Multidimensional Riemann-Hilbert Problems on Domains with Uniformly Rectifiable Interfaces *

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Abstract

This paper analyzes Riemann-Hilbert problems in the following multidimensional setting. Let M be a compact, n-dimensional Riemannian manifold, D a first order elliptic differential operator on M, acting on sections of a vector bundle. Let $\Omega_\pm$ be open subsets of M, with common boundary Σ. Under various conditions on the multiplier function Ψ, we seek functions $u_\pm$ on $\Omega_\pm$, annihilated by D, such that $\Psi u_+ - u_-$ is given in $L^p(\Sigma)$, and $u_\pm$ satisfy appropriate bounds, putting them in certain Hardy spaces.

We pursue this via singular integral operator techniques, when Ω_+ and Ω_- are uniformly rectifiable domains. This is essentially the maximal class of domains on which such singular integral operators act naturally on $L^p(\Sigma)$. The analysis brings in multidimensional Cauchy integrals, Hardy spaces, and Toeplitz operators.

Contents

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1 Introduction

In this paper we tackle Riemann-Hilbert problems in the following setting. Let M be a compact, n-dimensional Riemannian manifold, D a first order elliptic differential operator (between sections of vector bundles $\mathcal{E}_0, \mathcal{E}_1$) on M. Let $\Omega_\pm \subset M$ be open sets, with a common boundary $\Sigma := \partial\Omega_+ = \partial\Omega_-$. The formulation of the Riemann-Hilbert problem can be naturally nuanced as to take into account a desired level of regularity for the boundary data and solutions. For instance, one may seek determining two functions $u_\pm$ defined in $\Omega_\pm$ and taking values in $\mathcal{E}_0 \otimes \mathbb{C}^\ell$ (for some given $\ell \in \mathbb{N}$) such that

$$Du_\pm = 0 \ \text{ in } \ \Omega_\pm, \quad \Psi u_+\Big|_{\Sigma}^{\text{n.t.}} - u_-\Big|_{\Sigma}^{\text{n.t.}} = g \ \text{ on } \ \Sigma, \tag{1.0.1}$$

where $\cdot\big|_{\Sigma}^{\text{n.t.}}$ denotes the nontangential boundary trace on Σ (considered either from within Ω_+ or from within Ω_-, as appropriate; cf. (2.1.7)) and the multiplier function Ψ satisfies certain regularity and nondegeneracy conditions, such as

$$\Psi, \Psi^{-1} \in C^0(\Sigma, M(\ell, \mathbb{C})). \tag{1.0.2}$$

Here and below, $M(\ell, \mathbb{C})$ denotes the space of $\ell \times \ell$ complex matrices. Given $p \in (1, \infty)$, we may take $g \in L^p(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$ in which scenario it is natural to ask that

$$u_\pm \in \mathcal{H}^p(\Omega_\pm, D) \tag{1.0.3}$$

where, generally speaking,

$$\begin{aligned} \mathcal{H}^p(\Omega, D) := \big\{ u \in C^0(\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell) : Du = 0 \ \text{ in } \ \Omega, \ \mathcal{N}u \in L^p(\partial\Omega), \\ \text{and there exists } u\Big|_{\partial\Omega}^{\text{n.t.}} \ \text{a.e. on } \ \partial\Omega \big\}, \end{aligned} \tag{1.0.4}$$

with $\mathcal{N}u$ denoting the nontangential maximal function of u (see (2.1.7)).

Originally such problems were set in the complex plane (which one can compactify to the Riemann sphere) with $D = \overline{\partial}$, and Σ was a smooth curve. Treating such problems played a big role in the development of singular integral operators acting on functions on Σ. It became useful to work on more singular interfaces. For example, treating the Boussinesq equation by the inverse scattering method (cf. [13]) yields a Riemann-Hilbert problem on $\mathbb{C}$ (compactifiable to S^2) where Σ consists of 6 rays, meeting at the origin (and at ∞). Thus $M \setminus \Sigma$ has 6 connected components. One can write $M \setminus \Sigma = \Omega_+ \cup \Omega_-$, where Ω_+ and Ω_- each have 3 connected components.

We desire to work in multidimensional domains, under the hypothesis that Ω_+ and Ω_- are uniformly rectifiable domains. Uniformly rectifiable sets Σ, introduced in [10], form a maximal class of sets for which one has a viable theory of the sort of singular integral operators associated with layer potentials, in the category of $L^p(\Sigma)$.

Let us briefly define a uniformly rectifiable domain as follows. (See [16], [21] for more details.) Let Ω be a relatively compact open subset of an n-dimensional Riemannian manifold M. We assume Ω has finite perimeter. Concretely, if d stands for the exterior derivative operator on M, acting in the sense of distributions, then

$$d\chi_\Omega = \mu \tag{1.0.5}$$

is a finite, vector-valued measure. To avoid pathologies, we assume $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$, where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on M and $\partial_*\Omega$ is the measure-theoretic boundary of Ω. Then (thanks to fundamental results of De Giorgi and Federer) σ, the total variation measure associated to μ, is equal to $(n-1)$-dimensional Hausdorff measure on M restricted to $\partial\Omega$. In this situation, the Radon-Nikodym theorem permits one to write

$$\mu = -\nu\,\sigma, \tag{1.0.6}$$

where the T^*M-valued function ν is viewed as the (geometric measure theoretic) outward-pointing unit conormal to $\partial\Omega$, defined σ-a.e. on $\partial\Omega$. In fact, we shall impose what turns out to be a stronger condition on Ω than mere finite perimeter, namely we shall assume that $\partial\Omega$ is an Ahlfors regular set. This means that there exist $c_0, c_1 \in (0,\infty)$ such that

$$c_0 r^{n-1} \le \sigma(B_r(x)) \le c_1 r^{n-1}, \tag{1.0.7}$$

for all $x \in \partial\Omega$, $0 < r \le \operatorname{diam}\Omega$ (where $B_r(x)$ stands for the geodesic ball of radius r centered at x). Then we call Ω an Ahlfors regular domain provided $\Omega \subset M$ is open, $\partial\Omega$ is an Ahlfors regular set, and $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$.

Such a domain is a UR domain provided $\partial\Omega$ is a uniformly rectifiable set, meaning that it contains big pieces of Lipschitz images, at all length scales and all locations, satisfying uniform Lipschitz bounds. In more detail, there exist $\varepsilon, L \in (0,\infty)$ such that, for each $x \in \partial\Omega$, $R \in (0,1]$, there is a Lipschitz map $\varphi : B_R^{n-1} \to M$ (where B_R^{n-1} is a ball of radius R in $\mathbb{R}^{n-1}$) with Lipschitz constant $\le L$, such that

$$\mathcal{H}^{n-1}(\partial\Omega \cap B_R(x) \cap \varphi(B_R^{n-1})) \ge \varepsilon R^{n-1}. \tag{1.0.8}$$

The setting of UR domains, just described, allows for the following analytical results. Assume $E \in OPS^{-1}(M)$ is a pseudodifferential operator of order -1, with *odd* principal symbol, and integral kernel $E(x,y)$, so

$$Eu(x) = \int_M E(x,y)u(y)\,dV(y), \qquad u \in C_0^\infty(M), \tag{1.0.9}$$

where dV is the volume element on M. Consider the (boundary-to-boundary) "principal value" singular integral operator

$$\begin{aligned} Bf(x) &:= \mathrm{PV}\int_{\partial\Omega} E(x,y)f(y)\,d\sigma(y) \\ &:= \lim_{\varepsilon\to 0^+} \int_{\partial\Omega\setminus B_\varepsilon(x)} E(x,y)f(y)\,d\sigma(y), \qquad x \in \partial\Omega, \end{aligned} \tag{1.0.10}$$

sending functions defined on $\partial\Omega$ into functions defined on $\partial\Omega$. Then B induces a well-defined, linear and bounded mapping

$$B : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \qquad \forall\, p \in (1,\infty). \tag{1.0.11}$$

This was demonstrated in [10] when $M = \mathbb{R}^n$ and E is a convolution operator. Also [10] established associated L^p-estimates on the maximal operator sending f into the function

$$\sup_{0<\varepsilon\leq 1} \left| \int\limits_{\partial\Omega\setminus B_\varepsilon(x)} E(x,y)f(y)\,d\sigma(y) \right|, \qquad x \in \partial\Omega, \tag{1.0.12}$$

in the convolution setting. In [16] this was extended to the variable coefficient setting, and to manifolds. Also [16] studied the (boundary-to-domain) "double layer" potential

$$\mathcal{B}f(x) := \int\limits_{\partial\Omega} E(x,y)f(y)\,d\sigma(y), \qquad x \in \Omega, \tag{1.0.13}$$

complemented estimates on (1.0.12) with the nontangential maximal function estimate

$$\|\mathcal{N}(\mathcal{B}f)\|_{L^p(\partial\Omega)} \leq c_p \|f\|_{L^p(\partial\Omega)}, \qquad 1 < p < \infty, \tag{1.0.14}$$

and established the jump-formula

$$\mathcal{B}f\Big|_{\partial\Omega}^{\rm n.t.}(x) = \frac{1}{2i}\sigma_E(x,\nu(x))f(x) + Bf(x), \qquad \sigma\text{-a.e. } x \in \partial\Omega, \tag{1.0.15}$$

for each $f \in L^p(\partial\Omega)$, $1 < p < \infty$, where $\sigma_E(x,\xi)$ is the principal symbol of E and B is as in (1.0.10)–(1.0.11).

Here we apply certain layer potentials to the study of spaces (1.0.4) of solutions to $Du = 0$ in Ω, when Ω is a UR domain and D is a first-order elliptic differential operator on M, acting between sections of vector bundles $\mathcal{E}_j$, $j = 0, 1$. If D is invertible, say as a mapping

$$D : H^{1,2}(M,\mathcal{E}_0) \longrightarrow L^2(M,\mathcal{E}_1) \tag{1.0.16}$$

where, generally speaking, $H^{s,p}(M)$ stands for the L^p-based Sobolev space of fractional smoothness s, we can take $E = D^{-1}$ in (1.0.9). However, in many interesting cases, D will not be invertible, though the ellipticity of D implies the existence of a parametrix which serves as an inverse modulo compact operators for D in (1.0.16). Hence, while D may fail to be invertible, it is always a Fredholm operator in the context of (1.0.16), albeit it often has nonzero index. One the other hand, under mild conditions on D, one can construct $E \in OPS^{-1}(M)$ such that, for some neighborhood $\mathcal{O}$ of $\overline{\Omega}$,

$$\operatorname{supp} u \subset \mathcal{O} \Longrightarrow EDu = u. \tag{1.0.17}$$

Indeed, this is the case when D and D^* have the unique continuation property (henceforth abbreviated UCP). Specifically, D has UCP provided if $u \in H^{1,2}(M)$ is such that $Du = 0$ on M and u vanishes on some nonempty open subset of M then $u = 0$ everywhere on M. In the scenario when D and D^* have UCP, we can take $a \in C_0^\infty(M \setminus \overline{\Omega})$ satisfying $a \geq 0$ everywhere and $a > 0$ in a nonempty open set, which then implies that the auxiliary operator

$$\mathcal{D} := \begin{pmatrix} ia & D^* \\ D & ia \end{pmatrix} \tag{1.0.18}$$

is invertible as a mapping

$$\mathcal{D} : H^{1,2}(M, \mathcal{E}_0 \oplus \mathcal{E}_1) \longrightarrow L^2(M, \mathcal{E}_0 \oplus \mathcal{E}_1) \tag{1.0.19}$$

with inverse $\mathcal{D}^{-1} \in OPS^{-1}(M)$. Invertibility is a consequence of the following facts. First, since $\mathcal{D}$ is obtained as a lower order perturbation of an elliptic, self-adjoint operator, it is Fredholm of index zero. Second, elements in $\operatorname{Ker} \mathcal{D}$ vanish in the set where $a > 0$ given that for every $u \in H^{1,2}(M, \mathcal{E}_0 \oplus \mathcal{E}_1)$ we have

$$\operatorname{Im} (\mathcal{D}u, u)_{L^2(M)} = \int_M a\, |u|^2 \, dV. \tag{1.0.20}$$

In light of the structure of $\mathcal{D}$ and UCP for D and D^*, any function $u \in \operatorname{Ker} \mathcal{D}$ then necessarily vanishes on all of M (this is actually true even in the case when the coefficients of D have only a limited amount of smoothness; see [21, Corollary A.1.4]). From such $\mathcal{D}^{-1}$, we obtain

$$\mathcal{D}^{-1} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \Longrightarrow E_{12} D = I - iE_{11} a, \tag{1.0.21}$$

giving (1.0.17) with $E := E_{12}$. Similarly, $DE_{12} = I - iaE_{22}$, so

$$\operatorname{supp} u \subseteq M \setminus \Omega \Longrightarrow DEu = 0 \ \text{ on } \ \Omega. \tag{1.0.22}$$

See [21] for more details.

For applicability to the Riemann-Hilbert problem (1.0.1), it is convenient to pass from $\mathcal{B}$ and B to the "Cauchy integral operators" $\mathcal{C}$ and C, obtained by applying $\mathcal{B}$ and B to $i\sigma_D(\cdot, \nu(\cdot))f$ in place of just f. In §2 we analyze this and show that

$$\mathcal{C}f \Big|_{\partial\Omega}^{\text{n.t.}} = \frac{1}{2} f + Cf, \tag{1.0.23}$$

in the sense of nontangential a.e. convergence, and (with I denoting the identity)

$$\mathcal{P} := \frac{1}{2} I + C \Longrightarrow \mathcal{P}^2 = \mathcal{P}. \tag{1.0.24}$$

The operator $\mathcal{P}$ is a Calderón projector. It yields a projection of $L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$ onto the space $\mathcal{H}^p(\partial\Omega, D)$ of nontangential boundary values of functions from $\mathcal{H}^p(\Omega, D)$. Given also a multiplier function $\Phi \in C^0(\partial\Omega, M(\ell, \mathbb{C}))$, then we can define the Toeplitz operator

$$T_\Phi := \mathcal{P}\Phi\mathcal{P} + (I - \mathcal{P}). \tag{1.0.25}$$

In §2 we provide results on such Cauchy integrals, Calderón projectors, and Toeplitz operators. We show that such Toeplitz operators are Fredholm if Φ takes values in $\mathrm{GL}(\ell, \mathbb{C})$ (the general linear group, consisting of invertible elements from $M(\ell, \mathbb{C})$), and discuss properties of the index. We also consider the more general case when the continuity assumption on the multiplier function Φ is relaxed to

$$\Phi \in L^\infty \cap \operatorname{vmo}(\partial\Omega, \mathrm{GL}(\ell, \mathbb{C})), \tag{1.0.26}$$

treating simultaneously extensions to higher dimensions and to domains with rough boundary of index results of Brezis-Nirenberg [7]. Results of §2, which are distilled from the authors' paper [21], provide background for the analysis of the Riemann-Hilbert problem.

Here is a first indication of the relevance of Toeplitz operators to the Riemann-Hilbert problem. Assume D is invertible. Then one can take $f \in L^p(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$ and define $\mathcal{B}f(x)$ (and hence $\mathcal{C}f(x)$) for $x \in M \setminus \Sigma = \Omega_+ \cup \Omega_-$, and then

$$u_\pm = \mathcal{C}f \ \text{ in } \ \Omega_\pm \tag{1.0.27}$$

satisfies

$$u_+\big|_\Sigma^{\text{n.t.}} = \mathcal{P}f, \qquad u_-\big|_\Sigma^{\text{n.t.}} = (\mathcal{P} - I)f, \tag{1.0.28}$$

so the transmission condition (1.0.1) becomes

$$\widetilde{T}_\Psi f := \Psi \mathcal{P} f + (I - \mathcal{P})f = g, \tag{1.0.29}$$

an integral equation which turns out to differ from $T_\Psi f = g$ by the application of a compact operator.

If D is not invertible, which, as mentioned above, is frequently the case, a more elaborate construction is needed. We start this in §3. We take a pair of non-negative functions $a^\pm$, supported in $M \setminus \overline{\Omega}_\pm$, to give, in place of (1.0.23), two Cauchy-type operators

$$\mathcal{C}_\pm : L^p(\Sigma) \longrightarrow \mathcal{H}^p(\Omega_\pm, D), \tag{1.0.30}$$

satisfying

$$\mathcal{C}_\pm f\big|_\Sigma^{\text{n.t.}} = \mathcal{P}_\pm f, \tag{1.0.31}$$

for projections $\mathcal{P}_\pm$ on $L^p(\Sigma, \mathcal{E}_0)$. The range $\mathcal{H}^p_\pm(\Sigma, D)$ of $\mathcal{P}_\pm$ is equal to the image of $\mathcal{H}^p(\Omega_\pm, D)$ under the nontangential boundary trace map $\tau(u) := u\big|_\Sigma^{\text{n.t.}}$. When D is invertible, the construction indicated in (1.0.23)–(1.0.28) yields $\mathcal{P}_- = I - \mathcal{P}_+$, and

$$L^p(\Sigma, \mathcal{E}_0) = \mathcal{H}^p_+(\Sigma, D) \oplus \mathcal{H}^p_-(\Sigma, D). \tag{1.0.32}$$

In general, this is not the case. The modification of (1.0.32) that results is studied in §3. This involves an analysis of the operators

$$\begin{aligned} \mathcal{J}_p &: \mathcal{H}^p_+(\Sigma, D) \oplus \mathcal{H}^p_-(\Sigma, D) \longrightarrow L^p(\Sigma, \mathcal{E}_0), \\ \Gamma_p &: L^p(\Sigma, \mathcal{E}_0) \longrightarrow \mathcal{H}^p(\Sigma, D) \oplus \mathcal{H}^p_-(\Sigma, D), \end{aligned} \tag{1.0.33}$$

given by

$$\mathcal{J}_p(f_+, f_-) := f_+ - f_- \quad \text{and} \quad \Gamma_p f := (\mathcal{P}_+ f, -\mathcal{P}_- f). \tag{1.0.34}$$

We show that $\mathcal{J}_p$ and Γ_p are Fredholm inverses of each other, and that

$$\operatorname{Index} \mathcal{J}_p = -\operatorname{Index} \Gamma_p = \operatorname{Index} D, \tag{1.0.35}$$

where D is viewed as a global mapping on M, say $D : H^{1,2}(M, \mathcal{E}_0) \to L^2(M, \mathcal{E}_1)$. A related operator is

$$\begin{aligned} &\mathcal{A}_p : L^p(\Sigma, \mathcal{E}_0) \longrightarrow \mathcal{H}^p(\Omega_+, D) \oplus \mathcal{H}^p(\Omega_-, D), \\ &\mathcal{A}_p f := (\mathcal{C}_+ f, \mathcal{C}_- f), \qquad \forall\, f \in L^p(\Sigma, \mathcal{E}_0), \end{aligned} \tag{1.0.36}$$

which we show is Fredholm, with $\operatorname{Index} \mathcal{A}_p = -\operatorname{Index} D$. If D is invertible, then $\mathcal{J}_p$ and Γ_p are actually genuine inverses of each other, and hence are isomorphisms in (1.0.33), and then $\mathcal{A}_p$ is an isomorphism in (1.0.36).

We also study analogues of (1.0.33)–(1.0.36), involving $L^p_1(\Sigma,\mathcal{E}_0)$, the L^p-based Sobolev spaces of order one on Σ, and the correspondingly more regular Hardy spaces

$$\mathcal{H}^{1,p}(\Omega_\pm,D) := \Big\{u \in C^1(\Omega_\pm,\mathcal{E}_0) : Du = 0 \text{ in } \Omega_\pm,\ \mathcal{N}(u),\mathcal{N}(\nabla u) \in L^p(\Sigma),\ \text{and} \\ \text{there exist } u\big|^{\text{n.t.}}_{\Sigma},\ (\nabla u)\big|^{\text{n.t.}}_{\Sigma}\ \sigma\text{-a.e. on } \Sigma\Big\}, \tag{1.0.37}$$

as well as the associated spaces $\mathcal{H}^{1,p}_\pm(\Sigma,D)$ of their nontangential boundary traces, which also turn out to be the images of $L^p_1(\Sigma,\mathcal{E}_0)$ under $\mathcal{P}_\pm$.

In §4 we apply results of §§2–3 to the Riemann-Hilbert problem. In operator terms, the problem (1.0.1) is the study of

$$R_\Psi : \mathcal{H}^p(\Omega_+,D) \oplus \mathcal{H}^p(\Omega_-,D) \longrightarrow L^p(\Sigma,\mathcal{E}_0 \otimes \mathbb{C}^\ell), \\ R_\Psi(u,v) := \Psi u\big|^{\text{n.t.}}_{\Sigma} - v\big|^{\text{n.t.}}_{\Sigma}, \qquad \forall\,(u,v) \in \mathcal{H}^p(\Omega_+,D) \oplus \mathcal{H}^p(\Omega_-,D). \tag{1.0.38}$$

We show that

$$R_\Psi\,\widetilde{\mathcal{A}}_p = \widetilde{T}_\Psi, \tag{1.0.39}$$

where $\widetilde{T}_\Psi$ is as in (1.0.29) and $\widetilde{\mathcal{A}}_p$, given by $\widetilde{\mathcal{A}}_p f = (\mathcal{C}_+ f, -\mathcal{C}_- f)$, is a slight variant of $\mathcal{A}_p$ in (1.0.36), also Fredholm with index equal to $-\operatorname{Index} D$, when acting on sections of $\mathcal{E}_0$. For the action of $\widetilde{\mathcal{A}}_p$ on sections of $\mathcal{E}_0 \otimes \mathbb{C}^\ell$, we multiply the index by ℓ. If Ψ satisfies (1.0.2), or more generally (1.0.26), then $\widetilde{T}_\Psi$ is Fredholm, with the same index as T_Ψ, and we have

$$\operatorname{Index} R_\Psi = \operatorname{Index} T_\Psi + \ell \cdot \operatorname{Index} D. \tag{1.0.40}$$

If D is invertible, then $\widetilde{\mathcal{A}}_p$ is an isomorphism, and one gets the tight relation

$$\operatorname{Ker} R_\Psi \approx \operatorname{Ker} \widetilde{T}_\Psi, \qquad \operatorname{Range} R_\Psi = \operatorname{Range} \widetilde{T}_\Psi. \tag{1.0.41}$$

In §4.2 we obtain such results in the L^p-Sobolev space context,

$$R_\Psi : \mathcal{H}^{1,p}(\Omega_+,D) \oplus \mathcal{H}^{1,p}(\Omega_-,D) \longrightarrow L^p_1(\Sigma,\mathcal{E}_0 \otimes \mathbb{C}^\ell), \tag{1.0.42}$$

given

$$\Psi \in L^q_1(\Sigma,\mathrm{GL}(\ell,\mathbb{C})), \quad q \in (n-1,\infty),\ p \in (1,q], \tag{1.0.43}$$

provided $\Omega_\pm$ satisfy a two-sided John condition (discussed in §A.2).

In §4.3 we show how to apply the results of §§4.1–4.2 to Riemann-Hilbert problems on UR domains in the complex plane $\mathbb{C}$, by compactifying $\mathbb{C}$ and passing to a Riemann-Hilbert problem on the Riemann sphere S^2. In this setting, it is natural to treat the unknowns as sections of a holomorphic line bundle over S^2. In §4.4 we consider Riemann-Hilbert problems on more general compact Riemann surfaces, involving sections of holomorphic vector bundles.

We end with some appendices, providing useful background material. In §A.1 we give definitions and basic properties of L^p-Sobolev spaces $L^p_1(\partial\Omega)$, in the setting that Ω is an Ahlfors regular domain. In §A.2 we discuss a subclass of domains, those satisfying a two-sided John condition, for which one has additional results on $L^p_1(\partial\Omega)$. In §A.3 we discuss a refined divergence theorem, of crucial value for several technical analytical results. In §A.4 we present material on holomorphic line bundles and vector bundles over a compact Riemann surface, the associated $\overline{\partial}$-operator, $\overline{\partial}_L$, and a formula for its index, the celebrated Riemann-Roch formula.

2 Cauchy integrals, Calderón projectors, and Toeplitz operators

Here we present basic ingredients for the analysis of the Riemann-Hilbert problem, starting with Cauchy integrals. The key results are distilled from [21], to which we refer for detailed proofs.

In §2.1 we pass from $\mathcal{B}$ and B, as in (1.0.13)–(1.0.15), to the multidimensional Cauchy integral operators $\mathcal{C}$ and C, obtained by applying $\mathcal{B}$ and B to $i\sigma_D(y, \nu(y))f(y)$; see (2.1.10) and (2.1.13). We approach these operators from two different perspectives, first via a reproducing formula, and then via the goal to obtain from (1.0.15) an operator such that the first term on the right side of (1.0.15) gets replaced by $(1/2)f(x)$. Comparison of these two approaches yields the basic result that

$$\mathcal{P} := \frac{1}{2}I + C \Longrightarrow \mathcal{P}^2 = \mathcal{P}. \tag{2.0.1}$$

The operator $\mathcal{P}$ is a Calderón projector. It is a projection of $L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$ onto the space $\mathcal{H}^p(\partial\Omega, D)$ of nontangential boundary values of functions from $\mathcal{H}^p(\Omega, D)$, defined as in (1.0.4). Using $\mathcal{P}$ we define Toeplitz operators

$$T_\Phi := \mathcal{P}\Phi\mathcal{P} + (I - \mathcal{P}). \tag{2.0.2}$$

In §2.2 we analyze (2.0.2) for $\Phi \in C^0(\partial\Omega, M(\ell, \mathbb{C}))$, obtain compactness of $T_{\Phi\Psi} - T_\Phi T_\Psi$ on $L^p(\partial\Omega)$, $1 < p < \infty$, whenever $\Psi \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$, and deduce that (2.0.2) is Fredholm on $L^p(\partial\Omega)$, $1 < p < \infty$, when $\Phi \in C^0(\partial\Omega, \mathrm{GL}(\ell, \mathbb{C}))$. We note that the index of T_Φ on $L^p(\partial\Omega)$ is independent of p and that $\iota(\Phi) = \operatorname{Index} T_\Phi$ produces a group homomorphism

$$\iota : [\partial\Omega; \mathrm{GL}(\ell, \mathbb{C})] \longrightarrow \mathbb{Z}, \tag{2.0.3}$$

where $[\partial\Omega; \mathrm{GL}(\ell, \mathbb{C})]$ is the group of homotopy classes of continuous maps from $\partial\Omega$ to $\mathrm{GL}(\ell, \mathbb{C})$.

In §2.3 we extend the scope of our analysis by relaxing the continuity assumption on the multiplier function Φ to

$$\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, M(\ell, \mathbb{C})). \tag{2.0.4}$$

If also Φ^{-1} satisfies (2.0.4), then T_Φ is Fredholm on $L^p(\partial\Omega)$ for $1 < p < \infty$. The appropriate homotopy invariance in this setting is more subtle than that in §2.2. We discuss a result that extends the scope of some work of [7], both to higher dimensions and to rough boundaries.

In §2.4 we extend the scope in another direction, allowing Φ to be a section of $\operatorname{End} \mathcal{C}$, when $\mathcal{C} \to M$ is a vector bundle, yielding "twisted" Toeplitz operators.

In §2.5 we introduce cobordism invariance as a useful tool to apply to the problem of computing the index of a Toeplitz operator. This is applied in §2.6, in conjunction with some topological results of Bott and index results of [33], [6], and [2], to compute the index for a certain interesting class of Toeplitz operators. In §2.7 we record results on Toeplitz operators acting on $L^p_1(\partial\Omega)$, the L^p-based Sobolev spaces of order one on Σ. See §A.1 for a description of these spaces.

2.1 Reproducing formulas, Cauchy integrals, and Calderón projectors

We start with a sequence of reproducing formulas, valid for progressively less smooth functions u and for progressively less rough domains Ω. To begin with, assume

$$u \in C^0(M, \mathcal{E}_0), \quad Du \in L^1(M, \mathcal{E}_1). \tag{2.1.1}$$

We let $f \in \mathrm{Lip}(M)$ be scalar and note the Leibniz type formula

$$D(fu) = f Du + (D_0 f)u, \quad D_0 f(x) = \frac{1}{i}\sigma_D(x) df(x), \tag{2.1.2}$$

where the principal symbol of the operator D is written as $\sigma_D(x,\xi) = \sigma_D(x)\xi$, linear in $\xi \in T_x^* M$. Assume $\operatorname{supp} f \subset \mathcal{O}$, with $\mathcal{O}$ as in (1.0.17). Then

$$fu(x) = E\big((D_0 f)u\big) + E(f\, Du). \tag{2.1.3}$$

Now, assume $\Omega \subset M$ is a finite perimeter domain, and replace f in (2.1.3) by a sequence $f_k \in \mathrm{Lip}(M)$, supported in $\mathcal{O}$, and satisfying

$$\begin{aligned} f_k &\longrightarrow \chi_\Omega, \quad \text{boundedly and a.e.,} \\ df_k &\longrightarrow d\chi_\Omega = \mu = -\nu\sigma, \quad \text{weak}^* \text{ as measures on } M. \end{aligned} \tag{2.1.4}$$

Passing to the limit then gives

$$\begin{aligned} u(x) = i \int_{\partial\Omega} & E(x,y)\, \sigma_D(y, \nu(y)) u(y)\, d\sigma(y) \\ & + \int_{\Omega} E(x,y) Du(y)\, dV(y), \quad x \in \Omega. \end{aligned} \tag{2.1.5}$$

This is our basic reproducing formula. Note that the second integral vanishes if $Du = 0$ on Ω. At this point we have (2.1.5) for Ω with finite perimeter, provided u satisfies (2.1.1). We will need this formula for much rougher functions u.

The following is established in §2.2 of [21], extending a Green formula given in §2.3 of [16]. To state it, we bring in the spaces

$$\begin{aligned} \mathfrak{L}^p(\Omega) := \big\{ u \in C^0(\Omega, \mathcal{E}_0) : \mathcal{N}u \in L^p(\partial\Omega), \text{ and} \\ \text{there exists } u\big|_{\partial\Omega}^{\text{n.t.}} \ \sigma\text{-a.e. on } \partial\Omega \big\}. \end{aligned} \tag{2.1.6}$$

Here and elsewhere, the nontangential maximal function and nontangential trace of u are defined for $x \in \partial\Omega$ as

$$(\mathcal{N}u)(x) := \sup_{y \in \Gamma(x)} |u(y)|, \qquad \Big(u\big|_{\partial\Omega}^{\text{n.t.}}\Big)(x) := \lim_{\Gamma(x) \ni z \to x} u(z). \tag{2.1.7}$$

where, for some fixed, sufficiently large number $c_o \in (1, \infty)$, the nontangential approach regions are given by

$$\Gamma(x) := \big\{ y \in \Omega : \operatorname{dist}(x,y) < c_o \operatorname{dist}(y, \partial\Omega) \big\}, \qquad x \in \partial\Omega. \tag{2.1.8}$$

Proposition 2.1.1 *Assume $\Omega \subset M$ is an Ahlfors regular domain and that, for some $p > 1$,*

$$u \in \mathfrak{L}^p(\Omega), \quad \text{and } Du \in L^1(\Omega, \mathcal{E}_1). \tag{2.1.9}$$

Then (2.1.5) *holds* (*with $u\big|_{\partial\Omega}^{\text{n.t.}}$ replacing u in the boundary integral*).

A sharper extension of Green's formula, allowing $p = 1$ in (2.1.9), is discussed in §A.3.

We now specialize to the case that $\Omega \subset M$ is a UR domain. As stated in §1, the layer potential operator $\mathcal{B}$ defined by (1.0.13) satisfies (1.0.14)–(1.0.15), with B as in (1.0.10)–(1.0.11). Given the nontangential limit result (1.0.15), it follows that if

$$\mathcal{C}f(x) := i \int_{\partial\Omega} E(x,y)\sigma_D(y,\nu(y))f(y)\,d\sigma(y), \qquad x \in \Omega, \tag{2.1.10}$$

then

$$\|\mathcal{N}(\mathcal{C}f)\|_{L^p(\partial\Omega)} \le c_p \|f\|_{L^p(\partial\Omega)}, \qquad 1 < p < \infty, \tag{2.1.11}$$

and, since $\sigma_E(x,\xi)\sigma_D(x,\xi) = I$, we have nontangential σ-a.e. convergence

$$\mathcal{C}f\Big|_{\partial\Omega}^{\text{n.t.}}(x) = \frac{1}{2}f(x) + Cf(x), \qquad x \in \partial\Omega, \tag{2.1.12}$$

where

$$Cf(x) := i\ \text{PV}\int_{\partial\Omega} E(x,y)\sigma_D(y,\nu(y))f(y)\,d\sigma(y), \qquad x \in \partial\Omega. \tag{2.1.13}$$

It follows that if $1 < p < \infty$ and $f \in L^p(\partial\Omega, \mathcal{E}_0)$, then

$$u = \mathcal{C}f \Longrightarrow u \in \mathcal{H}^p(\Omega, D), \tag{2.1.14}$$

defined in (1.0.4) (for now, with $\ell = 1$). Also (2.1.5) applies, with $Du = 0$ on Ω, hence

$$u = \mathcal{C}f \Longrightarrow u = \mathcal{C}\big(u\big|_{\partial\Omega}^{\text{n.t.}}\big). \tag{2.1.15}$$

Comparing (2.1.12), we deduce that

$$\mathcal{P} := \frac{1}{2}I + C \Longrightarrow \mathcal{P}^2 = \mathcal{P}. \tag{2.1.16}$$

By (1.0.11), we have

$$\mathcal{P} : L^p(\partial\Omega, \mathcal{E}_0) \longrightarrow L^p(\partial\Omega, \mathcal{E}_0), \quad 1 < p < \infty. \tag{2.1.17}$$

The integral (2.1.10) is a multidimensional generalization of the familiar Cauchy integral, obtained when $M = \mathbb{C}$ and $D = \overline{\partial}$.

When $\partial\Omega$ is smooth, $\mathcal{P}$ is a classical Calderón-type projector. By the definition of $\mathcal{H}^p(\Omega, D)$ in (1.0.4), there is a bounded trace map

$$\tau : \mathcal{H}^p(\Omega, D) \longrightarrow L^p(\partial\Omega, \mathcal{E}_0), \qquad \tau(u) := u\big|_{\partial\Omega}^{\text{n.t.}}, \tag{2.1.18}$$

and Proposition 2.1.1 together with (2.1.10)–(2.1.15) imply that, when Ω is a UR domain,

$$\tau : \mathcal{H}^p(\Omega, D) \longrightarrow \mathcal{H}^p(\partial\Omega, D), \tag{2.1.19}$$

with

$$\mathcal{H}^p(\partial\Omega, D) = \mathcal{P}L^p(\partial\Omega, \mathcal{E}_0). \tag{2.1.20}$$

It follows from Proposition 2.1.1 that τ in (2.1.18) is injective, hence τ in (2.1.19) is an isomorphism.

In [21] the authors also treat a "Calderón-Szegö projector" S, defined initially on $L^2(\partial\Omega, \mathcal{E}_0)$ as the orthogonal projection onto $\mathcal{H}^2(\partial\Omega, D)$. Extensions of S to $L^p(\partial\Omega, \mathcal{E}_0)$, for a range of p, and relations with $\mathcal{P}$, are discussed there. Space considerations motivate us to pass over this topic here, so we point the reader to §3.2 of [21].

Remark. It is natural to consider the following variant of (1.0.4):

$$\widetilde{\mathcal{H}}^p(\Omega, D) := \big\{u \in C^0(\Omega, \mathcal{E}_0) : Du = 0 \text{ in } \Omega,\ \mathcal{N}u \in L^p(\partial\Omega)\big\}, \tag{2.1.21}$$

dropping the hypothesis that the nontangential trace $u\big|_{\partial\Omega}^{\text{n.t.}}$ exists. The assertion that

$$\widetilde{\mathcal{H}}^p(\Omega, D) = \mathcal{H}^p(\Omega, D) \tag{2.1.22}$$

is known as a Fatou theorem. Such a result is classical when Ω is smoothly bounded. In [22] it is shown that (2.1.22) holds when Ω is a Lipschitz domain, and also when Ω is a regular SKT domain (a class of domains defined in [16, §4.1]).

2.2 Toeplitz operators – Fredholmness

Here, Ω will be a UR domain.

The maps $\mathcal{C}, C$, and $\mathcal{P}$, defined in (2.1.10), (2.1.13), and (2.1.16), extend naturally from acting on sections of $\mathcal{E}_0$ to acting on sections of $\mathcal{E}_0 \otimes \mathbb{C}^\ell$, giving rise to projections

$$\mathcal{P} : L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell) \longrightarrow L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell), \quad 1 < p < \infty, \tag{2.2.1}$$

and we have (2.1.19)–(2.1.20), with $\mathcal{H}^p(\Omega, D)$ as in (1.0.4) for general $\ell \geq 1$.

For notational simplicity, we will henceforth typically denote $L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$ by $L^p(\partial\Omega)$, unless we need to explicitly specify the relevant vector bundle.

If $\Phi \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$, then multiplication by Φ also naturally acts on sections of $\mathcal{E}_0 \otimes \mathbb{C}^\ell$, and we have the following Toeplitz operator:

$$T_\Phi := \mathcal{P}\Phi\mathcal{P} + (I - \mathcal{P}). \tag{2.2.2}$$

If also $\Psi \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$, then

$$T_\Phi T_\Psi - T_{\Phi\Psi} = \mathcal{P}\Phi[\mathcal{P}, \Psi]\mathcal{P} \tag{2.2.3}$$

which is then compact on $L^p(\partial\Omega)$ as long as the commutator $[\mathcal{P}, \Psi]$ is. Note that

$$[\mathcal{P}, \Psi]f(x) = [C, \Psi]f(x) = i\,\mathrm{PV}\int_{\partial\Omega} E(x, y)\{\Psi(y) - \Psi(x)\}g(y)\, d\sigma(y), \tag{2.2.4}$$

where $g := \sigma_D(\cdot, \nu(\cdot))f$. If the multiplier function Ψ is Hölder continuous, say

$$\Psi \in C^\alpha(\partial\Omega, M(\ell, \mathbb{C})), \quad \alpha > 0, \tag{2.2.5}$$

then the integral in (2.2.4) is weakly singular, and compactness on $L^p(\partial\Omega)$ for $1 < p < \infty$ is elementary. If mere continuity is assumed,

$$\Psi \in C^0(\partial\Omega, M(\ell, \mathbb{C})), \tag{2.2.6}$$

then we can take a sequence $\Psi_k \in C^\alpha(\partial\Omega, M(\ell, \mathbb{C}))$ such that $\Psi_k \to \Psi$ uniformly, and deduce that $[\mathcal{P}, \Psi]$ is compact, hence

$$\begin{aligned} &\Psi \in L^\infty(\partial\Omega, M(\ell, \mathbb{C})),\ \Phi \in C^0(\partial\Omega, M(\ell, \mathbb{C})) \\ &\Longrightarrow T_\Phi T_\Psi - T_{\Phi\Psi} \text{ compact on } L^p(\partial\Omega),\ 1 < p < \infty. \end{aligned} \tag{2.2.7}$$

From here we readily get the following result.

Proposition 2.2.1 *Let Ω be a* UR *domain in M, and assume*

$$\Phi : \partial\Omega \longrightarrow \mathrm{GL}(\ell, \mathbb{C}) \tag{2.2.8}$$

is continuous. Then $T_{\Phi^{-1}}T_\Phi - I$ and $T_\Phi T_{\Phi^{-1}} - I$ are compact on $L^p(\partial\Omega)$ for all $p \in (1, \infty)$, so

$$T_\Phi : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \textit{is Fredholm}, \quad \forall\, p \in (1, \infty). \tag{2.2.9}$$

In §4.1 of [21] it is shown that

$$\iota(\Phi) = \iota(\Phi; D) := \mathrm{Index}\, T_\Phi \ \text{ on } \ L^p(\partial\Omega) \tag{2.2.10}$$

is independent of p. In fact, $\mathrm{Ker}\, T_\Phi$ on $L^p(\partial\Omega)$ and $\mathrm{Ker}\, T_\Phi^*$ on $L^{p'}(\partial\Omega)$ are both independent of p. (This can be interpreted as a regularity result.) From (2.2.7), we have that, if also $\Psi : \partial\Omega \to \mathrm{GL}(\ell, \mathbb{C})$ is continuous, then

$$\iota(\Phi\Psi) = \iota(\Phi) + \iota(\Psi). \tag{2.2.11}$$

Here and below, $\mathrm{GL}(\ell, \mathbb{C})$ denotes the group of invertible $\ell \times \ell$ complex matrices.

Note that if Φ_t is a continuous family of elements of $C^0(\partial\Omega, \mathrm{GL}(\ell, \mathbb{C}))$, then T_{Φ_t} is a norm continuous family of Fredholm operators, so has a constant index. That is, Index T_Φ depends only on the homotopy class of Φ in $[\partial\Omega; \mathrm{GL}(\ell, \mathbb{C})]$, the group of homotopy classes of continuous maps $\partial\Omega \to \mathrm{GL}(\ell, \mathbb{C})$. By (2.2.11), we obtain a group homomorphism

$$\iota : [\partial\Omega; \mathrm{GL}(\ell, \mathbb{C})] \longrightarrow \mathbb{Z}. \tag{2.2.12}$$

We return to this in §2.6.

2.3 Toeplitz operators with coefficients in $L^\infty \cap \mathrm{vmo}(\partial\Omega)$

We begin by defining some relevant function spaces. We take Ω to be a relatively compact UR domain, with boundary $\partial\Omega$, and define $\mathrm{bmo}(\partial\Omega)$ and $\mathrm{vmo}(\partial\Omega)$. These definitions extend to a broader class of measured metric spaces; cf. [16, §2.4]. We have the BMO-seminorm

$$\|\Phi\|_{\mathrm{BMO}} := \sup_B \frac{1}{\sigma(B)} \|\Phi - \Phi_B\|_{L^1(B)}, \tag{2.3.1}$$

where B runs over all "surface" balls in $\partial\Omega$ and

$$\Phi_B := \frac{1}{\sigma(B)} \int_B \Phi \, d\sigma. \tag{2.3.2}$$

This is only a seminorm since $\|\Phi\|_{\rm BMO} = 0$ whenever Φ is constant. We use the norm

$$\|\Phi\|_{\rm bmo} := \|\Phi\|_{\rm BMO} + \|\Phi\|_{L^1(\partial\Omega)}. \tag{2.3.3}$$

The space $\mathrm{vmo}(\partial\Omega)$ is the closure in bmo-norm of $C^0(\partial\Omega)$.

Here we study Toeplitz operators T_Φ associated with a multiplier function Φ satisfying

$$\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega). \tag{2.3.4}$$

The following is proved in [28, p. 81], for scalar functions. It extends readily to functions with values in $\mathrm{End}(\mathbb{C}^\ell)$.

Lemma 2.3.1 *$L^\infty \cap \mathrm{vmo}(\partial\Omega)$ is a closed linear subspace of $L^\infty(\partial\Omega)$, closed under products, hence a closed $*$-subalgebra of the C^*-algebra $L^\infty(\partial\Omega)$.*

Generally, if $\mathcal{A}$ is a C^*-algebra with unit 1 and $\mathcal{B}$ a C^*-subalgebra containing 1, then an element $\varphi \in \mathcal{B}$ is invertible in $\mathcal{B}$ if and only if it is invertible in $\mathcal{A}$. This has the following consequence:

$$\begin{aligned} &\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{GL}(\ell,\mathbb{C})) \ \text{ and } \ \Phi^{-1} \in L^\infty(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell) \\ &\Longrightarrow \Phi^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell). \end{aligned} \tag{2.3.5}$$

When Φ satisfies the conditions in the first line of (2.3.5), we say

$$\Phi \in L^\infty_{\rm inv} \cap \mathrm{vmo}(\partial\Omega, \mathrm{GL}(\ell,\mathbb{C})). \tag{2.3.6}$$

In particular, if $U(\ell)$ denotes the group of unitary $\ell \times \ell$ complex matrices, we have

$$L^\infty \cap \mathrm{vmo}(\partial\Omega, U(\ell)) \subset L^\infty_{\rm inv} \cap \mathrm{vmo}(\partial\Omega, \mathrm{GL}(\ell,\mathbb{C})). \tag{2.3.7}$$

The following extends the compactness result on $[\mathcal{P}, \Psi]$ in §2.2.

Lemma 2.3.2 *If $\Psi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell)$, then the commutator*

$$[\mathcal{P}, \Psi] : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \text{ is compact}, \ \forall\, p \in (1,\infty). \tag{2.3.8}$$

Proof. The assertion is that (2.2.4) is compact on $L^p(\partial\Omega)$ for such Ψ. This is established in [16, §4.2], building on a fundamental commutator estimate of [9]. □

This leads to the following extension of Proposition 2.2.1.

Theorem 2.3.3 *If Ω is a* UR *domain and the multiplier function Φ satisfies* (2.3.6)*, then $T_{\Phi^{-1}}T_\Phi - I$ and $T_\Phi T_{\Phi^{-1}} - I$ are compact on $L^p(\partial\Omega)$ for all $p \in (1,\infty)$, so the operator*

$$T_\Phi : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \text{ is Fredholm}, \ \forall\, p \in (1,\infty). \tag{2.3.9}$$

Again the index $\iota(\Phi) = \iota(\Phi; D) = \ \mathrm{Index}\, T_\Phi$ is independent of $p \in (1,\infty)$. Also, we have

$$\iota(\Phi\Psi) = \iota(\Phi) + \iota(\Psi), \tag{2.3.10}$$

when $\Phi, \Psi \in L^\infty_{\rm inv} \cap \mathrm{vmo}(\partial\Omega, \mathrm{GL}(\ell,\mathbb{C}))$.

The appropriate homotopy invariance is a bit more subtle in this setting than in §2.2. As a first step, given Φ as in (2.3.6), bring in the polar decomposition

$$\begin{aligned}\Phi &= AU, \quad A = (\Phi\Phi^*)^{1/2}, \\ U &= A^{-1}\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, U(\ell)).\end{aligned} \tag{2.3.11}$$

Then

$$\iota(\Phi) = \iota(U) + \iota(A). \tag{2.3.12}$$

Now $(1-t)A + tI \in L^\infty_{\mathrm{inv}} \cap \mathrm{vmo}(\partial\Omega, \mathrm{GL}(\ell, \mathbb{C}))$ for every $t \in [0,1]$, and the identity

$$T_{(1-t)A+tI} = (1-t)T_A + tT_I \tag{2.3.13}$$

yields

$$\iota(A) = 0, \quad \text{hence } \iota(\Phi) = \iota(U). \tag{2.3.14}$$

Hence to examine the index of T_Φ, it suffices to consider $\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, U(\ell))$. The following two propositions are established in [21, §4.2].

Proposition 2.3.4 *Assume $\Phi_t \in L^\infty \cap \mathrm{vmo}(\partial\Omega, U(\ell))$ for each $t \in [0,1]$ and*

$$t \mapsto \Phi_t \ \textit{is continuous from } [0,1] \ \textit{to} \ \ \mathrm{bmo}(\partial\Omega, \mathrm{End}\, \mathbb{C}^\ell). \tag{2.3.15}$$

Then $\iota(\Phi_t)$ is independent of $t \in [0,1]$.

The following result reduces index computations for T_Φ to the continuous case.

Proposition 2.3.5 *Given $\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, U(\ell))$, there exists an explicit approximation procedure, producing*

$$\Phi_t \in C^0(\partial\Omega, U(\ell)), \quad t > 0, \tag{2.3.16}$$

such that

$$\|\Phi_t - \Phi\|_{\mathrm{bmo}} \longrightarrow 0, \quad \textit{as } t \to 0. \tag{2.3.17}$$

Moreover, there exists $\varepsilon_1 > 0$ such that

$$\iota(\Phi) = \iota(\Phi_t), \quad \forall\, t \in (0, \varepsilon_1]. \tag{2.3.18}$$

In the special case where $\Omega \subset \mathbb{C}$ is the unit disk (hence has smooth boundary) and $D = \overline{\partial}$ (and $\ell = 1$), such results are obtained in [7], making use of the homotopy theory of BMO maps $X \to Y$ obtained in [7] when X and Y are smooth compact manifolds. The analysis in [21] requires extending the homotopy theory to allow X to be a compact, Ahlfors regular set. Among other things, a somewhat more complicated approximation procedure is required to produce Φ_t in (2.3.16)–(2.3.18). The arguments needed to prove Propositions 2.3.4–2.3.5 are fairly elaborate, so we refer to [21, §4.2] for details.

2.4 Twisted Toeplitz operators

We extend the setting of Toeplitz operators from (2.2.1) to

$$T_\Phi : L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathcal{C}) \longrightarrow L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathcal{C}), \quad 1 < p < \infty, \tag{2.4.1}$$

where $\mathcal{C} \to M$ is a smooth vector bundle and

$$\Phi \in C^0(\partial\Omega, \operatorname{End} \mathcal{C}) \tag{2.4.2}$$

is a continuous section of $\operatorname{End}\mathcal{C}$ over $\partial\Omega$. The case treated in §2.2 amounts to taking $\mathcal{C}$ to be the trivial bundle of rank ℓ. In that setting, $\mathcal{P}$ was extended to act on sections of $\mathcal{E}_0 \otimes \mathbb{C}^\ell = \mathcal{E}_0 \oplus \cdots \oplus \mathcal{E}_0$ componentwise. The current setting requires a more elaborate construction.

To begin, we move from D to

$$D_{\mathcal{C}} : H^{s+1,2}(M, \mathcal{E}_0 \otimes \mathcal{C}) \longrightarrow H^{s,2}(M, \mathcal{E}_1 \otimes \mathcal{C}), \tag{2.4.3}$$

such that

$$\sigma_{D_{\mathcal{C}}}(x,\xi) = \sigma_D(x,\xi) \otimes I_{\mathcal{C}}. \tag{2.4.4}$$

To do this, we provide $\mathcal{C}$ with a smooth connection ∇. Then, to define (2.4.3), we take a cue from (2.1.2) and seek to set

$$D_{\mathcal{C}}(u \otimes v) = Du \otimes v + (D_0 v)u, \tag{2.4.5}$$

where u is a section of $\mathcal{E}_0$ and v a section of $\mathcal{C}$. We need to define $(D_0 v)u$, as a section of $\mathcal{E}_1 \otimes \mathcal{C}$, again taking a cue from (2.1.2). Now $\sigma_D(x,\xi) = \sigma_D(x)\xi$ is linear in ξ, and we have

$$\sigma_D(x) : T^*_x \longrightarrow \operatorname{Hom}(\mathcal{E}_{0x}, \mathcal{E}_{1x}), \tag{2.4.6}$$

or equivalently

$$\sigma_D(x) : \mathcal{E}_{0x} \otimes T^*_x \longrightarrow \mathcal{E}_{1x}. \tag{2.4.7}$$

Tensoring with $I_{\mathcal{C}}$ gives

$$\sigma_D(x) : \mathcal{E}_{0x} \otimes T^*_x \otimes \mathcal{C}_x \longrightarrow \mathcal{E}_{1x} \otimes \mathcal{C}_x, \tag{2.4.8}$$

and it is natural to set

$$(D_0 v)u(x) = \frac{1}{i}\sigma_D(x)\big(u(x) \otimes \nabla v(x)\big). \tag{2.4.9}$$

The symbol identity (2.4.4) is readily verified, and the analysis of §§2.1–2.2 is applicable to $D_{\mathcal{C}}$, yielding the projection

$$\mathcal{P}_{\mathcal{C}} : L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathcal{C}) \longrightarrow L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathcal{C}). \tag{2.4.10}$$

Actually, in light of (2.4.9), this operator depends on the choice of connection ∇ on $\mathcal{C}$, but we will not burden the notation with this. Instead, we lighten the notation and simply use $\mathcal{P}$ to denote (2.4.10), and again (usually) denote the L^p-spaces in (2.4.10) simply by $L^p(\partial\Omega)$. Thus we set

$$T_\Phi := \mathcal{P}\Phi\mathcal{P} + (I - \mathcal{P}), \tag{2.4.11}$$

acting on sections of $\mathcal{E}_0 \otimes \mathcal{C}$, then (2.4.1) holds, for Φ of the form (2.4.2), and more generally for

$$\Phi \in L^\infty \cap \operatorname{vmo}(\partial\Omega, \operatorname{End}\mathcal{C}). \tag{2.4.12}$$

Parallel to Lemma 2.3.2, we have:

Lemma 2.4.1 *If $\Psi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathcal{C})$, then*

$$[\mathcal{P}, \Psi] : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \;\; \text{is compact}, \quad \forall\, p \in (1, \infty). \tag{2.4.13}$$

This time, the identity (2.2.4) does not quite hold, but, via an argument involving (2.4.4), the difference between the left and the right sides of (2.2.4) is given by a weakly singular integral, whose compactness is elementary. See [21, §4.5] for details.

This leads to the following extension of Theorem 2.3.3.

Theorem 2.4.2 *Assume Ω is a* UR *domain and*

$$\Phi,\, \Phi^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathcal{C}), \tag{2.4.14}$$

a condition which we also abbreviate as

$$\Phi \in L^\infty_{\mathrm{inv}} \cap \mathrm{vmo}(\partial\Omega, \mathrm{GL}(\mathcal{C})). \tag{2.4.15}$$

Then $T_{\Phi^{-1}} T_\Phi - I$ and $T_\Phi T_{\Phi^{-1}} - I$ are compact on $L^p(\partial\Omega)$ for $p \in (1, \infty)$, so

$$T_\Phi : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \;\; \text{is Fredholm}, \quad \forall\, p \in (1, \infty). \tag{2.4.16}$$

Thus we can set

$$\iota(\Phi) := \mathrm{Index}\, T_\Phi \;\; \text{on} \;\; L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathcal{C}), \quad p \in (1, \infty). \tag{2.4.17}$$

As before, this index is independent of $p \in (1, \infty)$. If Ψ also satisfies (2.4.15), then

$$\iota(\Phi\Psi) = \iota(\Phi) + \iota(\Psi). \tag{2.4.18}$$

It is useful to have the following.

Proposition 2.4.3 *Given Φ satisfying* (2.4.15)*, the index of T_Φ is independent of the choice of connection on $\mathcal{C}$.*

Proof. Two connections on $\mathcal{C}$ give two elliptic operators $D_{\mathcal{C}}$ that differ by an operator of order zero. Hence the integral kernels of $E(x, y)$ differ by a weakly singular term, and so the two versions of T_Φ differ by a compact operator. □

2.5 Localization and cobordism invariance

Tools developed in [21] to analyze the index of T_Φ include localization and cobordism invariance. We describe these here. To begin, suppose

$$\partial\Omega = \bigcup_{j=1}^{J} \Gamma_j, \quad \text{disjoint, closed subsets.} \tag{2.5.1}$$

For each j define the operator $C_j : L^p(\Gamma_j) \to L^p(\Gamma_j)$ by restricting the integral (2.11) to Γ_j, and set $\mathcal{P}_j := (1/2)I + C_j$, so that $\mathcal{P}_j : L^p(\Gamma_j) \to L^p(\Gamma_j)$. We have

$$\mathcal{P} - \bigoplus_{j=1}^{J} \mathcal{P}_j \;\; \text{compact on} \;\; L^p(\partial\Omega), \quad \mathcal{P}_j^2 - \mathcal{P}_j \;\; \text{compact on} \;\; L^p(\Gamma_j). \tag{2.5.2}$$

Thus, with

$$T_{\Gamma_j,\Omega,\Phi} f = \mathcal{P}_j \Phi \mathcal{P}_j f + (I - \mathcal{P}_j) f, \quad f \in L^p(\Gamma_j), \tag{2.5.3}$$

we have

$$T_\Phi - \bigoplus_{j=1}^{J} T_{\Gamma_j,\Omega,\Phi} \text{ compact on } L^p(\partial\Omega), \tag{2.5.4}$$

for $p \in (1,\infty)$, if $\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathcal{C})$. Clearly $T_{\Gamma_j,\Omega,\Phi}$ depends only on $\Phi|_{\Gamma_j}$. If

$$\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{GL}(\mathcal{C})), \tag{2.5.5}$$

then each operator $T_{\Gamma_j,\Omega,\Phi}$ is Fredholm on $L^p(\Gamma_j, \mathcal{E}_0 \otimes \mathcal{C})$, and

$$\mathrm{Index}\, T_\Phi = \sum_{j=1}^{J} \mathrm{Index}\, T_{\Gamma_j,\Omega,\Phi}. \tag{2.5.6}$$

Here is a related localization. Given the UR domain $\Omega \subset M$, assume there is another Riemannian manifold $\widetilde{M}$, a neighborhood $\mathcal{O}$ of $\overline{\Omega}$ in M, and an open $\widetilde{\mathcal{O}} \subset \widetilde{M}$, isometric to $\mathcal{O}$. (From here on, we identify $\mathcal{O}$ and $\widetilde{\mathcal{O}}$.) Assume there exists a first order elliptic differential operator $\widetilde{D}$ on $\widetilde{M}$ acting on sections of $\widetilde{\mathcal{E}}_0 \otimes \widetilde{\mathcal{C}} \to \widetilde{M}$, these bundles agreeing with $\mathcal{E}_0 \otimes \mathcal{C}$ on $\widetilde{\mathcal{O}} = \mathcal{O}$, such that $\widetilde{D} = D$ on $\mathcal{O}$. Then we have the Toeplitz operator

$$T_{\widetilde{M},\Phi} : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \tag{2.5.7}$$

and

$$\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathcal{C}) \Longrightarrow T_\Phi - T_{\widetilde{M},\Phi} \text{ compact on } L^p(\partial\Omega), \tag{2.5.8}$$

for $p \in (1,\infty)$, so

$$\Phi \in L^\infty_{\mathrm{inv}} \cap \mathrm{vmo}(\partial\Omega, \mathrm{GL}(\mathcal{C})) \Longrightarrow \mathrm{Index}\, T_\Phi = \mathrm{Index}\, T_{\widetilde{M},\Phi}. \tag{2.5.9}$$

The following cobordism result is established in [21, §4.7].

Proposition 2.5.1 *If $\Phi \in C^0(\overline{\Omega}, \mathrm{GL}(\mathcal{C}))$, then*

$$\mathrm{Index}\, T_\Phi = 0. \tag{2.5.10}$$

This proposition applies in the following setting. Take an open set $\mathcal{O} \subset \Omega$ such that

$$\mathcal{O} \text{ is a UR domain, and } \partial\mathcal{O} = \partial\Omega \cup \Gamma, \text{ disjoint closed sets.} \tag{2.5.11}$$

Let

$$\Phi \in C^0(\overline{\mathcal{O}}, \mathrm{GL}(\mathcal{C})). \tag{2.5.12}$$

Then we have $T_\Phi : L^p(\partial\Omega) \to L^p(\partial\Omega)$. Also, we have an analogue, which we denote $T_{\mathcal{O},\Phi}$, defined by replacing Ω by $\mathcal{O}$. Proposition 6.1, with $\mathcal{O}$ in place of Ω, implies

$$\mathrm{Index}\, T_{\mathcal{O},\Phi} = 0. \tag{2.5.13}$$

Furthermore,

$$\widetilde{\Omega} = \Omega \setminus \overline{\mathcal{O}} \Longrightarrow \partial\widetilde{\Omega} = \Gamma, \tag{2.5.14}$$

and via (2.5.13) and a localization argument, one gets

$$\operatorname{Index} T_\Phi = \operatorname{Index} T_{\widetilde{\Omega},\Phi}. \tag{2.5.15}$$

See §4.7 of [21] for details.

The result (2.5.15) sometimes applies in cases where $\partial\Omega$ is rough but $\partial\widetilde{\Omega}$ is smooth. There are tools available for calculating the right side of (2.5.15), including the Atiyah-Singer index formula, when $\partial\widetilde{\Omega}$ is smooth, so the identity (2.5.15) provides a path for the calculation of the index of T_Φ, in many cases where $\partial\Omega$ is rough.

2.6 Further results on index computations

As usual, Ω is a relatively compact UR domain. For simplicity, we assume here that $\Phi \in C^0(\partial\Omega, \mathrm{GL}(\ell,\mathbb{C}))$. In fact, going further, as in (2.3.11)–(2.3.14), we may as well take

$$\Phi \in C^0(\partial\Omega, U(\ell)). \tag{2.6.1}$$

As in (2.2.12), the assignment $\Phi \mapsto \iota(\Phi) := \operatorname{Index} T_\Phi$ induces a group homomorphism

$$\iota : [\partial\Omega; U(\ell)] \longrightarrow \mathbb{Z}. \tag{2.6.2}$$

When (2.6.1) holds, we can write

$$\Phi(x) = \Phi_0(x)\Phi_1(x), \tag{2.6.3}$$

with

$$\Phi_0(x) = \begin{pmatrix} \varphi & \\ & I \end{pmatrix}, \quad \varphi(x) = \det \Phi(x), \quad \Phi_1 \in C^0(\partial\Omega, SU(\ell)), \tag{2.6.4}$$

and

$$\iota(\Phi) = \iota(\Phi_0) + \iota(\Phi_1) = \iota(\varphi) + \iota(\Phi_1), \tag{2.6.5}$$

with $\varphi \in C^0(\partial\Omega, S^1)$, $S^1 \subset \mathbb{C}$. Here, $SU(\ell)$ denotes the subgroup of $U(\ell)$ with determinant 1. We have

$$[\partial\Omega; S^1] = 0 \Longrightarrow \iota(\Phi) = \iota(\Phi_1), \tag{2.6.6}$$

and

$$[\partial\Omega; SU(\ell)] = 0 \Longrightarrow \iota(\Phi) = \iota(\varphi). \tag{2.6.7}$$

Note that the implication in (2.6.6) holds when $\partial\Omega$ is simply connected while the implication in (2.6.7) holds if $\ell = 2$ and $\dim \Omega \le 3$.

We now specialize to the case where $\partial\Omega$ is homeomorphic to a sphere:

$$\partial\Omega \approx S^m, \quad m = n-1 \quad (n = \dim \Omega). \tag{2.6.8}$$

In such a case, $[\partial\Omega; U(\ell)] \approx \pi_m(U(\ell))$, where, by definition, $\pi_m(Y)$ is the group of homotopy classes of maps from the sphere S^m to a space Y (and here $Y = U(\ell)$). Classical results of Bott (cf. [19]) imply

$$m = 2\mu - 1 \Longrightarrow \pi_m(U(\ell)) \approx \mathbb{Z}, \text{ if } \ell \ge \mu. \tag{2.6.9}$$

By contrast,

$$m \notin \{1, 3, \dots, 2\ell - 1\} \Longrightarrow \pi_m(U(\ell)) \text{ is finite}. \tag{2.6.10}$$

When (2.6.9) holds, let

$$\vartheta : [\partial\Omega; U(\ell)] \xrightarrow{\approx} \mathbb{Z} \tag{2.6.11}$$

denote the induced isomorphism (uniquely defined up to sign). We have the following.

Proposition 2.6.1 *Assume Ω is a* UR *domain and* (2.6.8) *holds. If $m = 2\mu - 1$ and $\ell \geq \mu$, there exists $\alpha = \alpha(\Omega, D) \in \mathbb{Z}$ such that*

$$\iota(\Phi; D) = \alpha\vartheta([\Phi]), \qquad \forall\, \Phi \in C^0(\partial\Omega, U(\ell)). \tag{2.6.12}$$

If $m \notin \{1, 3, \dots, 2\ell - 1\}$, then

$$\iota(\Phi; D) = 0, \qquad \forall\, \Phi \in C^0(\partial\Omega, U(\ell)). \tag{2.6.13}$$

An extra argument is required to show that α in (2.6.12) is independent of ℓ (up to sign, when ℓ satisfies $\ell \geq \mu$, $m = 2\mu - 1$). See [21, §4.8] for details. This argument also yields the following.

Corollary 2.6.2 *In the setting of Proposition 2.6.1, if $m = 2\mu - 1$ and $\ell_1 \geq \mu$, and if there exists $\Phi_1 \in C^0(\partial\Omega, U(\ell_1))$ such that*

$$\operatorname{Index} T_{\Phi_1} = 1, \tag{2.6.14}$$

then (2.6.12) *holds with $\alpha = \pm 1$, for all $\ell \geq \mu$.*

In fact, we see that α must be a nonzero integer of magnitude ≤ 1.

We aim to produce some cases where Corollary 2.6.2 applies. We begin with an apparent digression. Let $B \subset \mathbb{C}^\mu$ be the unit ball. Assume $\mu \geq 2$. Let $S_h : L^2(\partial B) \to L^2(\partial B)$ be the Szegö projector onto the space of boundary values of functions holomorphic on B. Since holomorphic functions satisfy an overdetermined elliptic system, this is a different sort of projector from what we have been considering. For example,

$$S_h \in OPS^0_{1/2,1/2}(\partial B). \tag{2.6.15}$$

This is sufficient to imply that operators $\tau_\Phi = S_h \Phi S_h + (I - S_h)$ are Fredholm provided $\Phi \in C^0(\partial B, U(\ell))$, and one has an analogue of (2.6.12):

$$\operatorname{Index} \tau_\Phi = \alpha_h \vartheta([\Phi]). \tag{2.6.16}$$

In [33], it is shown that (2.6.16) holds with $\alpha_h = \pm 1$. An alternative treatment of such an index formula, in a more general setting, was done by Boutet de Monvel in [6]. His formula, valid when $B \subset \mathbb{C}^\mu$ is a smoothly bounded, strongly pseudoconvex domain, can be described as follows. Consider

$$D = \overline{\partial} + \overline{\partial}^* : \Lambda^{0,\text{even}}(\mathbb{C}^\mu) \longrightarrow \Lambda^{0,\text{odd}}(\mathbb{C}^\mu). \tag{2.6.17}$$

This is an operator of Dirac type. Then

$$\operatorname{Index} \tau_\Phi = \iota(\Phi; D). \tag{2.6.18}$$

See also [2] for a proof of (2.6.18) using K-homology. We have the following consequence.

Proposition 2.6.3 *When $\Omega = B$ is the unit ball in $\mathbb{C}^\mu$ and D is given by* (2.6.17)*, then* (2.6.12) *holds with $\alpha = \pm 1$, provided $\ell \geq \mu$.*

From here, we obtain the following.

Proposition 2.6.4 *Let $\Omega \subset \mathbb{C}^\mu$ be a bounded* UR *domain and let D be given by* (2.6.17)*. Let $\ell \geq \mu$. Then*

$$\text{there exists } \Phi_1 \in C^0(\partial\Omega, U(\ell)) \text{ such that } \operatorname{Index} T_{\Phi_1} = 1. \tag{2.6.19}$$

Proof. We can assume $0 \in B \subset \overline{B} \subset \Omega$. Take $\Phi_1 \in C^0(\partial B, U(\ell))$ such that T_{B,Φ_1} has index 1, using Proposition 2.6.3. Then extend Φ_1 to an element of $C^0(\mathbb{C}^\mu \setminus 0, U(\ell))$, homogeneous of degree 0, and restrict to $\partial\Omega$. The cobordism argument of §2.5 implies

$$\operatorname{Index} T_{\Omega,\Phi_1} = \operatorname{Index} T_{B,\Phi_1}, \tag{2.6.20}$$

so we have (2.6.19). □

Corollary 2.6.5 *Let $\Omega \subset \mathbb{C}^\mu$ be a bounded* UR *domain and let D be given by* (2.6.17)*. If $\partial\Omega$ is homeomorphic to $S^{2\mu-1}$, then* (2.6.12) *holds, with $\alpha = \pm 1$.*

2.7 Toeplitz operators on L^p-Sobolev spaces

Here we record some results from §4.3 of [21], regarding the behavior of operators T_Φ on $L^p_1(\partial\Omega)$, the L^p-based Sobolev spaces of order one on $\partial\Omega$. One natural focus is on

$$\Phi \in \operatorname{Lip}(\partial\Omega, \operatorname{End} \mathbb{C}^\ell), \tag{2.7.1}$$

the space of Lipschitz continuous maps from $\partial\Omega$ to $\operatorname{End}\mathbb{C}^\ell = M(\ell, \mathbb{C})$. As shown in [21], for such multiplier functions Φ, the associated Toeplitz operator T_Φ induces a bounded mapping

$$T_\Phi : L^p_1(\partial\Omega) \longrightarrow L^p_1(\partial\Omega), \quad 1 < p < \infty. \tag{2.7.2}$$

We seek conditions under which

$$T_\Psi T_\Phi - T_{\Psi\Phi} \text{ is compact on } L^p_1(\partial\Omega), \tag{2.7.3}$$

given also $\Psi \in \operatorname{Lip}(\partial\Omega, \operatorname{End}\mathbb{C}^\ell)$. A related issue is when

$$T_\Psi T_\Phi - T_{\Psi\Phi} : L^p(\partial\Omega) \longrightarrow L^p_1(\partial\Omega). \tag{2.7.4}$$

Clearly (2.7.4) implies compactness on $L^p_1(\partial\Omega)$ provided the natural injection

$$L^p_1(\partial\Omega) \hookrightarrow L^p(\partial\Omega) \text{ is compact.} \tag{2.7.5}$$

As shown in [16, Corollary 4.31], (2.7.5) holds for each $p \in (1, \infty)$ provided Ω is a UR domain and, in addition,

$$\Omega \text{ satisfies a two-sided John condition.} \tag{2.7.6}$$

See §A.2 for the definition and some basic properties of this class of domains.

As shown in [21, Proposition 4.3.2], if $\Psi \in \operatorname{Lip}(\partial\Omega, \operatorname{End}\mathbb{C}^\ell)$ and $\Phi \in C^{1+r}(M, \operatorname{End}\mathbb{C}^\ell)$, with $r > 0$, then (2.7.4) holds, whenever Ω is a UR domain. Hence, if (2.7.5) holds, then (2.7.3) holds. Consequently, as noted in [21, Proposition 4.3.3], by a limiting argument, (2.7.3) holds provided

$$\Psi \in \operatorname{Lip}(\partial\Omega, \operatorname{End}\mathbb{C}^\ell), \quad \Phi \in C^1(\partial\Omega, \operatorname{End}\mathbb{C}^\ell) := \big\{ \widetilde{\Phi}\big|_{\partial\Omega} : \widetilde{\Phi} \in C^1(M, \operatorname{End}\mathbb{C}^\ell) \big\}, \tag{2.7.7}$$

as long as Ω is a UR domain for which (2.7.5) holds, in particular if Ω is an Ahlfors regular domain satisfying (2.7.6). As a corollary, for such a domain Ω, if

$$\Phi \in C^1(\partial\Omega, \mathrm{GL}(\ell, \mathbb{C})), \tag{2.7.8}$$

then, for each $p \in (1, \infty)$,

$$T_\Phi : L^p_1(\partial\Omega) \longrightarrow L^p_1(\partial\Omega) \ \text{ is Fredholm.} \tag{2.7.9}$$

This does not quite capture the desired Fredholm result for Φ as in (2.7.1). However, [21, §4.3] goes further, making use of results recalled in §A.1 of this paper. The following is established in [21, Proposition 4.3.5]. Recall that $n := \dim M$.

Proposition 2.7.1 *Assume $\Omega \subset M$ is Ahlfors regular and satisfies a two-sided John condition. Take $p \in (1, \infty)$ and assume*

$$\Phi, \Psi \in L^q_1(\partial\Omega, M(\ell, \mathbb{C})), \quad q \ge p, \ q \in (n-1, \infty). \tag{2.7.10}$$

Then (2.7.3) *holds. Consequently, if*

$$\Phi \in L^q_1(\partial\Omega, \mathrm{GL}(\ell, \mathbb{C})), \quad q \ge p, \quad q \in (n-1, \infty), \tag{2.7.11}$$

then T_Φ is Fredholm in (2.7.9)*, with Fredholm inverse $T_{\Phi^{-1}}$.*

As shown in [21, §4.3], for $p \in (1, q]$,

$$\begin{aligned} \operatorname{Index} T_\Phi \ \text{ on } \ L^p_1(\partial\Omega) &= \operatorname{Index} T_\Phi \ \text{ on } \ L^p(\partial\Omega) \\ &= \iota(\Phi), \end{aligned} \tag{2.7.12}$$

with $\iota(\Phi)$ as in §2.2. Also

$$\begin{aligned} \operatorname{Ker} T_\Phi \ \text{ on } \ L^p_1(\partial\Omega) &= \operatorname{Ker} T_\Phi \ \text{ on } \ L^p(\partial\Omega) \\ \operatorname{Ker}(T_\Phi)^* \ \text{ on } \ L^p_1(\partial\Omega)^* &= \operatorname{Ker}(T_\Phi)^* \ \text{ on } \ L^p(\partial\Omega)^*. \end{aligned} \tag{2.7.13}$$

Material of this section can also be developed for the action of twisted Toeplitz operators on $L^p_1(\partial\Omega)$. We omit the details.

3 Bojarski-type index formulas

Here we assume the compact Riemannian manifold M is partitioned:

$$M = \Omega_+ \cup \Omega_- \cup \Sigma, \tag{3.0.1}$$

where

$$\Omega_+ \text{ and } \Omega_- \text{ are UR domains,} \quad \partial\Omega_+ = \partial\Omega_- = \Sigma. \tag{3.0.2}$$

We study the interaction of $\mathcal{H}^p(\Omega_+, D)$ and $\mathcal{H}^p(\Omega_-, D)$, and of the associated spaces of nontangential boundary values $\mathcal{H}^p_+(\Sigma, D)$ and $\mathcal{H}^p_-(\Sigma, D)$. As usual, $D : \mathcal{E}_0 \to \mathcal{E}_1$ is a first order elliptic differential operator, hence

$$D : H^{s+1,p}(M, \mathcal{E}_0) \to H^{s,p}(M, \mathcal{E}_1) \tag{3.0.3}$$

boundedly for each $p \in (1,\infty)$ and $s \in \mathbb{R}$. We assume D and D^* satisfy UCP.

With $\nu_\pm$ denoting the (geometric measure theoretic) outward unit conormals for $\Omega_\pm$, and σ the surface measure on Σ, we have the Cauchy integral operators

$$\mathcal{C}_\pm f(x) := i \int_\Sigma E_\pm(x,y)\sigma_D(y,\nu_\pm(y))f(y)\,d\sigma(y), \qquad x \in M \setminus \Sigma, \tag{3.0.4}$$

satisfying

$$\begin{aligned} \mathcal{C}_+ f\Big|^{\text{n.t.}}_{\partial\Omega_\pm}(x) &= \pm\frac{1}{2}f(x) + C_+ f(x), \\ \mathcal{C}_- f\Big|^{\text{n.t.}}_{\partial\Omega_\mp}(x) &= \pm\frac{1}{2}f(x) + C_- f(x), \end{aligned} \tag{3.0.5}$$

at σ-a.e. point $x \in \Sigma$, with

$$C_\pm f(x) := i\,\mathrm{PV}\int_\Sigma E_\pm(x,y)\sigma_D(y,\nu_\pm(y))f(y)\,d\sigma(y), \qquad x \in \Sigma. \tag{3.0.6}$$

Here, to define $E_+(x,y)$, we take $a^+ \in C_0^\infty(M\setminus\overline{\Omega}_+)$, which is nonnegative and not identically zero, which then implies that the auxiliary operator

$$\mathcal{D}_+ := \begin{pmatrix} ia^+ & D^* \\ D & ia^+ \end{pmatrix} \tag{3.0.7}$$

is invertible (say, as a mapping from $H^{1,2}(M)$ into $L^2(M)$), as discussed in the introduction, and define $E_+(x,y)$ by the analogue of (1.0.21). We similarly consider some nonnegative function $a^- \in C_0^\infty(M \setminus \overline{\Omega}_-)$ which is not identically zero in order to define $E_-(x,y)$. In (3.0.4) and (3.0.6), ν_+ points outward from Ω_+ and ν_- points outward from Ω_-, so we have $\nu_- = -\nu_+$. In (3.0.5), $\mathcal{C}_\pm f\big|^{\text{n.t.}}_{\partial\Omega_+}$ denotes the nontangential limit from within Ω_+ and $\mathcal{C}_\pm f\big|^{\text{n.t.}}_{\partial\Omega_-}$ that from within Ω_-.

The maps

$$\mathcal{P}_\pm := \frac{1}{2}I + C_\pm \tag{3.0.8}$$

are projections of $L^p(\Sigma,\mathcal{E}_0)$ onto

$$\mathcal{H}^p_\pm(\Sigma, D), \tag{3.0.9}$$

the image of $\mathcal{H}^p(\Omega_\pm, D)$ under the nontangential boundary trace isomorphism τ. Note that the leading singularities of $E_+(x,y)$ and $E_-(x,y)$ agree, since $\mathcal{D}_+$ and $\mathcal{D}_-$ have the same principal symbol and hence the same leading parametrix, so $E_+(x,y) - E_-(x,y)$ has a weaker singularity. It follows that

$$C_+ + C_- = K : L^p(\Sigma) \longrightarrow L^p(\Sigma) \text{ is compact}, \tag{3.0.10}$$

for $1 < p < \infty$. Consequently,

$$\mathcal{P}_+ + \mathcal{P}_- = I + K, \quad K \text{ compact on } L^p(\Sigma),\ \forall\, p \in (1,\infty). \tag{3.0.11}$$

If D is invertible, we can take $a^\pm = 0$, so then $E_+ = E_-$ and hence $K = 0$ in (3.0.11). In such a case,

$$L^p(\Sigma,\mathcal{E}_0) = \mathcal{H}^p_+(\Sigma, D) \oplus \mathcal{H}^p_-(\Sigma, D). \tag{3.0.12}$$

Here we want to see how (3.0.12) is altered when D is not invertible, particularly when $\operatorname{Index} D \neq 0$.

We bring in the following notion of a Fredholm pair. Let X be a Banach space, and Y_1 and Y_2 closed linear subspaces. Assume that $\dim Y_1 \cap Y_2 < \infty$ and that $Y_1 + Y_2$ is closed in X and has finite codimension. Then we say (Y_1, Y_2) is a Fredholm pair of subspaces of X, and

$$\operatorname{Index}(Y_1, Y_2) := \dim(Y_1 \cap Y_2) - \operatorname{codim}(Y_1 + Y_2). \tag{3.0.13}$$

One goal here is to show that, for each $p \in (1, \infty)$, the spaces $\mathcal{H}^p_+(\Sigma, D)$ and $\mathcal{H}^p_-(\Sigma, D)$ form a Fredholm pair for $L^p(\Sigma, \mathcal{E}_0)$, of index equal to the index of D (viewed as a global mapping on M). We will go further, and separately establish isomorphisms

$$\begin{aligned} \operatorname{Ker} D &\approx \mathcal{H}^p_+(\Sigma, D) \cap \mathcal{H}^p_-(\Sigma, D) \\ \operatorname{Ker} D^* &\approx [\mathcal{H}^p_+(\Sigma, D) + \mathcal{H}^p_-(\Sigma, D)]^\perp, \end{aligned} \tag{3.0.14}$$

where if Z is a linear subspace of X, $Z^\perp$ denotes its annihilator in the dual space X', so if Z is closed, $\operatorname{codim} Z = \dim Z^\perp$.

The Fredholm pair property and (3.0.14) are proved in §3.2, following a duality argument in §3.1 that allows us to deduce the second isomorphism in (3.0.14) from the first. Section 3.3 continues these Fredholm results. Work in §§3.2–3.3 involves analysis of the operators

$$\begin{aligned} \mathcal{J}_p &: \mathcal{H}^p_+(\Sigma, D) \oplus \mathcal{H}^p_-(\Sigma, D) \longrightarrow L^p(\Sigma, \mathcal{E}_0), \\ \Gamma_p &: L^p(\Sigma, \mathcal{E}_0) \longrightarrow \mathcal{H}^p_+(\Sigma, D) \oplus \mathcal{H}^p_-(\Sigma, D), \end{aligned} \tag{3.0.15}$$

given by

$$\mathcal{J}_p(f_+, f_-) = f_+ - f_- \quad \text{and} \quad \Gamma_p f = (\mathcal{P}_+ f, -\mathcal{P}_- f). \tag{3.0.16}$$

We show that $\mathcal{J}_p$ and Γ_p are Fredholm inverses of each other, and

$$\operatorname{Index} \mathcal{J}_p = \operatorname{Index} D, \qquad \operatorname{Index} \Gamma_p = -\operatorname{Index} D. \tag{3.0.17}$$

In §3.4 we derive analogous results in the L^p-Sobolev space setting. In this case, we consider

$$\begin{aligned} \mathcal{J}_{1,p} &: \mathcal{H}^{1,p}_+(\Sigma, D) \oplus \mathcal{H}^{1,p}_-(\Sigma, D) \longrightarrow L^p_1(\Sigma, \mathcal{E}_0), \\ \Gamma_{1,p} &: L^p_1(\Sigma, \mathcal{E}_0) \longrightarrow \mathcal{H}^{1,p}_+(\Sigma, D) \oplus \mathcal{H}^{1,p}_-(\Sigma, D), \end{aligned} \tag{3.0.18}$$

given as in (3.0.16). Here

$$\mathcal{H}^{1,p}_\pm(\Sigma, D) = \mathcal{P}_\pm L^p_1(\Sigma, \mathcal{E}_0), \tag{3.0.19}$$

are the spaces of boundary values of

$$\begin{aligned} \mathcal{H}^{1,p}(\Omega_\pm, D) := \big\{ u \in C^1(\Omega_\pm, \mathcal{E}_0) : Du = 0 \text{ in } \Omega_\pm,\ \mathcal{N}(u), \mathcal{N}(\nabla u) \in L^p(\Sigma), \text{ and} \\ \text{there exist } u\big|^{\text{n.t.}}_\Sigma,\ (\nabla u)\big|^{\text{n.t.}}_\Sigma\ \sigma\text{-a.e. on } \Sigma \big\}. \end{aligned} \tag{3.0.20}$$

Again we show that $\mathcal{J}_{1,p}$ and $\Gamma_{1,p}$ are Fredholm inverses of each other, under an additional condition, such as that $L^p_1(\Sigma) \hookrightarrow L^p(\Sigma)$ is compact, which holds, for example, if $\Omega_\pm$ satisfy a two-sided John condition (reviewed in §A.2). In such a case, we have a parallel to (3.0.17). The proof of this requires a certain regularity result, given in §3.4.

3.1 Polarity of $\mathcal{H}^p(\partial\Omega, D^*)$ and $\mathcal{H}^{p'}(\partial\Omega, D)$

Here, $\Omega \subset M$ is a UR domain with (geometric measure theoretic) outward unit conormal ν and surface measure σ, D is as in §1, and we assume D and D^* have UCP.

Theorem 3.1.1 *For $p \in (1,\infty)$, $p' := p/(1-p)$, the map γ_D, defined by*

$$\gamma_D f(x) := i\sigma_{D^*}(x,\nu(x))f(x), \qquad x \in \partial\Omega, \tag{3.1.1}$$

gives an isomorphism

$$\gamma_D : \mathcal{H}^p(\partial\Omega, D^*) \longrightarrow [\mathcal{H}^{p'}(\partial\Omega, D)]^\perp. \tag{3.1.2}$$

Here, if X is a Banach space and $Y \subset X$ a closed linear subspace, we define $Y^\perp \subset X'$ by

$$Y^\perp := \big\{g \in X' : \langle f, g\rangle = 0, \forall\, f \in Y\big\}. \tag{3.1.3}$$

Proof. To show that γ_D has the mapping property (3.1.2), pick two arbitrary functions $f \in \mathcal{H}^p(\partial\Omega, D^*)$ and $g \in \mathcal{H}^{p'}(\partial\Omega, D)$. Then pick $u \in \mathcal{H}^p(\Omega, D^*)$ and $v \in \mathcal{H}^{p'}(\Omega, D)$ such that

$$u\Big|_{\partial\Omega}^{\text{n.t.}} = f, \qquad v\Big|_{\partial\Omega}^{\text{n.t.}} = g, \qquad \sigma\text{-a.e. on } \partial\Omega. \tag{3.1.4}$$

We can then apply Proposition A.3.2 to justify the sequence of formulas

$$\begin{aligned}\int_{\partial\Omega} \langle g, \gamma_D f\rangle\, d\sigma &= \frac{1}{i}\int_{\partial\Omega} \langle \sigma_D(x,\nu)v, u\rangle\, d\sigma \\ &= \int_\Omega \big[\langle Dv, u\rangle - \langle v, D^*u\rangle\big]\, dV \\ &= 0,\end{aligned} \tag{3.1.5}$$

so we have the mapping property (3.1.2). The definition (3.1.1) and ellipticity of D^* clearly imply this map is injective. It remains to establish surjectivity.

It is convenient to reduce the proof of this surjectivity to the case of an elliptic operator $\mathbb{D}$ that is invertible and self-adjoint. Thus, with $\mathcal{D}$ as in (1.0.18), (with $a \in C_0^\infty(M \setminus \overline{\Omega})$ which is ≥ 0 and not $\equiv 0$), we set

$$\mathbb{D} := \begin{pmatrix} 0 & \mathcal{D}^* \\ \mathcal{D} & 0 \end{pmatrix}. \tag{3.1.6}$$

With obvious notation,

$$\begin{aligned}\mathcal{H}^p(\Omega, \mathbb{D}) &= \mathcal{H}^p(\Omega, \mathcal{D}) \oplus \mathcal{H}^p(\Omega, \mathcal{D}^*) \\ &= \mathcal{H}^p(\Omega, \mathcal{D}) \oplus \mathcal{H}^p(\Omega, \mathcal{D}),\end{aligned} \tag{3.1.7}$$

the last identity because $\mathcal{D} = \mathcal{D}^*$ on a neighborhood of $\overline{\Omega}$. Using the trace isomorphism (2.1.19), we then get

$$\begin{aligned}\mathcal{H}^p(\partial\Omega, \mathbb{D}) &= \mathcal{H}^p(\partial\Omega, \mathcal{D}) \oplus \mathcal{H}^p(\partial\Omega, \mathcal{D}) \\ &= \mathcal{H}^p(\partial\Omega, D) \oplus \mathcal{H}^p(\partial\Omega, D^*) \oplus \mathcal{H}^p(\partial\Omega, D) \oplus \mathcal{H}^p(\partial\Omega, D^*).\end{aligned} \tag{3.1.8}$$

Hence, for $p \in (1, \infty)$,

$$\begin{aligned}\mathcal{H}^p(\partial\Omega, \mathbb{D})^\perp = &[\mathcal{H}^p(\partial\Omega, D)]^\perp \oplus [\mathcal{H}^p(\partial\Omega, D^*)]^\perp \\ &\oplus [\mathcal{H}^p(\partial\Omega, D)]^\perp \oplus [\mathcal{H}^p(\partial\Omega, D^*)]^\perp.\end{aligned} \tag{3.1.9}$$

Note also that

$$\sigma_{\mathbb{D}}(x, \nu) = \begin{pmatrix} 0 & 0 & 0 & \sigma_{D^*}(x,\nu) \\ 0 & 0 & \sigma_D(x,\nu) & 0 \\ 0 & \sigma_{D^*}(x,\nu) & 0 & 0 \\ \sigma_D(x,\nu) & 0 & 0 & 0 \end{pmatrix}. \tag{3.1.10}$$

It follows that the mapping

$$\gamma_{\mathbb{D}} : \mathcal{H}^p(\partial\Omega, \mathbb{D}) \longrightarrow [\mathcal{H}^{p'}(\partial\Omega, \mathbb{D})]^\perp, \tag{3.1.11}$$

given by

$$\gamma_{\mathbb{D}} f = i\sigma_{\mathbb{D}}(x, \nu) f, \qquad \forall\, f \in \mathcal{H}^p(\partial\Omega, \mathbb{D}), \tag{3.1.12}$$

decomposes as

$$\gamma_{\mathbb{D}} = \begin{pmatrix} 0 & 0 & 0 & \gamma_D \\ 0 & 0 & \gamma_{D^*} & 0 \\ 0 & \gamma_D & 0 & 0 \\ \gamma_{D^*} & 0 & 0 & 0 \end{pmatrix}, \tag{3.1.13}$$

where γ_{D^*} is defined as in (3.1.1), with D^* in place of D. Thus the surjectivity of $\gamma_{\mathbb{D}}$ would imply surjectivity of γ_D.

To proceed, let $\mathbb{E}$ be the integral kernel of $\mathbb{D}^{-1}$, and form

$$\begin{aligned} C_{\mathbb{D}} f(x) &:= i\,\mathrm{PV}\int_{\partial\Omega} \mathbb{E}(x,y)\sigma_{\mathbb{D}}(y, \nu(y)) f(y)\, d\sigma(y), \quad x \in \partial\Omega, \\ \mathcal{P}_{\mathbb{D}} &:= \frac{1}{2} I + C_{\mathbb{D}}. \end{aligned} \tag{3.1.14}$$

Since $\mathbb{D} = \mathbb{D}^*$, hence $\sigma_{\mathbb{D}}(y,\xi)^* = \sigma_{\mathbb{D}}(y,\xi)$, it follows that the adjoint of $C_{\mathbb{D}}$ is

$$C_{\mathbb{D}}^* = -\sigma_{\mathbb{D}}(\cdot, \nu)\, C_{\mathbb{D}}\, \sigma_{\mathbb{D}}(\cdot, \nu)^{-1}. \tag{3.1.15}$$

Since $\mathcal{P}_{\mathbb{D}} = (1/2)I + C_{\mathbb{D}}$ entails $I - \mathcal{P}_{\mathbb{D}} = (1/2)I - C_{\mathbb{D}}$, we have

$$\mathcal{P}_{\mathbb{D}}^* = \sigma_{\mathbb{D}}(\cdot, \nu)(I - \mathcal{P}_{\mathbb{D}})\sigma_{\mathbb{D}}(\cdot, \nu)^{-1}. \tag{3.1.16}$$

Hence, on $L^p(\partial\Omega)$,

$$\begin{aligned} \operatorname{Ker} \mathcal{P}_{\mathbb{D}}^* &= \operatorname{Ker}(I - \mathcal{P}_{\mathbb{D}})\sigma_{\mathbb{D}}(\cdot, \nu)^{-1} \\ &= \sigma_{\mathbb{D}}(\cdot, \nu)\ \operatorname{Ker}(I - \mathcal{P}_{\mathbb{D}}) \\ &= \sigma_{\mathbb{D}}(\cdot, \nu)\mathcal{H}^p(\partial\Omega, \mathbb{D}). \end{aligned} \tag{3.1.17}$$

It follows that the range of $\gamma_{\mathbb{D}}$ in (3.1.11) is

$$\begin{aligned} \sigma_{\mathbb{D}}(\cdot, \nu)\mathcal{H}^p(\partial\Omega, \mathbb{D}) &= \operatorname{Ker} \mathcal{P}_{\mathbb{D}}^* \\ &= [\mathcal{P}_{\mathbb{D}} L^{p'}(\partial\Omega)]^\perp \\ &= [\mathcal{H}^{p'}(\partial\Omega, \mathbb{D})]^\perp. \end{aligned} \tag{3.1.18}$$

This yields the surjectivity of the mapping $\gamma_{\mathbb{D}}$ in (3.1.11) and completes the proof of Theorem 3.1.1. □

3.2 $\mathcal{H}^p_+(\Sigma, D)$ and $\mathcal{H}^p_-(\Sigma, D)$ as a Fredholm pair

We are in the setting of (3.0.1)–(3.0.2), and we assume D and D^* satisfy UCP. Our goal is to prove the following.

Theorem 3.2.1 *For each $p \in (1, \infty)$,*

$$(\mathcal{H}^p_+(\Sigma, D), \mathcal{H}^p_-(\Sigma, D)) \text{ is a Fredholm pair for } L^p(\Sigma, \mathcal{E}_0), \tag{3.2.1}$$

and, with D viewed as a mapping from $H^{1,2}(M)$ into $L^2(M)$,

$$\text{Index}(\mathcal{H}^p_+(\Sigma, D), \mathcal{H}^p_-(\Sigma, D)) = \text{Index}\, D. \tag{3.2.2}$$

In fact, there are natural isomorphisms

$$\mathcal{H}^p_+(\Sigma, D) \cap \mathcal{H}^p_-(\Sigma, D) \approx \text{Ker}\, D, \tag{3.2.3}$$

and

$$L^p(\Sigma, \mathcal{E}_0)/(\mathcal{H}^p_+(\Sigma, D) + \mathcal{H}^p_-(\Sigma, D)) \approx \text{Ker}\, D^*. \tag{3.2.4}$$

Consequently, the decomposition (3.0.12) *holds if and only if D is invertible.*

Proof. Take $\mathcal{C}_\pm, C_\pm$, and $\mathcal{P}_\pm$ as in (3.0.4)–(3.0.8). As seen in (3.0.11),

$$\mathcal{P}_+ + \mathcal{P}_- = I + K, \quad K \text{ compact on } L^p(\Sigma), \quad \forall\, p \in (1, \infty). \tag{3.2.5}$$

Clearly

$$(\mathcal{P}_+ + \mathcal{P}_-)L^p(\Sigma) \subset \mathcal{H}^p_+(\Sigma, D) + \mathcal{H}^p_-(\Sigma, D). \tag{3.2.6}$$

By (3.2.5), the range of $\mathcal{P}_+ + \mathcal{P}_-$ on $L^p(\Sigma)$ is closed and has finite codimension. It follows that the right side of (3.2.6) has finite codimension, and also that it is a closed subspace of $L^p(\Sigma, \mathcal{E}_0)$.

Next, if $f \in \mathcal{H}^p_+(\Sigma, D) \cap \mathcal{H}^p_-(\Sigma, D)$, then there exist unique functions

$$u_\pm \in \mathcal{H}^p(\Omega_\pm, D) \text{ such that } u_+\big|^{\text{n.t.}}_{\Sigma} = f = u_-\big|^{\text{n.t.}}_{\Sigma}. \tag{3.2.7}$$

Define $u_f \in L^p(M, \mathcal{E}_0)$ by

$$u_f := \begin{cases} u_+ \text{ in } \Omega_+, \\ u_- \text{ in } \Omega_-. \end{cases} \tag{3.2.8}$$

We claim

$$Du_f = 0 \text{ in } \mathcal{D}'(M). \tag{3.2.9}$$

(Clearly Du_f is supported on Σ.) In fact, given $\psi \in C^1(M, \mathcal{E}_1)$, we have

$$\begin{aligned} (Du_f, \psi) &= (u_f, D^*\psi) \\ &= \int_{\Omega_+} \langle u_+, D^*\psi \rangle\, dV + \int_{\Omega_-} \langle u_-, D^*\psi \rangle\, dV. \end{aligned} \tag{3.2.10}$$

Now, parallel to (3.1.5), we can apply Proposition A.3.2, to write

$$\begin{aligned}\int_{\Omega_\pm} \langle u_\pm, D^*\psi\rangle\, dV &= \int_{\Omega_\pm} \langle u_\pm, D^*\psi\rangle\, dV - \int_{\Omega_\pm} \langle Du_\pm, \psi\rangle\, dV \\ &= \int_{\Sigma} \langle u_\pm, \sigma_{D^*}(\cdot,\nu_\pm)\psi\rangle\, d\sigma.\end{aligned} \tag{3.2.11}$$

Since $\nu_- = -\nu_+$, this gives

$$\begin{aligned}(Du_f, \psi) &= \int_{\Sigma} \langle f, \sigma_{D^*}(\cdot,\nu_+)\psi\rangle\, d\sigma - \int_{\Sigma} \langle f, \sigma_{D^*}(\cdot,\nu_+)\psi\rangle\, d\sigma \\ &= 0,\end{aligned} \tag{3.2.12}$$

as desired. Given UCP for D, a similar argument shows that

$$\operatorname{Ker} D \ni u \mapsto u\big|_{\Sigma}^{\text{n.t.}} \text{ is injective,} \tag{3.2.13}$$

and this proves (3.2.3). Since $\operatorname{Ker} D$ is finite dimensional, at this point we also have (3.2.1).

The argument proving (3.2.3), with D^* in place of D and p' in place of p, gives

$$\operatorname{Ker} D^* \approx \mathcal{H}_+^{p'}(\Sigma, D^*) \cap \mathcal{H}_-^{p}(\Sigma, D^*). \tag{3.2.14}$$

Now Theorem 3.1.1, applied to Ω_+ and to Ω_-, gives

$$\gamma_D : \mathcal{H}_\pm^{p'}(\Sigma, D^*) \xrightarrow{\approx} \mathcal{H}_\pm^{p}(\Sigma, D)^\perp, \tag{3.2.15}$$

so

$$\begin{aligned}\operatorname{Ker} D^* &\approx \mathcal{H}_+^{p}(\Sigma, D)^\perp \cap \mathcal{H}_-^{p}(\Sigma, D)^\perp \\ &= [\mathcal{H}_+^{p}(\Sigma, D) + \mathcal{H}^{p}(\Sigma, D)]^\perp.\end{aligned} \tag{3.2.16}$$

This proves (3.2.4), and (3.2.2) follows. □

Here is a restatement of Theorem 3.2.1. Define the operator

$$\begin{aligned}&\mathcal{J}_p : \mathcal{H}_+^{p}(\Sigma, D) \oplus \mathcal{H}_-^{p}(\Sigma, D) \longrightarrow L^p(\Sigma, \mathcal{E}_0), \\ &\mathcal{J}_p(f_+, f_-) := f_+ - f_-, \quad \forall\, (f_+, f_-) \in \mathcal{H}_+^{p}(\Sigma, D) \oplus \mathcal{H}_-^{p}(\Sigma, D).\end{aligned} \tag{3.2.17}$$

Then

$$\operatorname{Ker} \mathcal{J}_p \approx \mathcal{H}_+^{p}(\Sigma, D) \cap \mathcal{H}_-^{p}(\Sigma, D), \tag{3.2.18}$$

and

$$\operatorname{Range} \mathcal{J}_p = \mathcal{H}_+^{p}(\Sigma, D) + \mathcal{H}_-^{p}(\Sigma, D). \tag{3.2.19}$$

Thus Theorem 3.2.1 yields:

Corollary 3.2.2 *For each* $p \in (1,\infty)$*, the operator* $\mathcal{J}_p$ *in* (3.2.17) *is Fredholm and*

$$\operatorname{Index} \mathcal{J}_p = \operatorname{Index} D. \tag{3.2.20}$$

In fact,

$$\operatorname{Ker} \mathcal{J}_p \approx \operatorname{Ker} D \quad \textit{and} \quad \operatorname{Coker} \mathcal{J}_p \approx \operatorname{Ker} D^*. \tag{3.2.21}$$

3.3 Implications for $\mathcal{H}^p(\Omega_+, D)$ and $\mathcal{H}^p(\Omega_-, D)$

Theorem 3.2.1 has a natural application to the following transmission problem.

Proposition 3.3.1 *In the setting of Theorem 3.2.1, given $p \in (1, \infty)$, the transmission problem*

$$u_\pm \in \mathcal{H}^p(\Omega_\pm, D), \quad u_+\big|_\Sigma^{\text{n.t.}} - u_-\big|_\Sigma^{\text{n.t.}} = f \in L^p(\Sigma, \mathcal{E}_0) \tag{3.3.1}$$

is Fredholm solvable, and its index is equal to the index of the Fredholm pair

$$(\mathcal{H}^p_+(\Sigma, D), \mathcal{H}^p_-(\Sigma, D)), \tag{3.3.2}$$

hence to Index D. *Furthermore,* (3.3.1) *is uniquely solvable for each $f \in L^p(\Sigma, \mathcal{E}_0)$ if and only if D is invertible.*

The proof is straightforward from the arguments of §3.2.

The problem (3.3.1) is a special case of the more general Riemann-Hilbert problem that we will tackle in §4. For now, we use Theorem 3.2.1 to prove the following Fredholmness result, which will be useful in §4. Let $\Omega_\pm, \Sigma, D$, and $\mathcal{C}_\pm$ be as in (3.0.1)–(3.0.7), assuming D and D^* have UCP. For $p \in (1, \infty)$, define

$$\begin{aligned} &\mathcal{A}_p : L^p(\Sigma, \mathcal{E}_0) \longrightarrow \mathcal{H}^p(\Omega_+, D) \oplus \mathcal{H}^p(\Omega_-, D), \\ &\mathcal{A}_p f := (\mathcal{C}_+ f, \mathcal{C}_- f), \qquad \forall\, f \in L^p(\Sigma, \mathcal{E}_0). \end{aligned} \tag{3.3.3}$$

Proposition 3.3.2 *For each $p \in (1, \infty)$, the operator $\mathcal{A}_p$ in* (3.3.3) *is Fredholm, and*

$$\operatorname{Index} \mathcal{A}_p = -\operatorname{Index} D. \tag{3.3.4}$$

Proof. We bring in $\mathcal{J}_p$, defined by (3.2.17), and also the nontangential boundary trace isomorphisms

$$\tau_\pm : \mathcal{H}^p(\Omega_\pm, D) \xrightarrow{\approx} \mathcal{H}^p_\pm(\Sigma, D), \tag{3.3.5}$$

yielding

$$\tau_+ \oplus (-\tau_-) : \mathcal{H}^p(\Omega_+, D) \oplus \mathcal{H}^p(\Omega_-, D) \xrightarrow{\approx} \mathcal{H}^p_+(\Sigma, D) \oplus \mathcal{H}^p_-(\Sigma, D). \tag{3.3.6}$$

We have

$$\mathcal{J}_p \circ (\tau_+ \oplus (-\tau_-)) \circ \mathcal{A}_p = \mathcal{P}_+ + \mathcal{P}_- = I + K, \tag{3.3.7}$$

with K compact on $L^p(\Sigma)$. Hence the left side of (3.3.7) is Fredholm of index zero. The Fredholmness of the operator $\mathcal{A}_p$ and index calculation (3.3.4) then follow from (3.3.6) and Corollary 3.2.2. □

To forge another tie to Corollary 3.2.2, for $p \in (1, \infty)$ let us define

$$\Gamma_p : L^p(\Sigma, \mathcal{E}_0) \longrightarrow \mathcal{H}^p_+(\Sigma, D) \oplus \mathcal{H}^p_-(\Sigma, D) \tag{3.3.8}$$

by setting $\Gamma_p := (\tau_+ \oplus (-\tau_-)) \circ \mathcal{A}_p$, i.e.,

$$\Gamma_p f = (\mathcal{P}_+ f, -\mathcal{P}_- f), \qquad \forall\, f \in L^p(\Sigma, \mathcal{E}_0). \tag{3.3.9}$$

Then (3.3.7) is equivalent to

$$\mathcal{J}_p \Gamma_p = \mathcal{P}_+ + \mathcal{P}_- = I + K. \tag{3.3.10}$$

Note also that $\Gamma_p \mathcal{J}_p$, acting on $\mathcal{H}^p_+(\Sigma, D) \oplus \mathcal{H}^p_-(\Sigma, D)$, is given by

$$\begin{aligned}\Gamma_p \mathcal{J}_p(f_+, f_-) &= (\mathcal{P}_+(f_+ - f_-), -\mathcal{P}_-(f_+ - f_-)) \\ &= (f_+ - \mathcal{P}_+\mathcal{P}_- f_-, f_- - \mathcal{P}_-\mathcal{P}_+ f_+),\end{aligned} \tag{3.3.11}$$

hence

$$\Gamma_p \mathcal{J}_p(f_+, f_-) = (f_+, f_-) - (K f_-, K f_+), \tag{3.3.12}$$

since

$$\begin{aligned}\mathcal{P}_+\mathcal{P}_- &= \mathcal{P}_+ K = K\mathcal{P}_-, \\ \mathcal{P}_-\mathcal{P}_+ &= \mathcal{P}_- K = K\mathcal{P}_+.\end{aligned} \tag{3.3.13}$$

These considerations establish the following result.

Proposition 3.3.3 *If D is invertible, so one can arrange $K = 0$ in* (3.3.10) *and* (3.3.12), *then $\mathcal{J}_p$ and Γ_p are two-sided inverses of each other.*

3.4 L^p-Sobolev variants

Here we work on the L^p-based Sobolev spaces of order one, defined and discussed in §A.1. The first step in this regard is to note that that Cauchy operator considered earlier behaves naturally on this scale.

Theorem 3.4.1 *Assume that $\Omega \subset M$ is an* UR *domain satisfying $\partial\Omega = \partial(\overline{\Omega})$. Then for each $p \in (1, \infty)$ there exists a constant $c_p \in (0, \infty)$ such that for each function $f \in L^p_1(\partial\Omega)$ one has*

$$\|\mathcal{N}(\nabla \mathcal{C} f)\|_{L^p(\partial\Omega)} \leq c_p \|f\|_{L^p_1(\partial\Omega)}. \tag{3.4.1}$$

Moreover, for each $f \in L^p_1(\partial\Omega)$, $1 < p < \infty$, the nontangential boundary limit

$$(\nabla \mathcal{C} f)\Big|^{\text{n.t.}}_{\partial\Omega} \text{ exists } \sigma\text{-a.e. on } \partial\Omega. \tag{3.4.2}$$

Proof. Recall the Cauchy operator $\mathcal{C}$ from (2.1.10). For the present purposes, let us agree to regard the fundamental solution E as a $\mathcal{E}_1 \otimes \mathcal{E}_0$-valued function. With this convention, we may express

$$\mathcal{C} f(x) = i \int_{\partial\Omega} \big\langle E(x, y), \sigma_D(y, \nu(y)) f(y) \big\rangle \, d\sigma(y), \qquad x \in \Omega. \tag{3.4.3}$$

Assume now that some $f \in L^p_1(\partial\Omega) = L^p_1(\partial\Omega, \mathcal{E}_0)$, with $1 < p < \infty$, has been fixed, and pick an arbitrary vector field X on M. Then, from (3.4.3) we have that for each $x \in \Omega$,

$$\begin{aligned}
\nabla_X(\mathcal{C}f)(x) &= -i \int_{\partial\Omega} \big\langle \sigma_{D^\top}(y,\nu(y))\nabla_{X,x}E(x,y), f(y)\big\rangle\, d\sigma(y) \\
&= i \int_{\partial\Omega} \big\langle \sigma_{D^\top}(y,\nu(y))\nabla_{X,y}E(x,y), f(y)\big\rangle\, d\sigma(y) \\
&\quad - i \int_{\partial\Omega} \big\langle \sigma_{D^\top}(y,\nu(y))\big(\nabla_{X,x}E(x,y) + \nabla_{X,y}E(x,y)\big), f(y)\big\rangle\, d\sigma(y) \\
&:= \mathrm{I}_f(x) + \mathrm{II}_f(x),
\end{aligned} \tag{3.4.4}$$

where $D^\top$ stands for the transposed of D, and we have written $\nabla_{X,x}, \nabla_{X,y}$ in order to indicate the variable in which the directional derivative ∇_X is taken. Bearing in mind that

$$D^\top_y E(x,y) = 0 \quad \text{for} \quad x \neq y, \quad \text{near} \quad \overline{\Omega}, \tag{3.4.5}$$

and relying on the boundary integration by parts formula (A.3.17), presently used with $P := D^\top$ (see the remark following the proof of Proposition A.3.4), for each fixed $x \in \Omega$ we may write

$$\begin{aligned}
\mathrm{I}_f(x) &= i \int_{\partial\Omega} \big\langle \big(\sigma_{D^\top}(y,\nu(y))\nabla_y - \sigma_{\nabla_X}(y,\nu(y))D^\top_y\big)E(x,y), f(y)\big\rangle\, d\sigma(y) \\
&= -\int_{\partial\Omega} \big\langle E(x,y), \big(\sigma_D(y,\nu(y))\nabla^\top_X - \sigma_{\nabla^\top_X}(y,\nu(y))D\big)f(y)\big\rangle\, d\sigma(y) \\
&\quad - \int_{\partial\Omega} \big\langle E(x,y), \sigma_{[D^\top,\nabla_X]}(y,\nu(y))f(y)\big\rangle\, d\sigma(y).
\end{aligned} \tag{3.4.6}$$

In this context, the fact that $f \in L^p_1(\partial\Omega)$ ensures

$$\big(\sigma_D(\cdot,\nu)\nabla^\top_X - \sigma_{\nabla^\top_X}(\cdot,\nu)D\big)f \in L^p(\partial\Omega), \tag{3.4.7}$$

given that $Q := \sigma_D(\cdot,\nu)\nabla^\top_X - \sigma_{\nabla^\top_X}(\cdot,\nu)D$ is a tangential first-order differential operator on $\partial\Omega$, as its principal symbol vanishes at ν:

$$\sigma_Q(\cdot,\nu) = \sigma_D(\cdot,\nu)\sigma_{\nabla^\top_X}(\cdot,\nu) - \sigma_{\nabla^\top_X}(\cdot,\nu)\sigma_D(\cdot,\nu) = 0. \tag{3.4.8}$$

Consequently, (3.4.6) permits us to conclude that

$$\|\mathcal{N}(\mathrm{I}_f)\|_{L^p(\partial\Omega)} \leq c_p\|f\|_{L^p_1(\partial\Omega)} \quad \text{and} \quad \mathrm{I}_f\Big|^{\text{n.t.}}_{\partial\Omega} \text{ exists } \sigma\text{-a.e. on } \partial\Omega, \tag{3.4.9}$$

To handle $\mathrm{II}_f(x)$ in (3.4.4), we claim that

$$k(x,y) := \nabla_{X,x}E(x,y) + \nabla_{X,y}E(x,y) \tag{3.4.10}$$

is a (variable coefficient) Calderón-Zygmund kernel, of the sort to which the theory outlined in §1 applies. Specifically, with

$$\mathcal{T}g(x) := \int_{\partial\Omega} \langle k(x,y), g(y)\rangle\, d\sigma(y), \qquad x \in \Omega, \tag{3.4.11}$$

then for each $g \in L^p(\partial\Omega)$ we have

$$\|\mathcal{N}(\mathcal{T}g)\|_{L^p(\partial\Omega)} \leq c_p\|g\|_{L^p(\partial\Omega)} \quad \text{and} \quad \mathcal{T}g\Big|_{\partial\Omega}^{\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\Omega. \tag{3.4.12}$$

To see that this is the case, consider first the case when D is invertible. In such a scenario, the first observation is that the Schwartz kernels of $D^{-1}\nabla_X$ and $\nabla_X D^{-1}$ are, respectively,

$$\nabla_{X,x}E(x,y) \text{ and } \nabla_{X,y}^{\top}E(x,y), \tag{3.4.13}$$

and the latter differs from $-\nabla_{X,y}E(x,y)$ by a zero-th order operator acting on $E(x,y)$ in the variable y, which suits our purposes. Consequently, if $[\cdot,\cdot]$ stands for the usual commutator bracket, up to a harmless additive adjustment, $k(x,y)$ may be regarded as the Schwartz kernel of $\big[D^{-1},\nabla_X\big]$. Now, if $q(x,\xi) \in S^{-1}_{\text{cl}}$ is the principal symbol of D^{-1} and if

$$\big\{p_1,p_2\big\} := \sum_j \big(\partial_{\xi_j}p_1\partial_{x_j}p_2 - \partial_{x_j}p_1\partial_{\xi_j}p_2\big) \tag{3.4.14}$$

denotes the Poisson bracket, then for each fixed $j \in \{1,\dots,n\}$ the principal symbol of $P_j := \big[D^{-1},\nabla_{\partial_j}\big] \in \text{OP}S^{-1}_{\text{cl}}$ is (cf., e.g., [30, Vol. 2, pp. 13])

$$p_j(x,\xi) = i\,\big\{\xi_j, q(x,\xi)\big\} = i(\partial_{x_j}q)(x,\xi). \tag{3.4.15}$$

Since $p_j(x,\xi) \in S^{-1}_{\text{cl}}$ is odd in ξ, the results established in [16, § 3.5] apply, finishing the proof of the claim made in connection with (3.4.10)-(3.4.12) when D is invertible.

At this stage, (3.4.1)-(3.4.2) follow from (3.4.4), (3.4.9), and (3.4.12), at least when D is invertible. The general case is handled similarly working with the auxiliary operator $\mathcal{D}$ from (1.0.18) in place of the original D, and paying attention to what happens to the individual components in this matrix formalism. $\square$

From Theorem 3.4.1 it follows that when $\Omega_\pm$ are UR domains with a common boundary Σ, we have

$$\|\mathcal{N}(\nabla\mathcal{C}_\pm f)\|_{L^p(\Sigma)} \leq c_p\|f\|_{L^p_1(\Sigma)}, \tag{3.4.16}$$

for $1 < p < \infty$, and consequently

$$\mathcal{C}_\pm : L^p_1(\Sigma) \longrightarrow \mathcal{H}^{1,p}(\Omega_\pm, D), \tag{3.4.17}$$

where

$$\begin{aligned}\mathcal{H}^{1,p}(\Omega_\pm,D) := \big\{u \in C^1(\Omega_\pm,\mathcal{E}_0) : Du = 0 \text{ in } \Omega_\pm,\ \mathcal{N}(u),\mathcal{N}(\nabla u) \in L^p(\Sigma), \text{ and}\\ \text{there exist } u\big|_{\Sigma}^{\text{n.t.}},\ (\nabla u)\big|_{\Sigma}^{\text{n.t.}}\ \sigma\text{-a.e. on } \Sigma\big\}.\end{aligned} \tag{3.4.18}$$

It follows that

$$\mathcal{P}_\pm : L^p_1(\Sigma) \longrightarrow L^p_1(\Sigma), \tag{3.4.19}$$

for $1 < p < \infty$. As for the weaker singularity for $E_+ - E_-$, described below (3.0.8) and leading to (3.0.11), let us note that, since a^+ and a^- are supported away from Σ, local elliptic regularity implies that there is a neighborhood $\mathcal{O}$ of Σ in M such that $E_+ - E_- \in C^\infty(\mathcal{O}\times\mathcal{O})$. Hence, in addition to (3.0.11), we have

$$\mathcal{P}_+ + \mathcal{P}_- = I + K, \quad K : L^p(\Sigma) \to L^p_1(\Sigma), \quad \forall\, p \in (1,\infty). \tag{3.4.20}$$

If $\Omega_\pm$ satisfy a two-sided John condition, then (3.4.20) implies (3.0.11). However, these two results are independently true whenever Ω_+ and Ω_- are mere UR domains.

Now recall the operators $\mathcal{J} = \mathcal{J}_p$ and $\Gamma = \Gamma_p$, given by (3.2.17) and (3.3.9). We see that they also give

$$\begin{aligned} \mathcal{J} = \mathcal{J}_{1,p} &: \mathcal{H}^{1,p}_+(\Sigma, D) \oplus \mathcal{H}^{1,p}_-(\Sigma, D) \longrightarrow L^p_1(\Sigma, \mathcal{E}_0), \\ \Gamma = \Gamma_{1,p} &: L^p_1(\Sigma, \mathcal{E}_0) \longrightarrow \mathcal{H}^{1,p}_+(\Sigma, D) \oplus \mathcal{H}^{1,p}_-(\Sigma, D), \end{aligned} \tag{3.4.21}$$

again by

$$\begin{aligned} \mathcal{J}_{1,p}(f_+, f_-) &:= f_+ - f_-, \\ \Gamma_{1,p} f &:= (\mathcal{P}_+ f, -\mathcal{P}_- f). \end{aligned} \tag{3.4.22}$$

Here $\mathcal{H}^{1,p}_\pm(\Sigma, D)$ is the image of $\mathcal{H}^{1,p}(\Omega_\pm, D)$ under the nontangential boundary trace isomorphism, or equivalently

$$\mathcal{H}^{1,p}_\pm(\Sigma, D) = \mathcal{P}_\pm L^p_1(\Sigma, \mathcal{E}_0). \tag{3.4.23}$$

In view of the mapping properties (3.4.19)–(3.4.20), the identities (3.3.10) and (3.3.12) restrict to the spaces in (3.4.21), i.e.,

$$\mathcal{J}_{1,p}\Gamma_{1,p} = \mathcal{P}_+ + \mathcal{P}_- = I + K, \tag{3.4.24}$$

on $L^p_1(\Sigma, \mathcal{E}_0)$, and

$$\Gamma_{1,p}\mathcal{J}_{1,p} = (f_+, f_-) - (Kf_-, Kf_+), \tag{3.4.25}$$

on $\mathcal{H}^{1,p}_+(\Sigma, D) \oplus \mathcal{H}^{1,p}_-(\Sigma, D)$.

We have the following immediate consequence.

Proposition 3.4.2 *Assume $\Omega_\pm$ are* UR *domains, with boundary Σ, and that D in invertible. Then we can take $K = 0$ in* (3.4.20)*, so, for $p \in (1, \infty)$, the operators $\mathcal{J}_{1,p}$ and $\Gamma_{1,p}$, defined by* (3.4.21)–(3.4.22)*, are two-sided inverses of each other. In particular, $\mathcal{J}_{1,p}$ in* (3.4.21) *is an isomorphism.*

If we know that the natural inclusion

$$L^p_1(\Sigma) \hookrightarrow L^p(\Sigma) \text{ is compact,} \tag{3.4.26}$$

then (3.4.20) implies $K : L^p_1(\Sigma) \to L^p_1(\Sigma)$ is compact, and we have the following.

Proposition 3.4.3 *Let $\Omega_\pm \subset M$ be* UR *domains, with boundary Σ, and assume* (3.4.26) *holds. Then, for each $p \in (1, \infty)$, the operators $\mathcal{J}_{1,p}$ and $\Gamma_{1,p}$ in* (3.4.21)–(3.4.22) *are Fredholm inverses of each other. As such,*

$$\operatorname{Index} \mathcal{J}_{1,p} = -\operatorname{Index} \Gamma_{1,p}. \tag{3.4.27}$$

In the situation of Proposition 3.4.3, the argument implying (3.2.21) also implies

$$\operatorname{Ker} \mathcal{J}_{1,p} \approx \operatorname{Ker} D. \tag{3.4.28}$$

We also have that the range of $\mathcal{J}_{1,p}$ in (3.4.21) is a closed linear subspace of $L^p_1(\Sigma, \mathcal{E}_0)$, of finite codimension. Its annihilator is then a finite dimensional linear subspace

$$V_{1,p} \subset L^{p'}_{-1}(\Sigma, \mathcal{E}_0) := \big(L^p_1(\Sigma, \mathcal{E}_0)\big)' \tag{3.4.29}$$

where $1/p + 1p' = 1$, consisting of elements v such that

$$\langle f_+ - f_-, v\rangle = 0, \quad \forall\, f_\pm \in H^{1,p}_\pm(\Sigma, D). \tag{3.4.30}$$

By comparison, the range of $\mathcal{J}_p$ in (3.2.17) is a closed linear subspace of $L^p(\Sigma, \mathcal{E}_0)$, of finite codimension, whose annihilator is

$$V_p \subset L^{p'}(\Sigma, \mathcal{E}_0), \tag{3.4.31}$$

consisting of elements v such that

$$\langle f_+ - f_-, v\rangle = 0, \quad \forall\, f_\pm \in \mathcal{H}^p_\pm(\Sigma, D). \tag{3.4.32}$$

Clearly

$$V_p \subset V_{1,p}. \tag{3.4.33}$$

We will establish that these two spaces are equal:

$$V_p = V_{1,p}. \tag{3.4.34}$$

A key step will be to show that

$$V_{1,p} \subset L^{p'}(\Sigma, \mathcal{E}_0). \tag{3.4.35}$$

This is a consequence of the following.

Lemma 3.4.4 *Assume $p, p' \in (1, \infty)$ satisfy $1/p + 1/p' = 1$. If $v \in L^{p'}_{-1}(\Sigma, \mathcal{E}_0)$ annihilates $(\mathcal{P}_+ + \mathcal{P}_-)L^p_1(\Sigma, \mathcal{E}_0)$, then $v \in L^{p'}(\Sigma, \mathcal{E}_0)$.*

Proof. The hypothesis is equivalent to

$$v \in L^{p'}_{-1}(\Sigma, \mathcal{E}_0), \quad \langle (I + K)f, v\rangle = 0, \quad \forall\, f \in L^p_1(\Sigma, \mathcal{E}_0), \tag{3.4.36}$$

hence to

$$(I + K^*)v = 0. \tag{3.4.37}$$

Now (3.4.20) implies

$$K^* : L^{p'}_{-1}(\Sigma, \mathcal{E}_0) \longrightarrow L^{p'}(\Sigma, \mathcal{E}_0), \tag{3.4.38}$$

which establishes $v = -K^*v \in L^{p'}(\Sigma, \mathcal{E}_0)$ and proves the lemma. ☐

Having (3.4.35), we can establish the following.

Proposition 3.4.5 *In the setting of Proposition 3.4.3, we have* (3.4.34)*. Hence, in addition to* (3.4.28)*, we also have*

$$\dim \operatorname{Coker} \mathcal{J}_{1,p} = \dim \operatorname{Coker} \mathcal{J}_p = \dim \operatorname{Ker} D^*. \tag{3.4.39}$$

Hence,

$$\operatorname{Index} \mathcal{J}_{1,p} = \operatorname{Index} D, \tag{3.4.40}$$

so

$$\operatorname{Index} \Gamma_{1,p} = -\operatorname{Index} D. \tag{3.4.41}$$

Proof. It remains to deduce (3.4.34) from (3.4.35). So take $v \in V_{1,p}$. We have $v \in L^{p'}(\Sigma, \mathcal{E}_0)$ and

$$\langle f_+ - f_-, v \rangle = 0 \tag{3.4.42}$$

for all $f_\pm \in H^{1,p}_\pm(\Sigma, D)$. It remains to show that (3.4.42) is true for all $f_\pm \in \mathcal{H}^p_\pm(\Sigma, D)$. This follows from the assertion that, for $p \in (1, \infty)$,

$$\mathcal{H}^{1,p}_\pm(\Sigma, D) \text{ is dense in } \mathcal{H}^p_\pm(\Sigma, D), \tag{3.4.43}$$

i.e., that

$$\mathcal{P}_\pm L^p_1(\Sigma, \mathcal{E}_0) \text{ is dense in } \mathcal{P}_\pm L^p(\Sigma, \mathcal{E}_0). \tag{3.4.44}$$

This in term follows from the fact that

$$L^p_1(\Sigma, \mathcal{E}_0) \text{ is dense in } L^p(\Sigma, \mathcal{E}_0), \tag{3.4.45}$$

which is elementary. In fact, for $p \in (1, \infty)$,

$$\mathrm{Lip}(\Sigma, \mathcal{E}_0) \text{ is dense in } L^p(\Sigma, \mathcal{E}_0), \tag{3.4.46}$$

and $\mathrm{Lip}(\Sigma) \subset L^p_1(\Sigma)$. □

The following is a restatement of the Fredholm properties of $\mathcal{J}_{1,p}$ given in Propositions 3.4.3 and 3.4.5.

Theorem 3.4.6 *The pair* $(\mathcal{H}^{1,p}_+(\Sigma, D), \mathcal{H}^{1,p}_-(\Sigma, D))$ *is a Fredholm pair for* $L^p_1(\Sigma, \mathcal{E}_0)$*, and*

$$\mathrm{Index}(\mathcal{H}^{1,p}_+(\Sigma, D), \mathcal{H}^{1,p}_-(\Sigma, D)) = \mathrm{Index}\, D. \tag{3.4.47}$$

For another perspective, we bring in

$$\begin{aligned} &\mathcal{A}_{1,p} : L^p_1(\Sigma, \mathcal{E}_0) \longrightarrow \mathcal{H}^{1,p}(\Omega_+, D) \oplus \mathcal{H}^{1,p}(\Omega_-, D), \\ &\mathcal{A}_{1,p} f = (\mathcal{C}_+ f, \mathcal{C}_- f), \qquad \forall\, f \in L^p_1(\Sigma, \mathcal{E}_0), \end{aligned} \tag{3.4.48}$$

parallel to (3.3.3). Parallel to (3.3.7), we have

$$\mathcal{J}_{1,p} \circ (\tau_+ \oplus (-\tau_-)) \circ \mathcal{A}_{1,p} = \mathcal{P}_+ + \mathcal{P}_- = I + K, \tag{3.4.49}$$

hence the following.

Proposition 3.4.7 *If* $\Omega_\pm$ *are* UR *domains with boundary* Σ *and* D *is invertible, then* $\mathcal{A}_{1,p}$ *is an isomorphism, for* $p \in (1, \infty)$.

Proposition 3.4.8 *In the setting of Proposition 3.4.3, if* $p \in (1, \infty)$*, the operator* $\mathcal{A}_{1,p}$ *is Fredholm, and*

$$\mathrm{Index}\, \mathcal{A}_{1,p} = -\,\mathrm{Index}\, D. \tag{3.4.50}$$

4 Riemann-Hilbert problems

As in §3, we assume M is a compact Riemannian manifold with a partition

$$M = \Omega_+ \cup \Omega_- \cup \Sigma, \tag{4.0.1}$$

where

$$\Omega_+ \text{ and } \Omega_- \text{ are UR domains, and } \partial\Omega_+ = \partial\Omega_- =: \Sigma. \tag{4.0.2}$$

We assume $D : \mathcal{E}_0 \to \mathcal{E}_1$ is a first order elliptic differential operator and that D and D^* satisfy UCP. We extend D to act componentwise on sections of $\mathcal{E}_0 \otimes \mathbb{C}^\ell$. For the Riemann-Hilbert problem we are given $\Phi \in L^\infty(\Sigma, M(\ell, \mathbb{C}))$ and $f \in L^p(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$, and we seek

$$u_\pm \in \mathcal{H}^p(\Omega_\pm, D) \text{ such that } \Phi u_+\big|_\Sigma^{\text{n.t.}} - u_-\big|_\Sigma^{\text{n.t.}} = f. \tag{4.0.3}$$

Here, the spaces $\mathcal{H}^p(\Omega_\pm, D)$ are defined as in (1.0.4).

For an operator formulation of the Riemann-Hilbert problem, we define

$$\begin{aligned} &R_\Phi : \mathcal{H}^p(\Omega_+, D) \oplus \mathcal{H}^p(\Omega_-, D) \longrightarrow L^p(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell), \\ &R_\Phi(u, v) := \Phi u\big|_\Sigma^{\text{n.t.}} - v\big|_\Sigma^{\text{n.t.}}, \quad u \in \mathcal{H}^p(\Omega_+, D),\ v \in \mathcal{H}^p(\Omega_-, D), \end{aligned} \tag{4.0.4}$$

so (4.0.3) becomes $R_\Phi(u_+, u_-) = f$. We are interested in obtaining conditions on Φ that imply R_Φ is Fredholm, and seek information on its index. Note that, corresponding to the case when $\Phi = I$, the identity, we have

$$R_I = \mathcal{J}_p \text{ on } L^p(\Sigma, \mathcal{E}_0), \tag{4.0.5}$$

where $\mathcal{J}_p$ is as in (3.2.17). Hence, by (3.2.20),

$$\operatorname{Index} R_I = \operatorname{Index} D. \tag{4.0.6}$$

More generally, we relate the Fredholm behavior of R_Φ to that of the Toeplitz operator

$$T_\Phi = \mathcal{P}_+ \Phi \mathcal{P}_+ + (I - \mathcal{P}_+), \tag{4.0.7}$$

introduced in §2, with $\Omega = \Omega_+$, $\mathcal{P} = \mathcal{P}_+$. In §4.1 we show that R_Φ is Fredholm if and only if T_Φ is Fredholm, and if so

$$\operatorname{Index} R_\Phi = \operatorname{Index} T_\Phi + \ell \cdot \operatorname{Index} D. \tag{4.0.8}$$

Here $\operatorname{Index} D$ denotes the index of D acting globally, on sections of $\mathcal{E}_0$. Acting on sections of $\mathcal{E}_0 \otimes \mathbb{C}^\ell$ multiplies its index by ℓ. We obtain extra information in case D is invertible.

In §4.2 we study Riemann-Hilbert problems on L^p-Sobolev spaces in cases where, in addition to (4.0.1)–(4.0.2), we assume $\Omega_\pm$ satisfy a two-sided John condition (a condition reviewed in §A.2).

In §4.3 we look at classical Riemann-Hilbert problems on domains in the complex plane $\mathbb{C}$ (but in the more general setting of UR domains). We show how one can transfer these to problems on the Riemann sphere S^2 and how results of §§4.1–4.2 apply. In this transformation, it is natural to view the unknowns $u_\pm$ as sections of a holomorphic line bundle over S^2. In §4.4 we take up Riemann-Hilbert problems on more general compact Riemann surfaces, involving sections of holomorphic line bundles, and also holomorphic vector bundles. Background material on Riemann surface theory, particularly the Riemann-Roch theorem, is given in §A.4.

4.1 Connection with Toeplitz operators

Here we compare the Fredholm properties of the Riemann-Hilbert operator R_Φ, defined by (4.0.4), and the Toeplitz operator T_Φ, defined by (4.0.7).

Proposition 4.1.1 *For each $p \in (1, \infty)$, the operator R_Φ is Fredholm in* (4.0.4) *if and only if T_Φ is Fredholm on $L^p(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$. In such a case,*

$$\operatorname{Index} R_\Phi = \operatorname{Index} T_\Phi + \ell \cdot \operatorname{Index} D. \tag{4.1.1}$$

Proof. We bring in the following variant of $\mathcal{A}_p$ from (3.3.3), namely

$$\begin{aligned} \widetilde{\mathcal{A}}_p &: L^p(\Sigma, \mathcal{E}_0) \longrightarrow \mathcal{H}^p(\Omega_+, D) \oplus \mathcal{H}^p(\Omega_-, D), \\ \widetilde{\mathcal{A}}_p f &:= (\mathcal{C}_+ f, -\mathcal{C}_- f), \qquad \forall\, f \in L^p(\Sigma, \mathcal{E}_0). \end{aligned} \tag{4.1.2}$$

As in Proposition 3.3.2, we have $\widetilde{\mathcal{A}}_p$ Fredholm, and

$$\operatorname{Index} \widetilde{\mathcal{A}}_p = -\operatorname{Index} D. \tag{4.1.3}$$

If $\widetilde{\mathcal{A}}_p$ acts on sections of $\mathcal{E}_0 \otimes \mathbb{C}^\ell$, multiply the index by ℓ. Now

$$\begin{aligned} R_\Phi \widetilde{\mathcal{A}}_p f &= \Phi \mathcal{C}_+ f\Big|_{\partial\Omega_+}^{\text{n.t.}} + \mathcal{C}_- f\Big|_{\partial\Omega_-}^{\text{n.t.}} \\ &= \Phi \mathcal{P}_+ f + \mathcal{P}_- f \\ &= \widetilde{T}_\Phi f, \end{aligned} \tag{4.1.4}$$

where

$$\widetilde{T}_\Phi = \Phi \mathcal{P}_+ + \mathcal{P}_-. \tag{4.1.5}$$

It follows that R_Φ is Fredholm on $L^p(\Sigma)$ if and only if $\widetilde{T}_\Phi$ is, and, if so,

$$\operatorname{Index} R_\Phi = \operatorname{Index} \widetilde{T}_\Phi + \ell \cdot \operatorname{Index} D. \tag{4.1.6}$$

The proof is completed by comparing $\widetilde{T}_\Phi$ and T_Φ, a step carried out separately, in the proposition below. □

Proposition 4.1.2 *Given $\Phi \in L^\infty(\Sigma, M(\ell, \mathbb{C}))$, $p \in (1, \infty)$, the operator T_Φ, defined by* (4.0.7)*, is Fredholm on $L^p(\Sigma)$ if and only if $\widetilde{T}_\Phi$, defined by* (4.1.5)*, is, and if so,*

$$\operatorname{Index} \widetilde{T}_\Phi = \operatorname{Index} T_\Phi. \tag{4.1.7}$$

Proof. By (3.0.11), T_Φ differs from

$$\widehat{T}_\Phi = \mathcal{P}_+ \Phi \mathcal{P}_+ + \mathcal{P}_- \tag{4.1.8}$$

by an operator that is compact on $L^p(\Sigma)$ for all $p \in (1, \infty)$. Thus it suffices to compare Fredholmness of $\widetilde{T}_\Phi$ and $\widehat{T}_\Phi$. To do this, we bring in

$$Q_\Phi^\pm = I \pm (I - \mathcal{P}_+) \Phi \mathcal{P}_+. \tag{4.1.9}$$

These operators are bounded on $L^p(\Sigma)$ for each $p \in (1, \infty)$, and

$$Q_\Phi^+ Q_\Phi^- = Q_\Phi^- Q_\Phi^+ = I. \tag{4.1.10}$$

Furthermore, a direct calculation gives

$$\widehat{T}_\Phi Q^+_\Phi = \widetilde{T}_\Phi + K(I-\mathcal{P}_+)\Phi\mathcal{P}_+, \tag{4.1.11}$$

where, as in (3.0.11), $K = \mathcal{P}_+ + \mathcal{P}_- - I$ is compact on $L^p(\Sigma)$ for all $p \in (1,\infty)$. Since Q^+_Φ is invertible, this establishes the equivalence of $\widehat{T}_\Phi$ and $\widetilde{T}_\Phi$ being Fredholm, and that

$$\operatorname{Index}\widetilde{T}_\Phi = \operatorname{Index}\widehat{T}_\Phi = \operatorname{Index}T_\Phi. \tag{4.1.12}$$

This finishes the proof of Proposition 4.1.2 (thus also completing the proof of Proposition 4.1.1). □

To the list of "Toeplitz operators" compared in (4.1.12) it is useful to add another, namely

$$\begin{aligned} &T^+_\Phi : \mathcal{H}^p_+(\Sigma, D) \longrightarrow \mathcal{H}^p_+(\Sigma, D), \\ &T^+_\Phi f := \mathcal{P}_+\Phi f, \qquad \forall\, f \in \mathcal{H}^p_+(\Sigma, D). \end{aligned} \tag{4.1.13}$$

It readily follows that T^+_Φ is Fredholm in (4.1.13) if and only if T_Φ is Fredholm on $L^p(\Sigma)$, and, if so,

$$\operatorname{Index}T^+_\Phi = \operatorname{Index}T_\Phi. \tag{4.1.14}$$

As seen in §2, the operator T_Φ is Fredholm on $L^p(\Sigma)$ for all $p \in (1,\infty)$ whenever

$$\Phi, \Phi^{-1} \in C^0(\Sigma, M(\ell, \mathbb{C})) \tag{4.1.15}$$

and, more generally, when

$$\Phi, \Phi^{-1} \in L^\infty \cap \mathrm{vmo}(\Sigma, M(\ell, \mathbb{C})), \tag{4.1.16}$$

which is actually equivalent to

$$\Phi \in L^\infty \cap \mathrm{vmo}(\Sigma, M(\ell, \mathbb{C})), \quad \Phi^{-1} \in L^\infty(\Sigma, M(\ell, \mathbb{C})). \tag{4.1.17}$$

In such cases, the index of T_Φ on $L^p(\Sigma)$ is independent of $p \in (1,\infty)$. Furthermore, $\operatorname{Ker}T_\Phi$ on $L^p(\Sigma)$ is independent of $p \in (1,\infty)$, as is $\operatorname{Ker}T^*_\Phi$ on $L^{p'}(\Sigma)$ (as usual, $1/p + 1/p' = 1$). The same holds for $\widetilde{T}_\Phi$, though $\operatorname{Ker}\widetilde{T}_\Phi$ might differ from $\operatorname{Ker}T_\Phi$.

We next record a refinement of Proposition 4.1.1, which holds if the elliptic operator D is invertible. In such a case, as noted in §3, we can take $E_+ = E_-$ to be the integral kernel of D^{-1} to define $\mathcal{C}_\pm$ in (3.0.5), and then $K = 0$ in (3.0.11), i.e., $\mathcal{P}_- = I - \mathcal{P}_+$, so (4.1.5) becomes

$$\widetilde{T}_\Phi = \Phi\mathcal{P}_+ + (I - \mathcal{P}_+). \tag{4.1.18}$$

Proposition 4.1.3 *In the setting of Proposition 4.1.1, if D is invertible, then there is a natural isomorphism*

$$\operatorname{Ker}R_\Phi \approx \operatorname{Ker}\widetilde{T}_\Phi. \tag{4.1.19}$$

Furthermore, for each $p \in (1,\infty)$,

$$\operatorname{Range}R_\Phi = \operatorname{Range}\widetilde{T}_\Phi \ \text{ in } \ L^p(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell). \tag{4.1.20}$$

Proof. Recall

$$\mathcal{J}_p : \mathcal{H}^p_+(\Sigma, D) \oplus \mathcal{H}^p_-(\Sigma, D) \longrightarrow L^p(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell), \tag{4.1.21}$$

from (3.2.17), extended componentwise to sections of $\mathcal{E}_0 \otimes \mathbb{C}^\ell$. By Corollary 3.2.2, if D is invertible, then $\mathcal{J}_p$ is an isomorphism. In the present case, (3.3.7) holds with $K = 0$. It follows that $\mathcal{A}_p$ in (3.3.3) is an isomorphism, and similarly $\widetilde{\mathcal{A}}_p$ in (4.1.2) is an isomorphism. Given this, (4.1.19) and (4.1.20) follow directly from (4.1.4). □

4.2 Riemann-Hilbert problems on L^p-Sobolev spaces

Here we take (with $n := \dim M$)

$$\Phi \in L^q_1(\Sigma, M(\ell, \mathbb{C})), \quad \text{with } q \in (n-1, \infty). \tag{4.2.1}$$

We assume (4.0.1)–(4.0.2) hold and

$$\Omega_\pm \text{ satisfy a two-sided John condition.} \tag{4.2.2}$$

Then, for $p \in (1, q]$, we have (cf. Proposition A.1.5)

$$\begin{aligned} &R_\Phi : \mathcal{H}^{1,p}(\Omega_+, D) \oplus \mathcal{H}^{1,p}(\Omega_-, D) \longrightarrow L^p_1(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell), \\ &R_\Phi(u, v) := \Phi u\big|^{\text{n.t.}}_{\Sigma} - v\big|^{\text{n.t.}}_{\Sigma}, \quad u \in \mathcal{H}^{1,p}(\Omega_+, D),\, v \in \mathcal{H}^{1,p}(\Omega_-, D), \end{aligned} \tag{4.2.3}$$

with $\mathcal{H}^{1,p}(\Omega_\pm, D)$ as in (3.4.18), but with $\mathcal{E}_0$ replaced by $\mathcal{E}_0 \otimes \mathbb{C}^\ell$.

We desire to study the Fredholm properties of this family of operators. Parallel to (4.1.4), we have

$$R_\Phi \widetilde{\mathcal{A}}_{1,p} = \widetilde{T}_\Phi, \tag{4.2.4}$$

where $\widetilde{\mathcal{A}}_{1,p}$ is the following variant of $\mathcal{A}_{1,p}$, from (3.4.48):

$$\begin{aligned} &\widetilde{\mathcal{A}}_{1,p} : L^p_1(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell) \longrightarrow \mathcal{H}^{1,p}(\Omega_+, D) \oplus \mathcal{H}^{1,p}(\Omega_-, D), \\ &\widetilde{\mathcal{A}}_{1,p} f := (\mathcal{C}_+ f, -\mathcal{C}_- f), \qquad \forall\, f \in L^p_1(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell). \end{aligned} \tag{4.2.5}$$

The operator

$$\widetilde{T}_\Phi : L^p_1(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell) \longrightarrow L^p_1(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell) \tag{4.2.6}$$

is given by the same formula as (4.1.5), i.e.,

$$\widetilde{T}_\Phi f = \Phi \mathcal{P}_+ f + \mathcal{P}_- f, \qquad \forall\, f \in L^p_1(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell). \tag{4.2.7}$$

By comparison with

$$\begin{aligned} &T_\Phi : L^p_1(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell) \longrightarrow L^p_1(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell), \\ &T_\Phi f = \mathcal{P}_+ \Phi \mathcal{P}_+ f + (I - \mathcal{P}_+) f, \quad \forall\, f \in L^p_1(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell), \end{aligned} \tag{4.2.8}$$

the analysis from §4.3 of [21] yielding Proposition 2.7.1 now gives

$$\Phi \mathcal{P}_+ - \mathcal{P}_+ \Phi \mathcal{P}_+ \text{ compact on } L^p_1(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell), \tag{4.2.9}$$

for $p \in (1, q]$, and, as in §3.4, the hypothesis (4.2.2) implies $\mathcal{P}_- - (I - \mathcal{P}_+) = K$ is compact on $L^p_1(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$, so

$$\widetilde{T}_\Phi - T_\Phi \text{ is compact on } L^p_1(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell), \text{ for } p \in (1, q]. \tag{4.2.10}$$

Thus Fredholm results given in §2.7 apply. We have the following.

Theorem 4.2.1 *Assume $\Omega_\pm$ are* UR *domains with boundary Σ and* (4.2.2) *holds. Also, suppose*

$$\Phi \in L^q_1(\Sigma, \mathrm{GL}(\ell, \mathbb{C})), \quad \textit{with } q \in (n-1, \infty). \tag{4.2.11}$$

Then, for each $p \in (1, q]$, the operator R_Φ in (4.2.3) *is Fredholm, and*

$$\operatorname{Index} R_\Phi = \operatorname{Index} T_\Phi + \ell \cdot \operatorname{Index} D. \tag{4.2.12}$$

Proof. It follows from Proposition 3.4.8 that $\widetilde{\mathcal{A}}_{1,p}$ in Fredholm in (4.2.5), with index equal to $-\ell \cdot \operatorname{Index} D$. Proposition 2.7.1 implies T_Φ is Fredholm in (4.2.8), and (4.2.10) then implies $\widetilde{T}_\Phi$ is also Fredholm, with the same index. The conclusion in (4.2.12) then follows by relying on (4.2.4). □

We also have the following analogue of Proposition 4.1.3.

Proposition 4.2.2 *In the setting of Theorem 4.2.1, if the operator D is invertible, then there is a natural isomorphism*

$$\operatorname{Ker} R_\Phi \approx \operatorname{Ker} \widetilde{T}_\Phi, \tag{4.2.13}$$

with $\widetilde{T}_\Phi$ *acting on* $L^p_1(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$ *as in* (4.2.6)*. Furthermore, for each* $p \in (1, q]$*,*

$$\operatorname{Range} R_\Phi = \operatorname{Range} \widetilde{T}_\Phi \ \text{ in } \ L^p_1(\Sigma, \mathcal{E}_0 \otimes \mathbb{C}^\ell). \tag{4.2.14}$$

Proof. Parallel to the proof of Proposition 4.1.3, this time we have $\mathcal{J}_{1,p}$ an isomorphism in (3.4.21), hence $\mathcal{A}_{1,p}$ an isomorphism in (3.4.48), hence $\widetilde{\mathcal{A}}_{1,p}$ an isomorphism in (4.2.5). Thus (4.2.13)–(4.2.14) follow from (4.2.4). □

4.3 Planar domains and domains in S^2

To start, let $\Omega = \Omega_+$ be a bounded UR domain in $\mathbb{C} \approx \mathbb{R}^2$. Assume

$$\partial\Omega = \Sigma = \partial(\mathbb{C} \setminus \overline{\Omega}). \tag{4.3.1}$$

We take $D := \overline{\partial} = \partial/\partial\overline{z}$ and, for $p \in (1, \infty)$, consider the Riemann-Hilbert problem

$$\Phi u_+\big|^{\text{n.t.}}_{\Sigma} - u_-\big|^{\text{n.t.}}_{\Sigma} = g \ \text{ on } \ \Sigma, \tag{4.3.2}$$

where g is given in $L^p(\Sigma, \mathbb{C}^\ell)$. We want to solve this problem for $u_+ \in \mathcal{H}^p(\Omega_+, \overline{\partial})$ and u_- in some version of $\mathcal{H}^p(\Omega_-, \overline{\partial})$, with $\Omega_- = \mathbb{C} \setminus \overline{\Omega}$. In particular, we want u_- to be holomorphic on Ω_-, and satisfy

$$\begin{aligned} &\mathcal{N}(u_-) \in L^p(\Sigma), \ \text{ and the non-tangential boundary} \\ &\text{limit } u_-\big|^{\text{n.t.}}_{\Sigma} \text{ exists at } \sigma\text{-a.e. point on } \Sigma. \end{aligned} \tag{4.3.3}$$

A standard attack (parallel to that in (1.0.27)–(1.0.29)) takes

$$u_\pm = \mathcal{C}f \ \text{ in } \ \Omega_\pm, \tag{4.3.4}$$

where $\mathcal{C}$ is the classical Cauchy integral operator,

$$\mathcal{C}f(z) := \frac{1}{2\pi i} \int_\Sigma \frac{f(\zeta)}{\zeta - z}\, d\zeta, \qquad \forall\, z \in \mathbb{C} \setminus \Sigma. \tag{4.3.5}$$

Here, as in (A.3.15),

$$d\zeta = i\nu(\zeta)\, d\sigma(\zeta). \tag{4.3.6}$$

It is clear that the right side of (4.3.5) vanishes as $|z| \to \infty$, so we are motivated to define

$$\begin{aligned} \mathcal{H}^p(\Omega_-, \overline{\partial}) := \big\{ &u \in C^0(\Omega_-, \mathbb{C}^\ell) : \overline{\partial} u = 0,\ u(z) \to 0 \text{ as } |z| \to \infty,\ \mathcal{N}(u) \in L^p(\Sigma), \\ &\text{and } u_-\big|^{\text{n.t.}}_{\Sigma} \text{ exists at } \sigma\text{-a.e. point on } \Sigma \big\}. \end{aligned} \tag{4.3.7}$$

Given $\Phi \in L^\infty(\Sigma, M(\ell,\mathbb{C}))$, we have, for $p \in (1,\infty)$,

$$
\begin{aligned}
&R_\Phi : \mathcal{H}^p(\Omega_+,\overline{\partial}) \oplus \mathcal{H}^p(\Omega_-,\overline{\partial}) \longrightarrow L^p(\Sigma,\mathbb{C}^\ell),\\
&R_\Phi(u,v) := \Phi u\big|_\Sigma^{\text{n.t.}} - v\big|_\Sigma^{\text{n.t.}}, \quad u \in \mathcal{H}^p(\Omega_+,\overline{\partial}),\ v \in \mathcal{H}^p(\Omega_-,\overline{\partial}).
\end{aligned}
\tag{4.3.8}
$$

Arguments parallel to those in §4.1 show that, if

$$
\Phi, \Phi^{-1} \in L^\infty \cap \mathrm{vmo}(\Sigma, M(\ell,\mathbb{C})), \tag{4.3.9}
$$

then the operator R_Φ is Fredholm, with

$$
\mathrm{Index}\, R_\Phi = \mathrm{Index}\, \widetilde{T}_\Phi, \tag{4.3.10}
$$

where

$$
\widetilde{T}_\Phi = \Phi\mathcal{P}_+ + (I - \mathcal{P}_+). \tag{4.3.11}
$$

Furthermore,

$$
\mathrm{Ker}\, R_\Phi \approx \mathrm{Ker}\, \widetilde{T}_\Phi, \quad \mathrm{Range}\, R_\Phi = \mathrm{Range}\, \widetilde{T}_\Phi. \tag{4.3.12}
$$

We will recast the current Riemann-Hilbert problem into one to which §4.1 applies directly. Before getting to this, we mention a variant of the problem formulated above. Namely, one often wants to solve (4.3.2), not with $u(z) \to 0$ as $|z| \to \infty$, but rather with

$$
u(z) \longrightarrow A \ \text{ as } \ |z| \to \infty, \tag{4.3.13}
$$

for some $A \in \mathbb{C}^\ell$. In such a case, observe that

$$
v := u - A \Longrightarrow v\big|_{\Omega_\pm} \in \mathcal{H}^p(\Omega_\pm,\overline{\partial}). \tag{4.3.14}
$$

Then the task is to solve

$$
\Phi v_+\big|_\Sigma^{\text{n.t.}} - v_-\big|_\Sigma^{\text{n.t.}} = g + (I - \Phi)A, \tag{4.3.15}
$$

with $v_\pm \in \mathcal{H}^p(\Omega_\pm,\overline{\partial})$.

We now transfer our Riemann-Hilbert problem to one for domains in the Riemann sphere $\mathbb{C} \cup \{\infty\} \approx S^2$, obtained as the one-point compactification of $\mathbb{C}$. This will serve not only to produce a problem to which the results of §4.1 (and §4.2) are directly applicable, but also to suggest further Riemann-Hilbert problems.

We now have

$$
S^2 = \Omega_+ \cup \Sigma \cup \Omega_-, \tag{4.3.16}
$$

where we have added p, the point at infinity, to Ω_-. In light of (4.3.7), we want to solve (4.3.2), not for scalar $u_\pm$, but for sections of the line bundle

$$
L = E_p, \tag{4.3.17}
$$

described in §A.4. Thus, in place of (4.3.8), we consider the operator

$$
\begin{aligned}
&R_\Phi : \mathcal{H}^p(\Omega_+,\overline{\partial}_L) \oplus \mathcal{H}^p(\Omega_-,\overline{\partial}_L) \longrightarrow L^p(\Sigma, L \otimes \mathbb{C}^\ell),\\
&R_\Phi(u,v) := \Phi u\big|_\Sigma^{\text{n.t.}} - v\big|_\Sigma^{\text{n.t.}}, \quad u \in \mathcal{H}^p(\Omega_+,\overline{\partial}_L),\ v \in \mathcal{H}^p(\Omega_-,\overline{\partial}_L),
\end{aligned}
\tag{4.3.18}
$$

with

$$\mathcal{H}^p(\Omega_\pm, \overline{\partial}_L) := \big\{u \in C^0(\Omega_\pm, L \otimes \mathbb{C}^\ell) : \overline{\partial}_L u = 0 \text{ in } \Omega_\pm,\ \mathcal{N}u \in L^p(\Sigma),\\ \text{and } u\big|_\Sigma^{\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \Sigma\big\}. \tag{4.3.19}$$

Here, as in §A.4,

$$\overline{\partial}_L : H^{s+1,p}(S^2, L) \longrightarrow H^{s,p}(S^2, L \otimes \overline{\kappa}). \tag{4.3.20}$$

We are trying to solve

$$R_\Phi(u_+, u_-) = g, \qquad g \in L^p(\Sigma, L \otimes \mathbb{C}^\ell), \tag{4.3.21}$$

with L as in (4.3.17). We continue to assume Φ satisfies (4.3.9). If we denote the functions of interest in (4.3.15) by $v_\pm$ and $\tilde{g} = g_0 + (I - \Phi)A$ (relabeling the g in (4.3.15) as g_0), they are related to $(u_\pm, g)$ in (4.3.21) by

$$u_\pm = v_\pm \psi, \qquad g = \tilde{g}\psi, \tag{4.3.22}$$

with $\psi \in \mathcal{M}(E_p)$ as in (A.4.19). Note that the pole of ψ at p cancels the zero of v_- at p.

Now, for $M = S^2$, Proposition A.4.2 gives

$$L = E_p \Longrightarrow \overline{\partial}_L \text{ is invertible.} \tag{4.3.23}$$

Consequently, Propositions 4.1.1–4.1.3 apply. We have

$$\operatorname{Index} R_\Phi = \operatorname{Index} \widetilde{T}_\Phi = \operatorname{Index} T_\Phi, \tag{4.3.24}$$

with

$$\widetilde{T}_\Phi = \Phi\mathcal{P}_+ + (I - \mathcal{P}_+), \quad T_\Phi = \mathcal{P}_+\Phi\mathcal{P}_+ + (I - \mathcal{P}_+), \tag{4.3.25}$$

and also

$$\operatorname{Ker} R_\Phi \approx \operatorname{Ker} \widetilde{T}_\Phi, \quad \operatorname{Range} R_\Phi = \operatorname{Range} \widetilde{T}_\Phi \text{ in } L^p(\Sigma, E_p \otimes \mathbb{C}^\ell), \tag{4.3.26}$$

for R_Φ in (4.3.18). Here, $\widetilde{T}_\Phi$ is not identical to the $\widetilde{T}_\Phi$ in (4.3.11), which acts on $L^p(\Sigma, \mathbb{C}^\ell)$, but they are intertwined by the action of multiplication by ψ, and so have the same index.

In particular, the index computation of [21, Proposition 4.1.6] applies. We describe that result. Assume that the bounded UR domain $\Omega \subset \mathbb{C}$ is connected and has the property that $\partial\Omega$ is a disjoint union

$$\partial\Omega = \bigcup_{j=0}^{\mu} \gamma_j, \quad \text{each } \gamma_j \text{ a simple closed curve.} \tag{4.3.27}$$

Say γ_0 is the outer boundary and γ_j for $j \ge 1$ enclose bounded components of $\mathbb{C} \setminus \overline{\Omega}$. Let each γ_j have the orientation induced as a boundary component of Ω (counterclockwise for γ_0, clockwise for the other γ_j). First, we assume $\ell = 1$, and

$$\Phi \in C^0(\partial\Omega, \mathbb{C} \setminus 0). \tag{4.3.28}$$

Let $w_j(\Phi)$ denote the winding number of $\Phi|_{\gamma_j}$ about 0. Then [21, Proposition 4.1.6] asserts that, for each $p \in (1, \infty)$, the index of T_Φ on $L^p(\partial\Omega)$ is given by

$$\operatorname{Index} T_\Phi = -\sum_{j=0}^{\mu} w_j(\Phi). \tag{4.3.29}$$

We record the consequence for the Riemann-Hilbert problem.

Proposition 4.3.1 *Let $\Omega \subset \mathbb{C}$ be a bounded, connected* UR *domain, satisfying* (4.3.27)*, and assume Φ satisfies* (4.3.28)*. Then*

$$\operatorname{Index} R_\Phi = -\sum_{j=0}^{\mu} w_j(\Phi), \tag{4.3.30}$$

both for R_Φ in (4.3.8) *and for R_Φ in* (4.3.18)*, with $L = E_p$.*

It is simple enough to extend this result to

$$\Phi \in C^0(\partial\Omega, \mathrm{GL}(\ell, \mathbb{C})). \tag{4.3.31}$$

Parallel to (2.6.4)–(2.6.5), we can write

$$\Phi(x) = \Phi_0(x)\Phi_1(x), \tag{4.3.32}$$

with

$$\Phi_0(x) = \begin{pmatrix} \varphi(x) & \\ & I \end{pmatrix}, \quad \varphi(x) = \det \Phi(x), \quad \Phi_1 \in C^0(\partial\Omega, \mathrm{SL}(\ell, \mathbb{C})), \tag{4.3.33}$$

where $\mathrm{SL}(\ell, \mathbb{C})$ consists of elements of $\mathrm{GL}(\ell, \mathbb{C})$ of determinant 1. Then

$$\operatorname{Index} T_\Phi = \operatorname{Index} T_{\Phi_0} + \operatorname{Index} T_{\Phi_1}.$$

But since $\mathrm{SL}(\ell, \mathbb{C})$ is simply connected, homotopy invariance yields $\operatorname{Index} T_{\Phi_1} = 0$. Hence

$$\operatorname{Index} T_\Phi = \operatorname{Index} T_\varphi, \tag{4.3.34}$$

so the conclusion (4.3.30) of Proposition 4.3.1 in this setting becomes

$$\operatorname{Index} R_\Phi = -\sum_{j=0}^{\mu} w_j(\det \Phi). \tag{4.3.35}$$

Methods discussed in §2.3 also allow one to extend Proposition 4.3.1 to the setting (4.3.9). It is also straightforward to generalize to the case when Ω has several connected components.

On the other hand, if $\Omega \subset \mathbb{C}$ is a bounded UR domain satisfying (4.3.27), and, in addition

$$\Omega \text{ satisfies a two-sided John condition,} \tag{4.3.36}$$

and

$$\Phi \in L^q_1(\partial\Omega, \mathrm{GL}(\ell, \mathbb{C})) \ \text{ for some } \ q \in (1, \infty), \tag{4.3.37}$$

then the results of §4.2 apply, and we have (4.3.35) for the action of R_Φ on the variant of (4.3.19),

$$R_\Phi : \mathcal{H}^{1,p}(\Omega_+, \overline{\partial}_L) \oplus \mathcal{H}^{1,p}(\Omega_-, \overline{\partial}_L) \longrightarrow L^p_1(\Sigma, L \otimes \mathbb{C}^\ell), \tag{4.3.38}$$

for $p \in (1, q]$, or equivalently on the variant of (4.3.8),

$$R_\Phi : \mathcal{H}^{1,p}(\Omega_+, \overline{\partial}) \oplus \mathcal{H}^{1,p}(\Omega_-, \overline{\partial}) \longrightarrow L^p_1(\Sigma, \mathbb{C}^\ell), \tag{4.3.39}$$

where, as in (4.3.7), one incorporates vanishing as $|z| \to \infty$ into the definition of $\mathcal{H}^{1,p}(\Omega_-, \overline{\partial})$.

It is useful to note that, in the setting of (4.3.18), with $L = E_p$, the analysis of the Fredholm properties of R_Φ, including the identities (4.3.24), works regardless of whether $p \in \Omega_-$. One might as well have $p \in \Sigma = \partial\Omega_\pm$. We will illustrate this with the following example, mentioned in the introduction.

Let $\Sigma \subset \mathbb{C}$ consist of six rays:

$$\Sigma = \bigcup_{k=0}^{5} \{re^{k\pi i/3} : 0 \le r < \infty\}. \tag{4.3.40}$$

Thus $\mathbb{C} \setminus \Sigma$ has six connected components. We can set

$$\mathbb{C} \setminus \Sigma = \Omega_+ \cup \Omega_-, \tag{4.3.41}$$

where Ω_+ and Ω_- each has three connected components. To wit, $\Omega_\pm = \bigcup_{j=1}^{3} \Omega_{\pm,j}$, with

$$\begin{aligned} 0 < \operatorname{Arg}\zeta < \frac{\pi}{3} \quad &\text{for } \zeta \in \Omega_{+,1}, \\ \frac{2\pi}{3} < \operatorname{Arg}\zeta < \pi \quad &\text{for } \zeta \in \Omega_{+,2}, \\ \frac{4\pi}{3} < \operatorname{Arg}\zeta < \frac{5\pi}{3} \quad &\text{for } \zeta \in \Omega_{+,3}, \end{aligned} \tag{4.3.42}$$

and

$$\Omega_{-,j} = e^{\pi i/3}\Omega_{+,j}. \tag{4.3.43}$$

We now compactify $\mathbb{C}$ to S^2, adding the point $p = \infty$, so $\Omega_\pm \subset S^2$. Abusing notation slightly, we add p to Σ, so

$$S^2 = \Omega_+ \cup \Omega_- \cup \Sigma. \tag{4.3.44}$$

It is clear that Ω_+ and Ω_- are Ahlfors regular and have big pieces of Lipschitz surfaces, so Ω_+ and Ω_- are UR domains.

The fact that $p \in \Sigma$ here does create an extra wrinkle in passing from a Riemann-Hilbert problem in $\mathbb{C}$ to one on S^2, which we now examine. Let us start with the problem on $\mathbb{C}$:

$$\Phi\tilde{u}_+\big|_{\Sigma}^{\text{n.t.}} - \tilde{u}_-\big|_{\Sigma}^{\text{n.t.}} = g_0, \tag{4.3.45}$$

with $\tilde{u}_\pm$ holomorphic on $\Omega_\pm$, $\mathcal{N}(\tilde{u})$ having certain bounds on Σ, and

$$\tilde{u}_\pm(z) \longrightarrow A, \quad \text{as } |z| \to \infty, \tag{4.3.46}$$

for some $A \in \mathbb{C}^\ell$. As in (4.3.14)–(4.3.15), we set

$$v_\pm := \tilde{u}_\pm - A, \tag{4.3.47}$$

and seek solutions to

$$\Phi v_+\big|_{\Sigma}^{\text{n.t.}} - v_-\big|_{\Sigma}^{\text{n.t.}} = g_0 + (I - \Phi)A, \tag{4.3.48}$$

with certain bounds on $\mathcal{N}(v_\pm)$ and having $v_\pm(z) \to 0$ as $|z| \to \infty$. The precise conditions placed on $v_\pm$ will be apparent once we pass to the transformed problem. Namely, as in (4.3.22), we set

$$u_\pm = v_\pm\psi_p, \quad g = (g_0 + (I - \Phi)A)\psi_p, \tag{4.3.49}$$

with $\psi_p \in \mathcal{M}(E_p)$ as in (A.4.19), and seek

$$u_\pm \in \mathcal{H}^p(\Omega_\pm, L), \quad L = E_p, \tag{4.3.50}$$

satisfying

$$R_\Phi(u_+, u_-) = g, \tag{4.3.51}$$

with R_Φ as in (4.3.18). For this, we need

$$g = (g_0 + (I - \Phi)A)\psi_p \in L^p(\Sigma, L \otimes \mathbb{C}^\ell). \tag{4.3.52}$$

In particular, since one wants to allow $g_0 = 0$, we need $(I - \Phi)A\psi_p \in L^p(\Sigma, L \otimes \mathbb{C}^\ell)$. Recall that ψ_p has a simple pole at p. Hence, a natural condition to place on Φ is

$$\|I - \Phi(z)\| \le c \operatorname{dist}(z, p), \tag{4.3.53}$$

the right side involving the distance in the standard metric on S^2. Returning to Φ defined on $\mathbb{C}$, this hypothesis is

$$\|I - \Phi(z)\| \le \frac{c}{|z|}, \quad \text{for } |z| \ge 1. \tag{4.3.54}$$

Under these hypotheses, the equation (4.3.51) is again within the framework of §4.1. The operator R_Φ has the form (4.3.18), again with $L = E_p$, and it is Fredholm, as long as

$$\Phi \in C^0(\Sigma, \mathrm{GL}(\ell, \mathbb{C})), \tag{4.3.55}$$

or more generally Φ satisfies (4.3.9). Keep in mind that in the current setting we do also want to require (4.3.53).

It is interesting to transform the Riemann-Hilbert problem (4.3.49) one more time. Namely, pick $q \in \Omega_- \subset S^2$, and set

$$w_\pm = u_\pm \psi_q^{-1} = v_\pm \psi_p \psi_q^{-1}, \qquad h = g\psi_q^{-1}, \tag{4.3.56}$$

with $\psi_q \in \mathcal{M}(E_q)$, the counterpart to ψ_p, as in (A.4.19), but this time with q in place of p. Note that $\psi_q^{-1} \in \mathcal{O}(E_{-q})$, and

$$w_\pm \in \mathcal{H}^p(\Omega_\pm, \overline{\partial}_{E_{p-q}}), \qquad h \in L^p(\Sigma, E_{p-q} \otimes \mathbb{C}^\ell). \tag{4.3.57}$$

Here (4.3.56) becomes $R_\Phi(w_+, w_-) = h$, with

$$R_\Phi : \mathcal{H}^p(\Omega_+, \overline{\partial}_{E_{p-q}}) \oplus \mathcal{H}^p(\Omega_-, \overline{\partial}_{E_{p-q}}) \longrightarrow L^p(\Sigma, E_{p-q} \otimes \mathbb{C}^\ell). \tag{4.3.58}$$

The significance of this transformation is enhanced by the fact that

$$E_{p-q} \text{ is holomorphically trivial,} \tag{4.3.59}$$

by (A.4.29). Hence after multiplying $w_\pm$ by the inverse of a nontrivial (necessarily nowhere vanishing) holomorphic section of E_{p-q} over S^2, in essence obtaining

$$v_\pm(z)(z - q), \tag{4.3.60}$$

we get a *scalar* Riemann-Hilbert problem. If we map S^2 to $\mathbb{C} \cup \{\infty\}$ so that $q = \infty$, then Ω_+ is transformed to a bounded UR domain in $\mathbb{C}$. Thus we are in the setting where this section started.

We now produce results on Index R_Φ when $\Omega_\pm$ are as in (4.3.42)–(4.3.43), mapped to a bounded domain in $\mathbb{C}$ via such a transformation as indicated above, and Φ satisfies (4.3.55). As we have seen, this is equivalent to analyzing Index T_Φ. We can do this by applying the cobordism invariance result, described in §2.5.

In detail, given $\Phi \in C^0(\Sigma, \mathrm{GL}(\ell, \mathbb{C}))$, we can extend Φ to

$$\Phi \in C^0(U, \mathrm{GL}(\ell, \mathbb{C})), \tag{4.3.61}$$

where U is a neighborhood of Σ in $\mathbb{C}$. Now we can take

$$\widetilde{\Omega} \supset \Omega_+ \tag{4.3.62}$$

to be smoothly bounded, $\partial\widetilde{\Omega}$ having three connected components, such that $\widetilde{\Omega} \setminus \Omega_+ \subset U$, let $\widetilde{\Phi} = \Phi|_{\partial\widetilde{\Omega}}$, defining

$$T_{\widetilde{\Phi}} : L^p(\partial\widetilde{\Omega}, \mathbb{C}^\ell) \longrightarrow L^p(\partial\widetilde{\Omega}, \mathbb{C}^\ell), \tag{4.3.63}$$

and conclude from (2.5.15) that

$$\operatorname{Index} T_\Phi = \operatorname{Index} T_{\widetilde{\Phi}}. \tag{4.3.64}$$

Now, as in the proof of Proposition 4.3.1, we may apply [21, Proposition 4.1.6] to Index $T_{\widetilde{\Phi}}$. On the other hand, the winding number of $\det \widetilde{\Phi}$ about the three boundary components of $\partial\widetilde{\Omega}$ coincides with the winding numbers of $\det \Phi$ about the three loops $\partial\Omega_{-,j}$, $1 \le j \le 3$. We deduce that

$$\operatorname{Index} R_\Phi = -\sum_{j=1}^{3} w_j(\det \Phi), \tag{4.3.65}$$

where $w_j(\varphi)$ denotes the winding number of $\varphi|_{\partial\Omega_{-,j}}$ about 0, with $\partial\Omega_{-,j}$ given the appropriate orientation. Note that the neighborhood U of Σ and the smoothly bounded domain $\widetilde{\Omega}$ need to be chosen in a way that depends on Φ, but the formula (4.3.65) does not depend on these choices.

We close this section with a brief discussion of results to the effect that, for a certain class of bounded UR domains $\Omega_+ \subset \mathbb{C}$, and $\Phi \in C^0(\partial\Omega_+, \mathbb{C} \setminus 0)$ (so $\ell = 1$),

$$\begin{aligned}
&\operatorname{Index} R_\Phi = 0 \Longrightarrow R_\Phi \text{ invertible}, \\
&\operatorname{Index} R_\Phi < 0 \Longrightarrow R_\Phi \text{ injective}, \\
&\operatorname{Index} R_\Phi > 0 \Longrightarrow R_\Phi \text{ surjective}.
\end{aligned} \tag{4.3.66}$$

In these cases,

$$R_\Phi : \mathcal{H}^p(\Omega_+, \overline{\partial}) \oplus \mathcal{H}^p(\Omega_-, \overline{\partial}) \longrightarrow L^p(\Sigma), \tag{4.3.67}$$

and $\mathcal{H}^p(\Omega_-, \overline{\partial})$ is as in (4.3.7), with $\ell = 1$. Here, we show that the first implication of (4.3.66) holds when

$$\Omega \subset \mathbb{C} \text{ is smoothly bounded and simply connected}, \tag{4.3.68}$$

and

$$\Phi \in C^\alpha(\partial\Omega_+, \mathbb{C} \setminus 0), \quad \text{with } \alpha > 0. \tag{4.3.69}$$

The argument in this setting is classical. It will serve here to advertise further work of the authors, which will appear elsewhere.

Here is a formal statement.

Proposition 4.3.2 *If $\Omega_+ \subset \mathbb{C}$ is a bounded domain satisfying* (4.3.68)*, and if the function $\Phi \in C^\alpha(\partial\Omega_+, \mathbb{C}\backslash 0)$ has winding number* 0 *about the origin of the complex plane, the operator R_Φ is bijective in* (4.3.67)*.*

Proof. It suffices to show that R_Φ is surjective in (4.3.67). To this end, note that the hypothesis on Φ implies we can define

$$\log \frac{1}{\Phi} \in C^\alpha(\partial\Omega_+). \tag{4.3.70}$$

Now set

$$\omega_\pm(z) = \frac{1}{2\pi i} \int\limits_{\partial\Omega_+} \log \frac{1}{\Phi(\zeta)} \frac{d\zeta}{\zeta - z}, \quad z \in \Omega_\pm. \tag{4.3.71}$$

Under our hypotheses on Ω_+ and Φ, it is classical that $\omega_\pm$ are holomorphic on $\Omega_\pm$ and piecewise continuous on $\overline{\Omega}_+$ and $\overline{\Omega}_-$, up to the boundary, that $\omega_-(z) \to 0$ as $|z| \to \infty$, and that

$$\omega_+\big|_\Sigma^{\text{n.t.}} - \omega_-\big|_\Sigma^{\text{n.t.}} = \log \frac{1}{\Phi}, \tag{4.3.72}$$

where $\Sigma = \partial\Omega_+$. Now, to solve $R_\Phi(u_+, u_-) = g \in L^p(\Sigma)$, we take

$$u_\pm(z) = \frac{e^{\omega_\pm(z)}}{2\pi i} \int\limits_{\partial\Omega_+} \frac{g(\zeta) e^{-\omega_+(\zeta)}}{\Phi(\zeta)} \frac{d\zeta}{\zeta - z}, \quad z \in \Omega_\pm, \tag{4.3.73}$$

i.e.,

$$u_\pm(z) = e^{\omega_\pm(z)} \mathcal{C}\Big(\Phi^{-1} g e^{-\omega_+}\Big)(z). \tag{4.3.74}$$

Clearly

$$g \in L^p(\Sigma) \Longrightarrow \Phi^{-1} g e^{-\omega_+} \in L^p(\Sigma), \tag{4.3.75}$$

from which we deduce that

$$\mathcal{N}(u_\pm) \in L^p(\Sigma). \tag{4.3.76}$$

Also (4.3.73) readily gives

$$u_-(z) \longrightarrow 0 \quad \text{as } |z| \to \infty. \tag{4.3.77}$$

Furthermore,

$$\mathcal{C}\Big(\Phi^{-1} g e^{-\omega_+}\Big)\Big|_{\partial\Omega_\pm}^{\text{n.t.}} = \pm\frac{1}{2}\Phi^{-1} g e^{-\omega_+} + C(\Phi^{-1} g e^{-\omega_+}), \tag{4.3.78}$$

so

$$\begin{aligned} \Phi u_+\big|_{\partial\Omega_+}^{\text{n.t.}} - u_-\big|_{\partial\Omega_-}^{\text{n.t.}} & \\ &= \frac{1}{2}\Phi e^{\omega_+} \Phi^{-1} g e^{-\omega_+} + \Phi e^{\omega_+} C(\Phi^{-1} g e^{-\omega_+}) \\ &\quad + \frac{1}{2} e^{\omega_-} \Phi^{-1} g e^{-\omega_+} - e^{\omega_-} C(\Phi^{-1} g e^{-\omega_+}) \\ &= g, \end{aligned} \tag{4.3.79}$$

since (4.3.72) implies

$$e^{\omega_-} e^{-\omega_+} = \Phi \text{ on } \partial\Omega_\pm. \tag{4.3.80}$$

This proves Proposition 4.3.2. □

The first implication in (4.3.66) for smoothly bounded domains $\Omega_+ \subset \mathbb{C}$ with several boundary components is amenable to a more elaborate argument, as are the other implications in (4.3.66), for such domains, and for Φ satisfying (4.3.69).

We have extended the scope of these implications in the following ways. First, they hold for bounded UR domains $\Omega_+ \subset \mathbb{C}$ of the class treated in Proposition 4.3.1, i.e., with boundaries as in (4.3.27). Furthermore, they hold not only for Φ as in (4.3.69), but more generally when

$$\Phi, \Phi^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega_+). \tag{4.3.81}$$

Details will be presented in the authors' work [22].

The implications in (4.3.66) fail for $\ell \times \ell$ systems with $\ell > 1$. For example, if φ_0 and φ_1 in $C^0(\partial\Omega_+, \mathbb{C} \setminus 0)$ have winding numbers $+1$ and -1, and

$$\Phi = \begin{pmatrix} \varphi_0 & \\ & \varphi_1 \end{pmatrix}, \tag{4.3.82}$$

then R_Φ has index 0, but is certainly not invertible.

One can examine R_Φ in (4.3.18) for many other holomorphic line bundles L, and also holomorphic vector bundles. We turn to this in the next section, in the setting of more general compact Riemann surfaces.

4.4 Riemann-Hilbert problems on compact Riemann surfaces

Let M be a compact Riemann surface, of genus g. Take $\Omega_\pm$, Σ as in (4.0.1)–(4.0.2), and let $L \to M$ be a holomorphic vector bundle. With

$$\Phi \in C^0(\Sigma, \mathrm{GL}(\ell, \mathbb{C})), \tag{4.4.1}$$

or, more generally,

$$\Phi, \Phi^{-1} \in L^\infty \cap \mathrm{vmo}(\Sigma, M(\ell, \mathbb{C})), \tag{4.4.2}$$

we consider the operator

$$\begin{aligned} &R_\Phi : \mathcal{H}^p(\Omega_+, \overline{\partial}_L) \oplus \mathcal{H}^p(\Omega_-, \overline{\partial}_L) \longrightarrow L^p(\Sigma, L \otimes \mathbb{C}^\ell), \\ &R_\Phi(u, v) = \Phi u\Big|_{\Sigma}^{\text{n.t.}} - v\Big|_{\Sigma}^{\text{n.t.}}, \quad u \in \mathcal{H}^p(\Omega_+, \overline{\partial}_L),\ v \in \mathcal{H}^p(\Omega_-, \overline{\partial}_L). \end{aligned} \tag{4.4.3}$$

Results of §4.1 imply that R_Φ is Fredholm for each $p \in (1, \infty)$, and

$$\mathrm{Index}\, R_\Phi = \mathrm{Index}\, \widetilde{T}_\Phi + \ell \cdot \mathrm{Index}\, \overline{\partial}_L, \tag{4.4.4}$$

with $\widetilde{T}_\Phi$ as in (4.1.5). Recall also that $\mathrm{Index}\, \widetilde{T}_\Phi = \mathrm{Index}\, T_\Phi$. Furthermore,

$$\overline{\partial}_L \text{ invertible } \Rightarrow \mathrm{Ker}\, R_\Phi \approx \mathrm{Ker}\, \widetilde{T}_\Phi \text{ and } \mathrm{Range}\, R_\Phi = \mathrm{Range}\, \widetilde{T}_\Phi. \tag{4.4.5}$$

As for the index of $\overline{\partial}_L$, the Riemann-Roch theorem, described in §A.4, gives

$$\mathrm{Index}\, \overline{\partial}_L = c_1(L) + r(1 - g), \tag{4.4.6}$$

if L is a rank-r holomorphic vector bundle. In particular,

$$\mathrm{Index}\, \overline{\partial}_L = c_1(L) + 1 - g, \tag{4.4.7}$$

if L is a holomorphic line bundle. We see that

$$\overline{\partial}_L \text{ is invertible} \Longleftrightarrow c_1(L) = r(g-1) \text{ and } \mathcal{O}(L) = 0. \tag{4.4.8}$$

We can use (4.4.8) to produce line bundles L such that $\overline{\partial}_L$ is invertible. As seen in §A.4, if $g = 0$, $L = E_p$ works, for each $p \in M$, and if $g = 1$, $L = E_{p-q}$ works, for distinct p and $q \in M$. It is useful to know the following.

Proposition 4.4.1 *If M has genus g, and $r \in \mathbb{N}$ is given, there are many holomorphic vector bundles $L \to M$ of rank r such that*

$$c_1(L) = r(g-1) \ \text{ and } \ \mathcal{O}(L) = 0, \tag{4.4.9}$$

so $\overline{\partial}_L$ is invertible.

See §A.4 for particulars on this.

We next take up the following transmission problem. Recalling (4.0.1)–(4.0.2), take

$$p_1, \dots, p_{\ell+g-1}, q_1, \dots, q_\ell \in M \setminus \Sigma, \tag{4.4.10}$$

such that

$$\vartheta = p_1 + \cdots + p_{\ell+g-1} - q_1 - \cdots - q_\ell \Longrightarrow \mathcal{O}(E_{-\vartheta}) = 0. \tag{4.4.11}$$

Note that $c_1(E_{-\vartheta}) = g-1$, hence $\overline{\partial}_{E_{-\vartheta}}$ is invertible.

Theorem 4.4.2 *Let $\Omega_\pm$ and Σ be as in (4.0.1)–(4.0.2), and assume (4.4.10)–(4.4.11) hold. Let $U \subset M$ be a neighborhood of Σ disjoint from $\{p_1, \dots, p_{\ell+g-1}\}$. Take $p \in (1, \infty)$. Then, given $g \in L^p(\Sigma)$, there exist unique*

$$u_\pm \in \mathcal{M}(\Omega_\pm), \tag{4.4.12}$$

such that

$$\vartheta(u_\pm) \geq -\vartheta, \quad \mathcal{N}\big(u_\pm\big|_U\big) \in L^p(\Sigma), \tag{4.4.13}$$

and

$$u_+\Big|_\Sigma^{\text{n.t.}} - u_-\Big|_\Sigma^{\text{n.t.}} = g. \tag{4.4.14}$$

Proof. Pick $\psi \in \mathcal{M}(E_\vartheta)$, satisfying $\vartheta(\psi) = -\vartheta$, as in (A.4.19). The relation

$$v_\pm = u_\pm \psi^{-1} \tag{4.4.15}$$

sets up a one-to-one correspondence between $u_\pm$ satisfying (4.4.12)–(4.4.13) and

$$v_\pm \in \mathcal{H}^p(\Omega_\pm, E_{-\vartheta}). \tag{4.4.16}$$

Under this correspondence, (4.4.14) is equivalent to

$$v_+\Big|_\Sigma^{\text{n.t.}} - v_-\Big|_\Sigma^{\text{n.t.}} = g\psi^{-1} \in L^p(\Sigma, E_{-\vartheta}). \tag{4.4.17}$$

Multiplication by ψ^{-1} also produces an isomorphism between $L^p(\Sigma)$ and $L^p(\Sigma, E_{-\vartheta})$. Now, unique solvability of (4.4.16)–(4.4.17) follows directly from (4.4.5), with $\Phi = 1$, yielding $\widetilde{T}_\Phi = I$. Hence we have unique solvability of (4.4.12)–(4.4.14), as asserted. □

Remark. The hypothesis (4.4.11) is equivalent to

$$\{u \, \text{meromorphic on} \, M : \vartheta(u) \geq -\vartheta\} = 0, \tag{4.4.18}$$

by (A.4.21)–(A.4.22), with ϑ replaced by $-\vartheta$. This is clearly necessary for uniqueness of a solution to (4.4.12)–(4.4.14).

Note that (4.4.17) is equivalent to

$$\mathcal{J}_p(f_+, f_-) = g\psi^{-1}, \tag{4.4.19}$$

with $\mathcal{J}_p : \mathcal{H}^p_+(\Sigma, \overline{\partial}_{E_{-\vartheta}}) \oplus \mathcal{H}^p_-(\Sigma, \overline{\partial}_{E_{-\vartheta}}) \to L^p(\Sigma, E_{-\vartheta})$, as in (3.2.17). Hence Corollary 3.2.2 provides an alternative end to the proof of Theorem 4.4.2.

Using Corollary 3.2.2, we can expand the scope of Proposition 4.2.2 as follows. Let $L \to M$ be a holomorphic vector bundle, and consider the operator

$$\begin{aligned} &\mathcal{J}_p : \mathcal{H}^p_+(\Sigma, \overline{\partial}_L) \oplus \mathcal{H}^p_-(\Sigma, \overline{\partial}_L) \longrightarrow L^p(\Sigma, L), \\ &\mathcal{J}_p(f_+, f_-) := f_+ - f_-, \qquad \forall_\pm \in \mathcal{H}^p_\pm(\Sigma, \overline{\partial}_L). \end{aligned} \tag{4.4.20}$$

Then

$$\operatorname{Ker} \mathcal{J}_p \approx \operatorname{Ker} \overline{\partial}_L = \mathcal{O}(L), \quad \operatorname{Coker} \mathcal{J}_p \approx \operatorname{Coker} \overline{\partial}_L \approx \mathcal{O}(L' \otimes \kappa). \tag{4.4.21}$$

Meanwhile, by (4.4.6),

$$\operatorname{Index} \mathcal{J}_p = c_1(L) + r(1-g), \tag{4.4.22}$$

if L has rank r. This leads to such implications as

$$\begin{aligned} &c_1(L) > r(g-1) \Longrightarrow \operatorname{Index} \overline{\partial}_L > 0 \Longrightarrow \operatorname{Ker} \mathcal{J}_p \neq 0, \\ &c_1(L) < r(g-1) \Longrightarrow \operatorname{Index} \overline{\partial}_L < 0 \Longrightarrow \operatorname{Coker} \mathcal{J}_p \neq 0. \end{aligned} \tag{4.4.23}$$

In case L is a line bundle (so $r = 1$), we also know that

$$\begin{aligned} &c_1(L) < 0 \Longrightarrow \mathcal{O}(L) = 0 \Longrightarrow \operatorname{Ker} \mathcal{J}_p = 0, \\ &c_1(L) > 2g-2 \Longrightarrow \mathcal{O}(L' \otimes \kappa) = 0 \Longrightarrow \operatorname{Coker} \mathcal{J}_p = 0. \end{aligned} \tag{4.4.24}$$

We return to the setting of R_Φ and establish the following index computation.

Theorem 4.4.3 *Let M be a compact Riemann surface, $L \to M$ a holomorphic vector bundle of rank r, and take $\Omega_\pm$ and Σ as in* (4.0.1)–(4.0.2)*. Assume in addition that*

$$\Sigma = \bigcup_{j=0}^{\mu} \gamma_j, \quad \textit{each } \gamma_j \textit{ a simple closed curve.} \tag{4.4.25}$$

Orient each γ_j as a component of $\partial\Omega_+$. Let

$$\Phi \in C^0(\Sigma, \mathrm{GL}(\ell, \mathbb{C})). \tag{4.4.26}$$

Finally, define R_Φ as in (4.4.3)*. Then*

$$\operatorname{Index} R_\Phi = -r \sum_{j=1}^{\mu} w_j(\det \Phi) + \ell \cdot \operatorname{Index} \overline{\partial}_L, \tag{4.4.27}$$

where $w_j(\varphi)$ is the winding number of $\varphi : \gamma_j \to \mathbb{C} \setminus 0$ about the origin in the complex plane.

Proof. In light of (4.4.4), our remaining task is to show that

$$\operatorname{Index} T_\Phi = -r \sum_{j=1}^{\mu} w_j(\det \Phi). \tag{4.4.28}$$

To prove (4.4.28), we use the localization procedure described in §2.5, with particular attention to (2.5.6) and (2.5.9). In implementing this localization, we use the fact that each $\gamma_j \subset M$ has a neighborhood U_j that is homeomorphic to an annulus, $(0,1) \times S^1$. Now any complex vector bundle L over M trivializes over each such U_j, and each annulus U_j is biholomorphic to an annulus in $\mathbb{C}$. Thus, we obtain

$$\operatorname{Index} T_\Phi = \sum_{j=1}^{\mu} \operatorname{Index} T_{\widetilde{\Phi}_j}, \tag{4.4.29}$$

where each $\widetilde{\Phi}_j : \partial \mathcal{O}_j \to \mathrm{GL}(r\ell, \mathbb{C})$ is essentially $\Phi|_{\gamma_j} \otimes I_r$, and $\mathcal{O}_j \subset \mathbb{C}$ is a bounded UR domain, with boundary $\partial \mathcal{O}_j \approx \gamma_j$. We have

$$w_j(\det \widetilde{\Phi}_j) = r w_j(\Phi), \tag{4.4.30}$$

and the arguments proving Proposition 4.3.1, and remarks following that result, yield (4.4.28). □

A Auxiliary material

A.1 L^p-Sobolev spaces on $\partial\Omega$

Let $\Omega \subset M$ be a relatively compact, n-dimensional, Ahlfors regular domain. We recall some definitions and basic results on L^p-Sobolev spaces $L^p(\partial\Omega)$. Details are given in [16, §3.6] and [21, §A.2]. For simplicity, we take $\Omega \subset \mathbb{R}^n$. Passage to more general M is not difficult.

To start, given $\varphi \in C_0^1(\mathbb{R}^n)$, for each $j, k \in \{1, \dots, n\}$ we set

$$\partial_{\tau_{jk}} \varphi := \nu_k (\partial_j \varphi)\big|_{\partial\Omega} - \nu_j (\partial_k \varphi)\big|_{\partial\Omega}. \tag{A.1.1}$$

Here $\nu = (\nu_1, \dots, \nu_n)$ is the unit, outward, measure-theoretic normal to $\partial\Omega$. Given any function $\psi \in C_0^1(\mathbb{R}^n)$, an argument using Green's formula (cf. [21, (A.2.2)]) yields

$$\int_{\partial\Omega} (\partial_{\tau_{jk}} \varphi) \psi \, d\sigma = -\int_{\partial\Omega} \varphi (\partial_{\tau_{jk}} \psi) \, d\sigma. \tag{A.1.2}$$

To proceed, given $f \in L^p(\partial\Omega)$, $p \in [1, \infty]$, we say $f \in L_1^p(\partial\Omega)$ provided that, for each $j, k \in \{1, \dots, n\}$, there exists $f_{jk} \in L^p(\partial\Omega)$ such that

$$\int_{\partial\Omega} (\partial_{\tau_{jk}} \varphi) f \, d\sigma = -\int_{\partial\Omega} \varphi f_{jk} \, d\sigma, \qquad \forall\, \varphi \in C_0^1(\mathbb{R}^n). \tag{A.1.3}$$

In such a case, we shall employ the notation

$$\partial_{\tau_{jk}} f := f_{jk}. \tag{A.1.4}$$

By (A.1.2), if $f = \psi|_{\partial\Omega}$ with $\psi \in C_0^1(\mathbb{R}^n)$, then $f \in L_1^p(\partial\Omega)$ and $f_{jk} = \nu_k \partial_j \psi|_{\partial\Omega} - \nu_j \partial_k \psi|_{\partial\Omega}$. The following is [21, Proposition A.2.1].

Proposition A.1.1 *For each $p \in [1, \infty]$, $L^p_1(\partial\Omega)$ is a Banach space, when equipped with the norm*

$$\|f\|_{L^p_1(\partial\Omega)} := \|f\|_{L^p(\partial\Omega)} + \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{L^p(\partial\Omega)}. \tag{A.1.5}$$

As shown in [21, Proposition A.2.2],

$$\mathrm{Lip}(\partial\Omega) \subset L^\infty_1(\partial\Omega). \tag{A.1.6}$$

On this level of generality, the reverse inclusion need not hold.

The following is [21, Proposition A.2.3]. The proof makes use of the Gauss-Green theorem given in [16, §2.3].

Proposition A.1.2 *Assume the function $u \in C^1(\Omega)$ satisfies $\mathcal{N}(u), \mathcal{N}(\nabla u) \in L^p(\partial\Omega)$, for some $p \in (1, \infty)$, and the nontangential boundary limits*

$$f := u\big|^{\text{n.t.}}_{\partial\Omega}, \quad f_j := (\partial_j u)\big|^{\text{n.t.}}_{\partial\Omega}, \quad j \in \{1, \dots, n\}, \tag{A.1.7}$$

exist σ-a.e. on $\partial\Omega$. Then $f \in L^p_1(\partial\Omega)$ and

$$\partial_{\tau_{jk}} f = \nu_k f_j - \nu_j f_k, \qquad \forall\, j, k \in \{1, \dots, n\}. \tag{A.1.8}$$

The next result, Proposition A.2.4 of [21], extends the scope of (A.1.3)–(A.1.4).

Proposition A.1.3 *Given $f \in \mathrm{Lip}(\partial\Omega)$ and $g \in L^p_1(\partial\Omega)$, one has*

$$\int_{\partial\Omega} (\partial_{\tau_{jk}} f) g \, d\sigma = - \int_{\partial\Omega} f (\partial_{\tau_{jk}} g) \, d\sigma. \tag{A.1.9}$$

Proposition A.2.6 of [21] gives the following Leibniz formula.

Proposition A.1.4 *Given $f \in \mathrm{Lip}(\partial\Omega)$ and $g \in L^p_1(\partial\Omega)$, one has*

$$fg \in L^p_1(\partial\Omega) \tag{A.1.10}$$

and, for each $j, k \in \{1, \dots, n\}$,

$$\partial_{\tau_{jk}}(fg) = (\partial_{\tau_{jk}} f) g + f (\partial_{\tau_{jk}} g). \tag{A.1.11}$$

There is a refinement of Proposition A.1.4, established in [21, Proposition A.2.7], in case Ω also has the property

$$\Omega \text{ satisfies a two-sided John condition.} \tag{A.1.12}$$

This property is discussed further in §A.2.

Proposition A.1.5 *Assume the n-dimensional Ahlfors regular domain Ω also satisfies* (A.1.12)*. Also, suppose*

$$p \in (1, \infty), \quad q \in (n-1, \infty), \quad q \geq p. \tag{A.1.13}$$

Then

$$f \in L^q_1(\partial\Omega),\, g \in L^p_1(\partial\Omega) \Longrightarrow fg \in L^p_1(\partial\Omega), \tag{A.1.14}$$

and the Leibniz formula (A.1.11) *holds.*

A.2 Domains with the two-sided John condition

Results for Toeplitz operators and the Riemann-Hilbert problem with data in the Sobolev space $L^p_1(\partial\Omega)$ are mostly set in the special class of UR domains that satisfy a two-sided John condition, which we define and discuss here.

As in [16, §3.1], we say a relatively compact open set $\Omega \subset M$ satisfies a local John condition provided there exist $\theta \in (0,1)$ and $R > 0$ (called the John constants of Ω) with the following properties. For every $q \in \partial\Omega$ and $r \in (0,R)$, we can find $q_r \in B_r(q) \cap \Omega$, called a John center relative to $\Delta_r(q) = B_r(q) \cap \partial\Omega$, such that

$$B_{\theta r}(q_r) \subset \Omega, \tag{A.2.1}$$

and with the property that for each $x \in \Delta_r(q)$ one can find a rectifiable path

$$\gamma_x : [0,1] \longrightarrow \overline{\Omega}, \text{ of length } \le \frac{r}{\theta}, \tag{A.2.2}$$

such that

$$\gamma_x(0) = x, \quad \gamma_x(1) = q_r, \quad \operatorname{dist}(\gamma_x(t), \partial\Omega) \ge \theta \operatorname{dist}(\gamma_x(t), x), \quad \forall\, t \in (0,1]. \tag{A.2.3}$$

Finally, Ω satisfies a two-sided John condition provided both Ω and $M \setminus \overline{\Omega}$ satisfy a local John condition.

As shown in [16, §3.1], if Ω is Ahlfors regular and satisfies a two-sided John condition, then Ω is a UR domain.

The following result is established in [16, §4.3].

Proposition A.2.1 *If $\Omega \subset M$ is a relatively compact, n-dimensional Ahlfors regular domain that satisfies a two-sided John condition, then the following hold:*

$$L^p_1(\partial\Omega) \hookrightarrow \begin{cases} L^{p^*}(\partial\Omega) \ \text{for} \ p^* = \frac{(n-1)p}{n-n-p}, \ \text{if} \ p \in (1, n-1), \\ L^q(\partial\Omega) \ \text{for all} \ q \in (1,\infty), \ \text{if} \ p = n-1, \\ C^r(\partial\Omega) \ \text{for} \ r = 1 - \frac{n-1}{p}, \ \text{if} \ p \in (n-1, \infty). \end{cases} \tag{A.2.4}$$

Also,

$$L^p_1(\partial\Omega) \hookrightarrow L^p(\partial\Omega) \ \text{compactly, for each} \ p \in (1,\infty). \tag{A.2.5}$$

Furthermore, the natural map

$$C^\infty(M)\big|_{\partial\Omega} \hookrightarrow L^p_1(\partial\Omega) \tag{A.2.6}$$

is well-defined with dense range, for each $p \in (1,\infty)$.

A.3 Refined divergence theorem

Divergence theorems have the form

$$\int_\Omega \operatorname{div} F \, dV = \int_{\partial\Omega} \nu \cdot F \, d\sigma, \tag{A.3.1}$$

where Ω is an open, relatively compact subset of an n-dimensional Riemannian manifold M, with outward unit conormal ν and "surface measure" σ on $\partial\Omega$. Results of De Giorgi and Federer give (A.3.1) for any such Ω with finite perimeter, provided the vector field F is Lipschitz on M. It is easy to extend this to $F \in C^0(M)$, $\text{div}F \in L^1(M)$, but, as indicated in §2, such an identity is needed for much rougher F. Furthermore, we have such an identity for much rougher F, provided we restrict the class of domains Ω. To describe the result of use to us, we bring in the spaces

$$\begin{aligned}\mathfrak{L}^p(\Omega) := \big\{F \in C^0(\Omega) : \mathcal{N}(F) \in L^p(\partial\Omega), \text{ and} \\ \text{the nontangential limit } F\big|_{\partial\Omega}^{\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\Omega\big\}.\end{aligned} \tag{A.3.2}$$

Proposition A.3.1 *Assume Ω is a relatively compact, Ahlfors regular domain. Assume F is a vector field on Ω satisfying*

$$F \in \mathfrak{L}^1(\Omega), \qquad \text{div}F \in L^1(\Omega). \tag{A.3.3}$$

Then (A.3.1) *holds* (*with F on $\partial\Omega$ in the right side understood as $F\big|_{\partial\Omega}^{\text{n.t.}}$*).

This was established in [16, §2.3], under the stronger hypothesis that $F \in \mathfrak{L}^p(\Omega)$ for some $p > 1$, and $\text{div}F \in L^1(\Omega)$. Under the sharper hypothesis (A.3.3), Proposition A.3.1 is proved in [20]. The following consequence of Proposition A.3.1 is directly applicable to results in this paper.

Proposition A.3.2 *Let $D : \mathcal{E}_0 \to \mathcal{E}_1$ be a first order differential operator, acting between the sections of two vector bundles $\mathcal{E}_0, \mathcal{E}_1 \to M$, and let $\Omega \subset M$ a relatively compact, Ahlfors regular domain, with* (*geometric measure theoretic*) *outward unit conormal ν and surface measure σ. In this context, consider a section u of $\mathcal{E}_0$ on Ω along with a section v of $\mathcal{E}_1$ on Ω such that*

$$u \in \mathfrak{L}^p(\Omega), \quad v \in \mathfrak{L}^{p'}(\Omega), \tag{A.3.4}$$

for some $p \in [1, \infty]$, where p' denotes its dual exponent. Moreover, assume that, in the sense of distributions, $Du, D^*v \in L^1_{\text{loc}}(\Omega)$, and

$$\langle Du, v\rangle - \langle u, D^*v\rangle \in L^1(\Omega). \tag{A.3.5}$$

Then

$$\int\limits_{\Omega} \big\{\langle Du, v\rangle - \langle u, D^*v\rangle\big\}\, dV = \frac{1}{i} \int\limits_{\partial\Omega} \Big\langle \sigma_D(\cdot, \nu) u\big|_{\partial\Omega}^{\text{n.t.}}, v\big|_{\partial\Omega}^{\text{n.t.}} \Big\rangle\, d\sigma. \tag{A.3.6}$$

Proof. Define the vector field F on Ω via the requirement that at a.e. $x \in \Omega$ we have

$$\langle \xi, F(x)\rangle = \frac{1}{i}\langle \sigma_D(x, \xi)u(x), v(x)\rangle, \qquad \forall\, \xi \in T_x^*M, \tag{A.3.7}$$

where the second $\langle\,,\,\rangle$ is the inner product on $\mathcal{E}_{1x}$. The hypothesis (A.3.4) then readily implies that $F \in \mathfrak{L}^1(\Omega)$. We claim that, in the sense of distributions,

$$\text{div}F = \langle Du, v\rangle - \langle u, D^*v\rangle. \tag{A.3.8}$$

Given this, (A.3.6) then follows from Proposition A.3.1.

To establish (A.3.8), pick an arbitrary test function $\psi \in C_0^1(\Omega)$. Then

$$\begin{aligned}\int_\Omega \psi(x)\mathrm{div}F(x)\,dV(x) &= -\int_\Omega \langle d\psi(x), F(x)\rangle\,dV(x) \\ &= i\int_\Omega \langle \sigma_D(x,d\psi)u(x), v(x)\rangle\,dV(x).\end{aligned} \tag{A.3.9}$$

Now (as in (2.1.2))

$$D(\psi u) = \psi Du + \frac{1}{i}\sigma_D(x,d\psi)u, \tag{A.3.10}$$

so (A.3.9) is equal to

$$\begin{aligned}&\int_\Omega \big[\langle \psi Du, v\rangle - \langle D(\psi u), v\rangle\big]\,dV \\ &\qquad = \int_\Omega \psi\big[\langle Du, v\rangle - \langle u, D^* v\rangle\big]\,dV,\end{aligned} \tag{A.3.11}$$

giving (A.3.8), as desired. □

To give a significant example, let $\Omega \subset \mathbb{C}$ be a bounded, Ahlfors regular domain. In this context, consider $D = \overline{\partial} = (1/2)(\partial_x + i\partial_y)$, so $D^* = -(1/2)(\partial_x - i\partial_y)$. Let u be holomorphic on Ω, so $\overline{\partial}u = 0$, and take $v = 1$, so $D^* v = 0$. Assume

$$\mathcal{N}u \in L^1(\partial\Omega), \quad \text{and the nontangential limit } u\Big|_{\partial\Omega}^{\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\Omega. \tag{A.3.12}$$

Thus $u \in \mathfrak{L}^1(\Omega)$, and Proposition A.3.2 applies. As such, we obtain

$$\int_{\partial\Omega} \sigma_{\overline{\partial}}(z,\nu(z))u(z)\,d\sigma(z) = 0. \tag{A.3.13}$$

If we identify $\nu(z) \in T_z^*\mathbb{C}$ with an element of $\mathbb{C}$, we have

$$\sigma_{\overline{\partial}}(z,\nu(z)) = \frac{i}{2}\nu(z). \tag{A.3.14}$$

Extending the classical identity for C^1 curves, it is natural to endow $\partial\Omega$ with the complex measure denoted dz, defined by

$$dz = i\nu(z)\,d\sigma(z). \tag{A.3.15}$$

Hence we have the following general version of the Cauchy integral theorem.

Corollary A.3.3 *If $\Omega \subset \mathbb{C}$ is a bounded Ahlfors regular domain and u is holomorphic on Ω and satisfies* (A.3.12)*, then*

$$\int_{\partial\Omega} u(z)\,dz = 0. \tag{A.3.16}$$

We conclude by proving a versatile boundary integration by parts formula.

Proposition A.3.4 *Let $P : \mathcal{E} \to \mathcal{E}$ be a first order differential operator, acting on the sections of a vector bundle $\mathcal{E}$ on M. Let $\Omega \subset M$ be a relatively compact Ahlfors regular domain that satisfies a two-sided John condition, and denote by ν and σ the (geometric measure theoretic) outward unit conormal and surface measure on $\partial\Omega$.*

Then for each functions $\varphi \in L^p_1(\partial\Omega, \mathcal{E})$, $\psi \in L^{p'}_1(\partial\Omega, \mathcal{E})$, where $p, p' \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{p'} = 1$, and each vector field $X \in TM$, there holds

$$\begin{aligned}\int_{\partial\Omega} \big\langle \big(\sigma_P(\cdot,\nu)\nabla_X - \sigma_{\nabla_X}(\cdot,\nu)P\big)\varphi, \psi\big\rangle \, d\sigma \\ = -\int_{\partial\Omega} \big\langle \varphi, \big(\sigma_{P^\top}(\cdot,\nu)\nabla_X^\top - \sigma_{\nabla_X^\top}(\cdot,\nu)P^\top\big)\psi\big\rangle \, d\sigma \\ - \int_{\partial\Omega} \big\langle \varphi, \sigma_{[P,\nabla_X]}(\cdot,\nu)\psi\big\rangle \, d\sigma.\end{aligned} \tag{A.3.17}$$

In fact, the same is true for ∇_X replaced by a first order differential operator $Q : \mathcal{E} \to \mathcal{E}$ with the property that the principal symbols σ_Q and σ_P commute.

Proof. By the density result recorded in (A.2.6), it suffices to consider the case when $\varphi, \psi \in C^\infty(M, \mathcal{E})$. With this goal in mind, consider the vector field $F : \Omega \to T^*M$ defined by asking that for every $\xi \in T^*M$ we have

$$\begin{aligned}\langle \xi, F\rangle =& \big\langle i\sigma_P(\cdot,\xi)\nabla_X\varphi - i\sigma_{\nabla_X}(\cdot,\xi)P\varphi, \psi\big\rangle \\ &+ \big\langle \varphi, i\sigma_P(\cdot,\xi)\nabla_X\psi - i\sigma_{\nabla_X}(\cdot,\xi)P\psi\big\rangle.\end{aligned} \tag{A.3.18}$$

Next, fix a scalar function $\eta \in C^1_0(\Omega)$ and compute, in the sense of distributions,

$$\begin{aligned}(\operatorname{div} F, \eta) &= -\int_\Omega \langle d\eta, F\rangle \, d\sigma = -\int_\Omega \big\langle i\sigma_P(\cdot,d\eta)\nabla_X\varphi - i\sigma_{\nabla_X}(\cdot,d\eta)P\varphi, \psi\big\rangle \, dV \\ &\quad - \int_\Omega \big\langle \varphi, i\sigma_P(\cdot,d\eta)\nabla_X\psi - i\sigma_{\nabla_X}(\cdot,d\eta)P\psi\big\rangle \, dV \\ &= -\int_\Omega \big\langle [P,\eta]\nabla_X\varphi - [\nabla_X,\eta]P\varphi, \psi\big\rangle \, dV \\ &\quad - \int_\Omega \big\langle \varphi, [P^\top,\eta]\nabla_X^\top\psi - [\nabla_X^\top,\eta]P^\top\psi\big\rangle \, dV \\ &=: I + II.\end{aligned} \tag{A.3.19}$$

We may further integrate by parts and express the above two terms as

$$\begin{aligned}I &= -\int_\Omega \langle \eta\nabla_X\varphi, P^\top\psi\rangle \, dV + \int_\Omega \langle \eta P\nabla_X\varphi, \psi\rangle \, dV \\ &\quad + \int_\Omega \langle \eta P\varphi, \nabla_X^\top\psi\rangle \, dV - \int_\Omega \langle \eta\nabla_X P\varphi, \psi\rangle \, dV \\ &= \int_\Omega \langle [P,\nabla_X]\varphi, \eta\psi\rangle \, dV,\end{aligned} \tag{A.3.20}$$

and

$$\begin{aligned} II &= -\int_{\Omega} \langle \eta P\varphi, \nabla_X^{\top}\psi\rangle\, dV + \int_{\Omega} \langle \varphi, \eta P^{\top}\nabla_X^{\top}\psi\rangle\, dV \\ &\quad + \int_{\Omega} \langle \eta\nabla_X\varphi, P^{\top}\psi\rangle\, dV - \int_{\Omega} \langle \varphi, \eta\nabla_X^{\top}P^{\top}\psi\rangle\, dV \\ &= \int_{\Omega} \langle \eta\varphi, [P^{\top}, \nabla_X^{\top}]\psi\rangle\, dV = -\int_{\Omega} \langle \eta\varphi, [P, \nabla_X]^{\top}\psi\rangle\, dV. \end{aligned} \tag{A.3.21}$$

Combining (A.3.19), (A.3.20) and (A.3.21), we obtain that

$$\operatorname{div} F = \langle [P, \nabla_X]\varphi, \psi\rangle - \langle \varphi, [P, \nabla_X]^{\top}\psi\rangle. \tag{A.3.22}$$

Consequently,

$$\begin{aligned} \int_{\Omega} \operatorname{div} F\, dV &= \int_{\Omega} \langle [P, \nabla_X]\varphi, \psi\rangle\, dV - \int_{\Omega} \langle \varphi, [P, \nabla_X]^{\top}\psi\rangle\, dV \\ &= \int_{\partial\Omega} \langle \varphi, \sigma_{[P,\nabla_X]}(\cdot, \nu)\psi\rangle\, d\sigma. \end{aligned} \tag{A.3.23}$$

With this in hand, formula (A.3.17) now follows from Proposition A.3.1, bearing in mind (A.3.18). □

Remark. Any Ahlfors regular domain satisfying a two-sided John condition is a UR domain (as pointed out earlier; cf. [16, §3.1]) which satisfies

$$\partial\Omega = \partial(\overline{\Omega}). \tag{A.3.24}$$

As such, Proposition A.3.4 pertains to a subclass of UR domains satisfying (A.3.24). This being said, it is possible to establish the boundary integration by parts formula (A.3.17) for the *entire* class of UR domains satisfying (A.3.24) provided somewhat stronger conditions on the functions involved are imposed. Specifically, we claim that (A.3.17) holds whenever

$$\Omega \subset M \text{ is a relatively compact UR domain satisfying (A.3.24)} \tag{A.3.25}$$

granted that

$$\varphi \in C^1(M, \mathcal{E})\big|_{\partial\Omega} \text{ and } \psi \in L^p_1(\partial\Omega, \mathcal{E}) \text{ for some } p \in (1, \infty). \tag{A.3.26}$$

Indeed, in this context, $\Omega_+ := \Omega$ and $\Omega_- := M \setminus \overline{\Omega}$ are UR domains with a common boundary, $\Sigma := \partial\Omega_+ = \partial\Omega_-$. We can then find (see [16, (3.6.46)-(3.6.47), p. 2684]) two functions $u_\pm \in C^1(\Omega_\pm)$ enjoying the following properties:

$$\begin{aligned} &\mathcal{N}(u_\pm) \in L^p(\Sigma), \quad \mathcal{N}(\nabla u_\pm) \in L^p(\Sigma), \\ &\text{there exist } u_\pm\big|^{\rm n.t.}_{\Sigma} \text{ as well as } (\nabla u_\pm)\big|^{\rm n.t.}_{\Sigma}, \\ &\text{and } \psi = u_+\big|^{\rm n.t.}_{\Sigma} - u_-\big|^{\rm n.t.}_{\Sigma} \text{ at } \sigma\text{-a.e. point on } \Sigma. \end{aligned} \tag{A.3.27}$$

Since $\varphi \in C^1(M,\mathcal{E})\big|_{\partial\Omega}$, we may carry out the same type of argument that has produced (A.3.17) (working separately in Ω_+ and Ω_-) in order to conclude that

$$
\begin{aligned}
\int_{\partial\Omega} \Big\langle \big(\sigma_P(\cdot,\nu)\nabla_X - \sigma_{\nabla_X}(\cdot,\nu)P\big)\varphi, u_\pm\big|^{\text{n.t.}}_{\Sigma} \Big\rangle\, d\sigma \\
= -\int_{\partial\Omega} \Big\langle \varphi, \big(\sigma_{P^\top}(\cdot,\nu)\nabla_X^\top - \sigma_{\nabla_X^\top}(\cdot,\nu)P^\top\big)\big(u_\pm\big|^{\text{n.t.}}_{\Sigma}\big) \Big\rangle\, d\sigma \qquad \text{(A.3.28)} \\
- \int_{\partial\Omega} \Big\langle \varphi, \sigma_{[P,\nabla_X]}(\cdot,\nu) u_\pm\big|^{\text{n.t.}}_{\Sigma} \Big\rangle\, d\sigma.
\end{aligned}
$$

Subtracting these two identities then yields (A.3.17), on account of the last property in (A.3.27).

A.4 Dirac-type operators $\overline{\partial}_L$ on a compact Riemann surface

Here we collect some concepts and results regarding a natural $\overline{\partial}$ operator defined on sections of a holomorphic line bundle, or more generally a holomorphic vector bundle, L over a compact Riemann surface M. The material on line bundles follows [29, §9], to which we refer for further details. Other sources include [14], [25], and, for vector bundles, [15], [3] and [32].

Some special line bundles over M include the canonical bundle $\kappa \to M$, and its conjugate $\overline{\kappa}$. The bundle κ is the cotangent bundle T^*M, endowed by the complex structure of M with the structure of a holomorphic line bundle. The bundle $\overline{\kappa}$ is an anti-holomorphic line bundle. We can characterize a holomorphic line bundle as follows. Given a complex line bundle $L \to M$, let $\{U_j\}$ be a covering of M. A holomorphic structure on L consists of nowhere vanishing sections s_j of L over U_j such that

$$s_j = \sigma_{jk} s_k, \quad \text{on } U_{jk} = U_j \cap U_k, \tag{A.4.1}$$

with σ_{jk} holomorphic, nowhere vanishing, complex-valued functions, called the transition functions of this line bundle. To define a line bundle, one needs the family $\{\sigma_{jk}\}$ to satisfy the cocycle condition

$$\sigma_{jk}\sigma_{k\ell} = \sigma_{j\ell} \ \text{ on } U_j \cap U_k \cap U_\ell. \tag{A.4.2}$$

If the functions σ_{jk} are anti-holomorphic, L gets the structure of an anti-holomorphic line bundle. We mention that changing the transition functions to $\overline{\sigma}_{jk}$ defines the conjugate line bundle, $\overline{L}$.

If $L \to M$ is a holomorphic line bundle, we have

$$\overline{\partial}_L : C^\infty(M,L) \longrightarrow C^\infty(M, L\otimes\overline{\kappa}), \tag{A.4.3}$$

defined as follows. Pick a local coordinate patch U and a local (nowhere-vanishing) holomorphic section S of L over U. Then an arbitrary section u over U is of the form $u = vS$, with v complex valued, and we set

$$\overline{\partial}_L u = \frac{\partial v}{\partial \overline{z}}\, S \otimes d\overline{z}. \tag{A.4.4}$$

This is independent of the choice of local holomorphic section S and of local holomorphic coordinate system.

The operator $\overline{\partial}_L$ is a first order, elliptic differential operator. In addition to (A.4.3), we also have

$$\overline{\partial}_L : \mathcal{D}'(M, L) \longrightarrow \mathcal{D}'(M, L \otimes \overline{\kappa}), \tag{A.4.5}$$

and, for $s \in \mathbb{R}$, $p \in (1, \infty)$,

$$\overline{\partial}_L : H^{s+1,p}(M, L) \longrightarrow H^{s,p}(M, L \otimes \overline{\kappa}). \tag{A.4.6}$$

Standard elliptic theory implies $\overline{\partial}_L$ is Fredholm in (A.4.3), (A.4.5), and (A.4.6), and has the same index in all cases. The fact that $\overline{\partial}_L$ in (A.4.5) has range of finite codimension (say m) implies that any linear subspace $V \subset \mathcal{D}'(M, L \otimes \overline{\kappa})$ of dimension $> m$ has nonzero intersection with the range of $\overline{\partial}_L$. It follows that there exists a nonzero $v \in \mathcal{D}'(M, L \otimes \overline{\kappa})$, supported on a finite point set $\mathcal{F}$, and in the range of $\overline{\partial}_L$, so $v = \overline{\partial}_L u$. Such u is holomorphic on $M \setminus \mathcal{F}$, and in fact is a nontrivial meromorphic section of L:

Each holomorphic line bundle over M has a nontrivial meromorphic section. (A.4.7)

We denote by $\mathcal{M}(L)$ the space of meromorphic sections of L, and by $\mathcal{O}(L)$ the space of holomorphic sections of L, i.e.,

$$\mathcal{O}(L) = \operatorname{Ker} \overline{\partial}_L. \tag{A.4.8}$$

One also has

$$\operatorname{Ker} \overline{\partial}_L^* \approx \mathcal{O}(L' \otimes \kappa), \tag{A.4.9}$$

where L' is the dual bundle to L. Hence

$$\operatorname{Index} \overline{\partial}_L = \dim \mathcal{O}(L) - \dim \mathcal{O}(L' \otimes \kappa). \tag{A.4.10}$$

The following formula for the index of $\overline{\partial}_L$ is known as the Riemann-Roch formula. (Cf., e.g., [29, Theorem 9.1].)

Proposition A.4.1 *If L is a holomorphic line bundle over a compact Riemann surface M,*

$$\operatorname{Index} \overline{\partial}_L = c_1(L) + 1 - g. \tag{A.4.11}$$

Here g is the genus of M, related to the Euler characteristic $\chi(M)$ by

$$\chi(M) = 2 - 2g. \tag{A.4.12}$$

Thus the Riemann sphere S^2 has genus 0 and a torus $\mathbb{C}/\Lambda$ has genus 1.

The integer $c_1(L)$ is called the first Chern class of L. For our purposes, the following characterization will suffice. Take a nontrivial meromorphic section $u \in \mathcal{M}(L)$. It has a finite number of zeros and poles. If p is a zero of u, let $\nu_u(p)$ be the order of the zero. If p is a pole of u, let $-\nu_u(p)$ be the order of the pole. We define the *divisor* of $u \in \mathcal{M}(L)$ to be the formal sum

$$\vartheta(u) = \sum_p \nu_u(p) \cdot p. \tag{A.4.13}$$

In such a case (cf. [29, Proposition 9.3]),

$$c_1(L) = \sum_p \nu_u(p). \tag{A.4.14}$$

This is independent of the choice of nontrivial $u \in \mathcal{M}(L)$. As a corollary,

$$c_1(L) < 0 \Longrightarrow \mathcal{O}(L) = 0. \tag{A.4.15}$$

Note that if L_1 and L_2 are holomorphic line bundles over M, with meromorphic sections u_1 and u_2, then $u_1 \otimes u_2$ is a meromorphic section of $L_1 \otimes L_2$. The characterization (A.4.14) readily gives

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2). \tag{A.4.16}$$

Generalizing (A.4.13), we say a divisor on M is a finite formal sum

$$\vartheta = \sum_p \nu(p) \cdot p, \quad \nu(p) \in \mathbb{Z}. \tag{A.4.17}$$

To any divisor ϑ we can associate a holomorphic line bundle, denoted E_ϑ. To construct E_ϑ, it is convenient to use the method of transition functions, as in (A.4.1). Cover M with holomorphic coordinate charts U_j. Pick ψ_j, meromorphic on U_j, having a pole of order exactly $|\nu(p)|$ at p if $\nu(p) < 0$, a zero of order exactly $\nu(p)$ if $\nu(p) > 0$, and no other poles or zeros. The transition functions

$$\varphi_{jk} = \psi_k^{-1} \psi_j \ \text{ on } \ U_j \cap U_k \tag{A.4.18}$$

define the holomorphic line bundle E_ϑ. The collection $\{\psi_j, U_j\}$ defines a meromorphic section

$$\psi \in \mathcal{M}(E_\vartheta), \quad \vartheta(\psi) = -\vartheta. \tag{A.4.19}$$

Hence

$$c_1(E_\vartheta) = -\sum_p \nu(p). \tag{A.4.20}$$

There is a natural isomorphism

$$\mathcal{M}(L, \vartheta) \approx \mathcal{O}(L \otimes E_\vartheta), \tag{A.4.21}$$

where

$$\mathcal{M}(L, \vartheta) = \{u \in \mathcal{M}(L) : \vartheta(u) \geq \vartheta\}. \tag{A.4.22}$$

The isomorphism takes $u \in \mathcal{M}(L, \vartheta)$ to $u\psi$, with ψ as in (A.4.19). We also mention that if u is a nontrivial element of $\mathcal{M}(L)$, then we have holomorphically equivalent line bundles

$$L \approx E_{-\vartheta(u)}. \tag{A.4.23}$$

We now specify some examples where $\overline{\partial}_L$ is invertible.

Proposition A.4.2 *If M has genus $g = 0$ and $p \in M$, then*

$$L = E_p \Longrightarrow \overline{\partial}_L \ \text{ is invertible.} \tag{A.4.24}$$

Proof. By (A.4.20), $c_1(L) = -1$. Hence, by (A.4.11), $\operatorname{Index} \overline{\partial}_L = 0$, and by (A.4.15), $\operatorname{Ker} \overline{\partial}_L = 0$. □

As is well known, if $g = 0$, then M is holomorphically equivalent to the Riemann sphere S^2. A proof can be found in [29, §9]. We mention that one ingredient is the following.

Lemma A.4.3 *If M is a compact Riemann surface, $q \in M$, and there exists a meromorphic function u on M with just one pole, a simple pole at q, then M is holomorphically equivalent to S^2. In fact, $u : M \to \mathbb{C} \cup \{\infty\}$ provides the equivalence.*

We move on to genus $g = 1$:

Proposition A.4.4 *If M has genus $g = 1$, and p, q are distinct points in M, then*

$$L = E_{p-q} \Longrightarrow \overline{\partial}_L \text{ is invertible.} \tag{A.4.25}$$

Proof. This time, $c_1(L) = 0$, hence, by (A.4.11), $\operatorname{Index} \overline{\partial}_L = 0$. It remains to show that $\operatorname{Ker} \overline{\partial}_L = 0$, i.e., $\mathcal{O}(E_{p-q}) = 0$. By (A.4.21)–(A.4.22), with L the trivial line bundle, any non-trivial element of $\mathcal{O}(E_{p-q})$ would yield a meromorphic function u on M that vanishes at p and has at most a simple pole at q. If u actually had a simple pole at q, Lemma A.4.3 would imply $M \approx S^2$, a contradiction. Otherwise, u must be constant, and hence $\equiv 0$. ☐

If $g = 1$, then M is holomorphically equivalent to some torus $\mathbb{C}/\Lambda$. A proof of this can also be found in [29, §9].

There are more general classes of holomorphic line bundles L for which $\overline{\partial}_L$ is invertible. We mention the following easy generalization of Propositions A.4.2 and A.4.4.

Proposition A.4.5 *Let $L \to M$ be a holomorphic line bundle. If M has genus $g = 0$, then*

$$c_1(L) = -1 \Longrightarrow \overline{\partial}_L \text{ is invertible.} \tag{A.4.26}$$

If M has genus $g = 1$, then

$$c_1(L) = 0 \text{ and } \mathcal{O}(L) = 0 \Longrightarrow \overline{\partial}_L \text{ is invertible.} \tag{A.4.27}$$

For general genus g,

$$c_1(L) = g - 1 \text{ and } \mathcal{O}(L) = 0 \Longrightarrow \overline{\partial}_L \text{ is invertible.} \tag{A.4.28}$$

In case $L = E_\vartheta$, (A.4.20) specifies $c_1(E_\vartheta)$, so one easily sees when the hypothesis of (A.4.26) is satisfied. One also easily sees when $c_1(L) = 0$.

Given $c_1(L) = 0$, one may or may not have $\mathcal{O}(L) = 0$. For example, if L is the trivial bundle, constants belong to $\mathcal{O}(L)$, so $\mathcal{O}(L) \approx \mathbb{C}$. Conversely, if there is a nontrivial $u \in \mathcal{O}(L)$, then, by (A.4.14), u is nowhere vanishing. Hence u provides a holomorphic trivialization of L. In case $L = E_\vartheta$, with ϑ as in (A.4.17) and $\sum \nu(p) = 0$, one can deduce from (A.4.21)–(A.4.22) that $\mathcal{O}(E_\vartheta) \neq 0$ if and only if there exists a meromorphic function u on M such that $\vartheta(u) = \vartheta$. When this holds is settled by a result known as Abel's Theorem, which can be found in [25].

One elementary result is that, given p and q distinct points of M,

$$\mathcal{O}(E_{p-q}) \neq 0, \text{ hence } E_{p-q} \text{ is holomorphically trivial } \Longleftrightarrow M \approx S^2. \tag{A.4.29}$$

The proof of the right-pointing implication follows that of Proposition A.4.4. For the converse, if $p, q \in \mathbb{C}$, take $u(z) = (z - p)/(z - q)$, for which $\vartheta(u) = p - q$.

Regarding the condition (A.4.28), for $g \geq 2$, there are many line bundles satisfying this. In more detail, the space of equivalence classes of holomorphic line bundles $L \to M$ with $c_1(L) = g - 1$ is a Jacobi variety, equivalent to a g-dimensional complex torus:

$$J^{g-1}(M) \approx \mathbb{C}^g/\Lambda, \tag{A.4.30}$$

See [25]. Given $L \in J^{g-1}(M)$, if $\mathcal{O}(L) \neq 0$, then by (A.4.23), $L \approx E_{-(p_1+\cdots+p_{g-1})}$, for some points $p_j \in M$, $1 \leq j \leq g-1$. Now the image of the $(g-1)$-fold product of copies of M in $J^{g-1}(M)$,

$$(p_1, \dots, p_{g-1}) \mapsto E_{-(p_1+\cdots+p_{g-1})}, \tag{A.4.31}$$

is a $(g-1)$-dimensional variety, often denoted Θ, in the g-dimensional space $J^{g-1}(M)$. We have

$$\text{If } L \in J^{g-1}(M) \setminus \Theta, \text{ then (A.4.28) holds.} \tag{A.4.32}$$

Moving on to vector bundles, we next record the formula for Index $\overline{\partial}_L$ when $L \to M$ is a holomorphic vector bundle, of rank r (so each fiber is a complex vector space of dimension r). In such a case, σ_{jk} in (A.4.2) are $r \times r$ matrices. Identity (A.4.10) continues to hold, and the Riemann-Roch theorem in this setting takes the following form.

Proposition A.4.6 *If L is a holomorphic vector bundle of rank r over a compact Riemann surface M, of genus g,*

$$\text{Index}\, \overline{\partial}_L = c_1(L) + r(1-g). \tag{A.4.33}$$

A proof can be found in [15]. Another proof, using methods from [29], is given in [31]. Here,

$$c_1(L) = c_1(\Lambda^r L), \tag{A.4.34}$$

if L has rank r. We mention that, complementing (A.4.16), if $\widetilde{L}$ has rank $\tilde{r}$,

$$c_1(L \oplus \widetilde{L}) = c_1(L) + c_1(\widetilde{L}), \quad c_1(L \otimes \widetilde{L}) = \tilde{r} c_1(L) + r c_1(\widetilde{L}), \tag{A.4.35}$$

and, if L' is the dual bundle to L,

$$c_1(L') = -c_1(L). \tag{A.4.36}$$

Parallel to Proposition A.4.5, we have:

Proposition A.4.7 *In the setting of Proposition A.4.6, the operator $\overline{\partial}_L$ is invertible if and only if*

$$c_1(L) = r(g-1) \ \textit{and} \ \mathcal{O}(L) = 0. \tag{A.4.37}$$

In case $g = 0$ or $g = 1$, each holomorphic vector bundle of rank r is a sum of line bundles, $L = L_1 \oplus \cdots \oplus L_r$. See [15] for $g = 0$ and [1] for $g = 1$. In case $g = 0$, (A.4.37) holds if and only if $c_1(L_j) = -1$ for each j. In case $g = 1$, (A.4.37) holds if and only if $c_1(L_j) = 0$ and $\mathcal{O}(L_j) = 0$ for each j.

For $g \geq 2$, it is the case that most "semistable" holomorphic vector bundles L of rank r with $c_1(L) = r(g-1)$ satisfy (A.4.37). Explaining this requires a definition.

Definition. A holomorphic vector bundle of rank r, $L \to M$, is semistable provided that, if L_0 is a subbundle of rank r_0, then

$$\frac{c_1(L_0)}{r_0} \leq \frac{c_1(L)}{r}. \tag{A.4.38}$$

If one always has strict inequality in (A.4.38), we say L is stable.

Here is one reason to restrict to semistable vector bundles.

Lemma A.4.8 *If L is not semistable, then* (A.4.37) *fails.*

Proof. Say $r = \operatorname{rank} L$, $c_1(L) = r(g-1)$. If L is not semistable, there is a subbundle L_0, of rank r_0, such that (A.4.38) fails, hence

$$c_1(L_0) > \frac{c_1(L)}{r} r_0 = r_0(g-1). \tag{A.4.39}$$

Then the Riemann-Roch formula implies $\operatorname{Index} \overline{\partial}_{L_0} > 0$, so $\mathcal{O}(L_0) \neq 0$, hence $\mathcal{O}(L) \neq 0$. □

The class of semistable holomorphic vector bundles $L \to M$, of rank r, with $c_1(L) = k$, has a well-studied moduli space $\mathcal{M}(r,k)$ of equivalence classes. The equivalence at stable bundles is the usual holomorphic equivalence. It is a bit more subtle at the vector bundles that are not stable, and we refer to [32] for details. It has been shown that $\mathcal{M}(r,k)$ is a complex projective variety, smooth at the stable bundles (which form a dense open subset). It has further been shown (cf. [3]) that there is a "theta divisor" $\Theta \subset \mathcal{M}(r, r(g-1))$, of complex codimension 1, such that

$$L \in \mathcal{M}(r, r(g-1)) \setminus \Theta \Longrightarrow \text{(A.4.37) holds.} \tag{A.4.40}$$

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