the sake of clarity, we once again choose to divide the treatment of (7.1.39) into a series of steps. To state our first preliminary result, recall the definition of the truncated nontangential maximal function from (2.3.26).

`Step 1.` *If u is a harmonic function in Ω which satisfies*

$$\mathcal{N}(\nabla u) \in L^p(\partial\Omega, d\sigma) \textit{ for some } p \in (1, \infty), \tag{7.1.43}$$

then

$$\begin{aligned} &\exists\, u\Big|_{\partial\Omega} \in L^p_1(\partial\Omega, d\sigma), \quad \exists\, \nabla u\Big|_{\partial\Omega} \in L^p(\partial\Omega, d\sigma), \qquad \textit{and} \\ &u(X) = \mathcal{D}(u|_{\partial\Omega})(X) - \mathcal{S}(\partial_\nu u)(X) \ \ \textit{for every } X \in \Omega. \end{aligned} \tag{7.1.44}$$

The existence of boundary traces is justified by the same type of argument as in Step 1 of the proof of Theorem 7.1.2. With this in hand, the integral representation formula (7.1.44) can be established in an analogous fashion to the Cauchy reproducing formula (4.7.13), by using Theorem 2.3.1.

`Step 2.` *Let u be a solution for the homogeneous version of (7.1.39). If Ω is connected, then u is constant in Ω.* From Step 1, we know that

$$u = \mathcal{D}(u|_{\partial\Omega}) - \mathcal{S}(\partial_\nu u) = \mathcal{D}(u|_{\partial\Omega}) \quad \text{ in } \ \Omega, \tag{7.1.45}$$

since $\partial_\nu u = 0$. Going nontangentially to the boundary, this now gives $(-\frac{1}{2}I + K)(u|_{\partial\Omega}) = 0$. Hence, from the claim made in Theorem 6.4.1 about the operator (6.3.49), we may conclude that $u|_{\partial\Omega}$ is a constant function. Now, $\mathcal{D}$ maps constants on $\partial\Omega$ to constants in Ω, so (7.1.45) shows that u is a constant in Ω.

`Step 3.` *Assume that $f \in L^p_0(\partial\Omega, d\sigma)$ and that Ω is connected. Then the problem (7.1.39) has a solution u which obeys the estimate (7.1.41).* Given $f \in L^p_0(\partial\Omega, d\sigma)$, we may take

$$u(X) = \mathcal{S}\Big((-\tfrac{1}{2}I + K^*)^{-1} f\Big)(X), \qquad X \in \Omega. \tag{7.1.46}$$

From the claim made in Theorem 6.3.6 about the operator (6.3.44) we know that this is meaningful. Also, (7.1.41) holds. Finally, (3.3.40) shows that $\partial_\nu u = f$ so that u solves (7.1.39).

`Step 4.` *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain as in the statement of the theorem, with arbitrary topology. Then for each $f \in L^p(\partial\Omega, d\sigma)$ satisfying (7.1.40) the problem (7.1.39) admits a solution which obeys the estimate (7.1.41). Furthermore, this solution is unique modulo functions which are locally constant in Ω.* Based on Steps 1-3, this is readily seen by working in each connected component Ω_ℓ of Ω.

Steps 1-4 conclude the treatment of the interior Neumann problem (7.1.39).

The argument for the exterior Neumann problem starts by observing that, in the same spirit as before, matters can be reduced to working in each connected component of Ω_- separately. For each such component which happens to be bounded, our earlier reasoning applies. The only remaining case is when the bounded chord arch domain $\Omega \subset \mathbb{R}^{n+1}$ is such that $\mathbb{R}^{n+1} \setminus \bar{\Omega}$ is connected. In this scenario, $D_\infty = \Omega_-$ and the compatibility conditions (7.1.40) are void. Also, in place of (7.1.46) we now take

$$u(X) = (\mathcal{S}g)(X), \quad X \in \mathbb{R}^{n+1} \setminus \bar{\Omega}, \quad \text{where } \ g := (\tfrac{1}{2}I + K^*)^{-1} f. \tag{7.1.47}$$

This gives a solution for which the estimate (7.1.41) holds. Also, clearly, $u(X) = O(|X|^{1-n})$ as $|X| \to \infty$, if $n \geq 2$. Let us also observe that, if $v := \mathcal{S}g$ in $\Omega_+ := \mathbb{R}^{n+1} \setminus \overline{\Omega_-}$, then

$$\begin{aligned} \int_{\partial\Omega_-} g \, d\sigma &= \int_{\partial\Omega_-} (\tfrac{1}{2}I + K^*) g \, d\sigma - \int_{\partial\Omega_-} (-\tfrac{1}{2}I + K^*) g \, d\sigma \\ &= \int_{\partial\Omega_-} f \, d\sigma - \int_{\partial\Omega_+} \partial_\nu v \, d\sigma = \int_{\partial D_\infty} f \, d\sigma, \end{aligned} \tag{7.1.48}$$

since v is harmonic in Ω_+ and satisfies $\mathcal{N}(\nabla v) \in L^2(\partial\Omega_+, d\sigma)$. Hence, when $n = 1$, we have

$$\begin{aligned} u(X) &= (\mathcal{S}g)(X) = \frac{1}{2\pi} \Big(\int_{\partial\Omega_-} g \, d\sigma \Big) \log|X| + O(|X|^{-1}) \\ &= \frac{1}{2\pi} \Big(\int_{\partial D_\infty} f \, d\sigma \Big) \log|X| + O(|X|^{-1}) \quad \text{as } \ |X| \to \infty, \end{aligned} \tag{7.1.49}$$

which shows that this particular solution satisfies an even stronger decay condition than the one specified in the statement of the theorem.

Therefore, we are left with the task of proving uniqueness (modulo locally constant functions) for the exterior Neumann problem. Let then u be a function which which decays as in (7.1.42) and solves the version of (7.1.39) in which $f = 0$ and Ω has been replaced by $\Omega_- = \mathbb{R}^{n+1} \setminus \bar{\Omega}$. By reasoning as before, there is no loss of generality is assuming that the latter is a connected domain. Also, by subtracting a constant from u if necessary, it can be assumed that u is harmonic at infinity, in which case (7.1.32) holds. When $n \geq 2$, this suffices to prove the following integral representation formula

$$u(X) = -\mathcal{D}(u|_{\partial\Omega})(X) + \mathcal{S}(\partial_\nu u)(X), \qquad X \in \Omega_-. \tag{7.1.50}$$

Using $\partial_\nu u = 0$, then going nontangentially to the boundary, we arrive at $(\frac{1}{2}I + K)(u|_{\partial\Omega}) = 0$. Hence, since the operator (6.3.47) in Theorem 6.4.1 is invertible in our setting, this gives $u|_{\partial\Omega} = 0$. Returning with this back in (7.1.50) we obtain $u = 0$ in Ω_-.

When $n = 1$, the decay exhibited by u does not, generally speaking, permit us to write (7.1.50). However, writing (7.1.50) with Ω_- replaced by $B(R, X^*) \cap \Omega_-$, with X^* as before and R large, then for each $j \in \{1, ..., n+1\}$ taking ∂_j of both sides (in order to enhance the decay) and, finally, passing to the limit $R \to \infty$ yields (keeping in mind that $\partial_\nu u = 0$ on $\partial\Omega$) the weaker version

$$\nabla\Big[u+\mathcal{D}(u|_{\partial\Omega})\Big](X)=0 \qquad \forall\, X\in\Omega_-. \tag{7.1.51}$$

Consequently, since Ω_- is connected,

$$u=-\mathcal{D}(u|_{\partial\Omega})+c \quad \text{ in } \ \Omega_-, \tag{7.1.52}$$

for some constant c. Once again, going to the nontangentially to the boundary in (7.1.52) gives $(\frac{1}{2}I+K)(u|_{\partial\Omega})=c$, hence zero in $L^p_1(\partial\Omega,d\sigma)/\mathbb{R}$. Invoking the fact that (6.3.50) is invertible, now allows us to conclude that $u|_{\partial\Omega}$ is a constant. Utilizing this back in (7.1.52) ultimately gives $u\equiv$ constant in Ω_-, as desired. This finishes the proof of the Theorem 7.1.3. $\square$

Theorem 7.1.4 *Let $p\in(1,\infty)$ and $\mu\in(0,1)$ be given and assume that $\Omega\subset\mathbb{R}^{n+1}$ is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. As usual, set $\Omega_+:=\Omega$, $\Omega_-:=\mathbb{R}^{n+1}\setminus\bar{\Omega}$.*

Then there exists $\varepsilon>0$ which depends only on p, μ, n and the John and Ahlfors regularity constants of Ω having the following role. If Ω is an ε-regular SKT domain then the transmission boundary value problem, concerned with finding two functions $u^{\pm}\in C^0(\Omega_{\pm})$ satisfying

$$\begin{cases} \Delta u^{\pm}=0 \ \textit{in } \Omega_{\pm}, \\ \mathcal{N}(\nabla u^{\pm})\in L^p(\partial\Omega,d\sigma), \\ u^+\Big|_{\partial\Omega}-u^-\Big|_{\partial\Omega}=f\in L^p_1(\partial\Omega,d\sigma), \quad \textit{given}, \\ \partial_\nu u^+-\mu\,\partial_\nu u^-=g\in L^p(\partial\Omega,d\sigma), \quad \textit{given}, \end{cases} \tag{7.1.53}$$

and the decay condition

$$u^-(X)=\begin{cases} O(|X|^{1-n}) \ \textit{as } |X|\to\infty, \quad \textit{if} \ \ n\geq 2, \\ -\frac{1}{2\pi\mu}\Big(\int_{\partial\Omega}g\,d\sigma\Big)\log|X|+O(|X|^{-1}) \ \textit{as } |X|\to\infty, \quad \textit{if } n=1, \end{cases} \tag{7.1.54}$$

has a unique solution. In addition, there exists $C>0$ such that

$$\|\mathcal{N}(\nabla u^+)\|_{L^p(\partial\Omega,d\sigma)}+\|\mathcal{N}(\nabla u^-)\|_{L^p(\partial\Omega,d\sigma)}\leq C\|f\|_{L^p_1(\partial\Omega,d\sigma)}+C\|g\|_{L^p(\partial\Omega,d\sigma)}. \tag{7.1.55}$$

Proof. To begin with, we would like to point out that, from the proof of Theorem 7.1.3, the nontangential traces $u^{\pm}|_{\partial\Omega}$, $(\nabla u^{\pm})|_{\partial\Omega}$ exist. Hence, the formulation of the transmission problem is meaningful. Next, given that under our current assumptions on Ω, the operator (6.3.43) is invertible and that $\lambda:=\frac{1}{2}\frac{\mu+1}{\mu-1}\in(-\infty,\frac{1}{2})$ whenever $\mu\in(0,1)$, a solution for (7.1.53)-(7.1.54) can be written explicitly in the form

$$u^+ \ := \ \mathcal{D}^+f+\frac{1}{1-\mu}\mathcal{S}^+\Big[\Big(\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I+K^*\Big)^{-1}\Big(g-\partial_\nu\mathcal{D}^+f-\mu\,\partial_\nu\mathcal{D}^-f\Big)\Big] \ \text{ in } \ \Omega_+, \tag{7.1.56}$$

$$u^- \ := \ \mathcal{D}^-f+\frac{1}{1-\mu}\mathcal{S}^-\Big[\Big(\tfrac{1}{2}\tfrac{\mu+1}{\mu-1}I+K^*\Big)^{-1}\Big(g-\partial_\nu\mathcal{D}^+f-\mu\,\partial_\nu\mathcal{D}^-f\Big)\Big] \ \text{ in } \ \Omega_-. \tag{7.1.57}$$

Here, for an arbitrary $h\in L^p(\partial\Omega,d\sigma)$, we have denoted by $\mathcal{D}^\pm h$ and $\mathcal{S}^\pm h$, respectively, the double and single layer potentials associated with Ω viewed as functions defined in $\Omega_\pm$. Then a direct calculation shows that the conditions in (7.1.53) are satisfied and that the estimate (7.1.55) holds. Furthermore, it is clear that $u^-(X)=O(|X|^{1-n})$ as $|X|\to\infty$, when $n\geq 2$. To compute the decay at infinity for u^- in the case $n=1$, we first note that, much as in (7.1.48), the following general implication is valid

$$\lambda\in\mathbb{R},\;\; h_1,h_2\in L^p(\partial\Omega,d\sigma),\;\; (\lambda I+K^*)h_1=h_2 \Longrightarrow \int_{\partial\Omega}h_2\,d\sigma=(\lambda+\tfrac{1}{2})\int_{\partial\Omega}h_1\,d\sigma. \tag{7.1.58}$$

Set now $h_1:=\Big(\frac{1}{2}\frac{\mu+1}{\mu-1}I+K^*\Big)^{-1}\Big(g-\partial_\nu\mathcal{D}^+f-\mu\,\partial_\nu\mathcal{D}^-f\Big)$ and $h_2:=g-\partial_\nu\mathcal{D}^+f-\mu\,\partial_\nu\mathcal{D}^-f$ so that $\Big(\frac{1}{2}\frac{\mu+1}{\mu-1}I+K^*\Big)h_1=h_2$. Also, observe that

$$\begin{aligned}\int_{\partial\Omega}h_2\,d\sigma &= \int_{\partial\Omega}g\,d\sigma+\int_{\partial\Omega}\partial_\nu\mathcal{D}^+f\,d\sigma-\mu\int_{\partial\Omega}\partial_\nu\mathcal{D}^-f\,d\sigma\\ &= \int_{\partial\Omega}g\,d\sigma+(1-\mu)\int_{\partial\Omega}\partial_\nu\mathcal{D}^+f\,d\sigma\\ &= \int_{\partial\Omega}g\,d\sigma,\end{aligned} \tag{7.1.59}$$

since a calculation based on (3.6.31) shows that

$$\partial_\nu\mathcal{D}^+f=\partial_\nu\mathcal{D}^-f,\qquad \forall\,f\in L^p_1(\partial\Omega,d\sigma). \tag{7.1.60}$$

Hence, $\int_{\partial\Omega}h_1\,d\sigma=\frac{\mu-1}{\mu}\int_{\partial\Omega}g\,d\sigma$, by (7.1.58) and (7.1.60). Consequently, when $n=1$,

$$\begin{aligned}u^-(X) &= \mathcal{D}^-f(X)+\tfrac{1}{1-\mu}\mathcal{S}^-h_1(X)=\frac{1}{2\pi(1-\mu)}\Big(\int_{\partial\Omega}h_1\,d\sigma\Big)\log|X|+O(|X|^{-1})\\ &= -\frac{1}{2\pi\mu}\Big(\int_{\partial\Omega}g\,d\sigma\Big)\log|X|+O(|X|^{-1}),\qquad \text{as }\;|X|\to\infty,\end{aligned} \tag{7.1.61}$$

in agreement with the case $n=1$ of (7.1.54).

There remains the issue of proving uniqueness for (7.1.53)-(7.1.54). To this end, assume that $u^\pm$ solve the homogeneous version of (7.1.53) and that

$$u^-(X)=\begin{cases} O(|X|^{1-n})\text{ as }|X|\to\infty, & \text{if }\;n\geq 2,\\ O(|X|^{-1})\text{ as }|X|\to\infty, & \text{if }\;n=1.\end{cases} \tag{7.1.62}$$

This shows that the harmonic functions $u^\pm$ are sufficiently well-behaved so that the following Green's formulas hold

$$u^\pm=\pm\mathcal{D}^\pm\Big(u^\pm\Big|_{\partial\Omega}\Big)\mp\mathcal{S}^\pm(\partial_\nu u^\pm)\quad\text{ in }\;\Omega_\pm. \tag{7.1.63}$$

Taking the normal derivatives of both sides of (7.1.63), then adding the two versions corresponding to choice plus or minus for the sign, gives, keeping in mind that $u^+|_{\partial\Omega} = u^-|_{\partial\Omega}$ and $\partial_\nu u^+ = \mu\partial_\nu u^-$, that $(\frac{1}{2}\frac{\mu+1}{\mu-1}I + K^*)(\partial_\nu u^-) = 0$. Thus, $\partial_\nu u^- = 0$ and, further, $\partial_\nu u^+ = 0$. In particular, by Proposition 3.2.5, the function

$$u := \begin{cases} u^+ & \text{in} \ \ \Omega_+, \\ u^- & \text{in} \ \ \Omega_-, \end{cases} \tag{7.1.64}$$

belongs to $L^1_{loc}(\mathbb{R}^{n+1})$ and, in the distributional sense, satisfies $\Delta u = 0$ in $\mathbb{R}^{n+1}$. Furthermore, u decays at infinity. Hence, $u = 0$ in $\mathbb{R}^{n+1}$, as wanted. □

7.2 Boundary value problems for the Stokes system

In this subsection we present our main well-posedness results for the Stokes system of hydrostatics in ε-regular SKT domains. They include the three basic types of boundary conditions: Dirichlet and its regular version (cf. Theorem 7.2.1), Neumann (Theorem 7.2.2), and transmission (Theorem 7.2.3).

The proof of Theorem 7.2.1 largely parallels that of its harmonic counterpart, Theorem 7.1.2, and relies in an essential fashion on the invertibility results from Theorem 6.4.3, Theorem 6.3.8. A similar set of comments apply to Theorem 7.2.2, Theorem 7.2.3, in which we follow strategies similar to those utilized in the course of the proofs of Theorem 7.1.3, Theorem 7.1.4, as well as [94] where these problems have been treated in Lipschitz domains. We shall therefore omit including further details and, instead, confine ourselves to providing the statements of the aforementioned theorems.

Theorem 7.2.1 *Let $p \in (1,\infty)$ be given and assume that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. Then there exists $\varepsilon > 0$ which depends only on p, n and the John and Ahlfors regularity constants of Ω with the following property. If Ω is an ε-regular SKT domain then the interior Dirichlet boundary value problem*

$$\begin{cases} \Delta\vec{u} = \nabla\pi, \ \ \operatorname{div}\vec{u} = 0 \quad \textit{in} \ \ \Omega, \\ \mathcal{N}(\vec{u}) \in L^p(\partial\Omega, d\sigma), \\ \vec{u}\Big|_{\partial\Omega} = \vec{f} \in L^p_{\nu_+}(\partial\Omega, d\sigma), \end{cases} \tag{7.2.1}$$

has a solution, which is unique modulo adding functions which are locally constant in Ω to the pressure term. In addition, there exists a finite constant $C > 0$ such that

$$\|\mathcal{N}(\vec{u})\|_{L^p(\partial\Omega,d\sigma)} \leq C\|\vec{f}\|_{\left[L^p(\partial\Omega,d\sigma)\right]^{n+1}}, \tag{7.2.2}$$

and the solution satisfies the following regularity property:

$$\begin{aligned} &\vec{f} \in L^p_{1,\nu_+}(\partial\Omega, d\sigma) \Longrightarrow \mathcal{N}(\nabla\vec{u}), \mathcal{N}(\pi) \in L^p(\partial\Omega, d\sigma) \\ &\textit{and} \quad \|\mathcal{N}(\nabla\vec{u})\|_{L^p(\partial\Omega,d\sigma)} + \|\mathcal{N}(\pi)\|_{L^p(\partial\Omega,d\sigma)} \leq C\|\vec{f}\|_{\left[L^p_1(\partial\Omega,d\sigma)\right]^{n+1}}. \end{aligned} \tag{7.2.3}$$

Similar results (including (7.2.3)) are valid for the exterior Dirichlet problem, formulated much as (7.2.1) but for $\mathbb{R}^{n+1}\setminus\bar{\Omega}$, with the additional decay conditions

$$\vec{u}(X)=\begin{cases} O(|X|^{1-n}) \ \text{as } |X|\to\infty, & \text{if } \ n\geq 2,\\ E(X)\vec{A}+O(1) \ \text{as } |X|\to\infty, & \text{if } \ n=1,\end{cases} \tag{7.2.4}$$

$$\partial_j\vec{u}(X)=\begin{cases} O(|X|^{-n}) \ \text{as } |X|\to\infty, & \text{if } \ n\geq 2,\\ \partial_j E(X)\vec{A}+O(|X|^{-2}) \ \text{as } |X|\to\infty, & \text{if } \ n=1,\end{cases} \tag{7.2.5}$$

$$\pi(X)=\begin{cases} O(|X|^{-n}) \ \text{as } |X|\to\infty, & \text{if } \ n\geq 2,\\ \langle\nabla E_\Delta(X),\vec{A}\rangle+O(|X|^{-2}) \ \text{as } |X|\to\infty, & \text{if } \ n=1,\end{cases} \tag{7.2.6}$$

where E_Δ is the fundamental solution for the Laplacian from (3.3.24), $E(X)=(E_{jk}(X))_{1\leq j,k\leq n+1}$ with $E_{jk}(X)$ as in (6.1.20), for some a priori given constant $\vec{A}\in\mathbb{R}^2$. Also, the standard nontangential maximal operator in (7.2.2), (7.2.3) should be replaced by its truncated version.

Theorem 7.2.2 *Let $p\in(1,\infty)$ be given and assume that $\Omega\subset\mathbb{R}^{n+1}$, $n\geq 1$, is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. Then there exists $\varepsilon>0$ which depends only on p, n and the John and Ahlfors regularity constants of Ω with the following property. If $1-\varepsilon<\gamma\leq 1$ and Ω is an ε-regular SKT domain then the interior Neumann boundary value problem*

$$\begin{cases} \Delta\vec{u}=\nabla\pi, \ \operatorname{div}\vec{u}=0 \quad \text{in} \ \ \Omega,\\ \mathcal{N}(\nabla\vec{u}),\,\mathcal{N}(\pi)\in L^p(\partial\Omega,d\sigma),\\ \partial^\gamma_\nu(\vec{u},\pi)=\vec{f}\in\Big[L^p(\partial\Omega,d\sigma)\Big]^{n+1},\end{cases} \tag{7.2.7}$$

has a solution if and only if the datum $\vec{f}$ satisfies finitely many (necessary) linear compatibility conditions; more precisely, if and only if

$$\vec{f}\in\operatorname{Im}\Big(-\tfrac{1}{2}I+K^*_\gamma: L^p_{\Psi^\gamma_+}(\partial\Omega,d\sigma)\to L^p_{\Psi^\gamma_+}(\partial\Omega,d\sigma)\Big). \tag{7.2.8}$$

Moreover, this solution is unique modulo adding to the velocity field functions from $\Psi^\gamma(\Omega)$. In addition, there exists a finite constant $C>0$ such that

$$\|\mathcal{N}(\nabla\vec{u})\|_{L^p(\partial\Omega,d\sigma)}+\|\mathcal{N}(\pi)\|_{L^p(\partial\Omega,d\sigma)}\leq C\|\vec{f}\|_{\big[L^p(\partial\Omega,d\sigma)\big]^{n+1}}. \tag{7.2.9}$$

Finally, a similar result holds for the exterior domain $\mathbb{R}^{n+1}\setminus\bar{\Omega}$ granted that one includes the decay conditions

$$\vec{u}(x) = \begin{cases} O(|X|^{1-n}) \quad \text{as} \quad |X| \to \infty, \quad \text{if} \quad n \geq 2, \\ E(X)\Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|X|^{-1}) \quad \text{as} \quad |X| \to \infty, \quad \text{if} \ n = 1, \end{cases} \tag{7.2.10}$$

$$\partial_j \vec{u}(X) = (\partial_j E)(X)\Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|X|^{-n-1}) \quad \text{as} \quad |X| \to \infty, \ 1 \leq j \leq n+1, \tag{7.2.11}$$

$$\pi(X) = \begin{cases} O(|X|^{-n}) \quad \text{as} \quad |X| \to \infty, \quad \text{if} \quad n \geq 2, \\ (-\nabla E_\Delta)(X) \cdot \Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|X|^{-2}) \quad \text{as} \ |X| \to \infty, \quad \text{if} \ n = 1, \end{cases} \tag{7.2.12}$$

where E_Δ is the fundamental solution for the Laplacian from (3.3.24), $E(X) = (E_{jk}(X))_{1 \leq j,k \leq n+1}$ with $E_{jk}(X)$ as in (6.1.20). More precisely, a solution to the exterior problem satisfying the above decay conditions exists if and only if

$$\vec{f} \in \operatorname{Im}\Big(\tfrac{1}{2}I + K_\gamma^* : L^p_{\Psi^\gamma_-}(\partial\Omega, d\sigma) \to L^p_{\Psi^\gamma_-}(\partial\Omega, d\sigma)\Big), \tag{7.2.13}$$

and solutions are unique modulo adding to the velocity field functions from $\Psi^\gamma(\mathbb{R}^{n+1} \setminus \bar{\Omega})$.

Theorem 7.2.3 *Let $p \in (1, \infty)$ and $\mu \in (0,1)$ be given and assume that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. Set $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$.*

Then there exists $\varepsilon > 0$ which depends only on p, n, μ and the John and Ahlfors regularity constants of Ω with the following property. If $1 - \varepsilon < \gamma \leq 1$ and Ω is an ε-regular SKT domain then the transmission problem, concerned with finding two pairs of functions $(\vec{u}_\pm, \pi_\pm)$ in $\Omega_\pm$ satisfying

$$\begin{cases} \Delta \vec{u}_\pm = \nabla \pi_\pm, \ \operatorname{div} \vec{u}_\pm = 0 \quad \text{in} \ \ \Omega_\pm, \\ \mathcal{N}(\nabla \vec{u}_\pm), \mathcal{N}(\pi_\pm) \in L^p(\partial\Omega, d\sigma), \\ \vec{u}_+\Big|_{\partial\Omega} - \vec{u}_-\Big|_{\partial\Omega} = \vec{g} \in \Big[L^p_1(\partial\Omega, d\sigma)\Big]^{n+1}, \\ \partial^\gamma_\nu(\vec{u}_+, \pi_+) - \mu\, \partial^\gamma_\nu(\vec{u}_-, \pi_-) = \vec{f} \in \Big[L^p(\partial\Omega, d\sigma)\Big]^{n+1}, \end{cases} \tag{7.2.14}$$

and the decay conditions

$$\vec{u}_-(X) = \begin{cases} O(|X|^{1-n}) \quad \text{as} \quad |X| \to \infty, \quad \text{if} \quad n \geq 2, \\ -\frac{1}{\mu} E(X)\Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|x|^{-1}) \quad \text{as} \ \ |x| \to \infty, \quad \text{if} \ n = 2, \end{cases} \tag{7.2.15}$$

$$\partial_j \vec{u}_-(X) = -\tfrac{1}{\mu}(\partial_j E)(X)\Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|X|^{-n-1}) \quad \text{as} \ \ |X| \to \infty, \tag{7.2.16}$$

$$\pi_-(X) = \begin{cases} O(|X|^{-n}) \quad \text{as} \quad |X| \to \infty, \quad \text{if} \quad n \geq 2, \\ \frac{1}{\mu}(\nabla E_\Delta)(X) \cdot \Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|X|^{-2}) \quad \text{as} \ \ |X| \to \infty, \quad \text{if} \ n = 1, \end{cases} \tag{7.2.17}$$

where E_Δ is the fundamental solution for the Laplacian from (3.3.24), $E(X) = (E_{jk}(X))_{1 \leq j,k \leq n+1}$ with $E_{jk}(X)$ as in (6.1.20), has a unique solution. In addition, there exists $C > 0$ such that

$$\|\mathcal{N}(\nabla\vec{u}_\pm)\|_{L^p(\partial\Omega,d\sigma)} + \|\mathcal{N}(\pi_\pm)\|_{L^p(\partial\Omega,d\sigma)} \leq C\|\vec{g}\|_{\left[L^p_1(\partial\Omega,d\sigma)\right]^{n+1}} + C\|\vec{f}\|_{\left[L^p(\partial\Omega,d\sigma)\right]^{n+1}}. \tag{7.2.18}$$

7.3 Boundary value problems for the Lamé system

Here we collect results for the Lamé system of elastostatics in ε-regular SKT domains in $\mathbb{R}^{n+1}$ which are analogous in spirit to those in § 7.2. Theorem 7.3.1, Theorem 7.3.2, Theorem 7.3.3 below deal, respectively, with Dirichlet (and regularity), Neumann and transmission problems for the Lamé system (6.1.5). Their proofs are analogous to their counterparts in § 7.2 (relying essentially on the invertibility results from Theorems 6.3.9, 6.4.3) and we omit the routine details.

Theorem 7.3.1 *Let λ, μ, as in (6.1.1), and $p \in (1,\infty)$ be given and assume that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. Then there exists $\varepsilon > 0$ which depends only on p, n, λ, μ and the John and Ahlfors regularity constants of Ω with the following property. If Ω is an ε-regular SKT domain then the interior Dirichlet boundary value problem*

$$\begin{cases} \mu\,\Delta\vec{u} + (\mu+\lambda)\nabla\mathrm{div}\,\vec{u} = 0 \quad \text{in } \Omega, \\ \vec{u}\Big|_{\partial\Omega} = \vec{f} \in \left[L^p(\partial\Omega,d\sigma)\right]^{n+1}, \\ \mathcal{N}(\vec{u}) \in L^p(\partial\Omega,d\sigma), \end{cases} \tag{7.3.1}$$

has a unique solution. In addition, there exists a finite constant $C > 0$ such that

$$\|\mathcal{N}(\vec{u})\|_{L^p(\partial\Omega,d\sigma)} \leq C\|\vec{f}\|_{\left[L^p(\partial\Omega,d\sigma)\right]^{n+1}}, \tag{7.3.2}$$

and the solution satisfies the following regularity property:

$$\begin{aligned} &\vec{f} \in \left[L^p_1(\partial\Omega,d\sigma)\right]^{n+1} \Longrightarrow \mathcal{N}(\nabla\vec{u}) \in L^p(\partial\Omega,d\sigma) \\ &\text{and} \quad \|\mathcal{N}(\nabla\vec{u})\|_{L^p(\partial\Omega,d\sigma)} \leq C\|\vec{f}\|_{\left[L^p_1(\partial\Omega,d\sigma)\right]^{n+1}}. \end{aligned} \tag{7.3.3}$$

Similar results (including (7.3.3)) are valid for the exterior Dirichlet problem, formulated much as (7.3.1) but for $\mathbb{R}^{n+1}\setminus\bar{\Omega}$, with the additional decay conditions

$$\vec{u}(X) = \begin{cases} O(|X|^{1-n}) \ \text{as } |X|\to\infty, & \text{if } \ n \geq 2, \\ E(X)\vec{A} + O(1) \ \text{as } |X|\to\infty, & \text{if } n = 1, \end{cases} \tag{7.3.4}$$

$$\partial_j\vec{u}(X) = \begin{cases} O(|X|^{-n}) \ \text{as } |X|\to\infty, & \text{if } \ n \geq 2, \\ \partial_j E(X)\vec{A} + O(|X|^{-2}) \ \text{as } |X|\to\infty, & \text{if } n = 1, \end{cases} \tag{7.3.5}$$

where $E(X) = (E_{jk}(X))_{1\leq j,k\leq n+1}$ with $E_{jk}(X)$ as in (6.1.6), for some a priori given constant $\vec{A} \in \mathbb{R}^2$. Also, the standard nontangential maximal operator in (7.3.2), (7.3.3) should be replaced by its truncated version.

Theorem 7.3.2 *Let λ, μ, as in (6.1.1), and $p \in (1, \infty)$ be given and assume that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. Then there exists $\varepsilon > 0$ which depends only on p, n, λ, μ and the John and Ahlfors regularity constants of Ω with the following property. If Ω is an ε-regular SKT domain and*

$$-\mu < r \leq \mu \quad \text{is such that} \quad \Big| r - \frac{\mu(\mu+\lambda)}{3\mu+\lambda} \Big| < \varepsilon, \tag{7.3.6}$$

then the interior Neumann boundary value problem

$$\begin{cases} \mu\,\Delta\vec{u} + (\mu+\lambda)\nabla \mathrm{div}\,\vec{u} = 0 \quad \text{in } \Omega, \\ \mathcal{N}(\nabla\vec{u}) \in L^p(\partial\Omega, d\sigma), \\ \partial_\nu^r \vec{u} = \vec{f} \in L^p(\partial\Omega, d\sigma), \end{cases} \tag{7.3.7}$$

has a solution if and only if the datum $\vec{f}$ satisfies finitely many (necessary) linear compatibility conditions. Moreover, this solution is unique modulo functions from $\Psi^r(\Omega)$. In addition, there exists a finite constant $C > 0$ such that

$$\|\mathcal{N}(\nabla\vec{u})\|_{L^p(\partial\Omega,d\sigma)} \leq C\|\vec{f}\|_{\big[L^p(\partial\Omega,d\sigma)\big]^{n+1}}. \tag{7.3.8}$$

Finally, a similar result holds for the exterior domain $\mathbb{R}^{n+1} \setminus \bar{\Omega}$ granted that one includes the decay conditions

$$\vec{u}(x) = \begin{cases} O(|X|^{1-n}) \quad \text{as } |X| \to \infty, \quad \text{if } n \geq 2, \\ E(X)\Big(\int_{\partial\Omega} \vec{f}\,d\sigma\Big) + O(|X|^{-1}) \quad \text{as } |X| \to \infty, \quad \text{if } n = 1, \end{cases} \tag{7.3.9}$$

$$\partial_j\vec{u}(X) = (\partial_j E)(X)\Big(\int_{\partial\Omega} \vec{f}\,d\sigma\Big) + O(|X|^{-n-1}) \quad \text{as } |X| \to \infty,\ 1 \leq j \leq n+1, \tag{7.3.10}$$

where $E(X) = (E_{jk}(X))_{1\leq j,k\leq n+1}$ with $E_{jk}(X)$ as in (6.1.6).

Theorem 7.3.3 *Let λ, μ, as in (6.1.1), and $p \in (1, \infty)$ be given and assume that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. Set $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$.*

Then there exists $\varepsilon > 0$ which depends only on p, n, λ, μ and the John and Ahlfors regularity constants of Ω with the following property. If Ω is an ε-regular SKT domain and (7.3.6) holds then the transmission problem, concerned with finding two pairs of functions $(\vec{u}_\pm, \pi_\pm)$ in $\Omega_\pm$ satisfying

$$\begin{cases} \mu\,\Delta\vec{u}_\pm + (\mu+\lambda)\nabla \mathrm{div}\,\vec{u}_\pm = 0 \quad \text{in } \Omega, \\ \mathcal{N}(\nabla\vec{u}_\pm) \in L^p(\partial\Omega, d\sigma), \\ \vec{u}_+\Big|_{\partial\Omega} - \vec{u}_-\Big|_{\partial\Omega} = \vec{g} \in \Big[L_1^p(\partial\Omega, d\sigma)\Big]^{n+1}, \\ \partial_\nu^r\vec{u}_+ - \eta\,\partial_\nu^r\vec{u}_- = \vec{f} \in \Big[L^p(\partial\Omega, d\sigma)\Big]^{n+1}, \end{cases} \tag{7.3.11}$$

and the decay conditions

$$\vec{u}_-(X) = \begin{cases} O(|X|^{1-n}) \quad \text{as} \quad |X| \to \infty, \quad \text{if} \quad n \geq 2, \\ -\frac{1}{\eta}E(X)\Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|x|^{-1}) \quad \text{as} \quad |x| \to \infty, \quad \text{if} \ n = 2, \end{cases} \tag{7.3.12}$$

$$\partial_j \vec{u}_-(X) = -\tfrac{1}{\eta}(\partial_j E)(X)\Big(\int_{\partial\Omega} \vec{f}\, d\sigma\Big) + O(|X|^{-n-1}) \quad \text{as} \quad |X| \to \infty, \tag{7.3.13}$$

where $E(X) = (E_{jk}(X))_{1 \leq j,k \leq n+1}$ *with* $E_{jk}(X)$ *as in (6.1.6), has a unique solution. In addition, there exists* $C > 0$ *such that*

$$\|\mathcal{N}(\nabla \vec{u}_\pm)\|_{L^p(\partial\Omega,d\sigma)} \leq C\|\vec{g}\|_{\big[L^p_1(\partial\Omega,d\sigma)\big]^{n+1}} + C\|\vec{f}\|_{\big[L^p(\partial\Omega,d\sigma)\big]^{n+1}}. \tag{7.3.14}$$

7.4 Boundary value problems for Maxwell's equations

For a given wave number $k \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Im} k \geq 0$, call a vector field E defined in a neighborhood of infinity in $\mathbb{R}^3$ and satisfying $(\Delta + k^2)E = 0$ there, *radiating at infinity* if

$$(\operatorname{curl} E)(X) \times \frac{X}{|X|} + (\operatorname{div} E)(X)\frac{X}{|X|} - \sqrt{-1}kE(X) = \mathrm{o}(|X|^{-1}) \quad \text{as} \quad |X| \to \infty, \tag{7.4.1}$$

uniformly for all directions $X/|X|$ in $\mathbb{R}^3$.

Theorem 7.4.1 *Let* $p \in (1, \infty)$ *be given and assume that* $\Omega \subset \mathbb{R}^3$ *is a bounded domain satisfying a two-sided John condition and whose boundary is Ahlfors regular. Then there exists* $\varepsilon > 0$ *which depends only on* p, n *and the John and Ahlfors regularity constants of* Ω *having the following role. If* Ω *is an* ε*-regular SKT domain and* $k \in \mathbb{C} \setminus \{0\}$ *with* $\operatorname{Im} k \geq 0$ *the exterior Maxwell boundary value problem*

$$\begin{cases} \operatorname{curl} E - \sqrt{-1}kH = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega}, \\ \operatorname{curl} H + \sqrt{-1}kE = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega}, \\ E, H \quad \text{radiate at infinity}, \\ \mathcal{N}(E), \mathcal{N}(H) \in L^p(\partial\Omega, d\sigma) \ \text{and} \ \exists\, E\Big|_{\partial\Omega}, \ \exists\, H\Big|_{\partial\Omega}, \\ \nu \times E\Big|_{\partial\Omega} = \vec{g} \in \Big[L^p(\partial\Omega, d\sigma)\Big]^3, \\ \Big\langle \nu, H\Big|_{\partial\Omega}\Big\rangle = h \in L^p(\partial\Omega, d\sigma), \end{cases} \tag{7.4.2}$$

is solvable if and only if

$$\vec{g} \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma) \quad \text{and} \quad \operatorname{Div} \vec{g} = -\sqrt{-1}k\, h. \tag{7.4.3}$$

Granted (7.4.3), the solution is unique, depends analytically on k *in* $\mathbb{R}^2_+$ *and continuously on* k *in* $\overline{\mathbb{R}^2_+} \setminus \{0\}$ *(the principle of limiting absorption), and satisfies*

$$\|\mathcal{N}(E)\|_{L^p(\partial\Omega,d\sigma)} + \|\mathcal{N}(H)\|_{L^p(\partial\Omega,d\sigma)} \leq C\|\vec{f}\|_{L^{p,\mathrm{Div}}_{\tan}(\partial\Omega,d\sigma)} \tag{7.4.4}$$

for some positive constant C depending only on k, p and Ω. Furthermore, there exists a sequence of real numbers $\{k_j\}_j$, with no finite accumulation point, with the property that if $k \in \overline{\mathbb{R}^2_+} \setminus \{k_j\}_j$ then a similar result is valid for the interior Maxwell boundary value problem

$$\begin{cases} \operatorname{curl} E - \sqrt{-1}kH = 0 \ \ \textit{in} \ \ \Omega, \\ \operatorname{curl} H + \sqrt{-1}kE = 0 \ \ \textit{in} \ \ \Omega, \\ \mathcal{N}(E), \mathcal{N}(H) \in L^p(\partial\Omega, d\sigma) \ \textit{and} \ \exists\, E\Big|_{\partial\Omega}, \ \exists\, H\Big|_{\partial\Omega}, \\ \nu \times E\Big|_{\partial\Omega} = \vec{g} \in \Big[L^p(\partial\Omega, d\sigma)\Big]^3, \\ \Big\langle \nu, H\Big|_{\partial\Omega}\Big\rangle = h \in L^p(\partial\Omega, d\sigma), \end{cases} \tag{7.4.5}$$

In the case in which $k \in \{k_j\}_j$ is a nonzero number, the problem (7.4.5) is solvable if and only (7.4.3) holds and the data $\vec{g}$, h satisfy finitely many linear conditions (in which case the solution is unique modulo a finite dimensional space).

Proof. Let us first deal with the exterior boundary value problem (7.4.2). The necessity of (7.4.3) from (7.4.2) and Lemma 6.5.2, so we turn to the sufficiency part. To show existence, we first remark that when either $\operatorname{Im} k > 0$, or when k is not of the exceptional values $\{\zeta_j\}_j$ from Theorem 6.5.7 corresponding to $\lambda = \pm\frac{1}{2}$, then the operator $-\frac{1}{2}I + M_k$ is invertible on $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$, so we may take $E := \operatorname{curl} \mathcal{S}_k \vec{f}$, $H := \frac{1}{\sqrt{-1}k} \operatorname{curl} E$ where we have set $\vec{f} := (-\frac{1}{2}I + M_k)^{-1}\vec{g} \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$.

The treatment of the case when k is one of the exceptional values described above requires an appropriate modification of this approach on which we now elaborate. The new difficulty is that we have to show that the problem (7.4.2) remains well-posed even when the integral operator $-\frac{1}{2}I+M_k$ is no longer invertible. Since this is, nonetheless, Fredholm with index zero, the idea is to add further source terms (of compact nature) in order to ensure that the resulting perturbed operator has trivial kernel. More specifically, assume that $k \in \mathbb{R} \setminus \{0\}$, and for each $\vec{f} \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$ consider

$$E := \operatorname{curl} \mathcal{S}_k \vec{f} + \sqrt{-1} \operatorname{curl}\operatorname{curl} \mathcal{S}_k(\nu \times S_0(S_0(\vec{f}))), \quad H := \tfrac{1}{\sqrt{-1}k} \operatorname{curl} E \quad \text{in} \ \ \mathbb{R}^3 \setminus \overline{\Omega}. \tag{7.4.6}$$

It is then not difficult to check that that (E, H) is a radiating solution of the Maxwell system in $\mathbb{R}^3 \setminus \overline{\Omega}$, for which $\mathcal{N}(E)$, $\mathcal{N}(H) \in L^p(\partial\Omega, d\sigma)$, the traces $E\Big|_{\partial\Omega}$, $H\Big|_{\partial\Omega}$ exist, and such that $\nu \times E = (-\frac{1}{2}I + M_k + \sqrt{-1}N_k \circ (\nu \times S_0^2))\vec{f}$, where

$$N_k \vec{f} := \nu \times \Big(\operatorname{curl}\operatorname{curl} \mathcal{S}_k \vec{f}\Big|_{\partial\Omega}\Big) = k^2 \nu \times S_k \vec{f} + \nu \times \nabla S_k(\operatorname{Div} \vec{f}). \tag{7.4.7}$$

Hence, matters are reduced to proving the invertibility of the operator

$$-\tfrac{1}{2}I + M_k + \sqrt{-1}N_k \circ (\nu \times S_0^2) : L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma) \longrightarrow L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma). \tag{7.4.8}$$

To this end, we note from Theorem 6.5.6 that the operator $-\frac{1}{2}I + M_k$ is Fredholm with index zero on $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$. Since, by Lemma 6.5.2, $\nu \times S_0^2$ also maps $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$ compactly into itself, it follows that that the operator in (7.4.8) is Fredholm with index zero. Consequently, we are left with showing that the operator (7.4.8) is injective on $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$. For $p=2$ this has been proved in [25] p. 167 (there it is assumed that $\partial\Omega \in C^2$ but for this particular purpose the reasoning in [25] can be adapted to the current setting, given the results we have proved so far. Consider now $\vec{f} \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$ such that $(-\frac{1}{2}I + M_k + \sqrt{-1}N_k \circ (\nu \times S_0^2))\vec{f} = 0$ and fix $k_o \in \mathbb{C}$ with $\mathrm{Im}\, k_o > 0$. It follows that

$$\vec{f} = -(-\tfrac{1}{2}I + M_{k_o})^{-1}[(M_{k_o} - M_k)\vec{f} + \sqrt{-1}N_k(\nu \times S_0(S_0(\vec{f})))]. \tag{7.4.9}$$

Recall next that the fractional integration theorem remains valid on spaces of homogeneous type. For example, a rather general version (as well as references to earlier work) can be found in [76]. In the context of (7.4.9), this gives that $\vec{f} \in L^{p+\varepsilon,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$ for some $\varepsilon > 0$. Iterating this sufficiently many times we finally arrive at $\vec{f} \in L^{2,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$ and, therefore, by the L^2-theory alluded to above, $\vec{f} = 0$. Thus, the existence part is proved.

To address the uniqueness issue, assume first that $p \geq 2$. The case $\mathrm{Im}\, k > 0$, when E, H have exponential decay at infinity, is easier and can be handled much as in [89], so we restrict attention to $k \in \mathbb{R} \setminus \{0\}$. First, the radiation condition (6.5.53) implies

$$\begin{aligned} 0 &= \lim_{R\to\infty} \int_{|X|=R} |H \times \nu - E|^2 \, d\sigma \\ &= \lim_{R\to\infty} \int_{|X|=R} \big(|H \times \nu|^2 + |E|^2 - 2\mathrm{Re}\,\langle \nu \times E, H^c\rangle\big) \, d\sigma. \end{aligned} \tag{7.4.10}$$

Since $\nu \times E = 0$ on $\partial\Omega$ and $k \in \mathbb{R}$, integrating by parts gives

$$\mathrm{Re} \int_{|X|=R} \langle \nu \times E, H^c\rangle \, d\sigma = \mathrm{Re}\left(\sqrt{-1}k \int_{B(0,R)\setminus\overline{\Omega}} \Big[|H|^2 - |E|^2\Big] \, dX\right) = 0. \tag{7.4.11}$$

Next, from this and (7.4.10) we may now deduce that $\int_{|X|=R} |E|^2 \, d\sigma = o(1)$ as $R \to \infty$. Rellich's lemma (cf. (6.5.55)) then gives that E and, hence, also H, vanishes in $\mathbb{R}^3 \setminus \overline{\Omega}$, at least if this domain in connected. Nonetheless, much as in the proof of Proposition 6.5.5, this argument can be further refined as to treat domains with arbitrary topology.

We now consider the remaining case, i.e., when $1 < p < 2$. First, from the Stratton-Chu formula (cf. Theorem 6.6 on p. 153 in [25]), which continues to hold in the current setting, we have that

$$E = \nabla \mathcal{S}_k(\langle \nu, E\rangle) - \sqrt{-1}k\, \mathcal{S}_k(\nu \times H) \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega}. \tag{7.4.12}$$

Thus, after applying curl to both sides of (7.4.12), we arrive at

$$H = -\mathrm{curl}\, \mathcal{S}_k(\nu \times H) \quad \text{in} \quad \mathbb{R}^3 \setminus \overline{\Omega}. \tag{7.4.13}$$

Going nontangentially to the boundary and applying $\nu\times$ to both sides of (7.4.13), then yields $(\frac{1}{2}I + M_k)(\nu \times H) = 0$. Hence, with k_o as before,

$$\nu \times H = -(-\tfrac{1}{2}I + M_{k_o})^{-1}[(M_k - M_{k_o})(\nu \times H)] \in L^{p+\varepsilon,\mathrm{Div}}_{\tan}(\partial\Omega) \tag{7.4.14}$$

for some positive ε. Iterating finitely many times then gives $\nu \times H \in L^{2,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$, so that $\mathcal{N}(H) \in L^2(\partial\Omega, d\sigma)$, by (7.4.13). Finally, since $\langle \nu, E\rangle = -\frac{1}{\sqrt{-1}k}\mathrm{Div}\,(\nu \times H) \in L^2(\partial\Omega, d\sigma)$, it follows from (7.4.12) that $\mathcal{N}(E) \in L^2(\partial\Omega, d\sigma)$. At this point, the conclusion follows from the case $p = 2$.

For the interior boundary value problem (7.4.5), Theorem 6.5.7 shows that there exists a sequence of real numbers $\{k_j\}_j$, with no finite accumulation points, such that the operator $\frac{1}{2}I + M_k$ is invertible on $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$ whenever $k \in \overline{\mathbb{R}^2_+} \setminus \{k_j\}_j$. In this situation, the well-posedness of (7.4.5) is handled as before. Assume now that $k \in \{k_j\}_j$ is a nonzero number and denote by $\mathcal{U}_k$ the collection of all vector fields of the form $\nu \times E$ where (E, H) satisfies the first three conditions in (7.4.5). Since $\mathcal{U}_k$ contains $(\frac{1}{2}I + M_k)\Big[L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)\Big]$, it follows that $\mathcal{U}_k$ is a closed, finite co-dimensional subspace of $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$. Clearly, the problem (7.4.5) is solvable for the boundary datum $\vec{g}$, if and only if $\vec{g} \in \mathcal{U}_k$.

Finally, if $k \in \{k_j\}_j$ is a nonzero number and (E, H) is a null-solution of (7.4.5), then writing the version of (7.4.12) for Ω and performing the same type of manipulations as before, we arrive at $(-\frac{1}{2}I + M_k)(\nu \times H) = 0$. Now, the application $(E, H) \mapsto \nu \times H$ is injective and takes values in a finite dimensional space, namely the null-space of the Fredholm operator $-\frac{1}{2}I + M_k$ on $L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma)$. Hence, the space of null-solutions for (7.4.5) is finite dimensional. The proof of the theorem is therefore finished. □

Our last results in this subsection deal with the the transmission problem for the Maxwell equations. This amounts to finding two pairs of vector fields $(E^{\pm}, H^{\pm})$ satisfying the following boundary value problem:

$$\begin{cases} \mathrm{curl}\, E^{\pm} - \sqrt{-1}kH^{\pm} = 0 \ \text{ in } \ \Omega_{\pm}, \\ \mathrm{curl}\, H^{\pm} + \sqrt{-1}kE^{\pm} = 0 \ \text{ in } \ \Omega_{\pm}, \\ \mathcal{N}(E^{\pm}), \mathcal{N}(H^{\pm}) \in L^p(\partial\Omega, d\sigma) \text{ and } \exists\, E^{\pm}\Big|_{\partial\Omega}, \ \exists\, H^{\pm}\Big|_{\partial\Omega}, \\ \nu \times E^{+}\Big|_{\partial\Omega} - \mu\, \nu \times E^{-}\Big|_{\partial\Omega} = \vec{g} \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma), \\ \nu \times H^{+}\Big|_{\partial\Omega} - \nu \times H^{-}\Big|_{\partial\Omega} = \vec{h} \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma), \\ E^{-},\, H^{-} \text{ radiate at infinity.} \end{cases} \tag{7.4.15}$$

The above problem models the scattering of electro-magnetic waves by a penetrable bounded obstacle Ω. See, e.g., [4], [78], [85], [96] for a more extensive discussion in this regard. Here we only wish to point out that physical considerations dictate that the transmission parameter μ belongs to the interval $(0, 1)$.

Theorem 7.4.2 *Let $\Omega \subset \mathbb{R}^3$ be a bounded regular SKT domain. Then for every $k \in \mathbb{C} \setminus \{0\}$, $\mu \in (0, 1)$ and $p \in (1, \infty)$, the transmission boundary value problem (7.4.15) is well-posed.*

Proof. Using the well-posedness of (7.4.2), it is easy to see that (7.4.15) has a solution if and only if the problem

$$
\begin{cases}
\operatorname{curl} E^{\pm} - \sqrt{-1}kH^{\pm} = 0 \quad \text{in} \quad \Omega_{\pm}, \\
\operatorname{curl} H^{\pm} + \sqrt{-1}kE^{\pm} = 0 \quad \text{in} \quad \Omega_{\pm}, \\
\mathcal{N}(E^{\pm}), \mathcal{N}(H^{\pm}) \in L^p(\partial\Omega, d\sigma) \text{ and } \exists\, E^{\pm}\Big|_{\partial\Omega}, \ \exists\, H^{\pm}\Big|_{\partial\Omega}, \\
\nu \times E^{+}\Big|_{\partial\Omega} - \mu\, \nu \times E^{-}\Big|_{\partial\Omega} = \vec{g} \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega, d\sigma), \\
\nu \times H^{+}\Big|_{\partial\Omega} = \nu \times H^{-}\Big|_{\partial\Omega}, \\
E^{-},\, H^{-} \text{ radiate at infinity}
\end{cases}
\tag{7.4.16}
$$

has a solution. In turn, thanks to Theorem 6.5.7, a solution for (7.4.16) can be found in the form

$$
E^{\pm} := \tfrac{1}{1-\mu} \operatorname{curl} \mathcal{S}_k (\lambda I + M_k)^{-1} \vec{g}, \qquad H^{\pm} := \tfrac{1}{\sqrt{-1}k} \operatorname{curl} E^{\pm} \quad \text{in} \quad \Omega_{\pm}, \tag{7.4.17}
$$

where $\lambda := \frac{1}{2}\frac{1+\mu}{1-\mu}$. Finally, uniqueness for (7.4.15) can be shown by proceeding much as in the proof of Proposition 6.5.5. $\square$

Acknowledments. S. Hofmann was supported in part by NSF grant DMS-0245401. M. Mitrea was supported in part by NSF grant DMS-0653180. Hofmann and Mitrea also had the support of NSF grant DMS-FRG0456306. M. Taylor was supported in part by NSF grant DMS-0456861.

References

[1] H. Aimar, *Elliptic and parabolic BMO and Harnack's inequality*, Trans. Amer. Math. Soc., 306 (1988), no. 1, 265–276.

[2] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Claderon Press, Oxford, 2000.

[3] J. Avarez, R.J. Bagby, D.S. Kurtz and C. Pérez, *Weighted estimates for commutators of linear operators*, Studia Math., 104 (1993), no. 2, 195–209.

[4] T. Angell and A. Kirsch, *The conductive boundary condition for Maxwell's equations*, SIAM J. Appl. Math., 52 (1992), 1597–1610.

[5] B. Bennewitz and J.L. Lewis, *On weak reverse Hölder inequalities for nondoubling harmonic measures*, Complex Var. Theory Appl., 49 (2004), no. 7-9, 571–582.

[6] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics, Vol. 129, Academic Press, Inc., Boston, MA, 1988.

[7] F. Brackx, R. Delanghe and F. Sommen, *Clifford Analysis*, Research Notes in Mathematics Vol. 76, Pitman, Boston, MA, 1982.

[8] M. Bramanti and C.M. Cerutti, *Commutators of singular integrals on homogeneous spaces*, Boll. Un. Mat. Ital. B (7), 10 (1996), no. 4, 843–883.

[9] Yu. D. Burago and V. G. Maz'ya, *Potential Theory and Function Theory for Irregular Regions*, Translated from Russian, Seminars in Mathematics, V.A. Steklov Mathematical Institute, Leningrad, Vol. 3, Consultants Bureau, New York, 1969.

[10] A.M. Caetano, *On fractals which are not so terrible*, Fund. Math., 171 (2002), no. 3, 249–266.

[11] A.P. Calderón, *Cauchy integrals on Lipschitz curves and related operators*, Proc. Nat. Acad. Sci. USA., 74 (1977), no. 4, 1324–1327.

[12] A.P. Calderón, *Estimates for singular integral operators in terms of maximal functions*, Studia Math., 44 (1972), 563–582.

[13] A.P. Calderón and R. Scott, *Sobolev type inequalities for* $p > 0$, Studia Math., 62 (1978), no. 1, 75–92.

[14] C.P. Calderón, *On commutators of singular integrals*, Studia Math., 53 (1975), no. 2, 139–174.

[15] L. Capogna, C.E. Kenig and L. Lanzani, *Harmonic Measure. Geometric and Analytic Points of View*, University Lecture Series Vol. 35, American Mathematical Society, Providence, RI, 2005.

[16] G.-Q. Chen, M. Torres, and W. Ziemer, *Measure-theoretical analysis and nonlinear conservation laws*, Pure Appl. Math. Q. 3 (2007), 841–879.

[17] M. Christ, *Lectures on Singular Integral Operators*, the American Mathematical Society CBMS Regional Conference Series in Mathematics, Vol. 77, Providence, RI, 1990.

[18] M. Christ, *A* $T(b)$ *theorem with remarks on analytic capacity and the Cauchy integral*, Colloq. Math., 60/61 (1990), no. 2, 601–628.

[19] R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math., 51 (1974), 241–250.

[20] R. Coifman, A. McIntosh and Y. Meyer, *L'intégrale de Cauchy definit un opérateur borné sur* L^2 *pour les courbes lipschitziennes*, Ann. Math., 116 (1982), 361–388.

[21] R.R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., 103 (1976), no. 3, 611–635.

[22] R.R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogénes. Etude de certaines intégrales singulières*, Lecture Notes in Mathematics, Vol. 242, Springer-Verlag, Berlin-New York, 1971.

[23] R.R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc., 83 (1977), no. 4, 569–645.

[24] D. Colton and R. Kress, *Integral Equation Methods in Scattering Theory*, Wiley, New York, 1983.

[25] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Applied Mathematical Sciences 93, Springer-Verlag, 1992.

[26] D. Cruz-Uribe, J. Martell and C. Pérez, *Extensions of Rubio de Francia's extrapolation theorem*, pp. 195–231 in Collect. Math., Vol. Extra, 2006.

[27] B.E.J. Dahlberg and C.E. Kenig, *Hardy spaces and the Neumann problem in* L^p *for Laplace's equation in Lipschitz domains*, Ann. of Math., 125 (1987), no. 3, 437–465.

[28] G. David, *Opérateurs d'intégrale singulière sur les surfaces régulières*, Ann. Scient. Ecole Norm. Sup., 21 (1988), 225–258.

[29] G. David, *Wavelets and Singular Integrals on Curves and Surfaces*, Lecture Notes in Mathematics, Vol. 1465, Springer-Verlag, Berlin, 1991.

[30] G. David, *Uniformly Rectifiable Sets*, preprint (2004).

[31] G. David and D. Jerison, *Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals*, Indiana Univ. Math. J., 39 (1990), no. 3, 831–845.

[32] G. David, J.-L. Journé and S. Semmes, *Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation*, Rev. Mat. Iberoamericana, 1 (1985), no. 4, 1–56.

[33] G. David and S. Semmes, *Singular Integrals and Rectifiable Sets in $\mathbb{R}^n$: Beyond Lipschitz Graphs*, Astérisque No. 193, 1991.

[34] G. David and S. Semmes, *Analysis of and on Uniformly Rectifiable Sets*, Mathematical Surveys and Monographs, AMS Series, 1993.

[35] M. Dindos, S. Petermichl, and J. Pipher, *The L^p Dirichlet problem for second order elliptic operators and a p-adapted square function,* J. Funct. Anal., 249 (2007), no. 2, 372–392.

[36] L.C. Evans and R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[37] E.B. Fabes, M. Jodeit Jr. and N.M. Rivière, *Potential techniques for boundary value problems on C^1-domains*, Acta Math., 141 (1978), no. 3-4, 165–186.

[38] E.B. Fabes, C.E. Kenig and G.C. Verchota, *The Dirichlet problem for the Stokes system on Lipschitz domains*, Duke Math. J., 57 (1988), no. 3, 769–793.

[39] H. Federer, *Measure and area*, Bull. AMS 58 (1952), 306–378.

[40] H. Federer, *The area of a nonparametric surface*, Proc. AMS 11 (1960), 436–439.

[41] H. Federer, *Geometric Measure Theory*, reprint of the 1969 edition, Springer-Verlag, 1996.

[42] C. Fefferman and E.M. Stein, *H^p spaces of several variables*, Acta Math., 129 (1972), no. 3-4, 137–193.

[43] J. Garcia-Cuerva and J. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North Holland, Amsterdam, 1985.

[44] H.G. Garnir, *Les problémes aux limites de la physique mathématique*, Basel, Birkhäuser Verlag, 1958.

[45] F. Gesztesy and M. Mitrea, *Generalized Robin boundary conditions and Robin-to-Dirichlet maps for Schrödinger operators on bounded Lipschitz domains*, preprint (2008).

[46] P. Hajłasz, *Sobolev spaces on an arbitrary metric space*, Potential Anal., 5 (1996), no. 4, 403–415.

[47] P. Hajłasz, *Sobolev spaces on metric-measure spaces*, pp. 173–218 in Contemp. Math., Vol. 338, Amer. Math. Soc., Providence, RI, 2003.

[48] P. Hajłasz and J. Kinnunen, *Hölder quasicontinuity of Sobolev functions on metric spaces*, Rev. Mat. Iberoamericana, 14 (1998), no. 3, 601–622.

[49] W. Hansen, *Uniform boundary Harnack principle and generalized triangle property*, J. Funct. Anal., 226 (2005), no. 2, 452–484.

[50] S. Hofmann, *On singular integrals of Calderón-type in $\mathbb{R}^n$, and* BMO, Rev. Mat. Iberoamericana, 10 (1994), no. 3, 467–505.

[51] S. Hofmann, *Parabolic singular integrals of Calderón-type, rough operators, and caloric layer potentials*, Duke Math. J., 90 (1997), no. 2, 209–259.

[52] S. Hofmann, E. Marmolejo-Olea, M. Mitrea, S. Pérez-Esteva and M. Taylor, *Hardy spaces, singular integrals and the geometry of Euclidean domains of locally finite perimeter*, GAFA, to appear.

[53] S. Hofmann, M. Mitrea and M. Taylor, *Geometric and transformational properties of Lipschitz domains, Semmes-Kenig-Toro domains, and other classes of finite perimeter domains*, J. Geometric Anal., 17 (2007), 593–647.

[54] D.S. Jerison and C.E. Kenig, *Hardy spaces, A_∞, and singular integrals on chord-arc domains*, Math. Scand., 50 (1982) 221–247.

[55] D.S. Jerison and C.E. Kenig, *Boundary behavior of harmonic functions in nontangentially accessible domains*, Adv. in Math., 46 (1982), no. 1, 80–147.

[56] D.S. Jerison and C.E. Kenig, *The logarithm of the Poisson kernel of a C^1 domain has vanishing mean oscillation*, Trans. Amer. Math. Soc., 273 (1982), no. 2, 781–794.

[57] F. John, *Rotation and strain*, Comm. Pure Appl. Math., 14 (1961) 391–413.

[58] P.W. Jones, *Square functions, Cauchy integrals, analytic capacity, and harmonic measure*, pp. 24–68 in "Harmonic Analysis and Partial Differential Equations," Lecture Notes in Math., Vol. 1384, Springer, Berlin, 1989.

[59] A. Jonsson and H. Wallin, *Function Spaces on Subsets of $\mathbb{R}^n$*, Math. Rep., 2 (1984), no. 1, xiv+221 pp.

[60] A. Kałamajska, *On compactness of embedding for Sobolev spaces defined on metric spaces*, Ann. Acad. Sci. Fenn. Math., 24 (1999), no. 1, 123–132.

[61] N. Kalton and M. Mitrea, *Stability of Fredholm properties on interpolation scales of quasi-Banach spaces and applications*, Transactions of the Amer. Math. Soc., Vol. 350, No. 10, (1998), 3837–3901.

[62] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1976.

[63] C.E. Kenig, *Weighted Hardy spaces on Lipschitz domains*, pp. 263–274 in Proc. Symp. Pure Math., Vol. XXXV, Part 1, Amer. Math. Soc., Providence, RI, 1979.

[64] C.E. Kenig and T. Toro, *Harmonic measure on locally flat domains*, Duke Math. J., 87 (1997), no. 3, 509–551.

[65] C.E. Kenig and T. Toro, *Free boundary regularity for harmonic measures and Poisson kernels*, Ann. of Math., 150 (1999), no. 2, 369–454.

[66] C.E. Kenig and T. Toro, *Poisson kernel characterization of Reifenberg flat chord arc domains*, Ann. Sci. École Norm. Sup., (4) 36 (2003), no. 3, 323–401.

[67] C.E. Kenig and T. Toro, *Free boundary regularity below the continuous threshold: 2-phase problems*, J. Reine Angew. Math. 596 (2006), 1–44.

[68] N. Kerzman and E. Stein, *The Szegö kernel in terms of Cauchy-Fantappié kernels*, Duke Math. J. 45 (1978), 197–224.

[69] J. Král, *Integral Operators in Potential Theory*, Lecture Notes in Mathematics, Vol. 823, Springer, Berlin, 1980.

[70] V.D. Kupradze, *Potential Methods in the Theory of Elasticity*, Daniel Davey & Co., Inc., New York 1965.

[71] O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach Science Publishers, New York-London, 1963.

[72] L. Lanzani and E.M. Stein, *Szegö and Bergman projections on non-smooth planar domains*, J. Geom. Anal., 14 (2004), no. 1, 63–86.

[73] M. Laurentiev, *Boundary problems in the theory of univalent functions*, Mat. Sb. (N.S.) 1 (43) (1936), 815–844. English translation: Amer. Math. Soc. Translations, ser. 2, 32 (1963), 1–36.

[74] R.A. Macias and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. in Math., 33 (1979), no. 3, 257–270.

[75] J.M. Martell, *Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications*, Studia Math., 161 (2004), no. 2, 113–145.

[76] J.M. Martell, *Fractional integrals, potential operators and two-weight, weak type norm inequalities on spaces of homogeneous type*, J. Math. Anal. Appl. 294 (2004), no. 1, 223–236.

[77] O. Martio and J. Sarvas, *Injectivity theorems in plane and space*, Ann. Acad. Sci. Fenn. Ser. A I Math., 4 (1979), no. 2, 383–401.

[78] P.A. Martin and P. Ola, *Boundary integral equations for the scattering of electromagnetic waves by a homogeneous dielectric obstacle*, Proc. of the Royal Soc. of Edinburgh, 123 (1993), 185–208.

[79] P. Mattila and D. Preiss, *Rectifiable measures in* $\mathbb{R}^n$ *and existence of principal values for singular integrals*, J. London Math. Soc. (2) 52 (1995), no. 3, 482–496.

[80] V.G. Maz'ya, *Boundary integral equations*, pp.127–222 in Analysis, IV, Encyclopaedia Math. Sci., Vol. 27, Springer, Berlin, 1991.

[81] N.G. Meyer, *Mean oscillation over cubes and Hölder continuity*, Proc. Amer. Math. Soc., 15 (1964) 717–721.

[82] Y. Meyer, *Ondelettes et Opérateurs II: Opérateurs de Calderón-Zygmund*, Hermann, Paris, 1990.

[83] Y. Meyer and R. Coifman, *Wavelets. Calderón-Zygmund and Multilinear Operators*, Cambridge University Press, Cambridge, 1997.

[84] D. Mitrea, *The method of layer potentials for non-smooth domains with arbitrary topology*, Integral Equations Operator Theory, 29 (1997), no. 3, 320–338.

[85] D. Mitrea and M. Mitrea, *Uniqueness for inverse conductivity and transmission problems in the class of Lipschitz domains*, Comm. Partial Differential Equations, 23 (1998), no. 7-8, 1419–1448.

[86] D. Mitrea, M. Mitrea and M. Taylor, *Layer potentials, the Hodge Laplacian, and global boundary problems in nonsmooth Riemannian manifolds*, Mem. Amer. Math. Soc., Vol. 150, No. 713, 2001.

[87] I. Mitrea, *On the spectra of elastostatic and hydrostatic layer potentials on curvilinear polygons*, J. Fourier Anal. Appl., 8 (2002), no. 5, 443-487.

[88] M. Mitrea, *Clifford Wavelets, Singular Integrals, and Hardy Spaces*, Lecture Notes in Mathematics, Vol. 1575, Springer-Verlag, Berlin, 1994.

[89] M. Mitrea, *The method of layer potentials in electro-magnetic scattering theory on non-smooth domains*, Duke Math. Journal, Vol. 7 (1995) no. 1, 111-133.

[90] M. Mitrea, *Generalized Dirac operators on nonsmooth manifolds and Maxwell's equations*, J. Fourier Anal. Appl., 7 (2001), no. 3, 207–256.

[91] M. Mitrea and M. Taylor, *Boundary layer methods for Lipschitz domains in Riemannian manifolds*, J. Funct. Anal., 163 (1999), no. 2, 181–251.

[92] M. Mitrea and M. Taylor, *Potential theory on Lipschitz domains in Riemannian manifolds: Hölder continuous metric tensors*, Comm. in PDE 25 (2000), 1487–1536.

[93] M. Mitrea and M. Taylor, *Potential theory on Lipschitz domains in Riemannian manifolds: the case of Dini metric tensors*, Trans. AMS 355 (2003), 1961–1985.

[94] M. Mitrea and M. Wright, *Boundary value problems for the Stokes system in arbitrary Lipschitz domains*, preprint (2007).

[95] J.-M. Morel and S. Solimini, *Variational Methods in Image Segmentation*, Birkhauser, Boston, 1995.

[96] C. Müller, *Foundations of the Mathematical Theory of Electromagnetic Waves*, Springer-Verlag, Berlin Heidelberg New York, 1969.

[97] D. Preiss, *Geometry of measures in* $\mathbb{R}^n$*: distribution, rectifiability, and densities*, Ann. of Math., (2) 125 (1987), no. 3, 537–643.

[98] D. Sarason, *Functions of vanishing mean oscillation*, Trans. Amer. Math. Soc., 207 (1975), 391–405.

[99] V.Yu. Shelepov, *On the index of an integral operator of potential type in the space* L_p, Dokl. Akad. Nauk SSSR, (6) 186 (1969), 1266–1268. English transl: Sov. Math. Dokl., 10/A (1969) 754–757.

[100] S.W. Semmes, *A criterion for the boundedness of singular integrals on hypersurfaces*, Trans. Amer. Math. Soc., 311 (1989), no. 2, 501–513.

[101] S. Semmes, *Chord-arc surfaces with small constant. I*, Adv. Math., 85 (1991), no. 2, 198–223.

[102] S. Semmes, *Chord-arc surfaces with small constant. II. Good parameterizations*, Adv. Math., 88 (1991), no. 2, 170–199.

[103] S. Semmes, *Hypersurfaces in* $\mathbb{R}^n$ *whose unit normal has small* BMO *norm*, Proc. Amer. Math. Soc., 112 (1991), no. 2, 403–412.

[104] S. Semmes, *Analysis vs. geometry on a class of rectifiable hypersurfaces in* $\mathbb{R}^n$, Indiana Univ. Math. J. , 39 (1990), no. 4, 1005–1035.

[105] S. Semmes, *Differentiable function theory on hypersurfaces in* $\mathbb{R}^n$ *(without bounds on their smoothness)*, Indiana Univ. Math. J., 39 (1990), no. 4, 985–1004.

[106] I. Ya.Šneiberg, *Spectral properties of linear operators in interpolation families of Banach spaces*, Mat. Issled., 9 (1974), 214–229.

[107] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J. 1970.

[108] E.M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, Vol. 43, Monographs in Harmonic Analysis, III, Princeton University Press, Princeton, NJ, 1993.

[109] E.M. Stein and A. Zygmund, *Boundedness of translation invariant operators on Hölder spaces and* L^p*-spaces*, Ann. of Math., 85 (1967), 337–349.

[110] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Mathematics Vol. 1391, Springer-Verlag, 1989.

[111] M. Taylor, *Measure Theory and Integration*, Graduate Studies in Math., American Mathematical Society, Providence, RI, 2006.

[112] M. Taylor, *Partial Differential Equations, Vols. 1–3*, Springer-Verlag, New York, 1996.

[113] X. Tolsa, *Principal values for Riesz transforms and rectifiability*, J. Funct. Anal. 254 (2008), 1811–1863.

[114] R. Tompson, *Areas of k-dimensional nonparametric surfaces in $k+1$ space*, Trans. AMS 77 (1954), 374–407.

[115] T. Toro, *Doubling and flatness: geometry of measures*, Notices Amer. Math. Soc., 44 (1997), no. 9, 1087–1094.

[116] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal., 59 (1984), no. 3, 572–611.

[117] J.H. Wells and L.R. Williams, *Embeddings and Extensions in Analysis*, Springer-Verlag, 1975.

[118] W. Ziemer, *Weakly Differentiable Functions*, Springer-Verlag, New York, 1989.

[119] A. Zygmund, *Trigonometric Series*, Vol. I, Cambridge University Press, 1959.

Steve Hofmann
Department of Mathematics
University of Missouri
Columbia, MO 65211, USA
e-mail: *hofmann@math.missouri.edu*

Marius Mitrea
Department of Mathematics
University of Missouri
Columbia, MO 65211, USA
e-mail: *marius@math.missouri.edu*

Michael Taylor
Mathematics Department
University of North Carolina
Chapel Hill, NC 27599, USA
e-mail: *met@email.unc.edu*