The Heat Kernel and the Wave Kernel

(Adapted from Appendix B of [PT])

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The solution to the wave equation

$$\frac{\partial^2 w}{\partial t^2} - \Delta w = 0, \quad w(0,x) = f(x),\ w_t(0,x) = g(x), \tag{1}$$

on $\mathbb{R} \times \mathbb{R}^n$, is given by $w(t,x) = u(t,x) + v(t,x)$, where

$$u(t,x) = \cos t\sqrt{-\Delta}\, f(x), \quad v(t,x) = \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}\, g(x). \tag{2}$$

We will derive formulas for these solution operators, in case

$$n = 2k+1, \tag{3}$$

by comparing two formulas for $e^{t\Delta} f(x)$.

The first formula for $e^{t\Delta}$ is

$$\begin{aligned} e^{t\Delta} f(x) &= (4\pi t)^{-n/2} \int\limits_{\mathbb{R}^n} e^{-|y|^2/4t} f(x-y)\, dy \\ &= (4\pi t)^{-n/2} A_{n-1} \int_0^\infty \overline{f}_x(r) r^{n-1} e^{-r^2/4t}\, dr, \end{aligned} \tag{4}$$

where A_{n-1} is the area of the unit sphere S^{n-1} in $\mathbb{R}^n$, and

$$\overline{f}_x(r) = \frac{1}{A_{n-1}} \int\limits_{S^{n-1}} f(x + r\omega)\, dS(\omega).$$

Note that $\overline{f}_x(r)$ is well defined for all $r \in \mathbb{R}$, and $\overline{f}_x(-r) = \overline{f}_x(r)$. The first identity in (4) follows, via Fourier analysis, from the evaluation of the Gaussian integral

$$\int\limits_{\mathbb{R}^n} e^{-t|\xi|^2 + ix\cdot\xi}\, d\xi = \Big(\frac{\pi}{t}\Big)^{n/2} e^{-|x|^2/4t}. \tag{5}$$

The second identity in (4) follows by switching to spherical polar coordinates, $y = r\omega$, and using $dy = r^{n-1}\, dr\, dS(\omega)$.

The second formula for $e^{t\Delta}$ is

$$
\begin{aligned}
e^{t\Delta} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{h}_t(s) \cos s\sqrt{-\Delta}\, f(x)\, ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{h}_t(s) u(s,x)\, ds,
\end{aligned} \tag{6}
$$

with $h_t(\sigma) = e^{-t\sigma^2}$, hence, by (5), $\hat{h}_t(s) = (2t)^{-1/2} e^{-s^2/4t}$. This follows from

$$
e^{-t|\xi|^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{h}_t(s)\, \cos s|\xi|\, ds,
$$

(itself an example of the Fourier inversion formula), by applying these Fourier multipliers to f.

Setting $4t = 1/\lambda$ and comparing the formulas (4) and (6), we have

$$
\int_0^{\infty} u(s,x) e^{-\lambda s^2}\, ds = \frac{A_{n-1}}{2} \Big(\frac{\lambda}{\pi}\Big)^{(n-1)/2} \int_0^{\infty} \overline{f}_x(r) r^{n-1} e^{-\lambda r^2}\, dr, \tag{7}
$$

for all $\lambda > 0$. The key to getting a formula for $u(s,x)$ from this is to make the factor $\lambda^{(n-1)/2}$ on the right side of (7) disappear.

Bringing in the hypothesis (3), we use the identity

$$
-\frac{1}{2r}\frac{d}{dr} e^{-\lambda r^2} = \lambda e^{-\lambda r^2} \tag{8}
$$

to write the right side of (7) as

$$
C_n \int_0^{\infty} r^{2k} \overline{f}_x(r) \Big(-\frac{1}{2r}\frac{d}{dr}\Big)^k e^{-\lambda r^2}\, dr. \tag{9}
$$

Repeated integration by parts shows that this is equal to

$$
C_n \int_0^{\infty} r \Big(\frac{1}{2r}\frac{d}{dr}\Big)^k \big[r^{2k-1} \overline{f}_x(r)\big]\, e^{-\lambda r^2}\, dr. \tag{10}
$$

Now it follows from uniqueness of Laplace transforms that

$$
u(t,x) = C_n t \Big(\frac{1}{2t}\frac{d}{dt}\Big)^k \big[t^{2k-1} \overline{f}_x(t)\big], \tag{11}
$$

for well behaved functions f on $\mathbb{R}^n$, when $n = 2k+1$. By (7), we have

$$
C_n = \frac{1}{2} \pi^{-(n-1)/2} A_{n-1}. \tag{12}
$$

We can also compute C_n directly in (11), by considering $f = 1$. Then $\overline{f}_x = 1$ and $u = 1$, so

$$1 = C_n t\Big(\frac{1}{2t}\frac{d}{dt}\Big)^k t^{2k-1} = C_n\Big(k - \frac{1}{2}\Big)\Big(k - \frac{3}{2}\Big)\cdots\frac{1}{2}, \tag{13}$$

i.e.,

$$C_n = \frac{1}{(k-\frac{1}{2})(k-\frac{3}{2})\cdots\frac{1}{2}}, \quad n = 2k+1. \tag{14}$$

This simply means

$$A_{2k} = \frac{2\pi^k}{(k-\frac{1}{2})(k-\frac{3}{2})\cdots\frac{1}{2}}, \tag{15}$$

a formula that is frequently derived by looking at Gaussian integrals.

To compute $(\sin t\sqrt{-\Delta})/\sqrt{-\Delta}$, we use

$$\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}g(x) = \int_0^t \cos s\sqrt{-\Delta}\, g(x)\, ds. \tag{16}$$

From (11), if $k \geq 1$,

$$\cos t\sqrt{-\Delta}\, g(x) = \frac{C_n}{2}\frac{d}{dt}\Big(\frac{1}{2t}\frac{d}{dt}\Big)^{k-1}\big[t^{2k-1}\overline{g}_x(t)\big], \tag{17}$$

so (16) becomes

$$\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}g(x) = \frac{C_n}{2}\Big(\frac{1}{2t}\frac{d}{dt}\Big)^{k-1}\big[t^{2k-1}\overline{g}_x(t)\big]. \tag{18}$$

The formulas (11) and (18) are for $t > 0$. For arbitrary $t \in \mathbb{R}$, use

$$\cos t\sqrt{-\Delta} = \cos(-t)\sqrt{-\Delta}, \quad \sin t\sqrt{-\Delta} = -\sin(-t)\sqrt{-\Delta}. \tag{19}$$

The case $k = 0$ is exceptional. Then (17) does not work. Instead, we have

$$\cos t\sqrt{-\Delta}\, g(x) = \frac{1}{2}[g(x+t) + g(x-t)], \tag{20}$$

and (16) gives

$$\begin{aligned} \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}\, g(x) &= \frac{1}{2}\int_0^t [g(x+s) + g(x-s)]\, ds \\ &= \frac{1}{2}\int_{-t}^t g(x+s)\, ds, \end{aligned} \tag{21}$$

for $n = 1$.

Let us turn to $k = 1$, when $n = 3$. From (14), $C_2 = 2$, and then (18) gives

$$\begin{aligned} \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}\, g(x) &= t\bar{g}_x(t) \\ &= \frac{t}{4\pi} \int\limits_{S^2} g(x + t\omega)\, dS(\omega) \\ &= \frac{1}{4\pi t} \int\limits_{|y|=|t|} g(x + y)\, dS(y). \end{aligned} \tag{22}$$

The formulas (17) and (18) (and also (22)) exhibit both finite propagation speed and the strong Huyghens principle: the left sides depend on $f(y)$ and $g(y)$ only for

$$y \in \mathbb{R}^n \text{ such that } |x - y| = |t|, \tag{23}$$

when n is odd and ≥ 3. For $n = 1$, (20) also depends on $f(y)$ only for $|x - y| = |t|$, but (21) depends on $g(y)$ for $|x-y| \leq |t|$. In this case, we still have finite propagation speed, but not the strong Huyghens principle.

Reference

[PT] M. Pinsky and M. Taylor, Pointwise Fourier inversion: a wave equation approach, J. Fourier Anal. and Appl. 3 (1997), 647–703.