Appendix G

Integration of Differential Forms

The calculus of differential forms provides a convenient setting for integration on manifolds, as we will explain in this appendix, due to the efficient way it keeps track of changes of variables.

A k-form β on an open set $\mathcal{O} \subset \mathbb{R}^n$ has the form

$$\beta = \sum_j b_j(x)\, dx_{j_1} \wedge \cdots \wedge dx_{j_k}. \tag{G.1}$$

Here $j = (j_1, \dots, j_k)$ is a k-multi-index. We write $\beta \in \Lambda^k(\mathcal{O})$. The wedge product used in (G.1) has the anti-commutative property

$$dx_\ell \wedge dx_m = -dx_m \wedge dx_\ell, \tag{G.2}$$

so that if σ is a permutation of $\{1, \dots, k\}$, we have

$$dx_{j_1} \wedge \cdots \wedge dx_{j_k} = (\operatorname{sgn} \sigma)\, dx_{j_{\sigma(1)}} \wedge \cdots \wedge dx_{j_{\sigma(k)}}. \tag{G.3}$$

In particular, an n-form α on $\Omega \subset \mathbb{R}^n$ can be written

$$\alpha = A(x)\, dx_1 \wedge \cdots \wedge dx_n. \tag{G.4}$$

If $A \in L^1(\mathcal{O}, dx)$, we write

$$\int_{\mathcal{O}} \alpha = \int_{\mathcal{O}} A(x)\, dx, \tag{G.5}$$

the right side being the usual Lebesgue integral, developed in Chapter 7.

Suppose now $\Omega \subset \mathbb{R}^n$ is open and there is a C^1 diffeomorphism $F : \Omega \to \mathcal{O}$. We define the *pull-back* $F^*\beta$ of the k-form β in (G.1) as

$$F^*\beta = \sum_j b_j(F(x))\,(F^*dx_{j_1}) \wedge \cdots \wedge (F^*dx_{j_k}), \tag{G.6}$$

where

$$F^*dx_j = \sum_\ell \frac{\partial F_j}{\partial x_\ell}\,dx_\ell, \tag{G.7}$$

the algebraic computation in (G.6) being performed using the rule (G.3).

If $B = (b_{\ell m})$ is an $n \times n$ matrix, then, by (G.3) and the formula for the determinant given in (7.77) (and (7.83)),

$$\begin{aligned} \Big(\sum_m b_{1m}\,dx_m\Big) \wedge \Big(\sum_m b_{2m}\,dx_m\Big) \wedge \cdots \wedge \Big(\sum_m b_{nm}\,dx_m\Big) \\ = \Big(\sum_\sigma (\operatorname{sgn}\sigma)\, b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)}\Big)\, dx_1 \wedge \cdots \wedge dx_n \\ = (\det B)\, dx_1 \wedge \cdots \wedge dx_n. \end{aligned} \tag{G.8}$$

Hence, if $F : \Omega \to \mathcal{O}$ is a C^1 map and α is an n-form on $\mathcal{O}$, as in (G.4), then

$$F^*\alpha = \det DF(x)\, A(F(x))\, dx_1 \wedge \cdots \wedge dx_n. \tag{G.9}$$

This formula is especially significant in light of the change of variable formula

$$\int_{\mathcal{O}} A(x)\,dx = \int_{\Omega} A(F(x))\,|\det DF(x)|\,dx, \tag{G.10}$$

when $F : \Omega \to \mathcal{O}$ is a C^1 diffeomorphism, given in Theorem 7.2. The only difference between the right side of (G.10) and $\int_\Omega F^*\alpha$ is the absolute value sign around $\det DF(x)$. We say a C^1 map $F : \Omega \to \mathcal{O}$ is *orientation preserving* when $\det DF(x) > 0$ for all $x \in \Omega$. In such a case, Theorem 7.2 yields

Proposition G.1. *If $F : \Omega \to \mathcal{O}$ is a C^1 orientation-preserving diffeomorphism and α an integrable n-form on $\mathcal{O}$, then*

$$\int_{\mathcal{O}} \alpha = \int_{\Omega} F^*\alpha. \tag{G.11}$$

In Appendix H we will present another proof of the change of variable formula, making direct use of basic results on differential forms developed in this appendix.

In addition to the pull-back, there are some other operations on differential forms. The wedge product of dx_ℓ's extends to a wedge product on forms as follows. If $\beta \in \Lambda^k(\mathcal{O})$ has the form (G.1) and if

$$\alpha = \sum_i a_i(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \in \Lambda^\ell(\mathcal{O}), \tag{G.12}$$

define

$$\alpha \wedge \beta = \sum_{i,j} a_i(x) b_j(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_\ell} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k} \tag{G.13}$$

in $\Lambda^{k+\ell}(\mathcal{O})$. We retain the equivalences (G.3). It follows that

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha. \tag{G.14}$$

It is also readily verified that

$$F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta). \tag{G.15}$$

Another important operator on forms is the *exterior derivative*:

$$d : \Lambda^k(\mathcal{O}) \longrightarrow \Lambda^{k+1}(\mathcal{O}), \tag{G.16}$$

defined as follows. If $\beta \in \Lambda^k(\mathcal{O})$ is given by (G.1), then

$$d\beta = \sum_{j,\ell} \frac{\partial b_j}{\partial x_\ell}\, dx_\ell \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}. \tag{G.17}$$

The antisymmetry $dx_m \wedge dx_\ell = -dx_\ell \wedge dx_m$, together with the identity $\partial^2 b_j / \partial x_\ell \partial x_m = \partial^2 b_j / \partial x_m \partial x_\ell$, implies

$$d(d\beta) = 0, \tag{G.18}$$

for any smooth differential form β. We also have a product rule:

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^j \alpha \wedge (d\beta), \quad \alpha \in \Lambda^j(\mathcal{O}),\ \beta \in \Lambda^k(\mathcal{O}). \tag{G.19}$$

The exterior derivative has the following important property under pull-backs:

$$F^*(d\beta) = dF^*\beta, \tag{G.20}$$

if $\beta \in \Lambda^k(\mathcal{O})$ and $F : \Omega \to \mathcal{O}$ is a smooth map. To see this, extending (G.19) to a formula for $d(\alpha \wedge \beta_1 \wedge \cdots \wedge \beta_\ell)$ and using this to apply d to $F^*\beta$, we have

(G.21)
$$\begin{aligned} dF^*\beta = &\sum_{j,\ell} \frac{\partial}{\partial x_\ell}\big(b_j \circ F(x)\big)\, dx_\ell \wedge \big(F^* dx_{j_1}\big) \wedge \cdots \wedge \big(F^* dx_{j_k}\big) \\ &+ \sum_{j,\nu} (\pm) b_j\big(F(x)\big)\big(F^* dx_{j_1}\big) \wedge \cdots \wedge d\big(F^* dx_{j_\nu}\big) \wedge \cdots \wedge \big(F^* dx_{j_k}\big). \end{aligned}$$

Now the definition (G.6)–(G.7) of pull-back gives directly that

$$F^* dx_i = \sum_\ell \frac{\partial F_i}{\partial x_\ell}\, dx_\ell = dF_i, \tag{G.22}$$

and hence $d(F^* dx_i) = ddF_i = 0$, so only the first sum in (G.21) contributes to $dF^*\beta$. Meanwhile,

$$F^* d\beta = \sum_{j,m} \frac{\partial b_j}{\partial x_m}\big(F(x)\big)\ (F^* dx_m) \wedge \big(F^* dx_{j_1}\big) \wedge \cdots \wedge \big(F^* dx_{j_k}\big), \tag{G.23}$$

so (G.20) follows from the identity

$$\sum_\ell \frac{\partial}{\partial x_\ell}\big(b_j \circ F(x)\big)\, dx_\ell = \sum_m \frac{\partial b_j}{\partial x_m}\big(F(x)\big)\, F^* dx_m,$$

which in turn follows from the chain rule.

Here is another important consequence of the chain rule. Suppose $F : \Omega \to \mathcal{O}$ and $\psi : \mathcal{O} \to U$ are smooth maps between open subsets of $\mathbb{R}^n$. We claim that for any form α of any degree,

$$\psi \circ F = \varphi \Longrightarrow \varphi^*\alpha = F^*\psi^*\alpha. \tag{G.24}$$

It suffices to check (G.24) for $\alpha = dx_j$. Then (G.7) gives the basic identity $\psi^*\ dx_j = \sum (\partial\psi_j/\partial x_\ell)\, dx_\ell$. Consequently,

$$F^*\psi^*\ dx_j = \sum_{\ell,m} \frac{\partial F_\ell}{\partial x_m}\frac{\partial \psi_j}{\partial x_\ell}\ dx_m, \quad \varphi^*\ dx_j = \sum_m \frac{\partial \varphi_j}{\partial x_m}\, dx_m; \tag{G.25}$$

but the identity of these forms follows from the chain rule:

$$D\varphi = (D\psi)(DF) \Longrightarrow \frac{\partial \varphi_j}{\partial x_m} = \sum_\ell \frac{\partial \psi_j}{\partial x_\ell}\frac{\partial F_\ell}{\partial x_m}. \tag{G.26}$$

One can define a k-form on an n-dimensional manifold M as follows. Say M is covered by open sets $\mathcal{O}_j$ and there are coordinate charts $F_j : \Omega_j \to \mathcal{O}_j$, with $\Omega_j \subset \mathbb{R}^n$ open. A collection of forms $\beta_j \in \Lambda^k(\Omega_j)$ is said to define a k-form on M provided the following compatibility condition holds. If $\mathcal{O}_i \cap \mathcal{O}_j \neq \emptyset$ and we consider $\Omega_{ij} = F_i^{-1}(\mathcal{O}_i \cap \mathcal{O}_j)$ and diffeomorphisms

$$\varphi_{ij} = F_j^{-1} \circ F_i : \Omega_{ij} \longrightarrow \Omega_{ji}, \tag{G.27}$$

we require

$$\varphi_{ij}^* \beta_j = \beta_i. \tag{G.28}$$

The fact that this is a consistent definition is a consequence of (G.24). For example, if $G : M \to R^m$ is a smooth map and γ is a k-form on $\mathbb{R}^m$, then there is a well-defined k-form $\beta = G^*\gamma$ on M, represented in such coordinate charts by $\beta_j = (G \circ F_j)^*\gamma$. Similarly, if β is a k-form on M as defined above and $G : U \to M$ is smooth, with $U \subset \mathbb{R}^m$ open, then $G^*\beta$ is a well-defined k-form on U.

We give an intrinsic definition of $\int_M \alpha$ when α is an n-form on M, provided M is *oriented*, i.e., there is a coordinate cover as above such that $\det D\varphi_{jk} > 0$. The object called an "orientation" on M can be identified as an equivalence class of nowhere vanishing n-forms on M, two such forms being equivalent if one is a multiple of another by a positive function in $C^\infty(\Omega)$. A member of this equivalence class, say ω, defines the orientation. The standard orientation on $\mathbb{R}^n$ is determined by $dx_1 \wedge \cdots \wedge dx_n$. The equivalence class of positive multiples $a(x)\omega$ is said to consist of "positive" forms. A smooth map $\psi : S \to M$ between oriented n-dimensional manifolds preserves orientation provided $\psi^*\sigma$ is positive on S whenever $\sigma \in \Lambda^n(M)$ is positive. We mention that there exist surfaces that cannot be oriented, such as the famous "Möbius strip."

We define the integral of an n-form over an oriented n-dimensional manifold as follows. First, if α is an n-form supported on an open set $\mathcal{O} \subset \mathbb{R}^n$, given by (G.4), then we define $\int_{\mathcal{O}} \alpha$ by (G.5).

More generally, if M is an n-dimensional manifold with an orientation, say the image of an open set $\mathcal{O} \subset \mathbb{R}^n$ by $\varphi : \mathcal{O} \to M$, carrying the natural orientation of $\mathcal{O}$, we can set

$$\int_M \alpha = \int_{\mathcal{O}} \varphi^*\alpha \tag{G.29}$$

for an n-form α on M. If it takes several coordinate patches to cover M, define $\int_M \alpha$ by writing α as a sum of forms, each supported on one patch.

We need to show that this definition of $\int_M \alpha$ is independent of the choice of coordinate system on M (as long as the orientation of M is respected). Thus, suppose $\varphi : \mathcal{O} \to U \subset M$ and $\psi : \Omega \to U \subset M$ are both coordinate patches, so that $F = \psi^{-1} \circ \varphi : \mathcal{O} \to \Omega$ is an orientation-preserving diffeomorphism. We need to check that, if α is an n-form on M, supported on U, then

$$\int\limits_{\mathcal{O}} \varphi^* \alpha = \int\limits_{\Omega} \psi^* \alpha. \tag{G.30}$$

To establish this, we use (G.24). This implies that the left side of (G.30) is equal to

$$\int\limits_{\mathcal{O}} F^*(\psi^* \alpha), \tag{G.31}$$

which is equal to the right side of (G.30), by (G.11) (with slightly altered notation). Thus the integral of an n-form over an oriented n-dimensional manifold is well defined.

We turn now to the Gauss-Green-Stokes formula for differential forms, commonly called simply the Stokes formula. This involves integrating a k-form over a k-dimensional manifold with boundary. We first define that concept. Let S be a smooth k-dimensional manifold, and let M be an open subset of S, such that its closure $\overline{M}$ (in $\mathbb{R}^N$) is contained in S. Its boundary is $\partial M = \overline{M} \setminus M$. We say $\overline{M}$ is a smooth surface with boundary if also ∂M is a smooth $(k-1)$-dimensional surface. In such a case, any $p \in \partial M$ has a neighborhood $U \subset S$ with a coordinate chart $\varphi : \mathcal{O} \to U$, where $\mathcal{O}$ is an open neighborhood of 0 in $\mathbb{R}^k$, such that $\varphi(0) = p$ and φ maps $\{x \in \mathcal{O} : x_1 = 0\}$ onto $U \cap \partial M$.

If S is oriented, then $\overline{M}$ is oriented, and ∂M inherits an orientation, uniquely determined by the following requirement: if

$$\overline{M} = \mathbb{R}^k_- = \{x \in \mathbb{R}^k : x_1 \le 0\}, \tag{G.32}$$

then $\partial M = \{(x_2, \dots, x_k)\}$ has the orientation determined by $dx_2 \wedge \cdots \wedge dx_k$.

We can now state the Stokes formula.

Proposition G.2. *Given a compactly supported $(k-1)$-form β of class C^1 on an oriented k-dimensional surface $\overline{M}$ (of class C^2) with boundary ∂M, with its natural orientation,*

$$\int\limits_{M} d\beta = \int\limits_{\partial M} \beta. \tag{G.33}$$

Proof. Using a partition of unity and invariance of the integral and the exterior derivative under coordinate transformations, it suffices to prove this when $\overline{M}$ has the form (G.32). In that case, we will be able to deduce (G.33) from the Fundamental Theorem of Calculus. Indeed, if

$$\beta = b_j(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k, \tag{G.34}$$

with $b_j(x)$ of bounded support, we have

$$d\beta = (-1)^{j-1} \frac{\partial b_j}{\partial x_j}\, dx_1 \wedge \cdots \wedge dx_k. \tag{G.35}$$

If $j > 1$, we have

$$\int\limits_M d\beta = (-1)^{j-1} \int \Big\{ \int_{-\infty}^{\infty} \frac{\partial b_j}{\partial x_j}\, dx_j \Big\}\, dx' = 0, \tag{G.36}$$

and also $\kappa^*\beta = 0$, where $\kappa : \partial M \to \overline{M}$ is the inclusion. On the other hand, for $j = 1$, we have

$$\begin{aligned} \int\limits_M d\beta &= \int \Big\{ \int_{-\infty}^{0} \frac{\partial b_1}{\partial x_1}\, dx_1 \Big\}\, dx_2 \cdots dx_k \\ &= \int b_1(0, x')\, dx' \\ &= \int\limits_{\partial M} \beta. \end{aligned} \tag{G.37}$$

This proves Stokes' formula (G.33).

The reason we required $\overline{M}$ to be a surface of class C^2 in Proposition G.2 is the following. Due to the formulas (G.6)–(G.7) for a pull-back, if β is of class C^j and F is of class C^ℓ, then $F^*\beta$ is generally of class C^μ, with $\mu = \min(j, \ell - 1)$. Thus, if $j = \ell = 1$, $F^*\beta$ might be only of class C^0, so there is not a well-defined notion of a differential form of class C^1 on a C^1 surface, though such a notion is well defined on a C^2 surface. This problem can be overcome, and one can extend Proposition G.2 to the case where $\overline{M}$ is a C^1 surface and β is a $(k-1)$-form with the property that both β and $d\beta$ are continuous. One can go further and formulate (G.33) for a $(k-1)$-form β with the property that

$$\beta,\ d\beta \in L^\infty(M), \quad \iota^*\beta \in L^\infty(\partial M), \tag{G.38}$$

where $\iota : \partial M \to \overline{M}$ is the natural inclusion, a class of forms that can be shown to be invariant under bi-Lipschitz maps. (It can be shown that the first two conditions in (G.38) imply $\iota^*\beta \in H^{1,1}(\partial M)'$.) We will not go into the details. However, in Appendix I we will present an elementary treatment of (G.33), stated in a more classical language, when M is an open domain in $\mathbb{R}^k$ whose boundary is locally the graph of a Lipschitz function. A far reaching extension, due to H. Federer, can be found in [**Fed**]; see also [**EG**].

The calculus of differential forms has many applications to differential equations, differential geometry, and topology. More on this can be found in [**Spi**] and also in [**T1**] (particularly Chapters 1, 5, and 10). To end this appendix, we make use of the calculus of differential forms to provide simple proofs of some important topological results of Brouwer. The first two results concern *retractions*. If Y is a subset of X, by definition a retraction of X onto Y is a map $\varphi : X \to Y$ such that $\varphi(x) = x$ for all $x \in Y$.

Proposition G.3. *There is no smooth retraction $\varphi : B \to S^{n-1}$ of the closed unit ball B in $\mathbb{R}^n$ onto its boundary S^{n-1}.*

In fact, it is just as easy to prove the following more general result. The approach we use is adapted from [**Kan**].

Proposition G.4. *If $\overline{M}$ is a compact oriented n-dimensional manifold with nonempty boundary ∂M, there is no smooth retraction $\varphi : \overline{M} \to \partial M$.*

Proof. You can pick $\omega \in \Lambda^{n-1}(\partial M)$ to be an $(n-1)$-form on ∂M such that $\int_{\partial M} \omega > 0$. Now apply Stokes' Theorem to $\beta = \varphi^*\omega$. If φ is a retraction, then $\varphi \circ j(x) = x$, where $j : \partial M \hookrightarrow \overline{M}$ is the natural inclusion. Hence $j^*\varphi^*\omega = \omega$, so we have

$$\int\limits_{\partial M} \omega = \int\limits_{M} d\varphi^*\omega. \tag{G.39}$$

But $d\varphi^*\omega = \varphi^* d\omega = 0$, so the integral (G.39) is zero. This is a contradiction, so there can be no retraction.

A simple consequence of this is the famous Brouwer Fixed-Point Theorem.

Theorem G.5. *If $F : B \to B$ is a continuous map on the closed unit ball in $\mathbb{R}^n$, then F has a fixed point.*

Proof. First, an approximation argument shows that if there is a continuous such F without a fixed point, then there is a smooth one, so assume $F : B \to B$ is smooth. We are claiming that $F(x) = x$ for some $x \in B$. If not, then for each $x \in B$ define $\varphi(x)$ to be the endpoint of the ray from $F(x)$ to x, continued until it hits $\partial B = S^{n-1}$. It is clear that φ would be a smooth retraction, contradicting Proposition G.3.

REMARK. Typical proofs of the Brouwer Fixed-Point Theorem use concepts of algebraic topology; cf. [**Spa**]. In fact, the proof of Proposition G.4 contains a germ of de Rham cohomology. See [**T1**], Chapter 1, §19 for more on this.

An integral calculus proof of the Brouwer Fixed-Point Theorem that does not involve differential forms is given in [**DS**], Vol. 1, pp. 467–470. One might compare it with the proof given above.