Isothermal Coordinates on Bounded Planar Domains
With L^∞ Metric Tensor

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Let $\overline{\Omega}$ be a smoothly bounded 2-dimensional domain, with an L^∞ metric tensor. Take $z_0 \in \partial\Omega$. We seek local isothermal coordinates centered at z_0, flattening out $\partial\Omega$. We do not expect C^1 coordinates, but we will obtain Hölder continuous coordinates that map a neighborhood of p in $\overline{\Omega}$ homeomorphically onto a neighborhood of the origin in the lower half plane, in a manner that is conformal.

Without loss of generality, we can take $\overline{\Omega} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \le 0\}$, $z_0 = 0$, and assume the metric tensor $G(z) = (g_{jk}(z))$ is given, in $L^\infty(\overline{\Omega})$, such that $G(z) \ge \lambda_0 I$ for all z and $G(z) = (\delta_{jk})$ for $|z| \ge R$. We take $z = x_1 + ix_2$, identifying $\mathbb{R}^2$ and $\mathbb{C}$. Extend G to all of $\mathbb{R}^2$ such that the map $(x_1, x_2) \mapsto (x_1, -x_2)$ is an isometry. Note that even if G is continuous on $\overline{\Omega}$, the extended metric tensor might have a jump across $\partial\Omega$.

The metric tensor G gives rise to an almost complex structure J on $\mathbb{R}^2$ (defined a.e.), defined as counterclockwise rotation by $90°$. A calculation gives

$$J = \frac{1}{\sqrt{g}} \begin{pmatrix} -g_{12} & -g_{22} \\ g_{11} & g_{12} \end{pmatrix}. \tag{1}$$

Here $g = \det G$. Of course, $J(z)^2 = -I$. For the flat metric tensor (δ_{jk}), we have

$$J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

The isothermal coordinates (u, v) (or $f = u + iv$) we want to construct will have the property that $Df \in L^p_{\text{loc}}(\mathbb{R}^2)$ for some $p > 1$ (in fact, for some $p > 2$) and intertwine J and J_0:

$$Df(z)J(z) = J_0 Df(z), \tag{3}$$

for a.e. $z \in \mathbb{C}$. In addition, we want $f : \mathbb{C} \to \mathbb{C}$ to be a homeomorphism, and

$$f : \mathbb{R} \longrightarrow \mathbb{R}. \tag{4}$$

Work of Ahlfors and Bers gives such a map.

To begin the description of this work, a calculation gives that (3) is equivalent to

$$f_{\overline{z}} = \mu f_z, \tag{5}$$

where

$$\mu = \frac{g_{11} - \sqrt{g} - ig_{12}}{g_{11} + \sqrt{g} + ig_{12}}. \tag{6}$$

Note that

$$|\mu|^2 = \frac{g_{11} + g_{22} - 2\sqrt{g}}{g_{11} + g_{22} + 2\sqrt{g}}.$$

The hypotheses give $\mu \in L^\infty(\mathbb{R}^2)$, satisfying

$$\|\mu\|_{L^\infty} = k < 1, \quad \mu(z) = 0 \ \text{ for } \ |z| > R. \tag{7}$$

The equation (5) is called the Beltrami equation. In [AB] (cf. also [A]) it is shown that if (7) holds, then (5) has a unique solution of the form

$$f(z) = z + g(z), \tag{8}$$

where

$$g = \frac{1}{\pi z} * h, \tag{9}$$

and

$$\operatorname{supp} h \in \{z : |z| \le R\}, \quad h \in L^p(\mathbb{R}^2), \tag{10}$$

for some $p > 2$.

In a little more detail, (5) gives $g_{\overline{z}} = \mu g_z + \mu$, and with $h = g_{\overline{z}}$, we want to solve

$$h = \mu R h + \mu, \tag{11}$$

for h, where R is Fourier multiplication by $(\xi_1 + i\xi_2)/(\xi_1 - i\xi_2)$, so $Rg_{\overline{z}} = g_z$. The operator R is unitary on $L^2(\mathbb{R}^2)$, and it has L^p-operator norm close to 1 for p close to 2 (by Calderon-Zygmund theory and interpolation estimates). Then μR has L^p-operator norm < 1 for p close to 2, and we find h as

$$h = (I - \mu R)^{-1}\mu, \tag{12}$$

solving (11), hence with compact support. Further arguments (see [A], [AB]) show that

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \ \text{ is a homeomorphism.} \tag{13}$$

In fact, f and its inverse are Hölder continuous (Such continuity for f follows from the result $f \in H^{1,p}_{\text{loc}}$). The inverse also satisfies a Beltrami equation as in (5) and (6).

In our current setting, uniqueness plus symmetry imply

$$f(\overline{z}) = \overline{f(z)}, \tag{14}$$

so (4) holds.

Remark 1. If in addition G is continuous on $\mathbb{R}^2$, or more generally if G belongs to $L^\infty \cap \mathrm{vmo}$, then one has $f \in H^{1,p}_{\mathrm{loc}}$ for all $p < \infty$ (cf. [T], Chapter 3, Proposition 2.10). However, typically G has a jump across $\mathbb{R}$, so this result is not applicable near $\mathbb{R}$. One might wonder if $f|_{\overline{\Omega}}$ belongs to $H^{1,p}(\overline{\Omega} \cap B_R)$ for large p. This is in fact the case; see Remark 3 below.

Remark 2. A partially successful resolution of the problem raised in Remark 1 goes as follows. Pick $p \in (2, \infty)$ (large). Then there exists $k_0 = k_0(p)$ such that if (7) holds with $k \le k_0$, then (12) produces $h \in L^p$, hence $f \in H^{1,p}_{\mathrm{loc}}$.

Now if you are given a continuous metric tensor G on $\overline{\Omega}$, you can make a linear change of coordinates (preserving $\partial\Omega = \mathbb{R}$) such that $g_{jk}(z_0) = \delta_{jk}$, and then alter G outside a small neighborhood of z_0 so that μ, given by (6), satisfies $\|\mu\|_{L^\infty} \le k_0$, thus obtaining local isothermal coordinates (in a small neighborhood of z_0), mapping $\partial\Omega$ to $\mathbb{R}$, and having the local regularity property $f \in H^{1,p}_{\mathrm{loc}}$.

Remark 3. Here is a complete resolution of said problem. If f_1 and f_2 are isothermal coordinates constructed as above, on overlapping regions, then $D(f_1 \circ f_2^{-1})$ commutes with J_0, hence $f_1 \circ f_2^{-1}$ is a holomorphic homeomorphism, preserving $\mathbb{R}$. Thus (by Schwartz reflection) it is a smooth diffeomorphism up to $\partial\Omega$. Hence f_1 and f_2 have the same regularity on their common domain. In light of Remarks 1–2, it follows that if G is continuous on $\overline{\Omega}$, all these isothermal coordinates belong to $H^{1,p}(\overline{\Omega} \cap B_R)$ for all $p < \infty$.

References

[A] L. Ahlfors, Lectures on Quasiconformal Mappings, Wadsworth, Monterey CA, 1987.

[AB] L. Ahlfors and L. Bers, Riemann's mapping theorem for variable metrics, Ann. of Math. 72 (1960), 385–404.

[T] M. Taylor, Tools for PDE, AMS, Providence RI, 2000.