Chapter 2

Lebesgue Measure on the Line

In this chapter we discuss the concept of Lebesgue measure of subsets of the real line $\mathbb{R}$. It is convenient to begin with a discussion of the measure of subsets of a bounded interval.

If S is a subset of an interval $I = [a, b]$, then, as indicated in Chapter 1, we define the *outer measure* of S by

$$m^*(S) = \inf\Big\{\sum_{k\geq 0} \ell(J_k) : S \subset \bigcup_{k\geq 0} J_k\Big\}, \quad J_k \text{ intervals.} \tag{2.1}$$

It is easy to see that the result is not affected if one insists that all the intervals J_k be open in I or that they all be closed (or half-open, etc.). We can let J_k be intervals in $\mathbb{R}$, or we can require $J_k \subset I$. In particular, if $\mathcal{O} \subset (a, b)$ is open, then $\mathcal{O}$ is a disjoint union of a countable collection of open intervals $\mathcal{O}_k$, and (see Exercise 1 at the end of this chapter)

$$m^*(\mathcal{O}) = \sum \ell(\mathcal{O}_k). \tag{2.2}$$

Furthermore, for any $S \subset (a, b)$,

$$m^*(S) = \inf\big\{m^*(\mathcal{O}) : \mathcal{O} \supset S,\ \mathcal{O} \text{ open}\big\}. \tag{2.3}$$

Replacing I by a slightly larger interval, we see that (2.3) holds for any $S \subset I$.

Obviously the outer measure of a single point $p \in I$ is zero. Under the most liberal allowance for intervals in (2.1), p itself is an interval, of length zero. If we insist on open intervals, then let J_ε be an interval of length ε centered at p. More generally, if $C = \{c_1, c_2, c_3, \dots\}$ is a *countable* subset of I, write $C \subset \bigcup J_k(\varepsilon)$, where $J_k(\varepsilon)$ is an open interval of length $2^{-k}\varepsilon$, centered at c_k. Thus $m^*(C) \le \sum 2^{-k}\varepsilon = \varepsilon$, so

$$C \subset I \text{ countable } \Longrightarrow m^*(C) = 0. \tag{2.4}$$

Note that, if $\{J_{1k} : k \ge 0\}$ covers S_1 and $\{J_{2k} : k \ge 2\}$ covers S_2, then $\{J_{1k}, J_{2k} : k \ge 0\}$ is a cover of $S_1 \cup S_2$, so

$$m^*(S_1 \cup S_2) \le m^*(S_1) + m^*(S_2). \tag{2.5}$$

This subadditivity property is shared by upper content, defined in Chapter 1, but outer measure is distinguished from upper content by also having the property of *countable subadditivity*:

Proposition 2.1. *If* $\{S_j : j \ge 0\}$ *is a countable family of subsets of* I, *then*

$$m^*\Bigl(\bigcup_j S_j\Bigr) \le \sum_j m^*(S_j). \tag{2.6}$$

Proof. Pick $\varepsilon > 0$. Each S_j has a countable cover $\{J_{jk} : k \ge 0\}$, by intervals, such that $m^*(S_j) \ge \sum_k \ell(J_{jk}) - 2^{-j}\varepsilon$. Then $\{J_{jk} : j, k \ge 0\}$ is a countable cover of $\bigcup_j S_j$ by intervals, so $m^*\bigl(\bigcup S_j\bigr) \le \sum m^*(S_j) + 2\varepsilon$, for all $\varepsilon > 0$. Letting $\varepsilon \searrow 0$, we get (2.6).

Our main goal in this section is to produce a large class (call it $\mathfrak{L}$) of subsets of I with the property that m^* is "countably additive" on $\mathfrak{L}$, in the sense that if $S_j \in \mathfrak{L}$ is a countable collection of mutually disjoint sets (indexed by $j \in \mathbb{Z}^+$), then

$$m^*\Bigl(\bigcup_{j\ge 1} S_j\Bigr) = \sum_{j\ge 1} m^*(S_j).$$

We will pursue this in stages, showing first that such an identity holds when $S_1 = K$ is compact and $S_2 = I \setminus K$. In preparation for this, we take a closer look at the outer measure of a compact set $K \subset I$. Since any open cover of K has a finite subcover, we can say that

$$K \text{ compact } \Longrightarrow m^*(K) = \inf\Bigl\{\sum_{k=1}^N \ell(J_k) : K \subset \bigcup_{k=1}^N J_k\Bigr\}, \tag{2.7}$$

where J_k are open intervals (which we are free to close up). This coincides with the definition of upper content given in Chapter 1. It implies that, given $\varepsilon > 0$, one can pick a finite collection of disjoint open intervals $\{J_k : 1 \le k \le N\}$ such that $\mathcal{O} = \bigcup_k J_k \supset K$ and such that we have

$$m^*(K) \le m^*(\mathcal{O}) = \sum_{k=1}^{N} \ell(J_k) \le m^*(K) + \varepsilon. \tag{2.8}$$

The following result is sharper than (2.8).

Lemma 2.2. *Given $\varepsilon > 0$, we can construct $\mathcal{O} = \bigcup_{k=1}^{N} J_k \supset K$ such that*

$$m^*(\mathcal{O} \setminus K) \le \varepsilon. \tag{2.9}$$

Proof. Start with the $\mathcal{O}$ described above. Then $\mathcal{O} \setminus K = \mathcal{A}$ is open, so write $\mathcal{A} = \bigcup_{k \ge 1} \mathcal{A}_k$, a countable disjoint union of open intervals. To achieve (2.9), we need to arrange that $\sum_{k \ge 1} \ell(\mathcal{A}_k) \le \varepsilon$, at least after possibly shrinking $\mathcal{O}$.

To do this, pick M large enough that

$$\sum_{k > M} \ell(\mathcal{A}_k) \le \frac{\varepsilon}{2}.$$

We want to replace $\mathcal{O}$ by $\mathcal{O} \setminus \bigcup_{k=1}^{M} \mathcal{A}_k^{\#}$, which should still be a cover of K by a finite number of open intervals. It would be tempting to take $\mathcal{A}_k^{\#} = \overline{\mathcal{A}}_k$, but note that the endpoints of $\mathcal{A}_k$ might belong to K. Instead, let $\mathcal{A}_k^{\#} \subset \mathcal{A}_k$ be a closed interval with the same center as $\mathcal{A}_k$, such that $\ell(\mathcal{A}_k^{\#}) \ge \ell(\mathcal{A}_k) - \varepsilon/2M$. With the new $\mathcal{O}$ we have the lemma.

Using Lemma 2.2, we can establish the following important result, advertised above.

Proposition 2.3. *If $K \subset I$ is compact, then*

$$m^*(K) + m^*(I \setminus K) = \ell(I). \tag{2.10}$$

Proof. To begin, we note that if $\mathcal{O} = \bigcup_{k=1}^{N} J_k$ is a cover of K satisfying the conditions of Lemma 2.2, then $I \setminus \mathcal{O}$ is a finite disjoint union of intervals, say $I \setminus \mathcal{O} = \bigcup_{j=1}^{\nu} J_j'$, and clearly $m^*(I \setminus \mathcal{O}) = \sum_{j=1}^{\nu} \ell(J_j')$, so

$$m^*(\mathcal{O}) + m^*(I \setminus \mathcal{O}) = \ell(I). \tag{2.11}$$

Furthermore, $I \setminus K = (I \setminus \mathcal{O}) \cup (\mathcal{O} \setminus K)$, so by (2.5) and (2.9),

$$m^*(I \setminus K) \le m^*(I \setminus \mathcal{O}) + \varepsilon. \tag{2.12}$$

It follows that

$$m^*(K) + m^*(I \setminus K) \le m^*(\mathcal{O}) + m^*(I \setminus \mathcal{O}) + \varepsilon = \ell(I) + \varepsilon, \tag{2.13}$$

for all $\varepsilon > 0$, which implies that the left side of (2.10) is $\le \ell(I)$. The reverse inequality is automatic from the subadditivity property (2.5).

The *additivity* property, that $m^*(S_1 \cup S_2) = m^*(S_1) + m^*(S_2)$ when S_1 and S_2 are disjoint, just verified for $S_1 = K$ compact and $S_2 = I \setminus K$, does not hold for all disjoint sets S_j, though it holds for the "measurable" ones, as we will see below. At this point we record three easy cases of additivity.

Lemma 2.4. *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two disjoint open sets, then*

$$m^*(\mathcal{O}_1 \cup \mathcal{O}_2) = m^*(\mathcal{O}_1) + m^*(\mathcal{O}_2). \tag{2.14}$$

If $S_1, S_2 \subset I$ and

$$\rho = \rho(S_1, S_2) = \inf\ \{|x_1 - x_2| : x_j \in S_j\} > 0, \tag{2.15}$$

then

$$m^*(S_1 \cup S_2) = m^*(S_1) + m^*(S_2). \tag{2.16}$$

Furthermore, if K_j, $1 \le j \le N$, is a finite collection of mutually disjoint compact subsets of I, then

$$m^*\Big(\bigcup_{j=1}^N K_j\Big) = \sum_{j=1}^N m^*(K_j). \tag{2.17}$$

Proof. The identity (2.14) is immediate from (2.2). To establish (2.16), given $\delta > 0$, pick an open set $\mathcal{O} \supset S_1 \cup S_2$ such that $m^*(\mathcal{O}) \le m^*(S_1 \cup S_2) + \delta$. Note that each open set

$$\mathcal{O}_j = \mathcal{O} \cap \Big\{x : \operatorname{dist}(x, S_j) < \frac{\rho}{2}\Big\}$$

contains S_j. We see that $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$, so (2.14) applies. Thus

$$m^*(S_1) + m^*(S_2) \le m^*(\mathcal{O}_1) + m^*(\mathcal{O}_2) = m^*(\mathcal{O}_1 \cup \mathcal{O}_2) \le m^*(S_1 \cup S_2) + \delta,$$

for all $\delta > 0$. Thus $m^*(S_1) + m^*(S_2) \le m^*(S_1 \cup S_2)$, when (2.15) holds. The reverse inequality follows from (2.5), so we have (2.16).

Finally, to establish (2.17), it suffices to treat the case $N = 2$, but two disjoint compact sets $S_j = K_j$ necessarily satisfy (2.15), so the lemma is proved.

Granted this result, we can establish the important property of *countable additivity* of m^* on disjoint compact sets.

Proposition 2.5. *If $K_j,\ j \ge 1$, is a countable collection of mutually disjoint compact sets in I, then*

$$m^*\Big(\bigcup_{j\ge 1} K_j\Big) = \sum_{j\ge 1} m^*(K_j). \tag{2.18}$$

Proof. The left side of (2.17) is $\le$ the left side of (2.18) for all N, while the right side of (2.17) converges to the right side of (2.18). Hence the left side of (2.18) is $\ge$ the right side. The reverse inequality follows from (2.6).

We now define *inner measure*: if $S \subset I$, we set

$$m_*(S) = \ell(I) - m^*(I \setminus S). \tag{2.19}$$

By (2.5), $m_*(S) \le m^*(S)$ for all S. Using the characterization (2.3) of outer measure together with Proposition 2.3, we have

$$m_*(S) = \sup\big\{m^*(K) : K \subset S, K \text{ compact}\big\}. \tag{2.20}$$

Definition. *$S \subset I$ is measurable if and only if $m^*(S) = m_*(S)$.*

If S is measurable, we set

$$m(S) = m^*(S) = m_*(S). \tag{2.21}$$

Clearly, by (2.19), $S \subset I$ is measurable if and only if $I \setminus S$ is measurable. In view of (2.20), we see that any compact $K \subset I$ is measurable. Hence any open $\mathcal{O} \subset I$ is measurable.

One useful measurability result follows easily from Proposition 2.5:

Proposition 2.6. *If $K_j,\ j \ge 1$, is a countable collection of mutually disjoint compact subsets of I, then $L = \bigcup_{j\ge 1} K_j$ is measurable, and*

$$m(L) = \sum_{j\ge 1} m(K_j). \tag{2.22}$$

Proof. Since $\bigcup_{j=1}^N K_j = L_N \subset L$ for all $N < \infty$, we have

$$m_*(L) \ge m\Big(\bigcup_{j=1}^N K_j\Big) = \sum_{j=1}^N m(K_j).$$

Hence $m_*(L) \ge \sum_{j\ge 1} m(K_j)$. In view of (2.18) and the inequality $m_*(L) \le m^*(L)$, we have the proposition.

In general, in counterpoint to countable subadditivity of outer measure, we have countable *superadditivity* of inner measure:

Proposition 2.7. *Let $S_k \subset I$ be a countable family of mutually disjoint sets. Then*

$$m_*\Big(\bigcup_{k\ge 1} S_k\Big) \ge \sum_{k\ge 1} m_*(S_k). \tag{2.23}$$

Proof. Pick $\varepsilon > 0$; then pick $K_k \subset S_k$, compact, such that $m(K_k) \ge m_*(S_k) - 2^{-k}\varepsilon$. Then $\bigcup_{k\ge 1} K_k = L \subset \bigcup_{k\ge 1} S_k$, and, by Proposition 2.6,

$$m_*(L) = m^*(L) = \sum_{k\ge 1} m(K_k).$$

Since $m_*(L) \le m_*(S)$, where $S = \bigcup S_k$, we have

$$m_*(S) \ge \sum_{k\ge 1} m(K_k) \ge \sum_{k\ge 1} m_*(S_k) - \varepsilon,$$

for all $\varepsilon > 0$, yielding (2.23).

We have the following important conclusion, extending Proposition 2.6.

Theorem 2.8. *If S_k, $k \ge 1$, is a countable family of mutually disjoint measurable subsets of I, then $S = \bigcup_{k\ge 1} S_k$ is measurable, and*

$$m(S) = \sum_{k\ge 1} m(S_k). \tag{2.24}$$

Proof. Using Proposition 2.1 and Proposition 2.7, we have

$$m^*(S) \le \sum_{k\ge 1} m^*(S_k) = \sum_{k\ge 1} m_*(S_k) \le m_*(S),$$

and since $m_*(S) \le m^*(S)$, we have $m^*(S) = m_*(S)$ and the identity (2.24).

The identity (2.24) asserts *countable additivity* of m and is at the heart of Lebesgue measure theory.

We will derive some further sufficient conditions for a set to be measurable. The following criterion for measurability is useful.

Lemma 2.9. *A set $S \subset I$ is measurable if and only if, for each $\delta > 0$, there exist a compact K and an open $\mathcal{O}$ such that $K \subset S \subset \mathcal{O}$ and $m(\mathcal{O} \setminus K) < \delta$.*

Since both K (compact) and $\mathcal{O} \setminus K$ (open) are measurable, we have $m(\mathcal{O}) - m(K) = m(\mathcal{O} \setminus K)$ by Theorem 2.8, so the proof of Lemma 2.9 is straightforward. Here is an application.

Proposition 2.10. *If S_1 and S_2 are measurable subsets of I, the following are also measurable:*

$$S_1 \cup S_2, \quad S_1 \cap S_2, \quad S_1 \setminus S_2. \tag{2.25}$$

Proof. Given $\delta > 0$, take K_j compact, $\mathcal{O}_j$ open, so that $K_j \subset S_j \subset \mathcal{O}_j$ and $m(\mathcal{O}_j \setminus K_j) < \delta/2$. Then $K = K_1 \cup K_2$ is compact, $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ is open, $K \subset S_1 \cup S_2 \subset \mathcal{O}$, and $\mathcal{O} \setminus K \subset \big(\mathcal{O}_1 \setminus K_1\big) \cup \big(\mathcal{O}_2 \setminus K_2\big)$, so subadditivity implies $m(\mathcal{O} \setminus K) < \delta$. This shows that $S_1 \cup S_2$ is measurable.

We know that S_j is measurable $\Leftrightarrow I \setminus S_j$ is measurable, so, by the last argument, $(I \setminus S_1) \cup (I \setminus S_2) = I \setminus (S_1 \cap S_2)$ is measurable; hence $S_1 \cap S_2$ is measurable.

Finally, $S_1 \setminus S_2 = S_1 \cap (I \setminus S_2)$, so this is measurable.

There is a more incisive countable counterpart.

Theorem 2.11. *If S_j, $j \geq 1$, is a countable collection of measurable subsets of I, the following are measurable:*

$$\bigcup_{j\geq 1} S_j, \quad \bigcap_{j\geq 1} S_j. \tag{2.26}$$

Proof. To treat $U = \bigcup_{j\geq 1} S_j$, let $S_j' = \bigcup_{\ell=1}^{j} S_j$, which is measurable by Proposition 2.10. Then $S_1' \subset S_2' \subset \cdots$ and $U = \bigcup_{j\geq 1} S_j'$. Now let $T_1 = S_1' = S_1$ and $T_j = S_j' \setminus S_{j-1}'$ for $j \geq 2$, also measurable by Proposition 2.10. Thus $U = \bigcup_{j\geq 1} T_j$. Since the T_j are mutually *disjoint*, we can apply Theorem 2.8, to conclude that U is measurable. Finally, $\bigcap_{j\geq 1} S_j$ is the complement in I of $\bigcup_{j\geq 1}(I \setminus S_j)$, so the theorem is proved.

A nonempty family $\mathfrak{F}$ of subsets of some set X is called a σ-algebra if it is closed under the formation of countable unions, countable intersections, and complements. Thus we see that the family $\mathfrak{L}$ of measurable subsets of I is a σ-algebra. The *smallest* σ-algebra of subsets of a topological space X containing all the closed sets is called the algebra of Borel sets. (Cf. Exercises 11–12 for the existence of such a smallest σ-algebra.) We see that each Borel set in I is measurable.

We record a criterion for measurability that, while first perhaps appearing curious, will actually serve as a convenient *definition* of measurability, in the more general construction of measures to be discussed in Chapter 5.

Proposition 2.12. *A set $S \subset I$ is measurable if and only if, for all $Y \subset I$ (measurable or not),*

$$m^*(Y) = m^*(Y \cap S) + m^*(Y \setminus S). \tag{2.27}$$

Proof. Since the validity of (2.27) for $Y = I$ was the definition of measurability, via (2.19), it remains only to show that (2.27) holds whenever S is measurable. Of course, we always have $m^*(Y) \le m^*(Y \cap S) + m^*(Y \setminus S)$. Now take $\varepsilon > 0$ and pick an open set $\mathcal{O} \supset Y$ such that $m(\mathcal{O}) \le m^*(Y) + \varepsilon$. We know that $\mathcal{O} \cap S$ and $\mathcal{O} \setminus S$ are measurable if S is, and thus

$$m(\mathcal{O}) = m(\mathcal{O} \cap S) + m(\mathcal{O} \setminus S) \ge m^*(Y \cap S) + m^*(Y \setminus S).$$

Hence $m^*(Y \cap S) + m^*(Y \setminus S) \le m^*(Y) + \varepsilon$ for all $\varepsilon > 0$, and (2.27) follows.

We now discuss Lebesgue measure on the line, which is very much like that on a bounded interval, except that now some sets can have measure $+\infty$. Let us partition $\mathbb{R}$ into a countable set of bounded intervals (of positive length):

$$\mathbb{R} = \bigcup_k I_k, \tag{2.28}$$

so two different intervals can intersect at most at one point. We say a subset $S \subset \mathbb{R}$ is measurable if and only if $S \cap I_k$ is measurable, for all k. If this holds, we then set

$$m(S) = \sum_k m(S \cap I_k). \tag{2.29}$$

Note that possibly $m(S) = +\infty$; for example, $\mathbb{R}$ is measurable and $m(\mathbb{R}) = +\infty$. It is easy to show that this characterization is independent of the choice of partition and to extend the results established above, to show that the family of measurable subsets of $\mathbb{R}$ is a σ-algebra and Lebesgue measure is countably additive, as in Theorem 2.8. We leave the job as an exercise.

We mention that (2.1) could still be used to define outer measure, and one has, for all $S \subset \mathbb{R}$,

$$m^*(S) = \sum_k m^*(S \cap I_k). \tag{2.30}$$

Also, (2.20) could be used to define inner measure. Furthermore, *provided* $m_*(S) < \infty$, then $S \subset \mathbb{R}$ is measurable $\Leftrightarrow$ $m_*(S) = m^*(S)$. However, if $m_*(S) = \infty$, then certainly $m^*(S) = \infty$, but S might possibly not be measurable.

There is also a natural Lebesgue measure on the circle $\mathbb{T}$, which can be identified with $\mathbb{R}/\mathbb{Z}$. In fact, we have a natural 1-1 correspondence between $\mathbb{T}$ and $I = [0,1]$, with 0 and 1 identified, and Lebesgue measure on $\mathbb{T}$ is simply the same as that on I. We note that $\mathbb{T}$ is a group, and it acts on itself in a measure-preserving fashion. That is, if $S \subset \mathbb{T}$ and $\alpha \in \mathbb{T}$, we can form $S_\alpha = S + \alpha = \{x + \alpha : x \in S\}$, where "+" denotes addition mod 1, and

$$m^*(S_\alpha) = m^*(S_\beta), \quad m_*(S_\alpha) = m_*(S_\beta), \tag{2.31}$$

for all $\alpha, \beta \in \mathbb{T}$.

We end this section with a standard example of a subset S of $I = [0,1]$ that is not measurable. Actually, it is convenient to construct S as a subset of $\mathbb{T}$. As noted above, $\mathbb{T}$ is a group. It has a subgroup $\mathfrak{Q} = \mathbb{Q}/\mathbb{Z}$, acting on $\mathbb{T}$ as a countable family of measure-preserving transformations. We form S by picking *one* element from each orbit of $\mathfrak{Q}$ in $\mathbb{T}$. Doing this requires the "axiom of choice." Now, for each $\alpha \in \mathfrak{Q}$, consider $S_\alpha = S + \alpha = \{x + \alpha : x \in S\}$, as in the previous paragraph. Note that

$$\bigcup_{\alpha \in \mathfrak{Q}} S_\alpha = \mathbb{T}, \quad \alpha \neq \beta \in \mathfrak{Q} \Rightarrow S_\alpha \cap S_\beta = \emptyset. \tag{2.32}$$

Also (2.31) holds, for all $\alpha, \beta \in \mathfrak{Q}$. Now respective applications of countable subadditivity of m^* and countable superadditivity of m_* to $\mathbb{T} = \bigcup_\alpha S_\alpha$ yield $1 \le \sum_\alpha m^*(S_\alpha)$ and $1 \ge \sum_\alpha m_*(S_\alpha)$. Hence

$$m^*(S_\alpha) > 0 \quad \text{and} \quad m_*(S_\alpha) = 0, \quad \forall\, \alpha \in \mathfrak{Q}. \tag{2.33}$$

Thus none of the sets S_α are measurable.

Exercises

1. In order to establish the identity (2.2), start by demonstrating the following:

   Claim. If $\mathfrak{I}$ is a bounded interval and $\{J_k : k \ge 1\}$ a countable cover of $\mathfrak{I}$ by intervals,

$$\sum_{k \ge 1} \ell(J_k) \ge \ell(\mathfrak{I}). \tag{2.34}$$

Hint. Reduce to the case where $\mathfrak{I}$ is *closed* and J_k are *open*. Consult Exercise 13 of Chapter 1.
From here finish the proof of (2.2).
Hint. The definition (2.1) clearly gives $m^*(\mathcal{O}) \le \sum \ell(\mathcal{O}_k)$. For the converse inequality, fix $\varepsilon > 0$ and pick a finite number of closed intervals $J_k \subset \mathcal{O}_k$, $1 \le k \le N$, such that $\sum_1^N \ell(J_k) \ge \sum \ell(\mathcal{O}_k) - \varepsilon$. Then show that $m^*(\mathcal{O}) \ge m^*(\bigcup_k J_k) = \sum_1^N \ell(J_k)$.

2. If S_j is an *increasing* sequence of measurable subsets of I, i.e., $S_1 \subset S_2 \subset S_3 \subset \cdots$, and $S = \bigcup_{j\ge 1} S_j$, show that

$$m(S_j) \nearrow m(S), \text{ as } j \to \infty. \tag{2.35}$$

Hint. Examine the proof of Theorem 2.11.

3. If S_j is a *decreasing* sequence of measurable subsets of I, i.e., $S_1 \supset S_2 \supset S_3 \supset \cdots$, and $S = \bigcap_{j\ge 1} S_j$, show that

$$m(S_j) \searrow m(S), \text{ as } j \to \infty. \tag{2.36}$$

4. If S_j is an increasing sequence of measurable subsets of $\mathbb{R}$, with union S, show that (2.35) continues to hold. If S_j is a decreasing sequence of measurable subsets of $\mathbb{R}$, with intersection S, show that (2.36) holds, *provided* $m(S_j) < \infty$ for some j. Give a counterexample to (2.36) when this provision does not hold.

5. Show that a nonempty family $\mathfrak{F}$ of subsets of X which is closed under countable unions and complements is automatically closed under countable intersections and hence is a σ-algebra.

6. A nonempty family $\mathfrak{F}$ of subsets of X is called a σ-ring if it is closed under countable unions and under *differences* (i.e., $S_j \in \mathfrak{F} \Rightarrow S_1 \setminus S_2 \in \mathfrak{F}$). Show that a σ-ring is also closed under countable intersections. Show that such a σ-ring $\mathfrak{F}$ is a σ-algebra if and only if $X \in \mathfrak{F}$.

7. Show that every measurable set $X \subset \mathbb{R}$ with *positive* measure contains a non measurable subset.
Hint. If $X \subset I$, consider $X \cap S_\alpha$, with S_α as in (2.31)–(2.33). Show that $m_*(X \cap S_\alpha) = 0$ for all α, while $m^*(X \cap S_\alpha) > 0$ for some α.

8. Form the Cantor middle third set as follows. Let $K_0 = [0, 1]$. Form K_1 by removing the open interval in the middle of K_0, of length $1/3$. Then K_1 consists of two intervals, each of length $1/3$. Next, remove from

each of the intervals making up K_1 the open interval in the middle, of length $1/3^2$, to get K_2. Continue this process. Thus K_ν is a union of 2^ν disjoint closed intervals, each of length $3^{-\nu}$. Then

$$K_0 \supset K_2 \supset K_2 \supset K_3 \supset \cdots \searrow K,$$

and $K = \bigcap_\nu K_\nu$ is the Cantor middle third set. Show that

$$m(K) = 0.$$

9. Modify the construction in Exercise 8 as follows. Let $L_0 = [0, 1]$. Form L_1 by removing the open interval in the middle of L_0 of length $1/5$. Next, remove from each of the intervals making up L_1 the open intervals in the middle, of length $1/5^2$, to get L_2. Continue this process. Thus L_ν is obtained from $[0, 1]$ by omitting one open interval of length $1/5$, two of length $1/5^2$, and so on, up to omitting $2^{\nu-1}$ open intervals of length $5^{-\nu}$. Then $L_0 \supset L_1 \supset L_2 \supset \cdots \searrow L$, where $L = \bigcap_\nu L_\nu$ is a compact set. Show that
$$m(L) > 0.$$

10. With K and L as in Exercises 8–9, show that there is a homeomorphism $F : [0, 1] \to [0, 1]$ such that

$$F(L) = K.$$

In Exercises 11–12, let X be a nonempty set and $\mathcal{C}$ a nonempty collection of subsets of X.

11. Show that the collection of σ-algebras of subsets of X that contain $\mathcal{C}$ is nonempty.
Hint. Consider the set of all subsets of X.

12. If $\{\mathcal{F}_\alpha : \alpha \in \mathcal{A}\}$ is the collection of all σ-algebras of subsets of X that contain $\mathcal{C}$, show that
$$\bigcap_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha = \mathcal{F}$$
is a σ-algebra of subsets of X, containing $\mathcal{C}$, and is in fact the *smallest* such σ-algebra. One says $\mathcal{F}$ is the σ-algebra generated by $\mathcal{C}$ and writes $\mathcal{F} = \sigma(\mathcal{C})$.