Variations on Quantum Ergodic Theorems

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Notes available on my website, under Downloadable Lecture Notes

8. Seminar talks and AMS talks

See also

4. Spectral theory
7. Quantum mechanics connections

Basic quantization: a function on "phase space" is taken to an operator on a Hilbert space H.

Euclidean case. $H = L^2(\mathbb{R}^n)$.
Position: $x_j \mapsto Q_j, \quad Q_j f = x_j f$.
Momentum: $p_j \mapsto P_j, \quad P_j f = (1/i)\partial f/\partial x_j$.
Laplace operator: $|p|^2 \mapsto -\Delta$.

Quantization of motion in a force field

$$-\Delta + V(x).$$

Quantization of free motion on a Riemannian manifold M.
$H = L^2(M)$.
Use Laplace-Beltrami operator, i.e.,

$$\Delta f = g^{-1/2}\partial_j \Big(g^{1/2} g^{jk} \partial_k f \Big).$$

Classical free motion on M: geodesic flow

$$\frac{dx_j}{dt} = \frac{\partial E}{\partial \xi_j}, \quad \frac{d\xi_j}{dt} = -\frac{\partial E}{\partial x_j}$$
$$E(x, \xi) = g^{jk}\xi_j\xi_k = \text{ symbol of } -\Delta.$$

Get flow on cotangent bundle T^*M, preserving level sets of $E(x, \xi)$. Hence get flow

$$G_t : S^*M \to S^*M$$

.

The flow G_t preserves a natural Liouville measure dS on S^*M. We normalize this so that $\int_{S^*M} dS = 1$.

Basic problem: relate dynamical properties of the geodesic flow G_t to spectral properties of Δ.

Assume M is a compact Riemannian manifold. $L^2(M)$ has an orthonormal basis $\{\varphi_k : k \in \mathbb{N}\}$ of eigenfunctions of Δ:

$$\Delta\varphi_k = -\lambda_k^2\varphi_k, \quad \lambda_k \nearrow +\infty.$$

Weyl law:

$$\lambda_k \sim (Ck)^{1/n}, \quad \text{as} \quad k \to \infty,$$

where $n = \dim M$, and $C = \Gamma(n/2+1)(4\pi)^{n/2}/\mathrm{Vol}\, M$. Here and below, normalize the metric on M so that $\mathrm{Vol}\, M = 1$.

Mean equidistribution of eigenfunctions:

$$\frac{1}{N}\sum_{k=1}^{N} |\varphi_k(x)|^2 \longrightarrow 1,$$

uniformly in $x \in M$, as $N \to \infty$. One tool for these results: heat kernel asymptotics.

Question: when can lots of eigenfunctions concentrate on some subset of M?

Example: Unit sphere S^n in $\mathbb{R}^{n+1}$.

Theorem (Shnirelman, 1974) Assume the flow G_t on S^*M is ergodic. Then there is a subset $\mathcal{N} \subset \mathbb{N}$, of density 0, such that for all $b \in C(M)$,

$$\lim_{k\to\infty, k\notin\mathcal{N}} \int_M b(x)|\varphi_k(x)|^2 \, dV(x) = \int_M b(x) \, dV(x).$$

Further phase space localization brings in a quantization of $C^\infty(S^*M)$,

$$\mathrm{op} : C^\infty(S^*M) \longrightarrow OPS^0(M) \subset \mathcal{L}(L^2(M)).$$

Quantizations include

$$\begin{aligned} &\text{Kohn-Nirenberg quantization,} && \mathrm{op}_{KN}, \\ &\text{Weyl quantization,} && \mathrm{op}_W, \\ &\text{Friedrichs quantization,} && \mathrm{op}_F\,. \end{aligned}$$

These differ by maps from $C^\infty(S^*M)$ to $OPS^{-1}(M)$, a space of compact operators on $L^2(M)$.
Special property of op_F:

$$a \in C^\infty(S^*M),\ \ a \ge 0 \Longrightarrow \mathrm{op}_F(a) \ge 0.$$

Constructions involve oscillatory integrals. See, e.g., [T1].

Kohn-Nirenberg quantization on $\mathbb{R}^n$

$$a(x, D)u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} a(x, \xi)\hat{u}(\xi)e^{ix\cdot\xi}\, d\xi,$$

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(y)e^{-iy\xi}\, dy.$$

Theorem (Colin de Verdière, 1985) Assume the flow G_t on S^*M is ergodic. Then there is a subset $\mathcal{N} \subset \mathbb{N}$, of density 0, such that for all $a \in C^\infty(S^*M)$,

$$\lim_{k\to\infty, k\notin\mathcal{N}} (A\varphi_k, \varphi_k)_{L^2} = \overline{a},$$

where

$$A = \mathrm{op}(a), \quad \overline{a} = \int_{S^*M} a \, dS.$$

Ingredients in the proof.

Weyl law

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} (A\varphi_k, \varphi_k)_{L^2} = \overline{a}.$$

(Does not require ergodicity of G_t.)

Egorov theorem

$$\mathrm{op}(a \circ G_t) - U^t A U^{-t} \in OPS^{-1}(M),$$

where $U^t = e^{it\sqrt{-\Delta}}$.

Mean ergodic theorem

$$a_T \longrightarrow Pa \ \text{ in } \ L^2(S^*M, dS), \quad \text{as} \ \ T \to \infty,$$

given

$$a \in L^2(S^*M), \quad a_T = \frac{1}{T}\int_0^T a \circ G_t \, dt,$$

$P =$ orthogonal projection of $L^2(S^*M)$ onto G_t-invariant elements.

Theorem (Schrader-Taylor 1989, Taylor 2015) Do not assume the flow G_t on S^*M is ergodic. There is a subset $\mathcal{N} \subset \mathbb{N}$, of density 0, such that if $a \in C(S^*M)$, $A = \mathrm{op}_F(a)$, then

$$Pa = \overline{a} \Longrightarrow \lim_{k\to\infty, k\notin\mathcal{N}} (A\varphi_k, \varphi_k)_{L^2} = \overline{a}.$$

More generally,

$$\begin{aligned} &Pa \in C(S^*M) \\ &\Longrightarrow \lim_{k\to\infty, k\notin\mathcal{N}} (A\varphi_k, \varphi_k)_{L^2} - (\mathrm{op}_F(Pa)\varphi_k, \varphi_k)_{L^2} = 0. \end{aligned}$$

First part proved in [ST], for $a \in C^\infty(S^*M)$. Rest done in [T2].

Example Take $M = \mathbb{T}^n$, flat torus (so G_t is integrable). Then $Pa = \overline{a}$ for $a = a(x)$, i.e., for

$$Af(x) = a(x)f(x).$$

This theorem uses the fact that, thanks to positivity

$$a \ge 0 \Longrightarrow \mathrm{op}_F(a) \ge 0,$$

op_F has a unique continuous extension from $C^\infty(S^*M)$ to

$$\mathrm{op}_F : C(S^*M) \longrightarrow \mathcal{L}(L^2(M)),$$

still satisfying such positivity.

Example M is an "inner tube," a non-flat torus of revolution in $\mathbb{R}^3$. As shown in [T2],

$$P : C(S^*M) \longrightarrow C(S^*M),$$

but P does not map $C^\infty(S^*M)$ to $C^\infty(S^*M)$, or even to the space of Hölder continuous functions on S^*M.

Quantization of discontinuous symbols ([T3])
The map op_F has a unique extension from $C(S^*M)$ to

$$\mathrm{op}_F : L^\infty(S^*M) \longrightarrow \mathcal{L}(L^2(M)),$$

satisfying

$$a_\nu \to a \text{ weak}^* \text{ in } L^\infty(S^*M)$$
$$\Longrightarrow \mathrm{op}_F(a_\nu) \to \mathrm{op}_F(a) \text{ in the weak operator topology.}$$

Special case:

$$\mathrm{op}_F : \mathcal{R}(S^*M) \longrightarrow \mathcal{L}(L^2(M)),$$

where $\mathcal{R}(S^*M)$ is the space of bounded functions on S^*M that are Riemann integrable.

Theorem ([T3]) Assume the flow G_t on S^*M is ergodic. Then there is a subset $\mathcal{N} \subset \mathbb{N}$ of density 0 such that

$$\lim_{k\to\infty, k\notin\mathcal{N}} (A\varphi_k, \varphi_k)_{L^2} = \int_{S^*M} a\, dS,$$

whenever

$$a \in \mathcal{R}(S^*M), \quad A = \mathrm{op}_F(a).$$

Elementary special case:

$$\lim_{k\to\infty, k\notin\mathcal{N}} \int_M b(x)\, |\varphi_k(x)|^2\, dV(x) = \int_M b(x)\, dV(x),$$

provided $b \in \mathcal{R}(M)$.

Theorem ([T3]) Do not assume the flow G_t on S^*M is ergodic. There is a subset $\mathcal{N} \subset \mathbb{N}$, of density 0, such that

$$\lim_{k\to\infty, k\notin\mathcal{N}} (A\varphi_k, \varphi_k)_{L^2} - (A_p\varphi_k, \varphi_k)_{L^2} = 0,$$

whenever $A = \mathrm{op}_F(a),\ A_p = \mathrm{op}_F(Pa)$, with

$$a \in C(S^*M), \quad Pa \in \mathcal{R}(S^*M).$$

Local (in phase space) equidistribution result.
(See also [Riv], [Gal].)

Theorem ([T3]) Assume G_t acts ergodically on an open set $U \subset S^*M$. Then there is a subset $\mathcal{N} \subset \mathbb{N}$, of density 0, such that if

$$a, b \in \mathcal{R}(S^*M) \text{ are supported on a compact subset of } U,$$

then

$$\int_U a\, dS = \int_U b\, dS$$
$$\Longrightarrow \lim_{k\to\infty, k\notin\mathcal{N}} (A\varphi_k, \varphi_k)_{L^2} - (B\varphi_k, \varphi_k)_{L^2} = 0,$$

for $A = \mathrm{op}_F(a),\ B = \mathrm{op}_F(b)$.

Concentration of eigenfunctions on S^2
G_t periodic of period 2π. Take

$$\Lambda = \sqrt{-\Delta + \tfrac{1}{4}} - \tfrac{1}{4}, \quad \operatorname{Spec}\Lambda = \{k \in \mathbb{Z} : k \geq 0\}.$$

Rotation about x_3-axis, $R(t) = e^{itX}$.
k-eigenspace V_k, $\dim V_k = 2k+1$. Eigenfunctions $\varphi_{k\ell}$, $|\ell| \leq k$.

$$\Lambda\varphi_{k\ell} = k\varphi_{k\ell}, \quad X\varphi_{k\ell} = \ell\varphi_{k\ell}.$$

$$Pa(x,\xi) = \frac{1}{2\pi}\int_0^{2\pi} a(G_t(x,\xi))\,dt.$$

Take $Au(x) = a(x)u(x)$, $a \in C^\infty(S^2)$, rotationally symmetric.

$$\Pi(A) = \frac{1}{2\pi}\int_0^{2\pi} e^{-it\Lambda} A e^{it\Lambda}\,dt,$$

commutes with Λ and X.
(Λ, X) simple spectrum $\Rightarrow \Pi(A) = F(\Lambda, X)$.
Task: analyze F.

Egorov theorem $\Rightarrow \Pi(A) - \mathrm{op}_F(Pa) \in OPS^{-1}(S^2)$.
Proposition. (Via [T1], Chapter 12.)

$$\mathrm{op}_F(Pa) = F_0(\Lambda, X) \mod OPS^{-1}(S^2),$$

for

$$F_0(1,\lambda) = Pa(x_0, (\lambda, \sqrt{1-\lambda^2})) = g(\lambda),$$

x_0 on equator of S^2. Hence

$$\Pi(A) - F_0(\Lambda, X) \in OPS^{-1}(S^2).$$

Corollary. For rotationally symmetric $a \in C^\infty(S^2)$,

$$\begin{aligned}\int_{S^2} a(x)|\varphi_{k\ell}(x)|^2\, dS(x) &= (A\varphi_{k\ell}, \varphi_{k\ell}) \\ &= (\Pi(A)\varphi_{k\ell}, \varphi_{k\ell}) \\ &= g\Big(\frac{\ell}{k}\Big) + O(k^{-1}).\end{aligned}$$

Apply the corollary to a vanishing on $|x_3| \le \beta$, with $\beta \in (0,1)$.

Conclusion. The orthonormal eigenfunctions $\varphi_{k\ell}$ concentrate on the strip $|x_3| \le \beta$, for

$$\frac{|\ell|}{k} \ge \sqrt{1-\beta^2}.$$

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