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Review of "The Analysis of Linear Partial Differential Operators, I–II," by Lars Hörmander

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Calculus was perfected by Newton to formulate and solve differential equations arising in mechanics. Work on continua, modeling things such as moving fluids and vibrating solids, quickly led to the formulation of partial differential equations, and the subject of PDE has enjoyed a long and productive history. A number of important developments in analysis have been motivated by, and particularly effective in, the development of linear PDE, the subject of the volumes by Prof. Hörmander.

What makes linear equations special is that the superposition principle applies:

$$Pu = f, \quad Pv = g \Longrightarrow P(u + \alpha v) = f + \alpha g. \tag{1}$$

It was noticed early that numerous constant coefficient linear PDE that arose naturally had special soutions in terms of trigonometric fuunctions, or exponentials, seen to be equivalent in view of Euler's formula

$$e^{ix} = \cos x + i \sin x. \tag{2}$$

Thus it is tempting to exploit the superposition principle by representing general functions as superpositions of trigonometric functions. This idea was proposed by Daniel Bernoulli, as a tool for solving the wave equation

$$\Bigl(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\Bigr) u = 0, \tag{3}$$

arising in the study of vibrating strings (in the linear approximation). The idea was not accepted at the time, but it began to gain reluctant acceptance after being reintroduced by Fourier as a tool in his investigation of the heat equation

$$\Bigl(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\Bigr) u = 0. \tag{4}$$

Today, the Fourier transform, defined for integrable functions on $\mathbb{R}^n$ by

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int f(x) e^{-ix\cdot\xi}\, dx, \tag{5}$$

is a cornerstone of analysis. The Fourier inversion formula returns $f(x)$ from $\hat{f}$:

$$f(x) = (2\pi)^{-n/2} \int \hat{f}(\xi) e^{ix\cdot\xi}\, d\xi. \tag{6}$$

The Fourier transform intertwines differentiation and multiplication:

$$\mathcal{F}\,\frac{\partial}{\partial x_j} = i\xi_j\,\mathcal{F}, \quad \mathcal{F}\,x_j = i\frac{\partial}{\partial \xi_j}\,\mathcal{F}, \tag{7}$$

and hence it is often effective in reducing constant coefficient linear PDE to algebraic problems, or to ODEs with parameters. Note in particular that

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \Longrightarrow \mathcal{F}\Delta = -|\xi|^2\mathcal{F}; \tag{8}$$

Δ is called the Laplace operator. The wave and heat equations in n space variables are, respectively,

$$\Big(\frac{\partial^2}{\partial t^2} - \Delta\Big)u = 0, \tag{9}$$

and

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)u = 0. \tag{10}$$

Natural exploitation of the Fourier transform led people up against the limits of the notion of the integral as it was understood in the eighteenth and nineteenth centuries, and the production of natural classes of functions (or other objects) on which the Fourier transform operates was achieved only in the twentieth century. A milestone was Lebesgue's solution of the problem of producing the right notion of integral of functions. Out of this came the Banach spaces $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, particularly the Hilbert space $L^2(\mathbb{R}^n)$, which played a major part in the theory of functional analysis developing around the beginning of the twentieth century. One aspect of the Fourier inversion formula is that $\mathcal{F}$ is a unitary operator on $L^2(\mathbb{R}^n)$.

The development of theories of bounded operators on Banach and Hilbert spaces afforded only indirect attacks on partial differential equations, through the use of integral equations. One extension of this was the study of unbounded operators on Hilbert space, pursued by von Neumann, and later by K. O. Friedrichs and many others, which has been very influential. But the major advance was surely the creation of the theory of distributions. Distributions such as the Dirac delta function $\delta(x)$, defined by

$$\int f(x)\delta(x)\,dx = f(0), \tag{11}$$

had arisen as a convenient language for describing certain "ideal" solutions to PDE. It was realized that, in some sense, $\delta(x)$ is the derivative of the Heaviside function $H(x)$, defined to be 1 for $x \ge 0$, 0 for $x < 0$ (in case $n = 1$). L. Schwartz showed in a very elegant fashion that the methods of functional analysis could be applied to the study of such "generalized functions," to produce topological vector spaces (particularly $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{E}'(\mathbb{R}^n)$, and $\mathcal{S}'(\mathbb{R}^n)$) containing all functions in $L^1(\mathbb{R}^n)$ (compactly supported in the case of $\mathcal{E}'(\mathbb{R}^n)$), and also closed under the action of differentiation. The Schwartz kernel theorem provided the basis of treating general operators as being generalized integral operators. Also, the neatest presentation of Fourier analysis on $\mathbb{R}^n$ was seen to exploit the Fourier transform on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, defined by

$$u \in \mathcal{S}(\mathbb{R}^n) \Longleftrightarrow x^\beta D^\alpha u \in L^\infty(\mathbb{R}^n), \quad \forall\, \alpha, \beta. \tag{12}$$

Here $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, with $D_j = i\partial/\partial x_j$. One has $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, and the Fourier inversion formula implies $\mathcal{F}$ is an isomorphism. By duality, there is an isomorphism

$$\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n); \tag{13}$$

the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions is the space of continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$, i.e., its dual space. There are natural inclusions

$$\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n), \tag{14}$$

induced by

$$\langle u, v\rangle = \int_{\mathbb{R}^n} u(x)v(x)\, dx. \tag{15}$$

The space $\mathcal{D}'(\mathbb{R}^n)$ is dual to $C_0^\infty(\mathbb{R}^n)$, the space of smooth, compactly supported functions, and the space $\mathcal{E}'(\mathbb{R}^n)$ is dual to $C^\infty(\mathbb{R}^n)$.

With the machinery of distributions, it is convenient to formulate and exploit the notion of fundamental solution to a linear PDE:

$$PE_y = \delta(x - y). \tag{16}$$

If P has constant coefficients, one can take $E_y(x) = E(x - y)$. In general, E will be a distribution. One has a similar notion of fundamental solution of an initial value problem. For example, the fundamental solution to the heat equation would satisfy (10) for $t > 0$ and also

$$\lim_{t \searrow 0} E(t, x) = \delta(x). \tag{17}$$

Let us indicate how Fourier analysis produces this fundamental solution. If $\widehat{E}(t, \xi)$ denotes the partial Fourier transform with respect to x, the PDE (10) leads to

$$\frac{d}{dt}\widehat{E}(t, \xi) = -|\xi|^2 \widehat{E}(t, \xi), \tag{18}$$

and (17) yields

$$\widehat{E}(0,\xi) = (2\pi)^{-n/2}. \tag{19}$$

Hence

$$\widehat{E}(t,\xi) = (2\pi)^{-n/2} e^{-t|\xi|^2}. \tag{20}$$

Calculating $E(t,x)$ from (20) is equivalent to computing the Fourier transform of a Gaussian function, a basic computation in the subject; one obtains

$$E(t,x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}. \tag{21}$$

Similarly, the Poisson kernel $P(y,x)$, satisfying

$$\begin{aligned} \Big(\frac{\partial^2}{\partial y^2} + \Delta\Big) P(y,x) &= 0, \quad \text{for } \ y > 0, \ x \in \mathbb{R}^n, \\ \lim_{y \searrow 0} P(y,x) &= \delta(x), \end{aligned} \tag{22}$$

is characterized by its partial Fourier transform with respect to x:

$$\widehat{P}(y,\xi) = (2\pi)^{-n/2} e^{-y|\xi|}. \tag{23}$$

The computation of the inverse Fourier transform of this is easy if $n = 1$, but not so simple if $n > 1$. One way to obtain a formula for $P(y,x)$ is to exploit the "subordination identity":

$$e^{-y|\xi|} = \frac{y}{2\pi^{1/2}} \int_0^\infty e^{-y^2/4t} e^{-t|\xi|^2} t^{-3/2} \, dt. \tag{24}$$

Application to (23), together with (21), gives

$$P(y,x) = c_n y (y^2 + |x|^2)^{-(n+1)/2}. \tag{25}$$

In order to prove (24), one might note that it suffices to obtain an independent proof of its implication (25), for any one special case. As the direct verification that (23) implies (25) for $n = 1$ is straightforward, one has a proof of (24). The formula (25) for $P(y,x)$ can be analytically continued in y for Re $y > 0$, and one can pass to the limit of purely imaginary y, to obtain formulas for the fundamental solution to the wave equation:

$$\Big(\frac{\partial^2}{\partial t^2} - \Delta\Big) W = 0, \quad W(0,x) = 0, \quad \frac{\partial}{\partial t} W(0,x) = \delta(x), \tag{26}$$

which satisfies

$$\widehat{W}(t,\xi) = |\xi|^{-1} \sin t|\xi|. \tag{27}$$

If we note that the inverse Fourier transform of $(2\pi)^{-n/2}|\xi|^{-1}e^{-y|\xi|}$ is obtained by integrating (25) from y to ∞, so is equal to

$$c_n'(y^2 + |x|^2)^{-(n-1)/2}, \tag{28}$$

for $n \geq 2$, it follows that

$$W(t,x) = \lim_{\varepsilon \searrow 0} c_n' \mathrm{Im}\,(|x|^2 - (t - i\varepsilon)^2)^{-(n-1)/2}. \tag{29}$$

In particular, for $n = 3$, one obtains the formula (for $t > 0$)

$$W(t,x) = (4\pi t)^{-1}\delta(|x| - t). \tag{30}$$

($W(t,x)$ is odd in t.) Integral formulas equivalent to these formulas for fundamental solutions are classical, of course, but the calculations (18)–(30) might give a feel for the utility and power of distribution theory and Fourier analysis.

A precise study of constant coefficient equations often enables one to get a hold on variable coefficient PDE, treating

$$P(x,D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

as a perturbation of a constant coefficient operator obtained by freezing the coefficients. For example, if $P(x,D)$ is elliptic, which means

$$P_m(x,\xi) = \sum_{|\alpha| = m} a_\alpha(x)\xi^\alpha \neq 0, \quad \text{for} \quad \xi \neq 0, \tag{31}$$

then fundamental solutions for the constant coefficient elliptic operators can be pasted together to give the leading term in an inverse of $P(x,D)$. An iterative process will improve this approximation. For hyperbolic equations, such as the wave equation (9), in the variable coefficient case, more powerful methods are required. The volumes under review concentrate on the study of constant coefficient operators and some results on variable coefficient operators that one can deduce by such perturbations. More powerful methods, involving particularly pseudodifferential operators and Fourier integral operators, will be developed in following volumes by the author and applied to large classes of variable coefficient PDE.

The first of Hörmander's volumes is devoted primarily to the theory of distributions. It starts on a very basic level, with a detailed study of calculus of C^∞ functions of several variables, and proceeds with masterly organization to develop the theory to an advanced level. Specific fundamental solutions such as mentioned above are constructed. Actually, the author develops the material quite far before introducing the Fourier transform. Homogeneous distributions are introduced, and the behavior of Δ on $\mathbb{R}^n$ under dilations leads one to look for a homogeneous distribution as a fundamental solution (at least for $n \geq 3$); rotational symmetry settles what it has to be, up to a constant multiple. Change of variables enables one to treat any second order constant coefficient elliptic operator $\sum a_{jk} D_j D_k$, with $\sum a_{jk}\xi_j\xi_k$ positive definite. Analytic continuation is used to pass to fundamental solutions of second order hyperbolic operators. When the Fourier transform is introduced, some of these formulas are rederived, though by that point the emphasis is more on general classes of constant coefficient PDE, where analytical identities are not so available. One has for example the construction of a fundamental solution for an arbitrary constant coefficient operator $P(D)$. Use of Fourier analysis leads to studying $P(\xi)^{-1}$, for ξ in the complex domain, and one needs a detailed understanding of the behavior of polynomials near their zero sets.

In studying solutions to a PDE $Pu = f$, one thing that is important is a description of the singularities of the solution. In particular, one wants to describe the singularities of the fundamental solution. For the wave equation in n space variables, formula (29) shows the fundamental solution is singular on the cone $|x| = |t|$. A refinement of the notion of singular support has been introduced, the wave front set. Given a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$, $\mathrm{WF}\, u$ is a subset of the cotangent bundle $T^*\mathbb{R}^n$, projecting onto the singular support of u in $\mathbb{R}^n$, telling not only where u is singular, i.e., where one cannot introduce a cutoff ψ so that $\psi u \in C_0^\infty$, but also where the Fourier transform of ψu fails to tend rapidly to zero. In the case of a distribution with a simple singularity along a smooth surface, $\mathrm{WF}\, u$ typically consists of all nonzero cotangent vectors normal to the surface. It is shown that for elliptic P, even with variable coefficients, $\mathrm{WF}(Pu) = \mathrm{WF}\, u$. This contains the classical elliptic regularity theorem. Volume I also considers notions of wave front set associated to other categories than C^∞, for example the analytic wave front set, and ends with an introduction to hyperfunctions, an analogue of distributions, consisting of locally finite sums of analytic functionals. The theory of hyperfunctions has been developed quite far by M. Sato and collaborators, making heavy use of homological algebra. The introduction here simplifies some of the foundations of that subject.

With the basic theory of distributions and Fourier analysis thoroughly set down, Volume II proceeds deeper into the study of linear PDE, particularly with constant coefficients. Detailed analysis is made of the regularity of certain fundamental solutions of general constant coefficient operators $P(D)$, including a classification of when $P(D)$ is hypoelliptic, i.e., satisfies

$$\text{sing supp}\, Pu = \text{sing supp}\, u, \tag{32}$$

or even $\mathrm{WF}(Pu) = \mathrm{WF}\, u$. A special class of hypoelliptic operators is the class of elliptic operators; the heat operator is also hypoelliptic. The wave front set of a fundamental solution E of $P(D)$ is studied in great detail. Also questions of global solvability of an equation $P(D)u = f$ on a domain $\Omega \subset \mathbb{R}^n$ are studied. Topics on variable coefficients include operators with "constant strength," and also a chapter on scattering theory, for short range potentials. As mentioned, subsequent volumes will go more deeply into the study of variable coefficient linear PDE.

These volumes are to some extent a greatly expanded rewrite of Hörmander's *Linear Partial Differential Operators*, published by Springer-Verlag in 1964. They cover an enormous amount of analysis. In addition to topics mentioned above, there are self contained treatments of topics ranging from the Gauss-Green formula and Cauchy's integral formula, to a complete though brief discussion of manifolds, differential forms, and Hamiltonian vector fields, to the Malgrange preparation theorem and the Tarski-Seidenberg theorem. Some of these topics will be more completely appreciated when they are used in subsequent volumes. The two volumes that are out, and their companions, which will follow, will not likely serve as the texts for one's first brush with PDE, but the serious analyst will find here an elegant presentation of a vast amount of material on linear PDE, by a consummate master of the subject.