Work of Lars Hörmander

Michael Taylor

Lars Hörmander was one of the giants of twentieth century mathematics. He made a big splash with his Ph.D. thesis, much of which was published in a 1955 Acta Mathematica article. This dealt with questions of existence and regularity of solutions to general classes of linear partial differential equations (PDE). Previously, there had been separate developments of results for special types of equations, particularly eliptic, parabolic, and hyperbolic equations. Some of the masters of this classical theory, such as Petrovsky, had begun to ask if there were some unifying themes that could be developed. In the mid 1950s several people, most notably Hörmander and Malgrange, stepped up to address this question. Malgrange succeeded in constructing a fundamental solution to a general constant-coefficient PDE, thus giving local solvability of $Pu = f$. Hörmander's 1955 paper had a number of fundamental results on both constant-coefficient and variable-coefficient PDE. He introduced the notion of strength of a constant-coefficient differential operator, and characterized strength in turns of the symbol of the operator (the Fourier multiplier associated to the operator). He established an estimate of the form

$$\|Pu\|_{L^2} \geq C\|P^{(\alpha)}u\|_{L^2}, \quad \forall\, u \in C_0^\infty(\mathbb{R}^n),$$

where P is Fourier multiplication by $P(\xi)$ (called the "symbol" of P) and $P^{(\alpha)}$ is Fourier multiplication by $D^\alpha P(\xi)$. Via a duality argument, this leads to results more precise than those of Malgrange. This was only the beginning of an array of striking results. He gave a complete characterization in terms of algebraic properties of the symbol $P(\xi)$, of operators P with constant coefficients that are *hypoelliptic*, i.e., if $Pu = f$, then u is smooth wherever f is. Results on hypoellipticity constitute a far-reaching extension of the famous Weyl lemma, on elliptic regularity.

The 1955 paper also contained local existence results for variable-coefficient operators of *real principal type*. These are operators of order m with real principal symbol $p_m(x,\xi)$ having the property that $\nabla_\xi p_m(x,\xi) \neq 0$ for $\xi \neq 0$. He established the estimate

$$\|Pu\|_{L^2}^2 \geq C \sum_{|\alpha|<m} \|D^\alpha u\|_{L^2}^2, \quad \forall\, u \in C_0^\infty(\mathbb{R}^n),$$

Again a duality argument implies local existence results.

Subsequent works both built upon this work and transcended it, yielding both ground-breaking results on general classes of PDE and exquisitly detailed results on a number of important specific classes of PDE. I will discuss notable results from a selection of these papers.

He studied operators of "constant strength," establishing local solvability for such operators. In a 1958 article in Comm. Pure Appl. Math., he proved that if P has constant strength and is "formally hypoelliptic," i.e., the frozen coeffienent versions satisfy his previously established conditions for hypoellipticity, then P itself is hypoelliptic. Though basically a perturbation result from his constant-coefficient result mentioned above, it helps set the stage for some major developments in the calculus of pseudodifferential operators, to be mentioned below.

A 1959 Math. Scand. article treated the *uniqueness in the Cauchy problem*, (UCP) sharpening some very sucessful results of A. P. Calderon. Calderon, using the theory of singular integral operators (which was to give rise to the theory of pseudodifferential operators), established UCP for a PDE when the principal symbol is real, the characteristics are simple, and no bicharacteristic is tangent to the initial surface. This was a very significant result, for which he received a Steele prize in 1989. Hörmander, using Carleman-type estimates, made an important advance. He showed it is sufficient to require convexity in the direction of the tangential bicharacteristics. Also the assumption of real principal symbol can be relaxed. This paper has stimulated much further work on unique continuation results.

One of Hörmander's most surprising results appeared in his 1960 paper in Math. Ann., on differential equations without solutions, a paper that showed what was really going on in the example of H. Lewy of a PDE that was not locally solvable. Here, he showed that if $p(x, \xi)$ is he principle symbol of a differential operator P, and $c(x, \xi)$ that of $[P^*, P]$, then local solvability requires

$$p(x, \xi) = 0 \Longrightarrow c(x, \xi) = 0.$$

Whenever this condition fails, one does not have local solvability. Not only did this result vastly extend the base of unsolvable PDE given by Lewy, the details of the proof presaged future results involving geometrical optics with complex phase. A companion paper in Math. Ann. showed that a certain strengthening of this necessary condition was sufficient for local solvability. This would stimulate much future work, some of which will be mentioned below.

Hörmander made an important contribution to the theory of holomorphic functions of several complex variables in his 1965 Acta Math. paper on estimates and existence theorems for the $\overline{\partial}$ operator. This followed a breakthrough by J. Kohn on the $\overline{\partial}$-Neumann problem. Kohn worked in smoothly bounded strongly pseudoconvex domains, and obtained his results by producing subelliptic estimates, via a reduction to estimates for systems of singular integral operators (soon to be refined to pseudodifferential operators) on the boundary. Hörmander took a substantially different approach, establishing solvability via Carleman-type estimates. His approach bypassed problems of boundary regularity. As an added benefit, his approach directly dealt with weakly pseudoconvex domains, leading to a new solution to the Cousin problem without extra arguments. Also his method produced bounds on such solutions, of potential use in other problems involving overdetermined systems. Related results (except for these delicate bounds) were obtained by Andreotti and Vesentini.

A 1967 Acta Math. paper gave a close to definitive analysis of which second order differential operators of the form

$$P = \sum X_j^2 + X_0 + C,$$

on a region $\Omega \subset \mathbb{R}^n$, where X_j are real vector fields and C a multiplication operator, is hypoelliptic. Hörmander showed that such hypoellipticity holds, with loss of < 2 derivatives, provided these vector fields and their iterated Lie brackets span $\mathbb{R}^n$ at each point in Ω. This result showed what was behind a result of Kolmogorov on the smoothness of the integral kernel for the solution operator of a certain singular diffusion equation. Ever since, this has been celebrated in many papers as the Hörmander bracket condition guaranteeing hypoellipticity.

As has been indicated above, in several cases, Hörmander improved results previously obtained using singular integral operators, by work that avoided such methods. As the theory of singular integral operators morphed into the theory of pseudodifferential operators, during the 1960s, Hörmander turned his attention to this subject, and became a central figure in its development, from there becoming one of the founders of the area of microlocal analysis.

Actually, Hörmander had an early foray into this area, in his 1960 Acta Math. paper, giving L^p estimates on convolution operators. He improved results of Calderon-Zygmund and of Mikhlin, involving estimates on a Fourier multiplier $P(\xi)$ guaranteeing that $P(D)$ is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$. A central ingredient was an integral kernel estimate, which ever since has been dubbed the Hörmander estimate, and which for the past 50 years has been used and extended in ways that have proved invaluable, in such studies as operators on manifolds with weakly bounded geometry. In addition, he extended results of E. Stein on multidimensional Littlewood-Paley theory. The Stein-Hörmander extension of Littlewood-Paley theory has had an enormous influence on analysis, particularly involving operator estimates on L^p-Sobolev spaces and Besov spaces, and in turn this has had an important impact on nonlinear analysis, which will be discussed below.

Hömander's full participation in the development of the theory of pseudodifferential operators began with his 1965 CPAM paper, which worked with the calculus recently developed by Kohn and L. Nirenberg, and settled a point left open in their work, namely, the coordinate invariance of their class of operators. From there, he rapidly developed the theory to new heights. His 1966 Annals paper was largely devoted to an abstract approach to obtaining boundary regularity results of Nirenberg-Kohn type, via subelliptic estimates on a general class of systems of pseudodifferential operators. In addition, subordinate sections of this paper, not related to subelliptic estimates, have also had enduring significance. Here, Hörmander established the sharp Garding inequality, a result that has lead to a great body of work. He used this to establish some improvements of his 1960 work on sufficient conditions for loal solvability of PDEs. Again, spinoffs of this were substantial, and we discus some of them below.

Hörmander's next advance in this direction appeared in his 1967 paper in the AMS Symposium Proceedings. There he developed the calculus of pseudodifferential operators for symbols $p(x,\xi) \in S^n_{\rho,\delta}$, which means

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}.$$

The case $\rho = 1,\ \delta = 0$ essentially captured the Calderon-Zygmund and Kohn-Nirenberg operators, but Hörmander made a case for the extension to this broader class. Central estimates were established in case $0 \le \delta < \rho \le 1$. Among other results, he extended his regularity theory for constant-strength operators to those identified in this paper as having "slowly varying strength." However, the major advance of this paper was to enlarge the scope of what was to become microlocal analysis. It had an immediate impact on other researchers. One notable development was the work of Calderon and R. Vaillancourt, obtaining L^2 bounds on operators with symbols in $S^0_{\rho,\rho}$, for $0 \le \rho < 1$. Another was work of Stein, establishing bounds on Sobolev spaces H^s, when $s > 0$, for operators with symbol in $S^0_{1,1}$.

Another major step was taken in Hörmander's 1968 Acta paper on the spectral function of an elliptic operator. He obtained the sharp estimate in the remainder term for the Weyl estimate for the estimate for the number of eigenvalues $\le \lambda$ for a positive self-adjoint pseudodiferential operator A on a compact manifold M. The method took off from previous results for $\sqrt{-\Delta}$, which made use of the Hadamard parametrix for the wave equation. Hörmander used an extenson of the Lax construction of geometrical optics to effect this construction. This paper, in addition to that of P. Lax, can be viewed as the precursor to the theory of microlocal analysis.

The early 1970s saw the explosive birth of microlocal analysis. One seminal event was the publication of Hörmander's 1971 Acta paper on Fourier integral operators. This globalized the local theory from his 1968 paper, and in doing so systematized some important ideas of J. Keller, Yu. Egorov, and V. Maslov. A follow-up paper with J. Duistermaat applied the Fourier integral operator calculus to a number of results in PDE, including propagation of singularities and local solvability of equations, beyond the class of real principal type.

From this pair of papers, a flurry of work ensued. One notable result was that if the imaginary part of the principal symbol of a differential operator P of principal type does not change sign along any null bicharacteristic of the real part, then $Pu = f$ is locally solvable. This was proven by Nirenberg and F. Treves, under the additional hypothesis that the principal symbol was real analytic. The proof made heavy use of these two papers on Fourier integral operators, to reduce the problem to estimates on pseudodifferential operators of the form $D_t - a(t,x,D_x) - ib(t,x,D_x)$. A factorization was required, and the authors assumed analyticity to make this factorization. The result was then completed, in the C^∞ case, by R. Beals and C. Fefferman, who introduced a finer microlocalization and exploited the Calderon-Vaillancourt estimate, with $\rho = 1/2$. Other results that relied heavily on Hörmander's theory included estimates and parametrices for subelliptic operators, including those arising in the $\overline{\partial}$-Neumann problem, results on propagation of singularities, including reflection of singularities at a boundary, and results in scattering theory, including analyses of scattering amplitudes and scattering poles. Results on these problems had seemed out of reach before Hörmander's development of microlocal analysis.

In the late 1970s, Hörmander started another revolution in microlocal analysis, with the publication of his 1979 CPAM paper on the Weyl calculus. This associates a pseudodifferential operator to a symbol in a subtly different manner than the method used before. Such a method had been used before, in a paper of Grossman, Loupias, and Stein, but Hörmander took the theory much further. Advantages of the Weyl calculus include that the symbol of the adjoint of an operator is just the adjoint of its symbol, that the first term in a parametrix construction is twice as good in the Weyl calculus, and, somewhat related, that the symbol of a commutator is better captured by the Poisson bracket of the symbols. What is behind this is that the Weyl calculus makes manifest the action of the symplectic group on the operator calculus, a symmetry that is hidden in the symbol calculus used before. As mentioned above, Beals and Fefferman introduced a finer microlocalization than captured by symbols in $S^m_{\rho,\delta}$ in the course of their work on local solvability. In his 1979 paper, Hörmander extended the Beals-Fefferman symbol classes even further, making use of this symplectic symmetry.

Hörmander wrote many other papers, but the descriptions above will perhaps give a feel for the magnitude of his contributions. He also produced several books. The first, published by Springer in 1963, covered a lot of then current material in linear PDE, including much of his early work. In 1966 his book on several complex variables was published, giving an excellent survey of that field and placing his work on the $\overline{\partial}$ operator in context. In the mid 1980s his magnificent four volume series on linear PDE appeared, giving a systematic treatment ranging from basic distribution theory and Fourier analysis to the state of the art in microlocal analysis. Also in the 1980s, he developed an interest in nonlinear wave equations. He gave a course that resulted in a set of lecture notes in 1986, which became widely circulated in the analysis community, and, by popular demand, was published as a book by Springer in 1996. These books have been a tremendous gift to the mathematical community.