L^p Bounds on Functions of Generalized Laplacians On a Compact Manifold with Boundary

Michael Taylor
Informal Notes

Abstract. Given a second-order, strongly elliptic, negative, self-adjoint differential operator L on a compact Riemannian manifold $\overline{M}$ with smooth boundary, we establish conditions under which $\varphi(\sqrt{-L})$ is bounded on $L^p(M)$, for $p \in (1, \infty)$.

1. Introduction

Let $\overline{M}$ be a compact, n-dimensional, Riemannian manifold with smooth boundary, and let L be a strongly elliptic, second-order differential operator on $\overline{M}$ (possibly a system), with smooth coefficients. We assume a coercive boundary condition makes L a negative, self-adjoint operator, with domain $\mathcal{D}(L) \subset H^2(M)$. Then, given a bounded continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, the spectral theorem defines $\varphi(\sqrt{-L})$ as a bounded operator on $L^2(M)$. The purpose of this paper it to establish results of the form

$$\varphi(\sqrt{-L}) : L^p(M) \longrightarrow L^p(M), \quad \forall\, p \in (1, \infty), \tag{1.1}$$

and

$$\varphi(\sqrt{-L}) \ \text{ is of weak type } \ (1, 1). \tag{1.2}$$

We will deal with functions φ of symbol type,

$$\varphi \in S^0_1(\mathbb{R}), \ \text{ i.e., } \ |\varphi^{(k)}(\lambda)| \le C_k (1 + |\lambda|)^{-k}, \quad k = 0, 1, 2, \dots. \tag{1.3}$$

See (1.9) and (1.23) below for more general conditions on φ that will allow us to establish (1.1)–(1.2). We will also assume $\varphi(\lambda)$ is an even function of λ, which involves no loss of generality, since Spec $\sqrt{-L}$ is a discrete subset of $[0, \infty)$.

We mention some previous results giving rise to (1.1)–(1.2). In case $L = \Delta$, the Laplace operator on M (or more generally L has scalar principal symbol), results of [Str] and [T1] (see also [T2], Chapter 12) yield $\varphi(\sqrt{-L}) \in OPS^0_{1,0}(M)$, when M is compact without boundary and (1.3) holds. Such pseudodifferential operators satisfy (1.1)–(1.2). In such a setting, [SeS] established (1.1)–(1.2) under the following weaker hypothesis on φ: for some $s > n/2$,

$$\sup_{\mu > 0} \|\beta \varphi_\mu\|_{H^s(\mathbb{R})} < \infty, \tag{1.4}$$

given $\beta \in C_0^\infty((1/2,2))$, $\beta = 1$ on $[1,3/2]$, and $\varphi_\mu(\lambda) = \varphi(\mu\lambda)$. The condition (1.3) is equivalent to the hypothesis that φ is smooth and (1.4) holds for all $s < \infty$. An alternative proof was given in [Xu1], and another in [DOS]. (See Appendix B for more on the work of [DOS].)

Going further, [Xu2] and [Xu3] treated compact manifolds with boundary, in case $L = \Delta$, first for the Dirichlet boundary condition, and then for the Neumann boundary condition. In these papers, (1.1)–(1.2) were established for φ satisfying (1.4). We make further comments on the work of [Xu1]–[Xu3] in Appendix A, but here mention that maximum principle arguments play a major role. For many systems to which our results apply, maximum principle arguments would not be available.

In the works [T1], [T2], and [SeS] (following seminal work of [H2]), an important role was played by the representation of $\varphi(\sqrt{-L})$ as

$$\varphi(\sqrt{-L}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varphi}(t)\, \cos t\sqrt{-L}\, dt, \tag{1.5}$$

for even φ, where $\hat{\varphi}$ is the Fourier transform of φ, and $\cos t\sqrt{-L}$ is a solution operator to the wave equation:

$$u(t,x) = \cos t\sqrt{-L}\, f(x), \tag{1.6}$$

where

$$(\partial_t^2 - L)u = 0, \quad u(0,x) = f(x), \quad \partial_t u(0,x) = 0. \tag{1.7}$$

It is then useful to split $\hat{\varphi}(t)$ into two pieces, using a partition of unity. Given $a > 0$, take $\chi \in C_0^\infty((-a,a))$, even, such that $\chi(t) = 1$ for $|t| \le a/2$, and set

$$\hat{\varphi}^{\#}(t) = \chi(t)\hat{\varphi}(t), \quad \hat{\varphi}^b(t) = (1-\chi(t))\hat{\varphi}(t). \tag{1.8}$$

The hypothesis (1.3) implies φ^b is smooth and rapidly decreasing. More generally, we impose the hypothesis

$$|\varphi^b(\lambda)| \le C(1+|\lambda|)^{-m}, \quad \text{for some } \ m > \frac{n}{2}, \tag{1.9}$$

where $n = \dim M$. In such a case, the ellipticity hypothesis implies

$$\varphi^b(\sqrt{-L}) : L^2(M) \longrightarrow H^m(M) \subset C(\overline{M}), \tag{1.10}$$

and, by duality,

$$\varphi^b(\sqrt{-L}) : L^1(M) \longrightarrow L^2(M). \tag{1.11}$$

The considerations (1.10)–(1.11) apply to our current setting, with $\partial M \neq \emptyset$.

It remains to analyze

$$\varphi^{\#}(\sqrt{-L}) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \hat{\varphi}^{\#}(t)\, \cos t\sqrt{-L}\, dt. \tag{1.12}$$

When $L = \Delta$ (or more generally, L has scalar principal symbol) and $\partial M = \emptyset$, one can use the method of geometrical optics to construct a parametrix for the solution to the hyperbolic equation (1.7), valid for $|t| \le a$, given $a > 0$ suffciently small. Such a parametrix is given by an oscillatory integral, and the papers [T1] and [SeS] used analyses of such operators to establish desired results on $\varphi^{\#}(\sqrt{-L})$. When $\partial M \neq \emptyset$, such parametrix constructions (for $L = \Delta$) range from somewhat subtle (in the diffractive and gliding cases) to terra incognita (for more general boundary geometry). Furthermore if L does not have a scalar principal symbol, parametrix constructions for (1.7) are sometimes lacking, even when $\partial M = \emptyset$.

Before tackling the study of $\varphi^{\#}(\sqrt{-L})$ when $\overline{M}$ is compact with nonempty boundary, we mention some studies of $\varphi(\sqrt{-L})$ for various classes of noncompact, complete Riemannian manifolds (with $L = \Delta$), made in [CGT], [DST], [T3], [MMV], and [T4], as some techniques brought to bear in these papers will influence how we analyze the case of nonempty boundary. In these papers, one also splits φ as in (1.8). The analysis of $\varphi^{b}(\sqrt{-L})$ becomes somewhat more elaborate, because when M has infinite volume, (1.11) does not imply that $\varphi^{b}(\sqrt{-L})$ is bounded on $L^1(M)$. In fact, spaces treated in these papers include examples, such as hyperbolic space, whose balls of radius R grow exponentially in volume with R, and one requires that φ be holomorphic on a horizontal strip, satisfying bounds of the form (1.3). We omit details here, but mention that a critical role is played by finite propagation speed:

$$\operatorname{supp} f \subset K \Longrightarrow \operatorname{supp} \cos t\sqrt{-\Delta} f \subset K_{|t|}, \tag{1.13}$$

where

$$K_{|t|} = \{x \in M : \operatorname{dist}(x, K) \le |t|\}. \tag{1.14}$$

This enables one to get a good hold on $\varphi^{b}(\sqrt{-\Delta})$, when φ has such symbol properties on a horizontal strip, of appropriate width, about $\mathbb{R} \subset \mathbb{C}$. As for $\varphi^{\#}(\sqrt{-\Delta})$, in [CGT] this was analyzed as a pseudodifferential operator, when M has C^∞ bounded geometry, via a parametrix construction for (1.7).

The papers [MMV] and [T4] dealt with L^p estimates under much weaker geometric bounds on M, namely lower bounds on the Ricci tensor and injectivity radius. The analyses in these papers avoided the task of producing a parametrix for the wave equation (1.7). The key was to replace (1.5) by

$$\varphi(\sqrt{-L}) = \frac{1}{2} \int_{-\infty}^{\infty} \varphi_k(t) \mathcal{J}_{k-1/2}(t\sqrt{-L})\, dt, \tag{1.15}$$

where

$$\mathcal{J}_\nu(\lambda) = \lambda^{-\nu} J_\nu(\lambda), \tag{1.16}$$

$J_\nu(\lambda)$ denoting the standard Bessel function, and

$$\varphi_k(t) = \prod_{j=1}^{k} \Bigl(-t\frac{d}{dt} + 2j - 2\Bigr)\hat{\varphi}(t). \tag{1.17}$$

As is classical,

$$\mathcal{J}_{-1/2}(\lambda) = \sqrt{\frac{2}{\pi}}\,\cos\lambda, \tag{1.18}$$

and then (1.15) follows from (1.5) by an integration by parts argument, using the inductive formula

$$\Bigl(t\frac{d}{dt} + 2\nu\Bigr)\mathcal{J}_\nu(t\sqrt{-L}) = \mathcal{J}_{\nu-1}(t\sqrt{-L}). \tag{1.19}$$

Compare (3.7)–(3.9) of [T4].

Similarly, we have

$$\varphi^{\#}(\sqrt{-L}) = \frac{1}{2}\int_{-\infty}^{\infty} \psi_k(t)\mathcal{J}_{k-1/2}(t\sqrt{-L})\,dt, \tag{1.20}$$

with

$$\psi_k(t) = \prod_{j=1}^{k} \Bigl(-t\frac{d}{dt} + 2j - 2\Bigr)\hat{\varphi}^{\#}(t). \tag{1.21}$$

Given (1.8), we have

$$\operatorname{supp}\psi_k \subset [-a, a]. \tag{1.22}$$

Furthermore, the hypothesis (1.3) implies

$$\bigl|(t\partial_t)^j\hat{\varphi}(t)\bigr| \le C_j|t|^{-1}, \quad \forall\, j \in \Bigl\{0, 1, \dots, \Bigl[\frac{n}{2}\Bigr] + 2\Bigr\}, \tag{1.23}$$

in fact, for all $j \in \mathbb{Z}^+$, but we only need the range j given in (1.23). This in turn implies

$$|\psi_k(t)| \le C_k|t|^{-1}, \quad 0 \le k \le \Bigl[\frac{n}{2}\Bigr] + 2. \tag{1.24}$$

Other important ingredients for the analysis of $\varphi^{\#}(\sqrt{-L})$ arise from the classical integral representation

$$\mathcal{J}_\nu(\lambda) = c_\nu \int_{-1}^{1} (1-s^2)^{\nu-1/2} \cos s\lambda \, ds, \quad \nu > -\frac{1}{2}. \tag{1.25}$$

One consequence is the estimate

$$|\mathcal{J}_{k-1/2}(\lambda)| \le C_k (1+|\lambda|)^{-k}, \quad k > 0. \tag{1.26}$$

Another follows from

$$\mathcal{J}_{k-1/2}(t\sqrt{-L}) = c_{k-1/2} \int_{-1}^{1} (1-s^2)^{k-1} \cos st\sqrt{-L} \, ds. \tag{1.27}$$

We will posit that finite propagation speed holds for (1.7), so, parallel to (1.13), for some $\alpha < \infty$,

$$\operatorname{supp} f \subset K \Longrightarrow \operatorname{supp} \cos t\sqrt{-L} f \subset K_{\alpha|t|}. \tag{1.28}$$

Then (1.27) gives

$$\operatorname{supp} f \subset K \Longrightarrow \operatorname{supp} \mathcal{J}_{k-1/2}(t\sqrt{-L}) \subset K_{\alpha|t|}. \tag{1.29}$$

We have assembled most of the ingredients needed to state our main result. The operator analysis of $\mathcal{J}_{k-1/2}(t\sqrt{-L})$ done in §2 will also make use of the following heat kernel bound:

$$\|\nabla e^{tL} f\|_{L^\infty} \le C(t^{-n/4-1/2} + 1)\|f\|_{L^2}, \quad t > 0, \tag{1.30}$$

accompanying the bound

$$\|e^{tL} f\|_{L^\infty} \le C(t^{-n/4} + 1)\|f\|_{L^2}. \tag{1.31}$$

Here is our main result.

Theorem 1.1. *Let $\overline{M}$ be a compact, n-dimensional, Riemannian manifold with smooth boundary ∂M, L a second-order, strongly elliptic differential operator, with a coercive boundary condition, defining L as a negative, self-adjoint operator, with domain $\mathcal{D}(L) \subset H^2(M)$. Assume the finite propagation speed result (1.28) and the heat kernel bound (1.30).*

Let φ be a smooth, bounded function on $\mathbb{R}$. Assume φ is even and that (with $\varphi = \varphi^{\#} + \varphi^b$ as in (1.8)) the estimates (1.9) and (1.23) hold. Then (1.1)–(1.2) hold.

The proof of Theorem 1.1 uses the decomposition $\varphi(\sqrt{-L}) = \varphi^{\#}(\sqrt{-L}) + \varphi^b(\sqrt{-L})$, and we already have the estimates (1.10)–(1.11) on $\varphi^b(\sqrt{-L})$. It remains to show that $\varphi^{\#}(\sqrt{-L})$ is weak type $(1,1)$. Details of this are given in §2. In §3, we discuss conditions guaranteeing that the finite propagation speed hypothesis (1.28) holds. In §4 we discuss heat kernel bounds, including (1.30)–(1.31). In §5 we establish estimates on $e^{-t\sqrt{-L}}$, which provide another proof of (1.30)–(1.31).

In Appendix A, we make some comments on the approach to L^p-boundedness given in [SeS] and [Xu1]–[Xu3]. In Appendix B we discuss some of the work of [DOS], which establishes results more general than Theorem 1.1.

2. Analysis of $\varphi^\#(\sqrt{-L})$ and proof of Theorem 1.1

Our goal in this section is to prove the following.

Proposition 2.1. *In the setting of Theorem 1.1, $\varphi^\#(\sqrt{-L})$ is of weak type* $(1,1)$.

Given this, it follows from (1.11) that $\varphi(\sqrt{-L})$ is of weak type $(1,1)$. Then, interpolation with the obvious L^2 estimate gives (1.1) for $1 < p \le 2$, and duality gives it for $2 \le p < \infty$.

The approach to Proposition 2.1 is to analyze the integral kernel $K^\#(x,y)$ for $\varphi^\#(\sqrt{-L})$, given by

$$\varphi^\#(\sqrt{-L})f(x) = \int\limits_M K^\#(x,y)f(y)\,dV(y), \tag{2.1}$$

and show that it satisfies certain Hörmander-type estimates.

To simplify the notation, we scale L to arrange that (1.28) holds with $\alpha = 1$. Also, we take $a = 1$ in the specification of χ in (1.8), and hence in (1.22). Thus, from (1.20),

$$K^\#(x,y) = \int_0^1 \psi_k(t)B_k(t,x,y)\,dt, \tag{2.2}$$

where $B_k(t,x,y)$ is the integral kernel of $\mathcal{J}_{k-1/2}(t\sqrt{-L})$:

$$\mathcal{J}_{k-1/2}(t\sqrt{-L})f(x) = \int\limits_M B_k(t,x,y)f(y)\,dV(y). \tag{2.3}$$

The following result is analogous to Proposition 2.2 of [MMV].

Lemma 2.2. *If $k > n/2+1$,*

$$\|\nabla_y B_k(t,\cdot,y)\|_{L^2(M)} \le Ct^{-n/2-1}, \quad \text{for } \ t \in (0,1], \ y \in \overline{M}. \tag{2.4}$$

Let us see how (2.4) leads to desired estimates on $K^\#(x,y)$. By (1.29) (with $\alpha = 1$), for $s \in (0,1]$,

$$\int_0^s \psi_k(t)B_k(t,x,y)\,dt \ \text{ is supported on } \ \{(x,y) \in \overline{M} \times \overline{M} : \mathrm{dist}(x,y) \le s\}. \tag{2.5}$$

Hence

$$K^\#(x,y) = \int_s^1 \psi_k(t)B_k(t,x,y)\,dt \ \text{ on } \ \{(x,y) \in \overline{M} \times \overline{M} : \mathrm{dist}(x,y) \ge s\}. \tag{2.6}$$

Using (2.4) plus the fact that $B_k(t,\cdot,y)$ is supported on the ball $B_{|t|}(y)$ in $\overline{M}$ of radius $|t|$, centered at y (by (1.29)), we have

$$\|\nabla_y B_k(t,\cdot,y)\|_{L^1} \le C\big(\mathrm{Vol}\big(B_{|t|}(y)\big)\big)^{1/2} t^{-n/2-1} \le \frac{C}{t}. \tag{2.7}$$

Hence, from (2.6) and (1.24), we have, for $k = [n/2]+2$,

$$\begin{aligned} \|\nabla_y K^{\#}(\cdot,y)\|_{L^1(B_1(y)\setminus B_s(y))} &\le \int_s^1 \frac{C}{t}\|\nabla_y B(t,\cdot,y)\|_{L^1}\,dt \\ &\le C\int_s^1 \frac{dt}{t^2} \\ &\le \frac{C}{s}. \end{aligned} \tag{2.8}$$

This leads to the following.

Lemma 2.3. *There exists $C<\infty$, independent of $s\in(0,1]$ and of $y,y'\in\overline{M}$, such that*

$$\mathrm{dist}(y,y') \le \frac{s}{2} \Longrightarrow \|K^{\#}(\cdot,y)-K^{\#}(\cdot,y')\|_{L^1(B_1(y)\setminus B_s(y))} \le C. \tag{2.9}$$

We now have:

Proof of Proposition 2.1. Given that $\varphi^{\#}(\sqrt{-L})$ is bounded on $L^2(M)$ and its integral kernel satisfies (2.9), the weak type (1,1) property is a consequence of Proposition 3.1 of [MMV], which in turn is a natural variant of Theorem 2.4 in Chapter III of [CW].

It remains to prove Lemma 2.2. In light of (1.26), this result is a special case of the following.

Lemma 2.4. *If $G:\mathbb{R}\to\mathbb{R}$ satisfies*

$$|G(\lambda)| \le C(1+|\lambda|)^{-\gamma-1}, \quad \gamma > \frac{n}{2}, \tag{2.10}$$

then

$$\|G(t\sqrt{-L})\|_{\mathcal{L}(L^2,\mathrm{Lip})} \le Ct^{-n/2-1}, \quad t\in(0,1]. \tag{2.11}$$

From Lemma 2.4, we have that, if

$$G(t\sqrt{-L})f(x) = \int_M g(t,x,y)f(y)\,dV(y), \tag{2.12}$$

then

$$\|\nabla_x g(t,x,\cdot)\|_{L^2} \le Ct^{-n/2-1}, \quad t \in (0,1],\ x \in \overline{M}. \tag{2.13}$$

To apply this to Lemma 2.2, note that $g(t,x,y) = g(t,y,x)^*$, so (2.13) yields

$$\|\nabla_y g(t,\cdot,y)\|_{L^2} \le Ct^{-n/2-1}, \quad t \in (0,1],\ y \in \overline{M}, \tag{2.14}$$

as asserted in (2.4).

Thus, it remains to prove Lemma 2.4. For this, we bring in the heat kernel estimate (1.30), i.e.,

$$\|e^{tL}\|_{\mathcal{L}(L^2,\mathrm{Lip})} \le C(t^{-n/4-1/2} + 1). \tag{2.15}$$

We can use

$$G(t\sqrt{-L}) = (1-t^2L)^{-\sigma}(1-t^2L)^{\sigma} G(t\sqrt{-L}), \quad 2\sigma = \gamma + 1, \tag{2.16}$$

to reduce our task to showing that

$$\|(1-t^2L)^{-\sigma}\|_{\mathcal{L}(L^2,\mathrm{Lip})} \le C(t^{-n/2-1} + 1), \quad \text{if } \sigma > \frac{n}{4} + \frac{1}{2}. \tag{2.17}$$

To prove (2.17), we use the identity

$$(1-t^2L)^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-s} e^{st^2L} s^{\sigma-1}\, ds. \tag{2.18}$$

Then, by (2.15),

$$\begin{aligned} \|(1-t^2L)^{-\sigma}\|_{\mathcal{L}(L^2,\mathrm{Lip})} &\le C \int_0^{t^{-2}} e^{-s} (st^2)^{-n/4-1/2} s^{\sigma-1}\, ds \\ &\qquad + C \int_{t^{-2}}^\infty e^{-s} s^{\sigma-1}\, ds \\ &\le C_1(t^{-n/2-1} + 1), \end{aligned} \tag{2.19}$$

with $C_1 < \infty$ if $\sigma > n/4 + 1/2$.

This proves Lemma 2.4, and hence completes the proof of Proposition 2.1.

3. Finite propagation speed

As before, L is a strongly elliptic, second-order, negative, self-adjoint operator on $\overline{M}$, a compact Riemannian manifold wih smooth boundary. We seek conditions that guarantee the finite propagation speed property (1.28). The situation when $\partial M = \emptyset$ is well known, as is the case $L = \Delta$ with the Dirichlet or Neumenn boundary condition. Here, we give further results, in case

$$-L = D^* D, \tag{3.1}$$

where $D : H^{s+1}(\overline{M}, \mathcal{E}_0) \to H^2(\overline{M}, \mathcal{E}_1)$ is a first-order differential operator (between sections of vector bundles). We assume that the symbol $\sigma_D(x, \xi) : \mathcal{E}_{0x} \to \mathcal{E}_{1x}$ is injective for each $x \in \overline{M}$, $\xi \in T^*_x\overline{M} \setminus 0$. One has the domain $\mathcal{D}(L) = \{u \in \mathcal{D}(D) : Du \in \mathcal{D}(D^*)\}$. We assume $\mathcal{D}(D)$ is given by a local boundary condition:

$$u \in \mathcal{D}(D) \Longrightarrow \beta(x)u(x) = 0, \quad \forall\, x \in \partial M. \tag{3.2}$$

We assume $\beta(x)$ is an orthogonal projection on $\mathcal{E}_{0x}$ for each $x \in \partial M$. In light of the identity

$$\int\limits_M \bigl[\langle Dv, w\rangle - \langle v, D^* w\rangle\bigr]\, dV = \frac{1}{i} \int\limits_{\partial M} \langle \sigma_D(x, \nu)v, w\rangle\, dS, \tag{3.3}$$

for sufficiently smooth v, w, we have, for smooth w,

$$w \in \mathcal{D}(D^*) \Longrightarrow (I - \beta(x))\sigma_D(x, \nu)^* w(x) = 0, \quad \forall\, x \in \partial M. \tag{3.4}$$

Here, $\nu(x) \in T^*_x\overline{M}$ is the outward unit normal to ∂M. Hence, for smooth v and w,

$$v \in \mathcal{D}(D),\ w \in \mathcal{D}(D^*) \Longrightarrow \langle \sigma_D(x, \nu)v, w\rangle = 0 \ \text{ on } \ \partial M. \tag{3.5}$$

Let $\overline{\Omega} \subset \mathbb{R} \times \overline{M}$. We assume $\partial\Omega$ has three pieces,

$$\partial\Omega = \mathcal{T} \cup \mathcal{B} \cup \mathcal{L}, \tag{3.6}$$

where $\mathcal{B} = \{(t, x) \in \overline{\Omega} : t = 0\}$ is the bottom part, $\mathcal{L} = \{(t, x) \in \overline{\Omega} : x \in \partial M\}$ is the lateral part, and $\mathcal{T}$ is the top part. We seek a condition guaranteeing that, for sufficiently smooth u on $\mathbb{R} \times \overline{M}$ satisfying

$$(\partial_t^2 - L)u = 0, \quad u(t), \partial_t u(t) \in \mathcal{D}(L), \tag{3.7}$$

we have the implication

$$u, u_t = 0 \;\text{ on }\; \mathcal{B} \Longrightarrow u = 0 \;\text{ on }\; \Omega. \tag{3.8}$$

The "energy integral" method we use to establish (3.8) goes as follows. We have, for sufficiently smooth u,

$$\begin{aligned}
&\int\limits_{\Omega} u_t(u_{tt} - Lu)\, dV\, dt \\
&= \frac{1}{2}\int\limits_{\Omega} \partial_t |u_t|^2\, dV\, dt + \int\limits_{\Omega} u_t\, D^* Du\, dV\, dt \\
&= \int\limits_{\Omega} \Bigl[\frac{1}{2}\partial_t|u_t|^2 + \langle Du_t, Du\rangle\Bigr]\, dV\, dt - \frac{1}{i}\int\limits_{\partial\Omega} \langle \sigma_D(x,\nu)u_t, Du\rangle\, dS_t\, dt \\
&= \frac{1}{2}\int\limits_{\partial\Omega} \Bigl[|u_t|^2 + |D_x u|^2\Bigr] N_t\, dS - \frac{1}{i}\int\limits_{\partial\Omega} \langle \sigma_D(x, N_x)u_t, D_x u\rangle\, dS.
\end{aligned} \tag{3.9}$$

Hence, for smooth u satisfying (3.7),

$$\frac{1}{2}\int\limits_{\partial\Omega} \Bigl[|u_t|^2 + |D_x u|^2\Bigr] N_t\, dS - \frac{1}{i}\int\limits_{\partial\Omega} \langle \sigma_D(x, N_x)u_t, D_x u\rangle\, dS = 0. \tag{3.10}$$

Here dS_t is the area element of $\partial\Omega_t \subset \overline{M}$, where $\Omega_t = \{(s,x) \in \Omega : s = t\}$, and dS is the area element on $\partial\Omega$. We have

$$dS = N_t\, dV\, dt = |N_x|\, dS_t\, dt. \tag{3.11}$$

where $N = (N_t, N_x)$ is the unit normal to $\partial\Omega \subset \mathbb{R} \times \overline{M}$.

If $u, u_t = 0$ on $\mathcal{B}$, the integrands in (3.10) vanish on $\mathcal{B}$. Also, since $N_t = 0$ on $\mathcal{L}$ and (3.5) holds, these integrands vanish on $\mathcal{L}$. Then the left side of (3.10) is

$$\geq \frac{1}{2}(1-\alpha)\int\limits_{\mathcal{T}} \Bigl[|u_t|^2 + |D_x u|^2\Bigr] N_t\, dS, \tag{3.12}$$

in absolute value, provided

$$|\langle \sigma_D(x, N_x)u_t, D_x u\rangle| \leq \frac{\alpha}{2}\Bigl[|u_t|^2 + |D_x u|^2\Bigr] N_t, \tag{3.13}$$

on $\mathcal{T}$, for some $\alpha \in (0,1)$. Now the left side of (3.13) is

$$\leq |\sigma_D(x, N_x)u_t| \cdot |D_x u|, \tag{3.14}$$

so (3.12) holds for some $\alpha \in (0,1)$ provided

$$|\sigma_D(x, N_x)| < N_t, \quad x \in \mathcal{T}. \tag{3.15}$$

This "spacelike" condition on $\mathcal{T}$ then yields the implication

$$u, u_t = 0 \;\text{ on }\; \mathcal{B} \Longrightarrow u_t, D_x u = 0 \;\text{ on }\; \mathcal{T}. \tag{3.16}$$

If Ω can be swept out by such spacelike surfaces, one gets $u, u_t = 0$ on $\mathcal{B} \Rightarrow u_t \equiv 0$ on Ω, hence (3.8).

4. Heat kernel estimates

We discuss estimates on e^{tL} that lead to (1.30)–(1.31). It suffices to treat the case $t \in (0,1]$. If we denote the integral kernel of e^{tL} by $p(t,x,y)$, so

$$e^{tL} f(x) = \int_M p(t,x,y) f(y)\, dV(y), \tag{4.1}$$

then (1.31) is a consequence of

$$\int_M |p(t,x,y)|^2 \, dV(y) \le C t^{-n/2}, \quad t \in (0,1],\ x \in \overline{M}, \tag{4.2}$$

and (1.30) is a consequence of

$$\int_M |\nabla_x p(t,x,y)|^2 \, dV(y) \le C t^{-n/2-1}, \quad t \in (0,1],\ x \in \overline{M}. \tag{4.3}$$

In turn, (4.2) is a consequence of the following pointwise estimate: for some $C, \kappa \in (0,\infty)$,

$$|p(t,x,y)| \le C t^{-n/2} e^{-\kappa\, \mathrm{dist}(x,y)^2/t}, \quad t \in (0,1],\ x,y \in \overline{M}, \tag{4.4}$$

and (4.3) is a consequence of

$$|\nabla_x p(t,x,y)| \le C t^{-n/2-1/2} e^{-\kappa\, \mathrm{dist}(x,y)^2/t}, \quad t \in (0,1],\ x,y \in \overline{M}. \tag{4.5}$$

We note that (4.5) also implies (A.4).

Modulo small error terms that do not affect (4.2)–(4.3), the pointwise estimates (4.4)–(4.5) follow from parametrix constructions for $p(t,x,y)$ on $(0,1] \times \overline{M} \times \overline{M}$, which can be carried out in substantial generality.

In case $\partial M = \emptyset$, such a parametrix construction can be done using the special class of non-isotropic pseudodifferential operators associated to the parabolic operator $\partial_t - L$. When $\partial M \neq \emptyset$, this is augmented by a theory of parabolic layer potentials. See, e.g., [Gr]. In the case emphasized here (with $m = 2$), wave equation methods can also be brought to bear on (4.4)–(4.5), as in [CGT]. We omit details, since a different argument, implying (1.30)–(1.31), is given in §5.

5. Poisson semigroup estimates

As before, we assume L is a second-order, strongly elliptic, negative, self-adjoint operator, on a compact manifold $\overline{M}$ with smooth boundary, with domain given by a coercive local boundary condition. Independently of the heat kernel estimates of §4, we establish the estimates

$$\|e^{-y\sqrt{-L}}\|_{\mathcal{L}(L^2,L^\infty)} \le Cy^{-n/2}, \quad 0 < y \le 1, \tag{5.1}$$

and

$$\|e^{-y\sqrt{-L}}\|_{\mathcal{L}(L^2,\mathrm{Lip})} \le Cy^{-n/2-1}, \quad 0 < y \le 1. \tag{5.2}$$

Having (5.2), we can use

$$(1+t\sqrt{-L})^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-s} e^{-st\sqrt{-L}} s^{\sigma-1}\, ds, \tag{5.3}$$

and an aregument similar to (2.19) to establish (via (5.2)) that

$$\|(1+t\sqrt{-L})^{-\sigma}\|_{\mathcal{L}(L^2,\mathrm{Lip})} \le Ct^{-n/2-1}, \quad \text{if } \ \sigma > \frac{n}{2}+1, \ \ t \in (0,1], \tag{5.4}$$

which can be used in place of (2.16) to prove Lemma 2.4. In turn, Lemma 2.4 contains (1.30). Similarly, (1.31) follows from (5.1).

We turn to a proof of (5.1)–(5.2). Given $f \in L^2(M)$, let us set

$$u(y,x) = e^{-y\sqrt{-L}} f(x), \quad y > 0, \ x \in \overline{M}. \tag{5.5}$$

Then u solves

$$\begin{aligned} (\partial_y^2 + L)u &= 0, \quad \text{on } \ (0,\infty)\times M, \\ B(x,\partial_x)u &= 0, \quad \text{on } \ (0,\infty)\times \partial M, \end{aligned} \tag{5.6}$$

where B provides the coercive boundary condition defining the domain of L. We have

$$\|u(y,\cdot)\|_{L^2(M)} \le \|f\|_{L^2(M)}, \quad \forall\, y > 0. \tag{5.7}$$

Let us pick $\delta \in (0,1]$, $y_0 = \delta$, and $x_0 \in \overline{M}$. Let $U = \{x \in \overline{M} : \mathrm{dist}(x,x_0) < 2\delta\}$. Let $U_0 = \{x \in \overline{M} : \mathrm{dist}(x,x_0) < \delta\}$. Let us scale the y variable and the x variable by a factor of $1/\delta$, and let $v(y,x)$ denote u in the scaled variables. Then v solves

$$\begin{aligned} (\partial_y^2 + \widetilde{L})v &= 0, \quad \text{on } \ (1/2,3/2)\times \widetilde{U}, \\ \widetilde{B}(x,\partial_x)v &= 0, \quad \text{on } \ (1/2,3/2)\times (\widetilde{U}\cap \partial M), \end{aligned} \tag{5.8}$$

which is a coercive elliptic system with uniformly smooth coefficients and uniform ellipticity bounds. Note that

$$\|v\|_{L^2((1/2,3/2)\times\widetilde{U})} \approx \delta^{-n/2}\|u\|_{L^2((\delta/2,3\delta/2)\times U)} \le C\delta^{-n/2}\|f\|_{L^2}. \tag{5.9}$$

Elliptic regularity gives bounds

$$\begin{aligned} \|v(1,\cdot)\|_{L^\infty(\widetilde{U}_0)} &\le C\delta^{-n/2}\|f\|_{L^2}, \\ \|\nabla v(1,\cdot)\|_{L^\infty(\widetilde{U}_0)} &\le C\delta^{-n/2}\|f\|_{L^2}, \end{aligned} \tag{5.10}$$

and scaling back gives

$$\begin{aligned} |u(y_0,x_0)| &\le C\delta^{-n/2}\|f\|_{L^2}, \\ |\nabla_x u(y_0,x_0)| &\le C\delta^{-n/2-1}\|f\|_{L^2}. \end{aligned} \tag{5.11}$$

These estimates prove (5.1)–(5.2).

Remark. In the setting of $w(t,x) = e^{tL}f(x)$, one can make a non-isotropic dilation of variables and appeal to regularity estimates for parabolic equations, with coercive boundary conditions. The elliptic setting (5.6) allows for the use of isotropic dilations of variables, done here.

A. Comments on the spectral projection approach

Here we comment on work done in [SeS] and in [Xu1]–[Xu3] on establishing (1.1)–(1.2), for compact $\overline{M}$, under the hypothesis (1.4) on φ, with $s > n/2$ ($n = \dim M$). One central ingredient in these works is the Hörmander-type estimate

$$\|\Pi_\lambda f\|_{L^\infty} \le C\lambda^{(n-1)/2}\|f\|_{L^2}, \quad \lambda \ge 1, \tag{A.1}$$

where Π_λ is the spectral projection of $\sqrt{-L}$ associated with $[\lambda, \lambda+1]$. This was established in [H2] when $\partial M = \emptyset$ and L has scalar principal symbol, in [Sog2] when $L = \Delta$ with the Dirichlet boundary condition, and in [Xu3] when $L = \Delta$ with the Neumann boundary condition. A derived ingredient is

$$\|\nabla \Pi_\lambda f\|_{L^\infty} \le C\lambda^{(n+1)/2}\|f\|_{L^2}. \tag{A.2}$$

This was deduced from (A.1) when $\partial M = \emptyset$ and $L = \Delta$ in [Xu1], when $L = \Delta$ with the Dirichlet boundary condition in [Xu2], and when $L = \Delta$ with the Neumann boundary condition in [Xu3]. In [Xu1]–[Xu3], the maximum principle played a key role. In these papers, (1.1)–(1.2) was deduced from (A.2).

Establishing (A.1) in more general situations, where the maximum principle would not apply, is a challenging task. Here, we show how to obtain (A.2) from (A.1), in a fairly general setting, without using maximum principle arguments. Our strategy will be to use a Littlewood-Paley approach, based on the "heat semigroup" e^{tL}. To wit, we hypothesize that

$$t^{1/2}\nabla e^{tL} : L^\infty(M) \longrightarrow L^\infty(M) \ \text{ is uniformly bounded, for } \ t \in (0,1]. \tag{A.3}$$

This is equivalent to the following estimate on the integral kernel $p(t,x,y)$ of e^{tL}:

$$\int_M |\nabla_x p(t,x,y)|\, dV(y) \le Ct^{-1/2}, \quad \forall\, t \in (0,1],\ x \in \overline{M}. \tag{A.4}$$

See §4 for results on this.

Here is how to get from (A.1) to (A.2) when we have (A.3). Take $t^{1/2} = 1/\lambda$. Then

$$\begin{aligned}
\|\nabla \Pi_\lambda f\|_{L^\infty} &= \|\nabla e^{tL}\Pi_\lambda e^{-tL}\Pi_\lambda f\|_{L^\infty} \\
&= \lambda \|t^{1/2}\nabla e^{tL}\Pi_\lambda e^{-tL}\Pi_\lambda f\|_{L^\infty} \\
&\le C\lambda \|\Pi_\lambda e^{-tL}\Pi_\lambda f\|_{L^\infty} \qquad \text{(by (A.3))} \\
&\le C\lambda^{(n+1)/2}\|e^{-tL}\Pi_\lambda f\|_{L^2} \qquad \text{(by (A.1))} \\
&\le C\lambda^{(n+1)/2}\|f\|_{L^2},
\end{aligned} \tag{A.5}$$

giving (A.2).

B. Comments on results of [DOS]

Here we discuss some results of [DOS]. These results are quite general, set in the context of an open subset X of a measured metric space of homogeneous type. For simplicity, we take M to be an open subset of a compact, n-dimensional Riemannian manifold. Let L be a negative, self-adjoint operator on $L^2(M)$. The following basic hypothesis is made on the integral kernel $p(t,x,y)$ of e^{tL}. With $C,\kappa \in (0,\infty)$ and some $m \geq 2$,

$$|p(t,x,y)| \leq Ct^{-n/m} e^{-\kappa \operatorname{dist}(x,y)^{m/(m-1)}/t^{1/(m-1)}}, \quad 0 < t \leq 1, \tag{B.1}$$

The case $m = 2$ coincides with (4.4). Note that (B.1) implies

$$\|e^{tL} f\|_{L^\infty} \leq C(t^{-n/2m} + 1)\|f\|_{L^2}, \quad t > 0, \tag{B.2}$$

which for $m = 2$ is (1.31). A consequence of Theorem 3.1 of [DOS] is that if $\varphi : \mathbb{R} \to \mathbb{R}$ is bounded and continuous and satisfies the following variant of (1.4),

$$\sup_{\mu > 0} \|\beta \varphi_\mu\|_{C^s(\mathbb{R})} < \infty, \tag{B.3}$$

for some $s > n/2$ (with β and φ_μ as in (1.4)), then $\varphi((-L)^{1/m})$ is of weak type $(1,1)$ and

$$\varphi((-L)^{1/m}) : L^p(M) \longrightarrow L^p(M), \quad \forall\, p \in (1,\infty). \tag{B.4}$$

Such a result includes Theorem 1.1 as a special case. Note that [DOS] avoids an hypothesis like (1.30). As explained there, the standard approach to showing that $\varphi((-L)^{1/m})$ is of weak type (1,1), involving a gradient estimate on the integral kernel of this operator, it replaced by the following result, from [DM]:

Lemma B.1. *Retain the hypotheses stated above. Let $K_t(x,y)$ denote the integral kernel of $\varphi((-L)^{1/m})(I - e^{tL})$. Assume $\varphi : \mathbb{R} \to \mathbb{R}$ is bounded and continuous, and*

$$\sup_{t>0} \sup_{y \in M} \int_{M \setminus B_{t^{1/m}}(y)} |K_t(x,y)| \, dV(x) \leq C_1 < \infty. \tag{B.5}$$

Then $\varphi((-L)^{1/m})$ of of weak type $(1,1)$.

The paper [DOS] mentions [F] as an antecendent to such a result.

In [DOS] there is a result (Theorem 3.2) guaranteeing (B.4) under an hypothesis on φ of the form

$$\sup_{\mu > 1} \|\beta \varphi_\mu\|_{H^{s,q}(\mathbb{R})} < \infty, \tag{B.6}$$

for some $q \in [2, \infty)$.

In §7 of [DOS] the estimate (A.1) is used to show that, if M is a compact Riemannian manifold (without boundary) and $L = \Delta$ $(m = 2)$, then this result appies, with $q = 2$, to yield an alternative proof of the result of [SeS] on boundedness of $\varphi(\sqrt{-L})$. On the other hand, the results of [DOS] as applied to $L = \Delta$ on a compact manifold with boundary, with the Dirichlet or Neumann boundary condition, are less sharp than those of [Xu2] and [Xu3].

References

[CGT] J. Cheeger, M. Gromov, and M. Taylor, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, J. Diff. Geom. 17 (1982), 15–53.

[CW] R. Coifman and G. Weiss, Analyse harmonique non commutative sur certains espaces homogenes, LNM #42, Springer-Verlag, New York, 1971.

[DST] E. B. Davies, B. Simon, and M. Taylor, L^p spectral theory of Kleinian groups, J. Funct. Anal. 78 (1988), 116–136.

[DM] X. Duong and A. McIntosh, Singular integral operators with non-smooth kernels on irregular domains, Math. Iberoam. 15 (1999), 233–265.

[DOS] X. Duong, E. Ouhabaz, and A. Sikora, Plancherel-type estimates and sharp spectral multipliers, J. Funct. Anal. 196 (2002), 443–485.

[F] C. Fefferman, Inequalities for strongly singular integral operators, Acta Math. 124 (1970), 9–36.

[Gr] P. Greiner, An asymptotic expansion for the heat equation, Arch. Rat. Mech. Anal. 41 (1971), 168–218.

[H1] L. Hörmander, Estimates for translation invariant operators on L^p spaces, Acta. Math. (1960), 93–140.

[H2] L. Hörmander, The spectral function of an elliptic operator, Acta. Math. 88 (1968), 341–370.

[KU] P. Kunstmann and M. Uhl, Spectral multipliers of Hörmander type on Hardy and Lebesgue spaces, Preprint, 2012.

[MMV] G. Mauceri, S. Meda, and G. Vallarino, Estimates for functions of the Laplacian on manifolds with bounded geometry, Math. Research Lett. 16 (2009), 861–879.

[SeS] A. Seeger and C. Sogge, On the boundedness of functions of pseudo-differential operators on compact manifolds, Duke Math. J. 59 (1989), 709–736.

[Sog] C. Sogge, Concerning the L^p norms of spectral clusters for second-order elliptic operators on compact manifolds, J. Funct. Anal. 77 (1988), 123–134.

[Sog2] C. Sogge, Eigenfunction and Bochner Riesz estimates on manifolds with boundary, Math. Res. Lett. 9 (2002), 205–216.

[Str] R. Strichartz, A functional calculus for elliptic pseudodifferential operators, Amer. J. Math. 94 (1972), 711–722.

[T1] M. Taylor, Fourier integral operators and harmonic analysis on compact manifolds, Proc. Symp. Pure Math. 35 (pt. 2) (1979), 115–136.

[T2] M. Taylor, Pseudodifferential Operators, Princeton Univ. Press, Princeton, NJ, 1981.

[T3] M. Taylor, L^p estimates on functions of the Laplace operator, Duke Math. J. 58 (1989), 773–793.

[T4] M. Taylor, Functions of the Laplace operator on manifolds with lower Ricci and injectivity bounds, Comm. PDE 34 (2009), 1114–1126.

[Xu1] X. Xu, New proof of Hörmander multiplier theorem on compact manifolds without boundary, Proc. AMS 135 (2007), 1585–1595.

[Xu2] X. Xu, Gradient estimates for the eigenfunctions on compact manifolds with boundary and Hörmander multiplier theorem, Forum Math. 21 (2009), 455–476.

[Xu3] X. Xu, Eigenfunction estimates for Neumann Laplacian and appications to multiplier problems, Proc. AMS 139 (2011), 3583–3599.