Lévy Processes

Michael Taylor

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1. Introduction

Wiener measure is a measure on the space of paths in $\mathbb{R}^n$ having the following property. Consider the Gaussian probability distribution

$$
\begin{aligned}
p(t,x) &= (2\pi)^{-n} \int e^{-t|\xi|^2} e^{ix\cdot\xi}\, d\xi \\
&= (4\pi t)^{-n/2} e^{-|x|^2/4t}.
\end{aligned} \tag{1.1}
$$

Given $0 < t_1 < t_2$ and given that $\omega(t_1) = x_1$, the probability density for the location of $\omega(t_2)$ is $p(t_2 - t_1, x - x_1)$. More generally, given $0 < t_1 < t_2 < \cdots < t_k$ and given Borel sets $E_j \in \mathbb{R}^n$, the probability that a path, starting at $x = 0$ at time $t = 0$, lies in E_j at time t_j for each $j \in \{1, \dots, k\}$ is

$$
\int_{E_1} \cdots \int_{E_k} p(t_k - t_{k-1}, x_k - x_{k-1}) \cdots p(t_1, x_1)\, dx_k \cdots dx_1. \tag{1.2}
$$

It takes some effort to prove that there is a countably additive measure characterized by these properties. This was first done by N. Wiener, who also proved that the associated measure, called Wiener measure, is supported on the space of continuous paths. One elegant approach to the construction of Wiener measure is due to [Nel].

Extensions of this theory to non-Gaussian distributions have been pursued by many people, notably P. Lévy. Our main purpose here is to extend the method of [Nel] to treat these Lévy processes. To start, we replace (1.1) by

$$
\begin{aligned}
p(t,x) &= (2\pi)^{-n} \int e^{-t\psi(\xi)} e^{ix\cdot\xi}\, d\xi \\
&= e^{-t\psi(D)} \delta(x).
\end{aligned} \tag{1.3}
$$

Here $\psi(\xi)$ is a function with the property that

$$
p(t,x) \ge 0, \quad \forall\, t > 0,\ x \in \mathbb{R}^n. \tag{1.4}
$$

We require $\psi(0) = 0$, so

$$
\int p(t,x)\, dx = 1, \quad \forall\, t > 0. \tag{1.5}
$$

Note also that

$$
\int p(t, x-y) p(s,y)\, dy = p(t+s, x). \tag{1.6}
$$

The example $\psi(\xi) = |\xi|^2$ gives the Gaussian case, as in (1.1). Other examples include $\psi_\alpha(\xi) = |\xi|^{2\alpha}$, for $\alpha \in (0,1)$. The function $p_\alpha(t,x) = e^{-t\psi_\alpha(D)}\delta(x) = e^{-t(-\Delta)^\alpha}\delta(x)$ is related to (1.1) by the subordination identity

$$e^{-tL^\alpha} = \int_0^\infty \Phi_{t,\alpha}(s) e^{-sL}\, ds, \quad 0 < \alpha < 1, \tag{1.7}$$

valid for any positive self adjoint operator L, where $\Phi_{t,\alpha}$ has the property

$$e^{-t\lambda^\alpha} = \int_0^\infty \Phi_{t,\alpha}(s) e^{-s\lambda}\, ds, \quad \lambda > 0. \tag{1.8}$$

The fact that

$$(-\partial_\lambda)^k e^{-t\lambda^\alpha} \ge 0, \quad \text{for} \ \ \lambda, t > 0, \ k \in \mathbb{C}^+, \tag{1.9}$$

implies

$$\Phi_{t,\alpha}(s) \ge 0, \quad \text{for} \ \ s \in [0,\infty), \tag{1.10}$$

given $t \in (0,\infty),\ \alpha \in (0,1)$. One also has

$$\int_0^\infty \Phi_{t,\alpha}(s)\, ds = 1. \tag{1.11}$$

This is discussed in a more general context in §IX.11 of [Y]. The most familiar case is the case $\alpha = 1/2$, where

$$\Phi_{t,1/2}(s) = \frac{t}{2\pi^{1/2}}\, e^{-t^2/4s}\, s^{-3/2}; \tag{1.12}$$

compare [T1], Chapter 3, (5.22)–(5.31).

The positivity (1.10) implies positivity in (1.4) when $\psi(\xi) = |\xi|^{2\alpha},\ \alpha \in (0,1)$. There is a good characterization of functions $\psi(\xi)$ for which (1.4) holds, the Lévy-Khinchin formula. We discuss this in Appendix A, with further details on such homogeneous cases as $|\xi|^{2\alpha}$ in Appendix B. Here we give another example, arising from applying (1.7) to $e^{t(\Delta-1)}$. This leads to

$$\varphi_\alpha(\xi) = (|\xi|^2 + 1)^\alpha - 1, \quad 0 < \alpha < 1. \tag{1.13}$$

We contrast $e^{-t\varphi_\alpha(D)}\delta(x)$ with $e^{-t\psi_\alpha(D)}\delta(x)$. The latter has a "heavy tail":

$$e^{-t\psi_\alpha(D)}\delta(x) \sim C_{n\alpha t}|x|^{-n-2\alpha}, \quad |x| \to \infty. \tag{1.14}$$

This is contrasted with the exponential decay:

$$e^{-t\varphi_\alpha(D)}\delta(x) \sim C'_{n\alpha t} e^{-|x|}, \quad |x| \to \infty, \tag{1.15}$$

which is much more rapid decay than in (1.14), though not as rapid as the decay in (1.1). Further results on heavy tails are given in Appendix C, and a detailed analysis of the long-time and short-time behavior of (1.15) in Appendix D.

Appendix E discusses vanishing and superexponential decay on cones. Appendix F treats regularity properties of the semigroup $e^{-t\psi(D)}$.

Results mentioned above all deal with Levy processes on Euclidean space. Appendix M extends the notion to Riemannian manifolds. Here we emphasize constructions that extend elements arising in the Levy-Khinchin formula, but such variants quickly lead further afield, and it becomes natural to include Appendix N, discussing the production of more general Markov semigroups. Material here includes continuous-time finite Markov chains and denumerable Markov chains, as well as more general cases.

2. Construction of the probability measures

We will anticipate that the stochastic processes to be constructed here are determined by their values at positive *rational* t. Thus we consider the set of “paths”

$$\mathfrak{P} = \prod_{t\in\mathbb{Q}^+} \dot{\mathbb{R}}^n. \tag{2.1}$$

Here $\dot{\mathbb{R}}^n$ is the one point compactificaqtion of $\mathbb{R}^n$. Thus $\mathfrak{P}$ is a compact metrizable space. For each $\psi(\xi)$ such that (1.3)–(1.5) holds, we associate a probability measure on $\mathfrak{P}$.

In order to construct this measure, we will construct a certain positive linear functional $E : C(\mathfrak{P}) \to \mathbb{R}$, on the space $C(\mathfrak{P})$ of real valued continuous functions on $\mathfrak{P}$, satisfying $E(1) = 1$, and a condition motivated by (1.2), which we give below. We first define E on the space $\mathcal{C}^\#$ consisting of continuous functions on $\mathfrak{P}$ of the form

$$\varphi(\omega) = F\big(\omega(t_1), \dots, \omega(t_k)\big), \quad t_1 < \cdots < t_k, \tag{2.2}$$

where F is continuous on $\prod_1^k \dot{\mathbb{R}}^n$, and $t_j \in \mathbb{Q}^+$. Motivated by (1.2), we take

$$\begin{aligned} E(\varphi) = \int \cdots \int p(t_1, x_1)p(t_2 - t_1, x_2 - x_1) \cdots p(t_k - t_{k-1}, x_k - x_{k-1}) \\ F(x_1, \dots, x_k)\, dx_k \cdots dx_1. \end{aligned} \tag{2.3}$$

If $\varphi(\omega)$ in (2.2) actually depends on $\omega(t_\nu)$ for some proper subset $\{t_\nu\}$ of $\{t_1, \dots, t_k\}$, there arises a formula for $E(\varphi)$ with a different appearance from (2.3). The fact that these two expressions are equal follows from the identity (1.6). From this it follows that $E : \mathcal{C}^\# \to \mathbb{R}$ is well defined. It is also a positive functional, satisfying $E(1) = 1$.

Now, by the Stone-Weierstrass Theorem, $\mathcal{C}^\#$ is dense in $C(\mathfrak{P})$. Since $E : \mathcal{C}^\# \to \mathbb{R}$ is a positive linear functional and $E(1) = 1$, it follows that E has a unique continuous extension to $C(\mathfrak{P})$, possessing these properties. The Riesz representation theorem associates to E a probability measure W. Therefore we have:

Theorem 2.1. *Given $p(t, x) = e^{-t\psi(D)}\delta(x)$ satisfying (1.4)–(1.5), there is a unique probability measure W on $\mathfrak{P}$ such that (2.3) is given by*

$$E(\varphi) = \int_{\mathfrak{P}} \varphi(\omega)\, dW(\omega), \tag{2.4}$$

for each $\varphi(\omega)$ of the form (2.2) with F continuous on $\prod_1^k \dot{\mathbb{R}}^n$. In such a case, (2.3) then holds for any bounded Borel function F, and also for any positive Borel function F, on $\prod_1^k \dot{\mathbb{R}}^n$.

Let us do some basic examples of calculations of (2.4). Define functions X_t on $\mathfrak{P}$, taking values in $\dot{\mathbb{R}}^n$, by

$$X_t(\omega) = \omega(t). \tag{2.5}$$

We see that if $0 < s < t,\ q \in \mathbb{R}$,

$$\begin{aligned} E(|X_t - X_s|^q) &= \iint p(s, x_1) p(t-s, x_2 - x_1)\, |x_2 - x_1|^q \, dx_2 \, dx_1 \\ &= \int p(t-s, y) |y|^q \, dy, \end{aligned} \tag{2.6}$$

making the change of variable $y = x_2 - x_1,\ z = x_1$ and using (1.5).

Let us specialize to $\psi(\xi) = \psi_\alpha(\xi) = |\xi|^{2\alpha}$, i.e.,

$$p(t, x) = e^{-t(-\Delta)^\alpha} \delta(x), \quad 0 < \alpha < 1. \tag{2.7}$$

Then $p(t, \cdot)$ is bounded and continuous on $\mathbb{R}^n$ for each $t > 0$ and we have the asymptotic behavior (1.14) as $|x| \to \infty$. We also have

$$p(t, x) = t^{-n/2\alpha} p(1, t^{-1/2\alpha} x), \tag{2.8}$$

and hence

$$\begin{aligned} \int p(t, y) |y|^q \, dy &= t^{-n/2\alpha} \int p(1, t^{-1/2\alpha} y) |y|^q \, dy \\ &= C_{n\alpha q}\, t^{q/2\alpha}, \end{aligned} \tag{2.9}$$

where

$$C_{n\alpha q} = \int p(1, y) |y|^q \, dy. \tag{2.10}$$

Since $p(1, y)$ is bounded and

$$p(1, y) \sim C_\alpha |y|^{-n-2\alpha}, \quad |y| \to \infty, \tag{2.11}$$

we have

$$C_{n\alpha q} < \infty \Longleftrightarrow -n < q < 2\alpha, \tag{2.12}$$

given $0 < \alpha < 1$. Of course, in the Gaussian case $\alpha = 1$ one has $C_{n\alpha q} < \infty$ for all $q \in (-n, \infty)$. In light of (2.6), we have

$$E(|X_t - X_s|^q) = C_{n\alpha q}|t - s|^{q/2\alpha}, \quad -n < q < 2\alpha, \ 0 < \alpha < 1. \tag{2.13}$$

If $\alpha = 1$ this extends to all $q \in (-n, \infty)$.

The identity (2.13) measures the distance from X_t to X_s in $L^q(\mathfrak{P}, W)$, provided $q > 0$ and the hypotheses hold to yield $C_{n\alpha q} < \infty$. Note that $L^q(\mathfrak{P}, W)$ is a Banach space for $q \in [1, \infty)$. For $q \in (0, 1)$, it is not a Banach space, but it is still a metric space. We see that $t \mapsto X_t$ extends continuously from $\mathbb{Q}^+$ to $\mathbb{R}^+$, yielding a continuous function of t with values in $L^q(\mathfrak{P}, W)$, for $q \in (0, \alpha/2)$.

The following is a useful generalization of (2.6); if $G : \mathbb{R}^n \to \mathbb{R}$ is positive or bounded (and Borel measurable) and $0 < s < t$,

$$\begin{aligned} E(G(X_t - X_s)) &= \iint p(s, x_1)p(t - s, x_2 - x_1)G(x_2 - x_1)\, dx_2\, dx_1 \\ &= \int p(t - s, y)G(y)\, dy \\ &= E(G(X_{t-s})). \end{aligned} \tag{2.14}$$

In other words, $X_t - X_s$ has the same statistical behavior as X_{t-s}. The following result asserts that if $t > s \geq 0$ then $X_t - X_s$ is independent of X_σ for $\sigma \leq s$.

Proposition 2.2. *Assume* $0 < s_1 < \cdots < s_k < s < t$ *and consider functions on* $\mathfrak{P}$ *of the form*

$$\varphi(\omega) = F\big(\omega(s_1), \ldots, \omega(s_k)\big), \quad \psi(\omega) = G\big(\omega(t) - \omega(s)\big). \tag{2.15}$$

Then

$$E(\varphi\psi) = E(\varphi)E(\psi). \tag{2.16}$$

Proof. Note that $E(\psi)$ is given by (2.14). Meanwhile, we have

$$\begin{aligned} E(\varphi\psi) = \int & p(s_1, x_1)p(s_2 - s_1, x_2 - x_1) \cdots p(s_k - s_{k-1}, x_k - x_{k-1}) \\ & p(s - s_k, y_1 - x_k)p(t - s, y_2 - y_1)F(x_1, \ldots, x_k) \\ & G(y_2 - y_1)\, dx_1 \cdots dx_k\, dy_1\, dy_2. \end{aligned} \tag{2.17}$$

If we change variables to $x_1, \ldots, x_k, y_1, z = y_2 - y_1$, then comparison with (2.14) shows that $E(\psi)$ factors out of (2.17). Then use of $\int p(s - s_k, y_1 - x_k)\, dy_1 = 1$ shows that the other factor is equal to $E(\varphi)$, so we have (2.16).

Note the characteristic function calculation

$$E(e^{i\xi \cdot X_t}) = \int p(t, y)e^{iy\cdot\xi}\, dy = e^{-t\psi(\xi)}. \tag{2.18}$$

Then, by (2.14), we have

$$E(e^{i\xi\cdot(X_t-X_s)}) = e^{-|t-s|\psi(\xi)}, \tag{2.19}$$

and an iterative use of (2.16) shows that if $0 < t_1 < \cdots < t_k$ and $\xi_j \in \mathbb{R}^n$, then

$$\begin{aligned} &E\big(e^{i\xi_1\cdot X_{t_1}+i\xi_2\cdot(X_{t_2}-X_{t_1})+\cdots+i\xi_k\cdot(X_{t_k}-X_{t_{k-1}})}\big) \\ &= e^{-t_1\psi(\xi_1)-(t_2-t_1)\psi(\xi_2)-\cdots-(t_k-t_{k-1})\psi(\xi_k)}. \end{aligned} \tag{2.20}$$

3. Stochastic continuity and regularity of paths

In §2 we constructed a probability space $(\mathfrak{P}, W)$ and a family X_t of random variables on $\mathfrak{P}$, given by (2.5), when $t \in \mathbb{Q}^+$. We indicated how to extend X_t to $t \in \mathbb{R}^+$ in case $\psi(\xi) = |\xi|^{2\alpha}$, $0 < \alpha \leq 1$. We begin this section by making such an extension for general $\psi(\xi)$ treated in Theorem 2.1, obtaining a *stochastically continuous* family of random variables on $\mathfrak{P}$.

This is obtained in a fashion parallel to (2.7)–(2.13), with $|y|^q$ replaced by a different function $G(y)$, namely

$$G(y) = 1 - e^{-|y|} = 1 - g(y). \tag{3.1}$$

By (2.14) we have (at first for $s, t \in \mathbb{Q}^+$),

$$\begin{aligned} E(G(X_t - X_s)) &= \int p(|t-s|, y) G(y)\, dy \\ &= 1 - \int p(|t-s|, y) g(y)\, dy. \end{aligned} \tag{3.2}$$

Note that $\hat{g} \in L^1(\mathbb{R}^n)$ and

$$\int p(t, y) g(y)\, dy = (2\pi)^{-n} \int e^{-t\psi(\xi)} \hat{g}(\xi)\, d\xi. \tag{3.3}$$

Also, the function ψ satisfies

$$\operatorname{Re} \psi(\xi) \geq 0, \tag{3.4}$$

so the Lebesgue dominated convergence theorem applies, to give $\lim_{t \searrow 0} \int p(t, y) g(y)\, dy = 1$, and hence

$$E(G(X_t - X_s)) = \vartheta(|t-s|), \quad \lim_{t \to 0} \vartheta(t) = 0. \tag{3.5}$$

Observe that

$$\rho(X, Y) = E(G(X - Y)) \tag{3.6}$$

yields a *metric* on the space $M(\mathfrak{P}, W)$ of equivalence classes of measurable $\mathbb{R}^n$-valued functions on $\mathfrak{P}$, as a consequence of the monotonicity and concavity of $r \to 1 - e^{-r}$ on $[0, \infty)$. This metric defines the topology of convergence in measure on $\mathfrak{P}$.

In fact, $M(\mathfrak{P}, W)$ is a complete metric space with the metric (3.6). Given a Cauchy sequence, one can take a subsequence (Y_j) satisfying $\rho(Y_j, Y_{j+k}) \leq 4^{-j}$, $\forall\, k \geq 1$. This sequence converges pointwise a.e. to a limit $Y \in M(\mathfrak{P}, W)$, by virtue of the estimate

$$W\Big(\{\omega \in \mathfrak{P} : |Y_j(\omega) - Y_{j+k}(\omega)| \geq 2^{-j}\}\Big) \leq C\, 2^{-j},$$

and convergence also takes place in ρ-metric.

Given this completeness, the estimate (3.5) implies there is a unique continuous extension of $t \mapsto X_t$ from $\mathbb{Q}^+ \to M(\mathfrak{P}, W)$ to $\mathbb{R}^+ \to M(\mathfrak{P}, W)$. There results a stochastically continuous process $\{X_t : t \in \mathbb{R}^+\}$.

Regarding the behavior of individual paths $t \mapsto X_t(\omega)$, there is the following result of Kolmogorov. For a proof see [Kry], p. 20.

Proposition 3.1. *Suppose there exist* $q, \beta > 0$, $C < \infty$ *such that*

$$E(|X_t - X_s|^q) \leq C|t - s|^{1+\beta}, \quad \forall\, s, t \geq 0. \tag{3.7}$$

Then the process $\{X_t\}$ *has a modification almost all of whose paths are continuous.*

Note that in (2.13) this estimate just barely fails, if one requires $q < 2\alpha$. As noted below (2.13), such an estimate works in the Gaussian case for all $q \in (-n, \infty)$, so (3.7) works there, which gives pathwise continuity for the Wiener process. For other Lévy processes, path continuity fails, but another result holds.

One says a path $t \mapsto \gamma(t)$ from $\mathbb{R}^+$ to $\mathbb{R}^n$ is cadlag provided that for each $t \in \mathbb{R}^+$,

$$\lim_{s \searrow t} \gamma(s) = \gamma(t), \quad \text{and} \quad \lim_{s \nearrow t} \gamma(s) \text{ exists}, \tag{3.8}$$

though the latter limit need not equal $\gamma(t)$. The following result is proven in [Kry], p. 136.

Proposition 3.2. *If* $\{X_t : t \in \mathbb{R}^+\}$ *is a stochastically continuous process with independent increments, then it admits a modification such that almost all paths are cadlag.*

4. Hausdorff dimension of Lévy paths and Lévy graphs

We restrict attention to Lévy processes on $\mathbb{R}^n$ generated by $(-\Delta)^\alpha$, with $\alpha \in (0,1]$. We estimate from below the Hausdorff dimension $\operatorname{Hdim} \omega(I)$ for a typical path $\omega(I) = \{\omega(t) : t \in I\}$, where $I = [0,T]$, $T \in (0,\infty)$. We will show that for each such I,

$$\operatorname{Hdim} \omega(I) \ge \min(2\alpha, n), \quad \text{for a.e.} \quad \omega. \tag{4.1}$$

Actually it is known that equality holds (see [Sat]), but we will not establish the reverse inequality. (For $\alpha = 1$ the reverse inequality is an immediate consequence of the modulus of continuity.) We will also estimate the Hausdorff dimension of a graph:

$$Z_\omega(t) = (t, \omega(t)). \tag{4.2}$$

With $Z_\omega(I) = \{Z_\omega(t) : t \in I\}$, we obtain the following estimates on $\operatorname{Hdim} Z_\omega(I)$. Namely,

$$n \ge 2, \ \frac{1}{2} \le \alpha \le 1 \Longrightarrow \operatorname{Hdim} Z_\omega(I) \ge 2\alpha, \tag{4.3}$$

for almost all ω, while

$$n = 1, \ \frac{1}{2} < \alpha \le 1 \Longrightarrow \operatorname{Hdim} Z_\omega(I) \ge 2 - \frac{1}{2\alpha}, \tag{4.4}$$

and for each $n \ge 1$,

$$0 < \alpha \le \frac{1}{2} \Longrightarrow \operatorname{Hdim} Z_\omega(I) \ge 1. \tag{4.5}$$

Perhaps one has equality in (4.3)–(4.5), for almost all ω, but we do not show this.

One tool we use to prove these estimates is the following (cf. [Fal], p. 78).

Lemma 4.1. *Let $K \subset \mathbb{R}^n$ be a compact set and take $b \in (0,\infty)$. Assume there is a positive Borel measure $\mu \ne 0$, supported on K, such that*

$$\iint \frac{d\mu(x)\, d\mu(y)}{|x-y|^b} < \infty. \tag{4.6}$$

Then the b-dimensional Hausdorff measure $\mathcal{H}^b(K) = \infty$, so $\operatorname{Hdim} K \ge b$.

To prove (4.1), we use the following consequence of (2.13):

$$E\big(|X_s - X_t|^{-b}\big) = C_{n\alpha b}|t-s|^{-b/2\alpha}, \quad 0 < b < n, \tag{4.7}$$

with $C_{n\alpha b} < \infty$ in this range. Consequently

$$E\left(\int_0^T \int_0^T \frac{ds\,dt}{|\omega(t)-\omega(s)|^b}\right) = C\int_0^T \int_0^T \frac{ds\,dt}{|t-s|^{b/2\alpha}} < \infty, \tag{4.8}$$

provided $0 < b < 2\alpha$, so

$$0 < b < \min(2\alpha, n) \Longrightarrow \int_0^T \int_0^T \frac{ds\,dt}{|\omega(t)-\omega(s)|^b} < \infty, \quad \text{for } W\text{-a.e. } \omega. \tag{4.9}$$

Now we define the measure μ^ω on $\omega([0,T])$ by

$$\mu^\omega(S) = m\big(\{t \in [0,T] : \omega(t) \in S\}\big), \tag{4.10}$$

where m denotes Lebesgue measure on $[0,T]$. Thus (4.9) becomes

$$\int\limits_{\omega(I)} \int\limits_{\omega(I)} \frac{d\mu^\omega(x)\,d\mu^\omega(y)}{|x-y|^b} < \infty, \quad \text{for a.e. } \omega, \quad \text{if } \ 0 < b < \min(2\alpha, n). \tag{4.11}$$

While $\omega(I)$ is not compact (unless $\alpha = 1$), a modification of Lemma 4.1 should apply, to yield (4.1). **(Check this!)**

Moving on to graphs, we have

$$\begin{aligned} E\big(|Z(s)-Z(t)|^{-b}\big) &= \int p(|t-s|, y)\big(|t-s|+|y|\big)^{-b}\,dy \\ &= |t-s|^{-n/2\alpha}\int p(1, |t-s|^{-1/2\alpha}y)\big(|t-s|+|y|\big)^{-b}\,dy \\ &= \int p(1,z)\big(|t-s|+|t-s|^{1/2\alpha}|z|\big)^{-b}\,dz. \end{aligned} \tag{4.12}$$

Hence, since $p(1,\cdot)$ is integrable,

$$\begin{aligned} E\big(|Z(s)-Z(t)|^{-b}\big) &= |t-s|^{-b/2\alpha}\int p(1,z)\big(|t-s|^{1-1/2\alpha}+|z|\big)^{-b}\,dz \\ &\le C|t-s|^{-b/2\alpha}\left[1+\int \big(|t-s|^{1-1/2\alpha}+|z|\big)^{-b}\,dz\right]. \end{aligned} \tag{4.13}$$

If $0 < b < n$ and $1 - 1/2\alpha \ge 0$, the last integral is bounded independently of $s, t \in [0,T]$, and one has

(4.14)

$$\begin{aligned} 0 < b < n,\ \frac{1}{2} \le \alpha \le 1 &\Longrightarrow E\big(|Z(s)-Z(t)|^{-b}\big) \le C|t-s|^{-b/2\alpha} \\ &\Longrightarrow E\left(\int_0^T \int_0^T \frac{ds\,dt}{|Z(t)-Z(s)|^b}\right) \le C\int_0^T \int_0^T \frac{ds\,dt}{|s-t|^{b/2\alpha}}, \end{aligned}$$

which is $< \infty$ provided $b < 2\alpha$. An argument parallel to that using (4.10)–(4.11) then yields

$$\text{Hdim}\, Z_\omega(I) \geq b, \quad \forall\, b < \min(2\alpha, n), \quad \text{if} \quad \alpha \geq \frac{1}{2}, \tag{4.15}$$

for almost all ω, which in turn gives (4.3).

Now assume that $1 = n < b$, while $\alpha \in (1/2, 1]$. To estimate the last integral in (4.3), write

$$\begin{aligned} \int_0^T \big(|t-s|^{1-1/2\alpha} + z\big)^{-b}\, dz &\leq \int_0^{|t-s|^\gamma} |t-s|^{-b(1-1/2\alpha)}\, dz + \int_{|t-s|^\gamma}^T z^{-b}\, dz \\ &\leq C + |t-s|^{\gamma - b(1-1/2\alpha)} + C|t-s|^{\gamma(1-b)}. \end{aligned} \tag{4.16}$$

Pick $\gamma = 1 - 1/2\alpha$ to make the exponents both equal to $(1-b)(1-1/2\alpha)$. Hence

$$\begin{aligned} E\big(|Z(s) - Z(t)|^{-b}\big) &\leq C|t-s|^{-b/2\alpha}\big(1 + |t-s|^{(1-b)(1-1/2\alpha)}\big) \\ &\leq C|t-s|^{1-b-1/2\alpha} + C|t-s|^{-b/2\alpha}. \end{aligned} \tag{4.17}$$

Thus

$$E\left(\int_0^T \int_0^T \frac{ds\, dt}{|Z(s) - Z(t)|^b}\right) \leq C \int_0^T \int_0^T \left[\frac{ds\, dt}{|t-s|^{b+1/2\alpha-1}} + \frac{ds\, dt}{|t-s|^{b/2\alpha}}\right], \tag{4.18}$$

which is $< \infty$ provided $b < 2 - 1/2\alpha$. (Note that $\alpha \in (1/2, 1] \Rightarrow 2 - 1/2\alpha < 2\alpha$.) This plus another application of Lemma 4.1 **(suitably modified)** gives (4.4).

We now turn to the case $\alpha \in (0, 1/2]$. In that case, replace (4.13) by

$$\begin{aligned} E\big(|Z(s) - Z(t)|^{-b}\big) &= |t-s|^{-b} \int p(1,z)\big(1 + |t-s|^{1/2\alpha - 1}|z|\big)^{-b}\, dz \\ &\leq C|t-s|^{-b}. \end{aligned} \tag{4.19}$$

Thus

$$E\left(\int_0^T \int_0^T \frac{ds\, dt}{|Z(s) - Z(t)|^b}\right) \leq C \int_0^T \int_0^T \frac{ds\, dt}{|t-s|^b}, \tag{4.20}$$

which is $< \infty$ provided $b < 1$. Then a third application of Lemma 4.1 **(suitably modified)** gives (4.5).

A. Generators of Lévy processes

Given a function $\psi(\xi)$ on $\mathbb{R}^n$, we say $\psi(D)$ generates a Lévy process if $p(t,x) = e^{-t\psi(D)}\delta(x)$ satisfies

$$p(t,x) \ge 0, \quad \int p(t,x)\,dx = 1, \tag{A.1}$$

for all $t > 0$. (We allow $p(t,\cdot)$ to be a positive measure.) Examples include $\xi \cdot A\xi = \sum a_{jk}\xi_j\xi_k$, when $A = (a_{jk})$ is a positive semi-definite matrix, yielding Gaussians. Another family is $\psi(\xi) = ib \cdot \xi$, generating translations. Still another type is

$$\psi(\xi) = c(1 - e^{iy\cdot\xi}), \tag{A.2}$$

given $c \in (0,\infty)$ and $y \in \mathbb{R}^n$, generating a "Poisson process." In such a case we have

$$e^{-t\psi(\xi)} = \sum_{k=0}^{\infty} \frac{(ct)^k}{k!}\, e^{-ct}\, e^{iky\cdot\xi}. \tag{A.3}$$

Hence $e^{-t\psi(D)}\delta(x)$ is a countable sum of point masses, supported on $\{ky : k = 0,1,2,\dots\}$.

In light of the identity

$$e^{-t(\psi_1(D)+\psi_2(D))} = e^{-t\psi_1(D)}e^{-t\psi_2(D)}, \tag{A.4}$$

it is clear that positive superpositions of the various generators described above are also generators of Lévy processors. P. Lévy showed that this class, suitably completed, yields all such generators. (His proof was simplified by Khinchin.) The resulting formula

$$\psi(\xi) = \xi \cdot A\xi + ib \cdot \xi + \int_{\mathbb{R}^n} \Bigl(1 - e^{iy\cdot\xi} + iy \cdot \xi\chi_B(y)\Bigr)\, d\mu(y) \tag{A.5}$$

is called the Lévy-Khinchin formula. Here χ_B is one on the unit ball B and zero on the complement, and μ is a positive measure on $\mathbb{R}^n \setminus 0$ satisfying

$$\int (|y|^2 \wedge 1)\, d\mu(y) < \infty. \tag{A.6}$$

Often it is useful to modify the term $iy\cdot\xi\chi_B$; sometimes one will drop it altogether (i.e., absorb it into the term $ib \cdot \xi$). Examples of such modifications are given in (B.2)–(B.3) of the next appendix, where we discuss homogeneous generators.

We end this section with a brief discussion of radial generators. If $\psi(\xi)$ is a radial function of the form (A.5), we have

$$\psi(\xi) = a|\xi|^2 + \int_0^\infty (1 - \psi_n(s|\xi|))\, d\rho(s), \tag{A.7}$$

where $a \geq 0$ and

$$\int_{S^{n-1}} e^{iy\cdot\xi}\, dS(y) = \psi_n(|\xi|) = (2\pi)^{n/2}|\xi|^{1-n/2}\, J_{n/2-1}(|\xi|). \tag{A.8}$$

Here ρ is a positive measure on $(0,\infty)$ satisfying $\int_0^\infty (s^2 \wedge 1)\, d\rho(s) < \infty$. In case $n = 1$, (A.7) takes the form

$$\psi(\xi) = a\xi^2 + \int_0^\infty (1 - \cos s|\xi|)\, d\rho(s). \tag{A.9}$$

B. Homogeneous Lévy generators

Here we construct functions homogeneous of degree $\alpha \in (0,2)$ for which $p(t,x) = e^{-t\psi(D)}\delta(x)$ satisfies (A.1). Of course

$$\psi(\xi) = |\xi|^\alpha, \quad 0 \le \alpha \le 2, \tag{B.1}$$

works, by the results of §1. We obtain further cases by specializing natural variants of the Lévy-Khinchin formula (A.5). In this way we obtain the following such homogeneous generators:

$$\Phi_{\alpha,g}(\xi) = -\int_{\mathbb{R}^n} (e^{iy\cdot\xi} - 1) g(y) |y|^{-n-\alpha}\, dy, \qquad 0 < \alpha < 1, \tag{B.2}$$

$$\Psi_{\alpha,g}(\xi) = -\int_{\mathbb{R}^n} (e^{iy\cdot\xi} - 1 - iy\cdot\xi) g(y) |y|^{-n-\alpha}\, dy, \quad 1 < \alpha < 2. \tag{B.3}$$

The function g is assumed to be positive, bounded, and homogeneous of degree 0, i.e.,

$$g \ge 0, \quad g \in L^\infty(\mathbb{R}^n), \quad g(ry) = g(y), \quad \forall\, r > 0. \tag{B.4}$$

It is easy to verify that both integrals in (B.2)–(B.3) are absolutely convergent, and, for $r > 0$,

$$\begin{aligned} \Phi_{\alpha,g}(r\xi) &= r^\alpha \Phi_{\alpha,g}(\xi), \quad 0 < \alpha < 1, \\ \Psi_{\alpha,g}(r\xi) &= r^\alpha \Psi_{\alpha,g}(\xi), \quad 1 < \alpha < 2. \end{aligned} \tag{B.5}$$

When $g \equiv 1$ we obtain a positive multiple of (B.1).

We now specialize to $n = 1$ and $g = \chi_{\mathbb{R}^+}$, so we look at

$$\begin{aligned} \varphi_\alpha(\xi) &= -\int_0^\infty (e^{iy\xi} - 1) y^{-1-\alpha}\, dy, \qquad 0 < \alpha < 1, \\ \psi_\alpha(\xi) &= -\int_0^\infty (e^{iy\xi} - 1 - iy\xi) y^{-1-\alpha}\, dy, \quad 1 < \alpha < 2. \end{aligned} \tag{B.6}$$

Clearly φ_α and ψ_α are holomorphic in $\{\xi \in \mathbb{C} : \operatorname{Im}\xi > 0\}$, and homogeneous of degree α in ξ. Also, for $\eta > 0$,

$$\begin{aligned} \varphi_\alpha(i\eta) &= -\int_0^\infty (e^{-y\eta} - 1) y^{-1-\alpha}\, dy > 0, \qquad 0 < \alpha < 1, \\ \psi_\alpha(i\eta) &= -\int_0^\infty (e^{-y\eta} - 1 + y\eta) y^{-1-\alpha}\, dy < 0, \quad 1 < \alpha < 2, \end{aligned} \tag{B.7}$$

since, for $r > 0,\ 1 - r < e^{-r} < 1$. It follows that $\varphi_\alpha(\xi)$ and $\psi_\alpha(\xi)$ are positive multiples of

(B.8)
$$\begin{aligned} \varphi^\#_\alpha(\xi) &= (-i\xi)^\alpha, \quad && 0 < \alpha < 1, \\ \psi^\#_\alpha(\xi) &= -(-i\xi)^\alpha, && 1 < \alpha < 2, \end{aligned}$$

restrictions to $\mathbb{R}$ of functions holomorphic on $\{\xi \in \mathbb{C} : \operatorname{Im} \xi > 0\}$. Taking instead $g = \chi_{\mathbb{R}^-}$, we obtain positive multiples of

(B.9)
$$\begin{aligned} \varphi^b_\alpha(\xi) &= (i\xi)^\alpha, \quad && 0 < \alpha < 1, \\ \psi^b_\alpha(\xi) &= -(i\xi)^\alpha, && 1 < \alpha < 2, \end{aligned}$$

restrictions to $\mathbb{R}$ of functions holomorphic on $\{\xi \in \mathbb{C} : \operatorname{Im} \xi < 0\}$, satisfying

(B.10)
$$\varphi^b_\alpha(-i\eta) > 0, \quad \psi^b_\alpha(-i\eta) < 0, \quad \forall\ \eta > 0.$$

The functions in (B.8) and (B.9) are well known examples of homogeneous functions $\psi(\xi)$ for which $e^{-t\psi(D)}$ satisfies (A.1). The associated operators $\psi(D)$ are fractional derivatives.

It is also useful to observe the explicit formulas

(B.11)
$$e^{-t\varphi^\#_\alpha(\xi)} = e^{-t(\cos \pi\alpha/2)|\xi|^\alpha} \Bigl[\cos\Bigl(t\bigl(\sin \frac{\pi\alpha}{2}\bigr)|\xi|^\alpha\Bigr) + i\sigma(\xi) \sin\Bigl(t\bigl(\sin \frac{\pi\alpha}{2}\bigr)|\xi|^\alpha\Bigr)\Bigr].$$

for $t > 0,\ 0 < \alpha < 1$, where

(B.12)
$$\sigma(\xi) = \operatorname{sgn} \xi,$$

and

(B.13)
$$e^{-t\psi^\#_\alpha(\xi)} = e^{t(\cos \pi\alpha/2)|\xi|^\alpha} \Bigl[\cos\Bigl(t\bigl(\sin \frac{\pi\alpha}{2}\bigr)|\xi|^\alpha\Bigr) - i\sigma(\xi) \sin\Bigl(t\bigl(\sin \frac{\pi\alpha}{2}\bigr)|\xi|^\alpha\Bigr)\Bigr],$$

fior $t > 0,\ 1 < \alpha < 2$. Note that

(B.14)
$$0 < \alpha < 1 \Rightarrow \cos \frac{\pi\alpha}{2} > 0, \quad 1 < \alpha < 2 \Rightarrow \cos \frac{\pi\alpha}{2} < 0,$$

so of course we have decaying exponentials in both (B.11) and (B.13). We get similar formulas with # replaced by b, since in fact

(B.15)
$$\varphi^b_\alpha(\xi) = \varphi^\#_\alpha(-\xi), \quad \psi^b_\alpha(\xi) = \psi^\#_\alpha(-\xi).$$

Returning to the general formulas (B.2)–(B.3), we can switch to polar coordinates and write

(B.16)
$$\begin{aligned} \Phi_{\alpha,g}(\xi) &= -\int_{S^{n-1}} \int_0^\infty (e^{is\omega\cdot\xi} - 1) g(\omega) s^{-1-\alpha}\, ds\, dS(\omega), \\ \Psi_{\alpha,g}(\xi) &= -\int_{S^{n-1}} \int_0^\infty (e^{is\omega\cdot\xi} - 1 - is\omega\cdot\xi) g(\omega) s^{-1-\alpha}\, ds\, dS(\omega), \end{aligned}$$

and hence

$$\Phi_{\alpha,g}(\xi) = \int_{S^{n-1}} \varphi_\alpha(\omega\cdot\xi)g(\omega)\,dS(\omega), \tag{B.17}$$
$$\Psi_{\alpha,g}(\xi) = \int_{S^{n-1}} \psi_\alpha(\omega\cdot\xi)g(\omega)\,dS(\omega).$$

We can extend the scope, replacing $g(\omega)\,dS(\omega)$ by a general positive, finite Borel measure on S^{n-1}. Taking into account the calculations yielding (B.8)–(B.9), we obtain homogeneous generators satisfying (A.1), of the form

$$\Phi^b_{\alpha,\nu}(\xi) = \int_{S^{n-1}} (i\omega\cdot\xi)^\alpha\,d\nu(\omega), \qquad 0<\alpha<1, \tag{B.18}$$
$$\Psi^b_{\alpha,\nu}(\xi) = -\int_{S^{n-1}} (i\omega\cdot\xi)^\alpha\,d\nu(\omega), \quad 1<\alpha<2,$$

where ν is a positive, finite Borel measure on S^{n-1}.

It remains to discuss the case $\alpha = 1$. For $n=1$ it is seen that positive multiples of

$$|\xi| + ia\xi, \quad a\in\mathbb{R}, \tag{B.19}$$

work. Hence the following functions on $\mathbb{R}^n$ work:

$$|\omega\cdot\xi| + ia\omega\cdot\xi, \quad \omega\in S^{n-1},\ a\in\mathbb{R}.$$

We can take positive superpositions of such functions and, in analogy with (B.18), obtain generators of diffusion semigroups whose negatives are Fourier multiplication by

$$ib\cdot\xi + \Xi_\nu(\xi), \tag{B.20}$$

where $b\in\mathbb{R}^n$ and

$$\Xi_\nu(\xi) = \int_{S^{n-1}} |\omega\cdot\xi|\,d\nu(\omega). \tag{B.21}$$

We now tie in results derived above with material given in Chapters 1–2 of [ST]. For such functions $\psi(\xi)$, homogeneous of degree $\alpha\in(0,2]$, as constructed above, the probability distributions

$$p_t(x) = e^{-t\psi(D)}\delta(x) \tag{B.22}$$

are known as α-stable distributions. In the notation (1.1.6) of [ST], consider

$$\psi(\xi) = \sigma^\alpha\,|\xi|^\alpha\Bigl(1 - i\beta(\operatorname{sgn}\xi)\tan\frac{\pi\alpha}{2}\Bigr), \quad \xi \in \mathbb{R}. \tag{B.23}$$

Here

$$\sigma \in (0,\infty), \quad \beta \in [-1,1], \tag{B.24}$$

and $\alpha \in (0,2)$ but $\alpha \neq 1$. Also, take $\mu \in \mathbb{R}$. Then $e^{-\psi(D)+i\mu D}\delta(x)$ is a probability distribution on the line called an α-stable distribution with scale parameter σ, skewness parameter β, and shift parameter μ. It is clear from (B.11)–(B.13) that each function of the form (B.23) is a positive linear combination of $\varphi^\#_\alpha(\xi)$ and $\varphi^b_\alpha(\xi)$ if $\alpha \in (0,1)$ and a positive linear combination of $\psi^\#_\alpha(\xi)$ and $\psi^b_\alpha(\xi)$ if $\alpha \in (1,2)$.

In case $\alpha = 1$, one goes beyond $\psi(\xi)$ homogeneous of degree 1 in ξ, to consider

$$\psi(\xi) = \sigma\,|\xi|\Bigl(1 + i\frac{2\beta}{\pi}(\operatorname{sgn}\xi)\log|\xi|\Bigr) + i\mu\xi, \quad \xi \in \mathbb{R}, \tag{B.25}$$

again with $\beta \in [-1,1]$, $\mu \in \mathbb{R}$. Then $e^{-\psi(D)}\delta(x)$ is a probability distribution on $\mathbb{R}$ called a 1-stable distribution, with scale parameter σ, skewness β, and shift μ. The cases arising from (B.19) all have skewness $\beta = 0$.

Similarly, functions $\psi(\xi)$ of the form (B.18) and (B.20)–(B.21) produce probability distributions $e^{-\psi(D)}\delta(x)$ on $\mathbb{R}^n$ that are α-stable. These, plus analogues with a shift incorporated, comprise all of them except when $\alpha = 1$, in which case one generalizes (5.21) to

$$\widetilde{\Xi}_\nu(\xi) = \int\limits_{S^{n-1}} |\omega\cdot\xi|\Bigl(1 + \frac{2i}{\pi}(\operatorname{sgn}\omega\cdot\xi)\log|\omega\cdot\xi|\Bigr)d\nu(\omega). \tag{B.26}$$

Compare (2.3.1)–(2.3.2) in [ST].

We return to the case $n = 1$ and make some more comments on the probability distributions

$$\begin{aligned} p^\alpha_t(x) &= e^{-t\varphi^\#_\alpha(D)}\delta(x), \quad 0 < \alpha < 1, \\ p^\alpha_t(x) &= e^{-t\psi^\#_\alpha(D)}\delta(x), \quad 1 < \alpha < 2, \end{aligned} \tag{B.27}$$

and their variants with # replaced by b, which are simply $p^\alpha_t(-x)$. Explicitly, we have

$$p^\alpha_t(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\cdot\xi - t\varphi^\#_\alpha(\xi)}\,d\xi, \tag{B.28}$$

for $0 < \alpha < 1$, with $\varphi^\#_\alpha(\xi)$ replaced by $\psi^\#_\alpha(\xi)$ for $1 < \alpha < 2$. Recall that $\varphi^\#_\alpha$ and $\psi^\#_\alpha$ are holomorphic in $\{\xi \in \mathbb{C} : \operatorname{Im}\xi > 0\}$. It follows from the Paley-Wiener theorem that, for each $t > 0$,

$$p^\alpha_t(x) = 0, \quad \text{for } x \in [0,\infty),\ 0 < \alpha < 1. \tag{B.29}$$

This theorem does not apply when $\alpha \in (1,2)$, but a shift in the contour of integration to $\{\xi + ib : \xi \in \mathbb{R}\}$, with arbitrary $b > 0$ yields

$$p_t^\alpha(x) = e^{-bx} \times \text{ bounded function of } x, \tag{B.30}$$

for $x \in \mathbb{R}$, whenever $1 < \alpha < 2$, hence

$$p_t^\alpha(x) = o(e^{-bx}), \quad \forall\, b > 0, \quad \text{as} \quad x \to +\infty, \quad \text{for} \quad 1 < \alpha < 2. \tag{B.31}$$

A more precise asymptotic behavior is stated in (1.2.11) of [ST]. See also results in §E.

We also note that, for $\alpha \in (1,2)$, $p_t^\alpha(x)$ is real analytic in $x \in \mathbb{R}$, and in fact extends to an entire holomorphic function in $x \in \mathbb{C}$, for each $t > 0$, due to rapidity with which $\operatorname{Re} \psi_\alpha^\#(\xi) \to +\infty$ as $|\xi| \to \infty$, which of course forbids (B.29) in this case.

C. Asymptotic behavior of a class of stable distributions (heavy tails)

A fair number of Lévy generators $\psi(D)$, producing probability distributions $p(t,x) = e^{-t\psi(D)}\delta(x)$, have the following properties:

$$\psi \in C^\infty(\mathbb{R}^n \setminus 0), \tag{C.1}$$

$$\operatorname{Re}\psi(\xi) \geq C|\xi|^\beta, \quad \text{for some} \ \ \beta \in (0,2), \ C > 0, \tag{C.2}$$

$$\psi(\xi) \sim \sum_{k\geq 0} a_k\Bigl(\frac{\xi}{|\xi|}\Bigr)\,|\xi|^{\gamma+k}, \quad |\xi| \to 0, \quad \text{for some} \ \ \gamma \in (0,2), \tag{C.3}$$

with $a_k \in C^\infty(S^{n-1})$, (C.3) implying that $\psi(\xi) - \sum_{k=0}^m a_k(\xi/|\xi|)|\xi|^{\gamma+k} \in C^m(\mathbb{R}^n)$ for each m. Our goal is to derive the asymptotic behavior of $p(t,x)$ as $|x| \to \infty$, for fixed $t > 0$, under these hypotheses.

To start, we can write

$$e^{-t\psi(\xi)} = A_t(\xi) + B_t(\xi), \tag{C.4}$$

where, for each $t > 0$,

$$A_t \in \mathcal{S}(\mathbb{R}^n), \quad \operatorname{supp} B_t \subset \{\xi : |\xi| \leq 1\}, \tag{C.5}$$

and

$$B_t(\xi) \sim 1 + \sum_{j\geq 1, k\geq 0} b_{jkt}\Bigl(\frac{\xi}{|\xi|}\Bigr)|\xi|^{j\gamma+k}, \quad |\xi| \to 0. \tag{C.6}$$

In such a case,

$$p(t,x) = \widehat{A}_t(x) + \widehat{B}_t(x), \quad \widehat{A}_t \in \mathcal{S}(\mathbb{R}^n), \tag{C.7}$$

so the asymptotic behavior of $p(t,x)$ as $|x| \to \infty$, for fixed $t > 0$, is given by that of $\widehat{B}_t(x)$. Now if we set

$$B_{jkt}(\xi) = b_{jkt}\Bigl(\frac{\xi}{|\xi|}\Bigr)|\xi|^{j\gamma+k}, \quad \xi \in \mathbb{R}^n, \tag{C.8}$$

then $B_{jkt} \in \mathcal{S}'(\mathbb{R}^n)$, and if $\Phi \in C_0^\infty(\mathbb{R}^n)$, $\Phi(\xi) = 1$ for $|\xi| \leq 1$, then

$$B_t(\xi) - \Phi(\xi)\sum_{j=1}^N \sum_{k=0}^N B_{jkt}(\xi) \tag{C.9}$$

has a Fourier transform bounded by $C|x|^{-M}$ as $|x| \to \infty$, with $M = M(N) \to \infty$ as $N \to \infty$. Meanwhile,

$$\big(1 - \Phi(\xi)\big) B_{jkt}(\xi) \in S_1^{j\gamma+k}(\mathbb{R}^n), \tag{C.10}$$

so its Fourier transform is rapidly decreasing as $|x| \to \infty$. (Cf. [T1], Chapter 3, Proposition 8.2.) Hence, for each $t > 0$,

$$\widehat{B}_t(x) \sim \sum_{j\geq 1, k\geq 0} \widehat{B}_{jkt}(x), \quad |x| \to \infty. \tag{C.11}$$

As for $\widehat{B}_{jkt}(x)$, since $B_{jkt}(\xi)$ is a homogeneous element of $\mathcal{S}'(\mathbb{R}^n)$, of degree $j\gamma + k$, and smooth on $\mathbb{R}^n \setminus 0$, i.e.,

$$B_{jkt} \in \mathcal{H}^{\#}_{j\gamma+k}(\mathbb{R}^n), \tag{C.12}$$

in the notation (8.8) of [T1], Chapter 3, we have

$$\widehat{B}_{jkt} \in \mathcal{H}^{\#}_{-n-j\gamma-k}(\mathbb{R}^n), \tag{C.13}$$

by Proposition 8.1 in [T1], Chapter 3, the proof using Proposition 8.2, cited above. In other words,

$$\widehat{B}_{jkt}(x) = b^{\#}_{jkt}\Big(\frac{x}{|x|}\Big)\, |x|^{-n-j\gamma-k}, \tag{C.14}$$

with

$$b^{\#}_{jkt} \in C^{\infty}(S^{n-1}). \tag{C.15}$$

There are integral formulas for $b^{\#}_{jkt}$ in terms of b_{jkt}, $j\gamma + k$, and n, which we will not record here. (See, e.g., calculations in [Zai].) We have the following conclusion.

Proposition C.1. *If $\psi(\xi)$ satisfies (C.1)–(C.3), then for each $t > 0$, $p(t,x) = e^{-t\psi(D)}\delta(x)$ is C^∞ in x and satisfies*

$$p(t,x) \sim \sum_{j\geq 1, k\geq 0} b^{\#}_{jkt}\Big(\frac{x}{|x|}\Big)\, |x|^{-n-j\gamma-k}, \quad |x| \to \infty. \tag{C.16}$$

In particular, the leading term is

$$b^{\#}_{10t}\Big(\frac{x}{|x|}\Big)\, |x|^{-n-\gamma}. \tag{C.17}$$

Note that

$$\text{(C.18)} \qquad \int_{|x|\ge 1} |x|^{-n-\gamma+\ell}\, dx = A_n \int_1^\infty r^{-1-\gamma+\ell}\, dr,$$

which is $+\infty$ for $\ell = 1$ if $\gamma \in (0,1]$ and is finite for $\ell = 1$ if $\gamma \in (1,2)$, but $+\infty$ for $\ell = 2$ for all $\gamma \in (0,2)$. Consequently, as long as $a_0(\xi/|\xi|)$ is not $\equiv 0$ in (C.3), we have, for each $t > 0$,

$$\text{(C.19)} \qquad \begin{aligned} \int p(t,x)|x|^\ell\, dx = \infty \quad &\text{if} \quad \ell = 1 \quad \text{and} \quad \gamma \in (0,1] \\ \infty \quad &\text{if} \quad \ell = 2 \quad \text{and} \quad \gamma \in (0,2). \end{aligned}$$

Note that the homogeneous generators of degree α considered in §B that satisfy (C.1) and $\operatorname{Re}\psi(\xi) > 0$ for $\xi \ne 0$ also satisfy (C.2)–(C.3) with $\beta = \gamma = \alpha$.

D. Short time and long time behavior of $e^{-t\psi(D)}\delta(x)$: examples

Here we examine the asymptotic behavior of

$$p(t,x) = e^{-t\psi(D)}\delta(x), \tag{D.1}$$

both as $t \nearrow \infty$ and as $t \searrow 0$, for some specific examples of $\psi(\xi)$.

The first example is

$$\psi(\xi) = (|\xi|^2 + 1)^\alpha - 1, \tag{D.2}$$

with $\alpha \in (0,1)$. As noted in (1.15) we have, for each fixed $t > 0$,

$$p(t,x) \le C_{n\alpha t}\, e^{-|x|}, \quad |x| \to \infty.$$

We first treat the large t behavior. In light of the estimate above, the Central Limit Theorem applies. We have $p(t,x)$ behaving like

$$q(t,x) = e^{\alpha t\Delta}\delta(x) \tag{D.3}$$

as $t \nearrow \infty$, over the region $|x| \le Kt^{1/2}$, for each $K \in (0,\infty)$. More precisely, set

$$p^\#(t,x) = t^{n/2} p(t, t^{1/2}x), \tag{D.4}$$

and compare it with

$$q^\#(t,x) = t^{n/2} q(t, t^{1/2}x) = q(1,x). \tag{D.5}$$

We have

$$p^\#(t,x) = (2\pi)^{-n} \int e^{-t[(1+|\xi|^2/t)^\alpha - 1]} e^{ix\cdot\xi}\, d\xi, \tag{D.6}$$

and standard arguments to establish versions of the Central Limit Theorem (cf. [T1], Chapter 3, §3, Exercises 7–13) yield

$$p^\#(t,x) \longrightarrow q(1,x), \quad \text{as} \;\; t \nearrow \infty, \tag{D.7}$$

both uniformly and in L^1-norm.

By contrast, we claim that as $t \searrow 0$, $p(t,x)$ behaves like

$$Q(t,x) = e^{-t(-\Delta)^\alpha}\delta(x). \tag{D.8}$$

To state this more precisely, in analogy with (D.4)–(D.7) we set

$$p^b(t,x) = t^{n/2\alpha} p(t, t^{1/2\alpha} x), \tag{D.9}$$

and compare it with

$$Q^b(t,x) = t^{n/2\alpha} Q(t, t^{1/2\alpha} x) = Q(1,x). \tag{D.10}$$

Then

$$\begin{aligned} p^b(t,x) &= (2\pi)^{-n} \int e^{-t[(1+|\xi|^2/t^{1/\alpha})^\alpha - 1]} e^{ix\cdot\xi}\, d\xi \\ &= (2\pi)^{-n} \int e^{-[(t^{1/\alpha}+|\xi|^2)^\alpha - t]} e^{ix\cdot\xi}\, d\xi. \end{aligned} \tag{D.11}$$

Now

$$e^{-[(t^{1/\alpha}+|\xi|^2)^\alpha - t]} \longrightarrow e^{-|\xi|^{2\alpha}}, \quad \text{as } t \searrow 0, \tag{D.12}$$

in $L^1(\mathbb{R}^n)$ and uniformly. The L^1-convergence implies

$$p^b(t,x) \longrightarrow Q(1,x), \quad \text{as } t \searrow 0, \tag{D.13}$$

uniformly, and the fact that $p^b(t,x)$ and $Q(1,x)$ are all positive functions of x integrating to 1 yields the convergence in L^1-norm.

We can say more. Note that

$$\begin{aligned} e^{-t} p^b(t,x) &= e^{-(-\Delta + t^{1/\alpha})^\alpha} \delta(x) \\ &= \int_0^\infty e^{-t^{1/\alpha} s}\, \Phi_{1,\alpha}(s)\, e^{s\Delta} \delta(x)\, ds, \end{aligned} \tag{D.14}$$

where we have used (1.7) with $L = -\Delta + t^{1/\alpha}$. Consequently, we also have

$$e^{-t} p^b(t,x) \nearrow Q(1,x), \quad t \searrow 0. \tag{D.15}$$

We can apply this to estimate the modulus of continuity of the stochastic process $\{X_t\}$ given in Theorem 2.1, with $\psi(D)$ given by (D.2). Recall (2.6):

$$E(|X_t - X_s|^q) = \int p(|t-s|, y) |y|^q\, dy. \tag{D.16}$$

Using (D.9) we have, for $-n < q < 2\alpha$,

$$\int p(t,y)|y|^q\, dy = t^{q/2\alpha} \int p^b(t,x) |x|^q\, dx, \tag{D.17}$$

and, by (D.15), as $t \searrow 0$,

$$e^{-t} \int p^b(t,x)|x|^q\,dx \nearrow \int Q(1,x)|x|^q\,dx, \tag{D.18}$$

which is a number $A \in (0,\infty)$, by arguments mentioned in (2.10)–(2.12). Thus, as $|t-s| \searrow 0$, $E(|X_t - X_s|^q)$ has the same asymptotic behavior for $\psi(\xi)$ given by (D.2) as it does for $\psi(\xi) = |\xi|^{2\alpha}$, given $\alpha \in (0,1)$.

Our second example is

$$\psi(\xi) = |\xi|^2 + |\xi|. \tag{D.19}$$

To treat the large time behavior, this time we examine

$$\begin{aligned} p^{\#}(t,x) &= t^n p(t,tx) \\ &= (2\pi)^{-n} \int e^{-t(t^{-2}|\xi|^2 + t^{-1}|\xi|)} e^{ix\cdot\xi}\,d\xi \\ &\to (2\pi)^{-n} \int e^{-|\xi|} e^{ix\cdot\xi}\,d\xi, \quad \text{as } \ t \nearrow \infty. \end{aligned} \tag{D.20}$$

In other words,

$$p^{\#}(t,x) \longrightarrow e^{-\sqrt{-\Delta}}\delta(x) = \frac{A_n}{(|x|^2+1)^{(n+1)/2}}, \quad \text{as } \ t \to \infty. \tag{D.21}$$

In this sense, $e^{t(\Delta - \sqrt{-\Delta})}\delta(x)$ behaves like $e^{-t\sqrt{-\Delta}}\delta(x)$ as $t \nearrow \infty$.

To treat small time behavior, we examine

$$\begin{aligned} p^b(t,x) &= t^{n/2} p(t, t^{1/2}x) \\ &= (2\pi)^{-n} \int e^{-t(t^{-1}|\xi|^2 + t^{-1/2}|\xi|)} e^{ix\cdot\xi}\,d\xi \\ &\to (2\pi)^{-n} \int e^{-|\xi|^2} e^{ix\cdot\xi}\,d\xi, \quad \text{as } \ t \searrow 0. \end{aligned} \tag{D.22}$$

In other words,

$$p^b(t,x) \longrightarrow e^{\Delta}\delta(x) = (4\pi)^{-n/2}\, e^{-|x|^2/4}, \quad \text{as } \ t \searrow 0. \tag{D.23}$$

In this sense, $e^{t(\Delta - \sqrt{-\Delta})}\delta(x)$ behaves like $e^{t\Delta}\delta(x)$ as $t \searrow 0$. However, one must be cautioned that the paths for the process associated to $e^{t(\Delta - \sqrt{-\Delta})}$ are not continuous, but rather cadlag, with jumps, so (D.23) does not tell the whole story about the short time behavior.

Let us now estimate $E(|X_t - X_s|^q)$ for the process $\{X_t\}$ generated by $\psi(D)$ with $\psi(\xi)$ given by (D.19). As in (D.16), we have

$$E(|X_t - X_s|^q) = \int p(|t-s|, y)|y|^q \, dy. \tag{D.24}$$

Using (D.22), we have the following analogue of (D.17):

$$\int p(t,y)|y|^q \, dy = t^{q/2} \int p^b(t,x)|x|^q \, dx. \tag{D.25}$$

However, from here the argument is different from (D.18). We have

$$p^b(t,x) = e^{\Delta - t^{1/2}\sqrt{-\Delta}} \delta(x) = h * q_t(x), \tag{D.26}$$

where

$$\begin{aligned} h(x) &= (4\pi)^{-n/2} \, e^{-|x|^2/4}, \quad q_t(x) = t^{-n/2} \, q_1(t^{-1/2}x), \\ q_1(x) &= \frac{C_n}{(1+|x|^2)^{(n+1)/2}}, \quad C_n = \pi^{-(n+1)/2} \, \Gamma\Big(\frac{n+1}{2}\Big). \end{aligned} \tag{D.27}$$

We deduce that

$$E(|X_t - X_s|^q) \le C_q |t-s|^{q/2}, \quad -n < q < 2. \tag{D.28}$$

As noted in (2.13) and the remark following it, for the process generated by Δ (i.e., Brownian motion) we have the estimate (D.28) over a larger range of q, namely $q \in (-n, \infty)$. Note that one can apply the Kolomogorov criterion (3.7) for sample path continuity as long as this estimate holds for some exponent $q/2 > 1$, but we do not get this in the product case, and this process is only cadlag.

On the other hand, using (D.28) with q close to $-n$, we obtain the following variant of (4.1), giving another respect in which the short time behavior of X_t in this case is like that of Brownian motion.

Proposition D.1. *For the process generated by* $\Delta - \sqrt{-\Delta}$*, if* $n \ge 2$ *we have for each interval* $I = [0,T],\ T > 0$*,*

$$\operatorname{Hdim} \omega(I) \ge 2, \quad \textit{for a.e. } \omega. \tag{D.29}$$

Here, as in (2.5), $\omega(t) = X_t(\omega)$. Presumably, one has equality in (D.29), but we do not have a proof of this.

E. Vanishing and super-exponential decay on cones

Let us set

$$\begin{aligned} \varphi_\alpha(x) &= (ix+i0)^\alpha, && 0<\alpha<1, \\ \psi_\alpha(x) &= -(ix+i0)^\alpha, && 1<\alpha<2, \end{aligned} \tag{E.1}$$

as in (B.9), i.e., φ_α is the boundary value on $\mathbb{R}$ of the function $(iz)^\alpha$ and ψ_α that of $-(iz)^\alpha$ on $\{z : \operatorname{Im} z < 0\}$, satisfying

$$\varphi_\alpha(-iy) > 0, \quad \psi_\alpha(-iy) < 0, \quad \forall\, y > 0. \tag{E.2}$$

As in (B.18), we consider

$$\begin{aligned} \Phi_{\alpha,\nu}(\xi) &= \int_{S^{n-1}} \varphi_\alpha(\xi\cdot\omega)\, d\nu(\omega), && 0<\alpha<1, \\ \Psi_{\alpha,\nu}(\xi) &= \int_{S^{n-1}} \psi_\alpha(\xi\cdot\omega)\, d\nu(\omega), && 1<\alpha<2, \end{aligned} \tag{E.3}$$

where ν is a positive, finite Borel measure on S^{n-1}, and we consider the associated probability distributions

$$\begin{aligned} P_{\alpha,\nu}(t,x) &= e^{-t\Phi_{\alpha,\nu}(D)}\delta(x), && 0<\alpha<1, \\ Q_{\alpha,\nu}(t,x) &= e^{-t\Psi_{\alpha,\nu}(D)}\delta(x), && 1<\alpha<2. \end{aligned} \tag{E.4}$$

Here we extend to n dimensions the vanishing result (B.29) (for $0<\alpha<1$) and the super-exponential decay result (B.31) (for $1<\alpha<2$), on a half-line, in case $n=1$ and the measure ν on $S^0=\{-1,1\}$ has support in one point. We start with the extended vanishing result, when $0<\alpha<1$.

Proposition E.1. *Assume ν is a positive measure supported on $\Sigma\subset S^{n-1}$, and let $K\subset\mathbb{R}^n$ be the convex hull of the cone over Σ. Then, for $\alpha\in(0,1)$, $t>0$,*

$$\mathit{supp}\, P_{\alpha,\nu}(t,\cdot) \subset K. \tag{E.5}$$

Proof. In this case we have (B.2), i.e.,

$$\Phi_{\alpha,\nu}(\xi) = \int_\Sigma \int_0^\infty (1-e^{iy\cdot\xi}) s^{-1-\alpha}\, ds\, d\nu(\omega), \quad y = s\omega. \tag{E.6}$$

This is a limit of finite, positive linear combinations of functions

$$\psi_y(\xi) = 1 - e^{iy\cdot\xi}, \quad y = s\omega,\ \omega \in \Sigma,\ s > 0. \tag{E.7}$$

Hence $e^{-t\Phi_{\alpha,\nu}(D)}$ is a limit of compositions

$$e^{-tc_1\psi_{y_1}(D)} \cdots e^{-tc_N\psi_{y_N}(D)}, \tag{E.8}$$

and $P_{\alpha,\nu}(t,\cdot)$ is a limit in $\mathcal{S}'(\mathbb{R}^n)$ of a sequence of distributions of the form

$$p_{tc_1,y_1} * \cdots * p_{tc_N,y_N}(x), \tag{E.9}$$

where

$$p_{tc_j,y_j}(x) = \sum_{k=0}^{\infty} \frac{(tc_j)^k}{k!}\, e^{-c_j t}\, \delta(x - ky_j). \tag{E.10}$$

Note that p_{tc_j,y_j} is supported on the ray through y_j in $\mathbb{R}^n$. Hence the support of (E.9) is contained in the convex hull of the set of rays through $\{y_1,\dots,y_N\} \subset \Sigma$. In the limit we get (E.5).

Next we establish super-exponential decay of $Q_{\alpha,\nu}(t,x)$, not on the complement of K, but on the dual cone:

$$L = \{x \in \mathbb{R}^n : x\cdot\omega < 0,\ \forall\,\omega \in \Sigma\}. \tag{E.11}$$

Proposition E.2. *Assume ν is a positive measure supported on $\Sigma \subset S^{n-1}$. Also assume that, for some $C > 0$,*

$$\mathit{Re}\,\Psi_{\alpha,\nu}(\xi) \ge C|\xi|^\alpha, \quad \xi \in \mathbb{R}^n. \tag{E.12}$$

Then, for $\alpha \in (1,2),\ t > 0$,

$$Q_{\alpha,\nu}(t,x) = o(e^{-b|x|}), \quad \forall\, b > 0,\ \text{as}\ \ |x| \to \infty,\ x \in L, \tag{E.13}$$

where L (which might be empty) is given by (E.11).

Proof. Under the stated hypotheses,

$$\Psi_{\alpha,\nu}(\xi + i\eta) = \int_\Sigma \psi_\alpha(\omega\cdot\xi + i\omega\cdot\eta)\, d\nu(\omega) \tag{E.14}$$

is well defined and holomorphic on

$$\{\xi + i\eta : \xi \in \mathbb{R}^n,\ \eta \in L\}. \tag{E.15}$$

Furthermore,

$$\operatorname{Re}\Psi_{\alpha,\nu}(\xi+i\eta)\ge C|\xi|^\alpha - C'|\eta|^\alpha, \tag{E.16}$$

with $C>0$. Hence, for each $\eta\in L$,

$$\begin{aligned} Q_{\alpha,\nu}(t,x) &= (2\pi)^{-n}\int e^{ix\cdot\xi - t\Psi_{\alpha,\nu}(\xi)}\,d\xi \\ &= (2\pi)^{-n}e^{-x\cdot\eta}\int e^{ix\cdot\xi-t\Psi_{\alpha,\nu}(\xi+i\eta)}\,d\xi. \end{aligned} \tag{E.17}$$

Hence

$$|Q_{\alpha,\nu}(t,x)|\le C_t(\eta)e^{-x\cdot\eta}. \tag{E.18}$$

If $x\in L$, we can pick $\eta = 2bx/|x|$ and deduce (E.13).

As for when (E.12) holds, note that, for $x\in\mathbb{R}$,

$$\operatorname{Re}\psi_\alpha(x) = \Bigl|\cos\frac{\pi\alpha}{2}\Bigr|\cdot|x|^\alpha, \tag{E.19}$$

(compare (B.13)–(B.14)), and hence, for $\xi\in\mathbb{R}^n,\ A_\alpha = |\cos\pi\alpha/2|$,

$$\operatorname{Re}\Psi_{\alpha,\nu}(\xi) = A_\alpha\int_\Sigma |\omega\cdot\xi|^\alpha\,d\nu(\omega). \tag{E.20}$$

Thus

$$\text{(E.12) holds} \Longleftrightarrow \int_\Sigma |\omega\cdot\xi|^\alpha\,d\nu(\omega)>0,\quad \forall\,\xi\in\mathbb{R}^n\setminus 0. \tag{E.21}$$

Remark. In light of (E.16), we can restate (E.18) more precisely as

$$|Q_{\alpha,\nu}(t,x)|\le C_t\Bigl(\frac{\eta}{|\eta|}\Bigr)e^{C'|\eta|^\alpha}e^{-x\cdot\eta}. \tag{E.22}$$

Hence, picking $\eta = bx/|x|$, we have

$$|Q_{\alpha,\nu}(t,x)|\le C_t\Bigl(\frac{x}{|x|}\Bigr)e^{-b|x|+C'tb^\alpha},\quad b\in(0,\infty). \tag{E.23}$$

Optimizing over b, we then have

$$|Q_{\alpha,\nu}(t,x)|\le C_t\Bigl(\frac{x}{|x|}\Bigr)e^{-\kappa|x|^{\alpha/(\alpha-1)}/t^{1/(\alpha-1)}},\quad x\in L. \tag{E.24}$$

F. Regularity properties of the semigroup $e^{-t\psi(D)}$

Let $e^{-t\psi(D)}$ be as in §1. In particular, we have (3.3)–(3.4). It is elementary that for each $t > 0$, $e^{-t\psi(D)}$ is a contraction on $L^p(\mathbb{R}^n)$ for each $p \in [1,\infty]$. It is also a contraction on $\mathrm{BC}(\mathbb{R}^n)$, the space of bounded continuous functions on $\mathbb{R}^n$, and on the closed linear subspace $\mathrm{UC}(\mathbb{R}^n)$ of bounded, uniformly continuous functions on $\mathbb{R}^n$, and on the space $C_*(\mathbb{R}^n)$ of continuous functions on $\mathbb{R}^n$ vanishing at infinity. It is positivity preserving on all these spaces.

The family $e^{-t\psi(D)}$ is a strongly continuous semigroup on most of these spaces, though not on $L^\infty(\mathbb{R}^n)$ or $\mathrm{BC}(\mathbb{R}^n)$. This continuity is quite elementary for $L^2(\mathbb{R}^n)$, by virtue of the identity

$$e^{-t\psi(D)}u(x) = (2\pi)^{-n} \int e^{-t\psi(\xi)}\hat{u}(\xi)e^{ix\cdot\xi}\,d\xi \tag{F.1}$$

and the Plancherel theorem. Also, if $u \in \mathcal{S}(\mathbb{R}^n)$, the integrand on the right side of (F.1) is a continuous function of $t \in [0,\infty)$ with values in $L^1(\mathbb{R}^n)$, so the Fourier integral is a continuous function of t with values in $C_*(\mathbb{R}^n)$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $C_*(\mathbb{R}^n)$, we have a strongly continuous semigroup on $C_*(\mathbb{R}^n)$. More generally, given $u \in \mathcal{S}(\mathbb{R}^n)$, $e^{-t\psi(\xi)}\hat{u}(\xi)$ is a continuous function of t with values in $L^p(\mathbb{R}^n)$ for each $p \in [1,2]$, so (F.1) is a continuous function of t with values in $L^{p'}(\mathbb{R}^n)$. The same density argument shows that we have a strongly continuous semigroup on $L^q(\mathbb{R}^n)$ for each $q \in [2,\infty)$.

We next argue that $e^{-t\psi(D)}$ is strongly continuous on $L^p(\mathbb{R}^n)$ for $p \in [1,2)$. To begin, take $u \in \mathcal{S}(\mathbb{R}^n)$ such that $u \geq 0$. Then $v(t) = e^{-t\psi(D)}u$ is ≥ 0 and $\int v(t,x)\,dx \equiv \int u(x)\,dx$. We already know $v(t) \to u$ uniformly as $t \searrow 0$. From these facts it follows that $v(t) \to u$ in L^1-norm. Hence for each $u \in \mathcal{S}(\mathbb{R}^n)$ (without the sign condition) we have $e^{-t\psi(D)}u \to u$ in L^1-norm as $t \searrow 0$. We also know this holds in L^2-norm, so it holds in L^p-norm for each $p \in [1,2]$. Again a density argument yields the asserted strong continuity on $L^p(\mathbb{R}^n)$, at $t = 0$. As is well known (cf. [Bob], p. 249) this suffices to establish strong continuity in $t \in [0,\infty)$.

Our next goal is to prove the following.

Proposition F.1. *The semigroup* $e^{-t\psi(D)}$ *is strongly continuous on* $\mathrm{UC}(\mathbb{R}^n)$.

Proof. As noted above, it suffices to prove strong continuity at $t = 0$, so we need to show that if $u \in \mathrm{UC}(\mathbb{R}^n)$, then

$$e^{-t\psi(D)}u \to u \quad \text{uniformly, as} \ \ t \searrow 0. \tag{F.2}$$

It suffices to show that (F.2) holds for u in a dense subspace of $\mathrm{UC}(\mathbb{R}^n)$, and we take $\mathrm{Lip}(\mathbb{R}^n)$, which a mollifier argument shows to be dense. So suppose u is bounded and

$$|u(x+y) - u(x)| \leq L(|y| \wedge 1), \tag{F.3}$$

for all $x, y \in \mathbb{R}^n$. As in (3.1), consider

$$G(y) = 1 - e^{-|y|} = 1 - g(y). \tag{F.4}$$

For each $y \in \mathbb{R}^n$,

$$u(y) - 2LG(x) \le u(x+y) \le u(y) + 2LG(x), \quad \forall\, x \in \mathbb{R}^n, \tag{F.5}$$

so for each $t > 0$,

$$u(y) - 2Le^{-t\psi(D)}G(x) \le e^{-t\psi(D)}u(x+y) \le u(y) + 2Le^{-t\psi(D)}G(x), \tag{F.6}$$

so

$$\bigl|e^{-t\psi(D)}u(y) - u(y)\bigr| \le 2Le^{-t\psi(D)}G(0). \tag{F.7}$$

As seen in (3.3)–(3.5), the right side of (F.7) tends to 0 as $t \searrow 0$, so the proof is complete.

M. Lévy processes on manifolds

Paralleling the study of translation invariant Lévy processes on Euclidean space $\mathbb{R}^n$, there is a theory of left (or right) invariant Lévy processes on Lie groups, initiated by G. Hunt. There are also studies of Lévy processes on more general Riemannian manifolds. Some articles in [BMR] discuss this, and give more references. Our goal here is to present various generalizations of the generators of Lévy processes given by the Lévy-Khinchin formula (A.5) that work in the manifold context.

The generators we seek are generators of semigroups $P^t = e^{tA}$ of positivity-preserving operators on $C_b(M)$ satisfying $P^t 1 = 1$. They have the form

$$P^t u(x) = \int_M p_t(x, dy)\, u(y), \tag{M.1}$$

where $p_t(x, \cdot)$ is a family of probability measures on M.

One general observation is that if A and B generate such semigroups and the Trotter product formula holds:

$$e^{t(A+B)} = \lim_{n\to\infty} \Bigl(e^{(t/n)A} e^{(t/n)B}\Bigr)^n, \tag{M.2}$$

then $A + B$ generates such a semigroup. Extending this, if $\{A(y) : y \in Y\}$ is a family of generators of such semigroups, then (frequently) so is

$$\int_Y A(y)\, d\mu(y), \tag{M.3}$$

given a positive measure μ on Y (perhaps with some sort of bound). Let us now get more specific.

The first two terms on the right side of (A.5) have well known generalizations to second order differential operators on M. In local coordinates,

$$L = \sum a^{jk}(x)\partial_j\partial_k + \sum b^j(x)\partial_j. \tag{M.4}$$

One makes various hypotheses, including $\sum a_{jk}(x)\xi_j\xi_k \geq 0$. There is a large literature on diffusion processes with such generators. See for example [Str].

We now point out various generators analogous to the last term in (A.5). To start, let $\varphi : M \to M$ be a continuous map, and consider

$$Tu(x) = u(\varphi(x)). \tag{M.5}$$

Now pick $c \in (0,\infty)$ and set

$$\text{(M.6)}\qquad \begin{aligned} P^t_{T,c}u(x) &= e^{-tc(I-T)}u(x) \\ &= \sum_{k=0}^{\infty} \frac{(ct)^k}{k!} e^{-ct}\, T^k u(x). \end{aligned}$$

This has the form (M.1) with

$$\text{(M.7)}\qquad p_t(x,\cdot) = \sum_{k=0}^{\infty} \frac{(ct)^k}{k!} e^{-ct}\, \delta_{\varphi^k(x)}.$$

One can impose further structure by requiring φ to be a diffeomorphism, or a volume preserving map, or an isometry (with respect to some Riemannian metric), etc. One can take a family $\varphi_y : M \to M$ of such maps and apply the process (M.3), obtaining generators of the form

$$\text{(M.8)}\qquad -\int_Y (I - T(y))\, d\mu(y), \quad T(y)u(x) = u(\varphi_y(x)).$$

To specialize this construction, let $X_1, \dots, X_m$ be smooth vector fields on M, and assume that for each $y = (y_1, \dots, y_m) \in \mathbb{R}^m$, $y \cdot X = y_1 X_1 + \cdots + y_m X_m$ generates a global flow on M. Given a positive measure μ on $\mathbb{R}^n$, one has the generator

$$\text{(M.9)}\qquad -\int_{\mathbb{R}^m} (I - e^{y\cdot X})\, d\mu(y),$$

given some bounds on μ. For example, one might require $\int (|y| \wedge 1)\, d\mu(y) < \infty$. Then (M.9) would be convergent if each $y \cdot X$ generated a volume preserving flow. Otherwise, further restrictions on μ might be needed. One can allow more singular behavior of μ near 0, i.e., $\int (|y|^2 \wedge 1)\, d\mu(y) < \infty$, upon replacing (M.9) by

$$\text{(M.10)}\qquad -\int_{\mathbb{R}^m} (I - e^{y\cdot X} + \chi_B(y)\, y \cdot X)\, d\mu(y).$$

The operators (M.9) and (M.10) are often pseudodifferential operators of order $2\alpha \in (0,2)$, for various measures $d\mu(y) = P(y)\, dy$, where $P(y)$ is smooth on $\mathbb{R}^m \setminus 0$, supported near 0 (this requirement can often be relaxed) and having a conormal singularity at 0, weaker than $|y|^{-m-1}$. One needs to require that $X_1^2 + \cdots + X_m^2$ be elliptic.

We mention the following problem. Suppose M has a Riemannian metric tensor, whose Laplace operator Δ generates a non-explosive diffusion, so $e^{t\Delta}1 = 1$ for $t > 0$.

For $\alpha \in (0,1)$, one would like to write $-(-\Delta)^\alpha$ in the form (M.9) or (M.10) (perhaps with the term $\chi_B(y) y \cdot X$ suitably modified). Surely this is well known in some cases, but it would be nice to have a general result. It would also be interesting to find such representations for variants, such as

$$1 - (1-\Delta)^\alpha, \quad \alpha \in (0,1).$$

Leaving (M.9)–(M.10), we note the following more general context for (M.6)–(M.8). Namely $T : C_b(M) \to C_b(M)$ could be any positivity preserving operator satisfying $T1 = 1$. Then

$$T^k u(x) = \int_M \mu_k(x, dy)\, u(y), \tag{M.11}$$

where $\mu_k(x,\cdot)$ is a family of probability measures, and one replaces (M.7) by

$$p_t(x,\cdot) = \sum_{k=0}^\infty \frac{(ct)^k}{k!} e^{-ct}\, \mu_k(x,\cdot). \tag{M.12}$$

The processes associated to the semigroups $e^{-tc(I-T)}$ for such T are called Feller's pseudo-Poisson processes; cf. [Ap], pp. 160–162.

N. Other Markov processes

The transition beyond Lévy processes in the Euclidean setting to Riemannian manifolds, discussed in §M, motivates us to go a step further, and present some general results about Markov semigroups. We say only a little about this big area, referring to [D] for a more thorough introduction.

We start with continuous-time Markov processes on a finite set X, say with n points, also called a finite Markov chain. We have $C(X)$ isomorphic to $\mathbb{R}^n$, and Markov semigroups are given by $n \times n$ matrices,

$$e^{tA}, \quad A \in M(n, \mathbb{R}), \quad t \geq 0. \tag{N.1}$$

To say this is a Markov semigroup is to say

$$e^{tA}\mathbf{1} \equiv \mathbf{1}, \quad \text{and} \ \ v \in \mathbb{R}^n, \ v \geq 0 \Rightarrow e^{tA}v \geq 0, \quad \text{for } t \geq 0. \tag{N.2}$$

Here $v \geq 0$ means each component is ≥ 0, and

$$\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{N.3}$$

We can restate the positivity condition as

$$e^{tA} = \Big(p_{jk}(t)\Big), \quad p_{jk}(t) \geq 0, \quad \text{for} \ \ t \geq 0, \quad j, k \in \{1, \dots, n\}. \tag{N.4}$$

The set of probability measures on X is given by $\mathcal{P}$, where, for $w \in \mathbb{R}^n$,

$$w \in \mathcal{P} \Longleftrightarrow w \geq 0 \ \ \text{and} \ \ w \cdot \mathbf{1} = 1. \tag{N.5}$$

Then the action of the Markov semigroup on $\mathcal{P}$ is given by

$$(e^{tA})^* : \mathcal{P} \longrightarrow \mathcal{P}, \quad \text{for} \ \ t \geq 0, \tag{N.6}$$

where, for $B \in M(n, \mathbb{R})$, B^* is the transpose of B.

The following result characterizes the generators A of all such semigroups.

Proposition N.1. *Given $A = \big(a_{jk}\big) \in M(n, \mathbb{R})$, e^{tA} satisfies (N.2) if and only if*

$$A\mathbf{1} = 0, \tag{N.7}$$

and

$$a_{jk} \geq 0 \ \ \textit{whenever} \ \ j \neq k. \tag{N.8}$$

Proof. Noting that

$$\frac{d}{dt} e^{tA}\Big|_{t=0} = A, \tag{N.9}$$

we see the relation $e^{tA}\mathbf{1} \equiv \mathbf{1}$ implies (N.7), and the positivity (N.8) follows from (N.4) plus $p_{jk}(0) = 0$ for $j \neq k$.

For the converse, if (N.8) is strengthened to $a_{jk} > 0$ whenever $j \neq k$, then, via

$$e^{tA} = I + tA + O(t^2), \tag{N.10}$$

one has $t_0 > 0$ such that $e^{tA} \geq 0$ for $0 \leq t \leq t_0$, and positivity for all $t \geq 0$ follows from $e^{ntA} = (e^{tA})^n$. Then the sufficiency of (N.7)–(N.8) in general can be established by a limiting argument. We leave the details to the reader. An alternative approach to the converse, valid in a much more general setting, is described below, in Proposition N.2.

Remark. Clearly the conditions (N.7)–(N.8) imply for the diagonal elements of A that

$$a_{jj} \leq 0, \quad \text{for } j \in \{1, \dots, n\}. \tag{N.11}$$

Denumerable Markov chains are associated to processes on a countably infinite set, such as $\mathbb{N} = \{1, 2, 3, \dots\}$. One might have a semigroup

$$e^{tA} : \ell^\infty(\mathbb{N}) \longrightarrow \ell^\infty(\mathbb{N}), \quad t \geq 0, \tag{N.12}$$

satisfying

$$f \in \ell^\infty(\mathbb{N}),\ f \geq 0 \Rightarrow e^{tA} f \geq 0 \ \text{ and } \ e^{tA} 1 \equiv 1. \tag{N.13}$$

Alternatively, one might consider sequences $f(n)$ that tend to a limit as $n \to \infty$, and

$$e^{tA} : C(\widehat{\mathbb{N}}) \longrightarrow C(\widehat{\mathbb{N}}), \quad t \geq 0, \tag{N.14}$$

where $\widehat{\mathbb{N}}$ is the one point compactification $\mathbb{N} \cup \{\infty\}$.

Extending the scope of (N.14), one can let X be a compact Hausdorff space, and consider semigroups

$$e^{tA} : C(X) \longrightarrow C(X), \quad t \geq 0, \tag{N.15}$$

satisfying

$$e^{tA} 1 \equiv 1, \quad \text{and } \ f \in C(X),\ f \geq 0 \Rightarrow e^{tA} f \geq 0. \tag{N.16}$$

The class (N.15)–(N.16) actually contains (N.12)–(N.13) as a special case. In fact, we can regard $\ell^\infty(\mathbb{N})$ as a commutative C^* algebra and take X to be its maximal ideal space. Then the Gelfand transform provides a positivity-preserving isometric isomorphism $\ell^\infty(\mathbb{N}) \approx C(X)$.

The following result yields a large class of Markov semigroups. In particular, it provides a far-reaching generalization of the result of Proposition N.1 that any $A \in M(n, \mathbb{R})$ satisfying (N.7)–(N.8) generates a Markov semigroup on $\mathbb{R}^n$.

Proposition N.2. *Let X be a compact Hausdorff space. Let*

$$B : C(X) \longrightarrow C(X) \tag{N.17}$$

be continuous and positivity-perserving, i.e.,

$$f \in C(X), f \geq 0 \Longrightarrow Bf \geq 0. \tag{N.18}$$

Set $\varphi = B1 \in C(X)$, and define

$$A : C(X) \longrightarrow C(X), \quad Au = -\varphi u + Bu. \tag{N.19}$$

Then $\{e^{tA} : t \geq 0\}$ is a Markov semigroup on $C(X)$.

Proof. First, clearly

$$A1 = 0, \quad \text{so} \quad e^{tA}1 \equiv 1. \tag{N.20}$$

It remains to show that, for $t \geq 0$, e^{tA} has the positivity property given in (N.16). This follows from the Trotter product formula,

$$e^{tA}f = \lim_{n\to\infty} \left(e^{-t\varphi/n}e^{(t/n)B}\right)^n f, \tag{N.21}$$

plus the fact that, for $s \geq 0$, $f \in C(X)$,

$$f \geq 0 \Longrightarrow e^{-s\varphi}f \geq 0 \quad \text{and} \quad e^{sB}f \geq 0, \tag{N.22}$$

the latter via the power series expansion

$$e^{sB} = \sum_{k=0}^{\infty} \frac{s^k}{k!} B^k. \tag{N.23}$$

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