Chapter 1

The Riemann Integral

The Riemann integral is a fundamental part of calculus and an essential precursor to the Lebesgue integral. In this chapter we define the Riemann integral of a bounded function on an interval $I = [a, b]$ on the real line. To do this, we partition I into smaller intervals. A *partition* $\mathcal{P}$ of I is a finite collection of subintervals $\{J_k : 0 \le k \le N\}$, disjoint except for their endpoints, whose union is I. We can order the J_k so that $J_k = [x_k, x_{k+1}]$, where

$$x_0 < x_1 < \cdots < x_N < x_{N+1}, \quad x_0 = a,\ x_{N+1} = b. \tag{1.1}$$

We call the points x_k the *endpoints* of $\mathcal{P}$. We set

$$\begin{aligned} \ell(J_k) = x_{k+1} - x_k, \quad \operatorname{maxsize}(\mathcal{P}) &= \max_{0\le k\le N} \ell(J_k), \\ \operatorname{minsize}(\mathcal{P}) &= \min_{0\le k\le N} \ell(J_k). \end{aligned} \tag{1.2}$$

We then set

$$\begin{aligned} \overline{I}_{\mathcal{P}}(f) &= \sum_k \sup_{J_k} f(x)\,\ell(J_k), \\ \underline{I}_{\mathcal{P}}(f) &= \sum_k \inf_{J_k} f(x)\,\ell(J_k). \end{aligned} \tag{1.3}$$

Note that $\underline{I}_{\mathcal{P}}(f) \le \overline{I}_{\mathcal{P}}(f)$. These quantities should approximate the Riemann integral of f, if the partition $\mathcal{P}$ is sufficiently "fine."

To be more precise, if $\mathcal{P}$ and $\mathcal{Q}$ are two partitions of I, we say $\mathcal{P}$ *refines* $\mathcal{Q}$, and we write $\mathcal{P} \succ \mathcal{Q}$, if $\mathcal{P}$ is formed by partitioning each interval in $\mathcal{Q}$. Equivalently, $\mathcal{P} \succ \mathcal{Q}$ if and only if all the endpoints of $\mathcal{Q}$ are also endpoints of $\mathcal{P}$. It is easy to see that any two partitions have a common refinement; just take the union of their endpoints to form a new partition. Note also that

$$\mathcal{P} \succ \mathcal{Q} \Longrightarrow \overline{I}_{\mathcal{P}}(f) \le \overline{I}_{\mathcal{Q}}(f) \text{ and } \underline{I}_{\mathcal{P}}(f) \ge \underline{I}_{\mathcal{Q}}(f). \tag{1.4}$$

Consequently, if $\mathcal{P}_j$ are any two partitions and $\mathcal{Q}$ is a common refinement, we have

$$\underline{I}_{\mathcal{P}_1}(f) \le \underline{I}_{\mathcal{Q}}(f) \le \overline{I}_{\mathcal{Q}}(f) \le \overline{I}_{\mathcal{P}_2}(f). \tag{1.5}$$

Now, whenever $f : I \to \mathbb{R}$ is bounded, the following quantities are well defined:

$$\overline{I}(f) = \inf_{\mathcal{P}\in\Pi(I)} \overline{I}_{\mathcal{P}}(f), \quad \underline{I}(f) = \sup_{\mathcal{P}\in\Pi(I)} \underline{I}_{\mathcal{P}}(f), \tag{1.6}$$

where $\Pi(I)$ is the set of all partitions of I. Clearly, by (1.5), $\underline{I}(f) \le \overline{I}(f)$. We then say that f is *Riemann integrable* provided $\overline{I}(f) = \underline{I}(f)$, and in such a case, we set

$$\int_I f(x)\, dx = \overline{I}(f) = \underline{I}(f). \tag{1.7}$$

We will denote the set of Riemann integrable functions on I by $\mathcal{R}(I)$.

We derive some basic properties of the Riemann integral.

Proposition 1.1. *If $f,\ g \in \mathcal{R}(I)$, then $f+g \in \mathcal{R}(I)$, and*

$$\int_I (f+g)\, dx = \int_I f\ dx + \int_I g\, dx. \tag{1.8}$$

Proof. If J_k is any subinterval of I, then

$$\sup_{J_k} (f+g) \le \sup_{J_k} f + \sup_{J_k} g,$$

so, for any partition $\mathcal{P}$, we have $\overline{I}_{\mathcal{P}}(f+g) \le \overline{I}_{\mathcal{P}}(f) + \overline{I}_{\mathcal{P}}(g)$. Also, using common refinements, we can *simultaneously* approximate $\overline{I}(f)$ and $\overline{I}(g)$ by $\overline{I}_{\mathcal{P}}(f)$ and $\overline{I}_{\mathcal{P}}(g)$. Thus the characterization (1.6) implies $\overline{I}(f+g) \le \overline{I}(f) + \overline{I}(g)$. A parallel argument implies $\underline{I}(f+g) \ge \underline{I}(f) + \underline{I}(g)$, and the proposition follows.

Next, there is a fair supply of Riemann integrable functions.

Proposition 1.2. *If f is continuous on I, then f is Riemann integrable.*

Proof. Any continuous function on a compact interval is uniformly continuous; let $\omega(\delta)$ be a modulus of continuity for f, so

$$|x-y| \le \delta \Longrightarrow |f(x)-f(y)| \le \omega(\delta), \quad \omega(\delta) \to 0 \text{ as } \delta \to 0. \tag{1.9}$$

Then

$$\operatorname{maxsize}(\mathcal{P}) \le \delta \Longrightarrow \overline{I}_{\mathcal{P}}(f) - \underline{I}_{\mathcal{P}}(f) \le \omega(\delta)\cdot \ell(I), \tag{1.10}$$

which yields the proposition.

This argument, showing that every continuous function is Riemann integrable, also provides a criterion on a partition $\mathcal{P}$ guaranteeing that $\overline{I}_{\mathcal{P}}(f)$ and $\underline{I}_{\mathcal{P}}(f)$ are close to $\int_I f\,dx$, when f is continuous. The following is a useful extension. Let $f \in \mathcal{R}(I)$, take $\varepsilon > 0$, and let $\mathcal{P}_0$ be a partition such that

$$\overline{I}_{\mathcal{P}_0}(f) - \varepsilon \le \int_I f\,dx \le \underline{I}_{\mathcal{P}_0}(f) + \varepsilon. \tag{1.11}$$

Let

$$M = \sup_I |f(x)|, \quad \delta = \operatorname{minsize}(\mathcal{P}_0). \tag{1.12}$$

Proposition 1.3. *Under the hypotheses above, if $\mathcal{P}$ is any partition of I satisfying*

$$\operatorname{maxsize}(\mathcal{P}) \le \frac{\delta}{k}, \tag{1.13}$$

then

$$\overline{I}_{\mathcal{P}}(f) - \varepsilon_1 \le \int_I f\,dx \le \underline{I}_{\mathcal{P}}(f) + \varepsilon_1, \ \textit{with } \varepsilon_1 = \varepsilon + \frac{2M}{k}\ell(I). \tag{1.14}$$

Proof. Consider on the one hand those intervals in $\mathcal{P}$ that are contained in intervals in $\mathcal{P}_0$, and on the other hand those intervals in $\mathcal{P}$ that are *not* contained in intervals in $\mathcal{P}_0$ (whose lengths sum to $\le \ell(I)/k$). Let $\mathcal{P}_1$ be the minimal common refinement of $\mathcal{P}$ and $\mathcal{P}_0$. We obtain

$$\overline{I}_{\mathcal{P}}(f) \le \overline{I}_{\mathcal{P}_1}(f) + \frac{2M}{k}\ell(I), \quad \underline{I}_{\mathcal{P}}(f) \ge \underline{I}_{\mathcal{P}_1}(f) - \frac{2M}{k}\ell(I).$$

Since also $\overline{I}_{\mathcal{P}_1}(f) \le \overline{I}_{\mathcal{P}_0}(f)$ and $\underline{I}_{\mathcal{P}_1}(f) \ge \underline{I}_{\mathcal{P}_0}(f)$, this implies (1.14).

The following corollary is sometimes called Darboux's Theorem.

Corollary 1.4. *Let $\mathcal{P}_\nu$ be any sequence of partitions of I into ν intervals $J_{\nu k},\ 1 \le k \le \nu$, such that*

$$\operatorname{maxsize}(\mathcal{P}_\nu) = \delta_\nu \to 0,$$

and let $\xi_{\nu k}$ be any choice of one point in each interval $J_{\nu k}$ of the partition $\mathcal{P}_\nu$. Then, whenever $f \in \mathcal{R}(I)$,

$$\int_I f(x)\, dx = \lim_{\nu\to\infty} \sum_{k=1}^{\nu} f(\xi_{\nu k})\, \ell(J_{\nu k}). \tag{1.15}$$

The sum on the right side of (1.15) is called a Riemann sum. One should be warned that, once such a specific choice of $\mathcal{P}_\nu$ and $\xi_{\nu k}$ has been made, the limit on the right side of (1.15) might exist for a bounded function f that is *not* Riemann integrable. This and other phenomena are illustrated by the following example of a function which is not Riemann integrable. For $x \in I$, set

$$\vartheta(x) = 1 \ \text{ if } \ x \in \mathbb{Q}, \quad \vartheta(x) = 0 \ \text{ if } \ x \notin \mathbb{Q}, \tag{1.16}$$

where $\mathbb{Q}$ is the set of *rational* numbers. Now every interval $J \subset I$ of positive length contains points in $\mathbb{Q}$ and points not in $\mathbb{Q}$, so for any partition $\mathcal{P}$ of I we have $\overline{I}_{\mathcal{P}}(\vartheta) = \ell(I)$ and $\underline{I}_{\mathcal{P}}(\vartheta) = 0$, and hence

$$\overline{I}(\vartheta) = \ell(I), \quad \underline{I}(\vartheta) = 0. \tag{1.17}$$

Note that, if $\mathcal{P}_\nu$ is a partition of I into ν equal subintervals, then we could pick each $\xi_{\nu k}$ to be rational, in which case the limit on the right side of (1.15) would be $\ell(I)$, or we could pick each $\xi_{\nu k}$ to be irrational, in which case this limit would be zero. Alternatively, we could pick half of them to be rational and half to be irrational, and the limit would be $\ell(I)/2$.

Let $f_k \in \mathcal{R}(I)$ be a uniformly bounded, monotonically increasing sequence of functions. Then there is a bounded function f on I such that, as $k \to \infty$,

$$f_k(x) \nearrow f(x), \quad \forall x \in I. \tag{1.18}$$

It would be desirable to conclude that f is integrable and

$$\int_I f_k(x)\, dx \to \int_I f(x)\, dx. \tag{1.19}$$

A shortcoming of the Riemann integral is that such a limit might not belong to $\mathcal{R}(I)$. For example, since $I \cap \mathbb{Q}$ is countable, let $I \cap \mathbb{Q} = \{c_1, c_2, c_3, \dots\}$, and let

$$\vartheta_k(x) = 1 \text{ if } x \in \{c_1, \dots, c_k\}, \quad 0 \text{ otherwise.} \tag{1.20}$$

It is easy to see that $\underline{I}(\vartheta_k) = \overline{I}(\vartheta_k) = 0$, so each $\vartheta_k \in \mathcal{R}(I)$. But, as $k \to \infty$,

$$\vartheta_k(x) \nearrow \vartheta(x), \tag{1.21}$$

defined by (1.16), which is not in $\mathcal{R}(I)$. The Lebesgue theory of integration remedies this defect. If f_k are Lebesgue integrable, and if one uses in (1.19) the Lebesgue integral (which coincides with the Riemann integral for functions in $\mathcal{R}(I)$), then (1.18) $\Rightarrow$ (1.19). This is known as the Monotone Convergence Theorem, and it will be seen to be a central result in the Lebesgue theory.

Associated to the Riemann integral is a notion of size of a set S, called *content*. If S is a subset of I, define the "characteristic function"

$$\chi_S(x) = 1 \text{ if } x \in S, \quad 0 \text{ if } x \notin S. \tag{1.22}$$

We define "upper content" cont^+ and "lower content" cont^- by

$$\mathrm{cont}^+(S) = \overline{I}(\chi_S), \quad \mathrm{cont}^-(S) = \underline{I}(\chi_S). \tag{1.23}$$

We say S "has content," or "is contented" if these quantities are equal, which happens if and only if $\chi_S \in \mathcal{R}(I)$, in which case the common value of $\mathrm{cont}^+(S)$ and $\mathrm{cont}^-(S)$ is

$$m(S) = \int_I \chi_S(x)\, dx. \tag{1.24}$$

It is easy to see that

$$\mathrm{cont}^+(S) = \inf\Big\{\sum_{k=1}^N \ell(J_k) : S \subset J_1 \cup \cdots \cup J_N\Big\}, \tag{1.25}$$

where J_k are intervals. Here, we require S to be in the union of a *finite* collection of intervals.

The key to the construction of Lebesgue measure is to cover a set S by a *countable* (either finite or *infinite*) set of intervals. The *outer measure* of $S \subset I$ will be defined by

$$m^*(S) = \inf\Big\{\sum_{k\ge 1} \ell(J_k) : S \subset \bigcup_{k\ge 1} J_k\Big\}. \tag{1.26}$$

Here $\{J_k\}$ is a finite or countably infinite collection of intervals. Clearly

$$m^*(S) \le \text{cont}^+(S). \tag{1.27}$$

Note that, if $S = I \cap \mathbb{Q}$, then $\chi_S = \vartheta$, defined by (1.16). In this case it is easy to see that $\text{cont}^+(S) = \ell(I)$, but $m^*(S) = 0$. Zero is the "right" measure of this set.

We develop a few more properties of the Riemann integral. It is useful to note that $\int_I f\ dx$ is additive in I, in the following sense.

Proposition 1.5. *If $a < b < c$, $f : [a,c] \to \mathbb{R}$, $f_1 = f\big|_{[a,b]}$, $f_2 = f\big|_{[b,c]}$, then*

$$f \in \mathcal{R}\big([a,c]\big) \Longleftrightarrow f_1 \in \mathcal{R}\big([a,b]\big) \quad \textit{and} \quad f_2 \in \mathcal{R}\big([b,c]\big), \tag{1.28}$$

and, if this holds,

$$\int_a^c f\,dx = \int_a^b f_1\ dx + \int_b^c f_2\,dx. \tag{1.29}$$

Proof. Since any partition of $[a,c]$ has a refinement for which b is an endpoint, we may as well consider a partition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, where $\mathcal{P}_1$ is a partition of $[a,b]$ and $\mathcal{P}_2$ is a partition of $[b,c]$. Then

$$\overline{I}_{\mathcal{P}}(f) = \overline{I}_{\mathcal{P}_1}(f_1) + \overline{I}_{\mathcal{P}_2}(f_2), \quad \underline{I}_{\mathcal{P}}(f) = \underline{I}_{\mathcal{P}_1}(f_1) + \underline{I}_{\mathcal{P}_2}(f_2), \tag{1.30}$$

so

$$\overline{I}_{\mathcal{P}}(f) - \underline{I}_{\mathcal{P}}(f) = \big\{\overline{I}_{\mathcal{P}_1}(f_1) - \underline{I}_{\mathcal{P}_1}(f_1)\big\} + \big\{\overline{I}_{\mathcal{P}_2}(f_2) - \underline{I}_{\mathcal{P}_2}(f_2)\big\}. \tag{1.31}$$

Since both terms in braces in (1.31) are ≥ 0, we have equivalence in (1.28). Then (1.29) follows from (1.30) upon taking sufficiently fine partitions.

Let $I = [a,b]$. If $f \in \mathcal{R}(I)$, then $f \in \mathcal{R}([a,x])$ for all $x \in [a,b]$, and we can consider the function

$$g(x) = \int_a^x f(t)\,dt. \tag{1.32}$$

If $a \le x_0 \le x_1 \le b$, then

$$g(x_1) - g(x_0) = \int_{x_0}^{x_1} f(t)\,dt, \tag{1.33}$$

so, if $|f| \le M$,

$$|g(x_1) - g(x_0)| \le M|x_1 - x_0|. \tag{1.34}$$

In other words, if $f \in \mathcal{R}(I)$, then g is Lipschitz continuous on I.

To finish this section, we want to relate the integral and the derivative. Recall from elementary calculus that a function $g : (a, b) \to \mathbb{R}$ is said to be differentiable at $x \in (a, b)$ provided there exists the limit

$$\lim_{h\to 0} \frac{1}{h}\big[g(x + h) - g(x)\big] = g'(x). \tag{1.35}$$

When such a limit exists, $g'(x)$, also denoted dg/dx, is called the derivative of g at x. Clearly g is continuous wherever it is differentiable.

The next result is part of the Fundamental Theorem of Calculus.

Theorem 1.6. *If $f \in C([a, b])$, then the function g, defined by (1.32), is differentiable at each point $x \in (a, b)$, and*

$$g'(x) = f(x). \tag{1.36}$$

Proof. Parallel to (1.33), we have, for $h > 0$,

$$\frac{1}{h}\big[g(x + h) - g(x)\big] = \frac{1}{h}\int_x^{x+h} f(t)\, dt. \tag{1.37}$$

If f is continuous at x, then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(t) - f(x)| \le \varepsilon$ whenever $|t - x| \le \delta$. Thus the right side of (1.37) is within ε of $f(x)$ whenever $h \in (0, \delta]$. Thus the desired limit exists as $h \searrow 0$. A similar argument treats $h \nearrow 0$.

The next result is the rest of the Fundamental Theorem of Calculus.

Theorem 1.7. *If G is differentiable and $G'(x)$ is continuous on $[a, b]$, then*

$$\int_a^b G'(t)\ dt = G(b) - G(a). \tag{1.38}$$

Proof. Consider the function

$$g(x) = \int_a^x G'(t)\, dt. \tag{1.39}$$

We have $g \in C([a, b])$, $g(a) = 0$, and, by Theorem 1.6,

$$g'(x) = G'(x), \quad \forall\, x \in (a, b).$$

Thus $f(x) = g(x) - G(x)$ is continuous on $[a, b]$, and

$$f'(x) = 0, \quad \forall\, x \in (a, b). \tag{1.40}$$

We claim that (1.40) implies f is *constant* on $[a, b]$. Granted this, since $f(a) = g(a) - G(a) = -G(a)$, we have $f(x) = -G(a)$ for all $x \in [a, b]$, so the integral (1.39) is equal to $G(x) - G(a)$ for all $x \in [a, b]$. Taking $x = b$ yields (1.38).

The fact that (1.40) implies f is constant on $[a, b]$ is a consequence of the following result, the Mean Value Theorem.

Theorem 1.8. *Let $f : [a, \beta] \to \mathbb{R}$ be continuous, and assume f is differentiable on (a, β). Then $\exists\, \xi \in (a, \beta)$ such that*

$$f'(\xi) = \frac{f(\beta) - f(a)}{\beta - a}. \tag{1.41}$$

Proof. Replacing $f(x)$ by $\widetilde{f}(x) = f(x) - \kappa(x - a)$, where κ is the right side of (1.41), we can assume without loss of generality that $f(a) = f(\beta)$. Then we claim that $f'(\xi) = 0$ for some $\xi \in (a, \beta)$. Indeed, since $[a, \beta]$ is compact, f must assume a maximum and a minimum on $[a, \beta]$. If $f(a) = f(\beta)$, *one* of these must be assumed at an interior point, ξ, at which f' clearly vanishes.

We now show that (1.40) implies f is constant on $[a, b]$. If not, $\exists\, \beta \in (a, b]$ such that $f(\beta) \neq f(a)$. Then just apply Theorem 1.8 to f on $[a, \beta]$. This completes the proof of Theorem 1.7.

We mention some useful notation. If a function G is differentiable on (a, b) and G' is continuous on (a, b), we say G is a C^1 function, and we write $G \in C^1\big((a, b)\big)$. Inductively, we say $G \in C^k\big((a, b)\big)$ provided $G' \in C^{k-1}\big((a, b)\big)$. Similarly define $C^k\big([a, b]\big)$. Note that the hypothesis of Theorem 1.7 is that $G \in C^1\big([a, b]\big)$.

Finally, we mention that there are more general versions of the Fundamental Theorem of Calculus involving the Riemann integal; see for example [**BS**]. Since the Riemann integral is not our main focus here, we have been content to present the simpler results above. (See, however, Exercise 14 below.) Our main motivation for taking the space to present these results (which the reader might reasonably be presumed to have seen before) is provided by their role in the study of weak derivatives, in Chapter 10. Furthermore, we present extensions of Theorems 1.6–1.7, involving the Lebesgue integral, in Chapter 11, particularly in Propositions 11.11–11.12.

Exercises

1. Show that, if $f_k \in \mathcal{R}(I)$ and $f_k \to f$ *uniformly* on I, i.e.,

$$\sup_I |f_k(x) - f(x)| \to 0 \text{ as } k \to \infty,$$

then $f \in \mathcal{R}(I)$, and (1.19) holds.

2. Establish the following "monotone convergence theorem" for the Riemann integral. Assume f_k and f are continuous on I. Then (1.18) $\Rightarrow$ (1.19).
Hint. Show that $f_k \to f$ uniformly on I, under these hypotheses. (This result is known as Dini's Theorem.)

3. If $I \cap \mathbb{Q} = \{c_1, c_2, c_3, \dots\}$ as in the construction of (1.20), set

$$\begin{aligned} \Theta(x) = \ & \frac{1}{k} \ \text{ if } \ x = c_k, \\ & 0 \ \text{ if } \ x \in I \setminus \mathbb{Q}. \end{aligned}$$

Show that $\Theta \in \mathcal{R}(I)$ and compute $\int_I \Theta(x)\, dx$.

4. Assume f is bounded on $I = [a, b]$ and *continuous* on $\overset{\circ}{I} = (a, b)$. Show that $f \in \mathcal{R}(I)$ and that $\int_I f(x)\, dx$ is independent of the values of f at $x = a$ and $x = b$.

5. Let $\mathcal{P}_\nu$ be a sequence of partitions of I satisfying the hypotheses of Corollary 1.4. Show that, when $f : I \to \mathbb{R}$ is bounded,

$$\lim_{\nu\to\infty} \overline{I}_{\mathcal{P}_\nu}(f) = \overline{I}(f) \quad \text{and} \quad \lim_{\nu\to\infty} \underline{I}_{\mathcal{P}_\nu}(f) = \underline{I}(f). \tag{1.42}$$

6. Let $C(I)$ denote the set of continuous real-valued functions on I. Show that, for any bounded function $f : I \to \mathbb{R}$,

$$\overline{I}(f) = \inf \Big\{ \int_I g\, dx : g \geq f,\ g \in C(I) \Big\}. \tag{1.43}$$

Similarly characterize $\underline{I}(f)$.

7. Let $c > 0$ and let $f : [ac, bc] \to \mathbb{R}$ be Riemann integrable. Working directly with the definition of integral, show that

$$\int_a^b f(cx)\,dx = \frac{1}{c}\int_{ac}^{bc} f(x)\,dx.$$

More generally, show that

$$\int_{a-d/c}^{b-d/c} f(cx+d)\,dx = \frac{1}{c}\int_{ac}^{bc} f(x)\,dx.$$

8. Let $f : I \times S \to \mathbb{R}$ be continuous, where $I = [a, b]$ and $S \subset \mathbb{R}^n$. Take $\varphi(y) = \int_I f(x, y)\,dx$. Show that φ is continuous on S.
Hint. If $f_j : I \to \mathbb{R}$ are continuous and $|f_1(x) - f_2(x)| \le \delta$ on I, then

$$\Bigl|\int_I f_1\,dx - \int_I f_2\,dx\Bigr| \le \ell(I)\delta.$$

9. With f as in Exercise 8, suppose $g_j : S \to \mathbb{R}$ are continuous and $a \le g_0(y) < g_1(y) \le b$. Take $\varphi(y) = \int_{g_0(y)}^{g_1(y)} f(x, y)\,dx$. Show that φ is continuous on S.
Hint. Make a change of variables, linear in y, to reduce this to Exercise 8.

10. Let $\varphi : [a, b] \to [A, B]$ be C^1 on a neighborhood J of $[a, b]$, with $\varphi'(x) > 0$ for all $x \in [a, b]$. Assume $\varphi(a) = A,\ \varphi(b) = B$. Show that the identity

$$\int_A^B f(y)\,dy = \int_a^b f\bigl(\varphi(t)\bigr)\varphi'(t)\,dt, \tag{1.44}$$

for any $f \in C(J)$, follows from the chain rule and the Fundamental Theorem of Calculus.
Hint. Replace b by x, B by $\varphi(x)$, and differentiate.
Note that this result contains that of Exercise 7.
Try to establish (1.44) directly by working with the definition of the integral as a limit of Riemann sums.

11. Show that, if f and g are C^1 on a neighborhood of $[a, b]$, then

$$\int_a^b f(s)g'(s)\,ds = -\int_a^b f'(s)g(s)\,ds + \bigl[f(b)g(b) - f(a)g(a)\bigr]. \tag{1.45}$$

This transformation of integrals is called “integration by parts.”

12. Let $f : (a, b) \to \mathbb{R}$ be a C^{k+1}-function, and take $y \in (a, b)$. Show that for $x \in (a, b)$

$$\begin{aligned} f(x) = f(y) + f'(y)(x-y) + \frac{f''(y)}{2}(x-y)^2 \\ + \cdots + \frac{f^{(k)}(y)}{k!}(x-y)^k + R_k(x, y), \end{aligned} \tag{1.46}$$

where

$$R_k(x, y) = \frac{1}{k!}\int_y^x (x-s)^k f^{(k+1)}(s)\, ds. \tag{1.47}$$

This is Taylor's formula with remainder.
Hint. Apply $\partial/\partial y$ to both sides of (1.46). The left side becomes 0 and there is considerable cancellation on the right side, yielding

$$\frac{\partial}{\partial y} R_k(x, y) = -\frac{f^{(k+1)}(y)}{k!}(x-y)^k, \quad R_k(x, x) = 0. \tag{1.48}$$

Integrating then gives (1.47).
Note that a reformulation of (1.47) is

$$R_k(x, y) = \frac{(x-y)^{k+1}}{(k+1)!}\int_0^1 f^{(k+1)}\big(y + (1 - t^{1/(k+1)})(x-y)\big)\, dt. \tag{1.49}$$

13. Suppose $[a, b]$ is covered by open intervals J_k, $1 \le k \le N$, of length $\ell(J_k)$. Show that

$$b - a \le \sum_{k=1}^{N} \ell(J_k).$$

Hint. Show that $\sum_k \chi_{J_k}(x) \ge \chi_{[a,b]}(x)$ and deduce consequences for the Riemann integrals of these functions.
Alternative. Try an induction on N.

14. Extend Theorem 1.7 to the case where G is differentiable and G' is Riemann integrable on $[a, b]$.
Hint. Use

$$\begin{aligned} G(b) - G(a) &= \sum_{k=0}^{n-1}\Big[G\Big(a + (b-a)\frac{k+1}{n}\Big) - G\Big(a + (b-a)\frac{k}{n}\Big)\Big] \\ &= \frac{1}{n}\sum_{k=0}^{n-1} G'(\xi_{kn}). \end{aligned}$$