Multidimensional Toeplitz Operators With Discontinuous Symbols

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Notes available on my website, under Downloadable Lecture Notes

8. Seminar talks and AMS talks

See also

2. Singular integral operators
16. Index theory

Discussion includes joint work with S. Hofmann, I. Mitrea, and M. Mitrea.

Reproducing Formula

First order elliptic differential operator
$D : C^\infty(M, \mathcal{E}_0) \to C^\infty(M, \mathcal{E}_1)$.
$\Omega \subset M$, UR domain (defined below).
E fundamental solution of D over neighborhood $\mathcal{O}$ of $\overline{\Omega}$.
Leibniz formula:

$$D(fu) = fDu + (D_0 f)u, \quad \operatorname{supp} f \subset \mathcal{O}. \tag{1}$$

$Du = \sum A_j \partial_j u + Bu$, $(D_0 f)u = \sum A_j(\partial_j f)u = i^{-1}\sigma_D(x, df)u$.
Apply E to (1).

$$fu = \int_M E(x,y) D_0 f(y)\, u(y)\, dV(y) + \int_M E(x,y) f(y) Du(y)\, dV(y). \tag{2}$$

Assume Ω has finite perimeter,
so $d\chi_\Omega = -\nu\, d\sigma$ is a finite measure.
Let $f = f_\nu \to \chi_\Omega$ boundedly, $df_\nu \to d\chi_\Omega$, weak* in measure.
Assume $u \in C(M)$, $Du \in L^1(M)$. Get **reproducing formula:**

$$u(x) = i \int_{\partial\Omega} E(x,y)\sigma_D(y,\nu(y))u(y)\, d\sigma(y) + \int_{\Omega} E(x,y)Du(y)\, dV(y), \tag{3}$$

for $x \in \Omega$. Last term vanishes if $Du = 0$ on Ω.
Assume now that Ω is Ahlfors regular, so, with $n = \dim\Omega$,
$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$ and $\sigma(\partial\Omega \cap B_r(q)) \approx Cr^{n-1}$, $q \in \partial\Omega$.

Hoffmann-Mitrea-Taylor: (3) holds provided $Du \in L^1(\Omega)$ and

$$u \in C(\Omega),\ \mathcal{N}u \in L^p(\partial\Omega),\quad \exists\,\text{nontangential limit } u\big|_{\partial\Omega},\ \text{a.e.} \tag{4}$$

Here, $\mathcal{N}u$ denotes the nontangential maximal function, $p > 1$.

Uniformly rectifiable domains

An Ahlfors regular domain Ω is a UR domain provided

$\partial\Omega$ contains big pieces of Lipschitz surfaces, at all length scales, satisfying uniform Lipschitz bounds.

That is, $\exists\, \varepsilon, L \in (0, \infty)$ such that for each $x \in \partial\Omega$, $R \in (0, 1]$, $\exists$ Lipschitz map $\varphi : B_R^{n-1} \to M$, with lipschitz constant $\le L$, such that

$$\mathcal{H}^{n-1}(\partial\Omega \cap B_R(x) \cap \varphi(B_R^{n-1})) \ge \varepsilon R^{n-1}. \tag{5}$$

Here, B_R^{n-1} is a ball of radius R in $\mathbb{R}^{n-1}$, $n = \dim \Omega$.

Layer potentials

$$\mathcal{B}f(x) = \int_{\partial\Omega} E(x,y) f(y)\, d\sigma(y), \quad x \in \Omega. \tag{6}$$

$$Bf(x) = \mathrm{PV} \int_{\partial\Omega} E(x,y) f(y)\, d\sigma(y), \quad x \in \partial\Omega. \tag{7}$$

G. David: If Ω is a UR domain,

$$B : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad 1 < p < \infty. \tag{8}$$

Hofmann-Mitrea-Taylor: If Ω is a UR domain,

$$\|\mathcal{N}\mathcal{B}f\|_{L^p(\partial\Omega)} \le C_p \|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty, \tag{9}$$

and there exists a nontangential limit a.e. on $\partial\Omega$,

$$\mathcal{B}f\big|_{\partial\Omega}(x) = \frac{1}{2i}\sigma_E(x,\nu(x)) f(x) + Bf(x). \tag{10}$$

Cauchy transform

Given $f \in L^p(\partial\Omega)$, set

$$\mathcal{C}f(x) = i\mathcal{B}(\vartheta f)(x), \quad Cf(x) = iB(\vartheta f)(x), \quad \vartheta(x) = \sigma_D(x, \nu(x)), \tag{11}$$

In particular,

$$\mathcal{C}f(x) = i \int_{\partial\Omega} E(x,y)\sigma_D(y,\nu(y))f(y)\, d\sigma(y), \quad x \in \Omega. \tag{12}$$

By (10), for a.e. $x \in \partial\Omega$,

$$\mathcal{C}f\big|_{\partial\Omega}(x) = \frac{1}{2}f(x) + Cf(x). \tag{13}$$

Note that $D\mathcal{C}f = 0$ on Ω, so $\mathcal{C} : L^p(\partial\Omega) \to \mathcal{H}^p(\Omega, D)$, for $1 < p < \infty$, where

$$\mathcal{H}^p(\Omega, D) = \{u \in C^1(\Omega) : Du = 0,\ \mathcal{N}u \in L^p(\partial\Omega),\ u|_{\partial\Omega} \text{ exists}\}. \tag{14}$$

Calderon projections

Comparing (12)–(13) and the reproducing formula (3) gives that

$$\mathcal{C}f\Big|_{\partial\Omega} = \mathcal{P}f, \tag{15}$$

where

$$\mathcal{P}f = \Big(\frac{1}{2}I + C\Big)f, \quad \mathcal{P} : L^p(\partial\Omega) \to L^p(\partial\Omega), \tag{16}$$

satisfies

$$\mathcal{P}^2 = \mathcal{P}. \tag{17}$$

This is a Calderon-type projection.

Toeplitz operators

Given a UR domain Ω, $\Phi \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$, $f \in L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$,

$$T_\Phi f = \mathcal{P}\Phi\mathcal{P}f + (I - \mathcal{P})f. \tag{18}$$

$$T_\Phi : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad 1 < p < \infty. \tag{19}$$

Theorem 1. ([MMT]) If Φ or $\Psi \in C(\partial\Omega)$, then

$$T_{\Psi\Phi} - T_\Psi T_\Phi : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \text{ is compact,} \tag{20}$$

for $1 < p < \infty$. More generally, such compactness holds for

$$\Phi \quad \text{or} \quad \Psi \in L^\infty \cap \text{vmo}(\partial\Omega). \tag{21}$$

Idea: difference in (20) is

$$\mathcal{P}[\mathcal{P}, \Psi]\Phi\mathcal{P} = \mathcal{P}\Psi[\Phi, \mathcal{P}]\mathcal{P}.$$

Fredholm properties

If

$$\Phi \in C(\partial\Omega, G\ell(\ell, \mathbb{C})), \tag{22}$$

or more generally if $\Phi,\ \Phi^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega)$, then $T_{\Phi^{-1}}$ inverts T_Φ, mod compacts, so

$$T_\Phi : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \text{ is Fredholm}, \tag{23}$$

for $p \in (1, \infty)$. We set

$$\iota(\Phi) = \text{Index}\, T_\Phi. \tag{24}$$

The index $\iota(\Phi)$ is independent of $p \in (1, \infty)$. ([MMT])
This implies some global regularity results, such as, for $1 < p < q < \infty$,

$$f \in L^p(\partial\Omega),\ T_\Phi f \in L^q(\partial\Omega) \Longrightarrow f \in L^q(\partial\Omega). \tag{25}$$

Homotopy properties of index

If $\Phi_t \in C(\partial\Omega, G\ell(\ell, \mathbb{C}))$ varies continuously with t, then $\iota(\Phi_t)$ is constant. So we get a group homomorphism (on the group of homotopy classes)

$$\iota : [\partial\Omega; G\ell(\ell, \mathbb{C})] \longrightarrow \mathbb{Z}. \tag{26}$$

Polar decomposition $\Phi = AU,\ A = (\Phi\Phi^*)^{1/2},\ U$ unitary. $\iota(\Phi) = \iota(U)$. More delicate result:

Theorem 2. ([MMT]) Assume $\Phi_t \in L^\infty \cap \mathrm{vmo}(\partial\Omega, U(\ell))$ for $t \in [0,1]$, and

$$t \mapsto \Phi_t \text{ continuous from } [0,1] \text{ to } \mathrm{bmo}(\partial\Omega, M(\ell, \mathbb{C})). \tag{27}$$

Then $\iota(\Phi_t)$ is independent of t.

Proof involves an extension of the bmo-homotopy theory of maps of Brezis-Nirenberg.

Cobordism invariance

Theorem 3. ([MMT]) If Ω is a UR domain and $\Phi \in C(\overline{\Omega}, G\ell(\ell, \mathbb{C}))$, then

$$\text{Index } T_\Phi = 0. \tag{28}$$

Key application. $\widetilde{\Omega} \subset\subset \Omega, \ \mathcal{O} = \Omega \setminus \widetilde{\Omega}, \ \partial\mathcal{O} = \partial\Omega \cup \partial\widetilde{\Omega}.$

$$\Phi \in C(\overline{\mathcal{O}}, G\ell(\ell, \mathbb{C})) \Longrightarrow \text{Index } T_\Phi = \text{Index } T_{\widetilde{\Omega},\Phi}. \tag{29}$$

Sometimes $\partial\widetilde{\Omega}$ is smooth, and the Atiyah-Singer theorem is available.

Calderon-Szego projectors

$S = \perp$ projection of $L^2(\partial\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$ onto $\mathcal{R}(\mathcal{P})$.

Key identities: $S\mathcal{P} = \mathcal{P}, \ \mathcal{P}S = S$, hence
$\mathcal{P} = S(I + A), \ A = \mathcal{P} - \mathcal{P}^* = C - C^*$.

A skew adjoint on L^2, $I + A$ invertible on $L^2(\partial\Omega)$, hence on $L^p(\partial\Omega)$, for a range $q_0 < p < q_1$. (Sneiberg)
Hence $S = \mathcal{P}(I + A)^{-1}$, bounded on $L^p(\partial\Omega)$, for $q_0 < p < q_1$.

Another Toeplitz class

Def. $\mathcal{T}_\Phi = S\Phi S + (I - S)$.

Key identities: Given $F, G \in L^\infty(\partial\Omega)$,

$$\mathcal{T}_{FG} - \mathcal{T}_F \mathcal{T}_G = S[S, F]GS,$$
$$[S, F](I + A) = [\mathcal{P}, F] - S[A, F].$$

Hence $[S, F]$ compact on $L^p(\partial\Omega)$, $p \in (q_0, q_1)$, if $F \in L^\infty \cap \mathrm{vmo}(\partial\Omega)$.

Theorem 4. ([MMT]) If $F, F^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega)$, then $\mathcal{T}_F$ is Fredholm on $L^p(\partial\Omega)$, for $q_0 < p < q_1$. Also

$$\mathrm{Index}\, \mathcal{T}_F = \mathrm{Index}\, T_F.$$

Idea: $\mathcal{P}_t = tS + (1 - t)\mathcal{P} \Rightarrow \mathcal{P}_t^2 = \mathcal{P}_t$.
Get Fredholm path from $\mathcal{T}_F$ to T_F.

More severely discontinuous symbols

Typically, $F, F^{-1} \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$ does **not** imply that $\mathcal{T}_F$ is Fredholm.

Def. Given $f \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$, we say f is **locally sectorial** provided that for each $y \in \partial\Omega$, $\exists$ neighborhood $\mathcal{O}_y$ of y in $\partial\Omega$ and $C_y \in M(\ell, \mathbb{C})$ such that

$$\mathrm{Re}\, C_y f(x) \geq bI > 0, \quad \forall\, x \in \mathcal{O}_y. \tag{30}$$

Here, $\mathrm{Re}\, T = (1/2)(T + T^*)$.

Theorem 5. ([T]) If $f \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$ is locally sectorial, then

$$\mathcal{T}_f \ \text{ is Fredholm on } L^p(\partial\Omega), \tag{31}$$

for a range $p_0 < p < p_1$ about $p = 2$.

For $\Omega =$ disk in $\mathbb{C}$ and f scalar, this was proved in [DW] (for $p = 2$). Argument there was very one-dimensional.

Factorization of locally sectorial $f \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$.

Cover $\partial\Omega$ by finitely many open sets $\mathcal{O}_j = \mathcal{P}_{y_j}$ such that (30) holds.
Let $\{\psi_j\}$ be a continuous partition of unity subordinate to this cover. Set

$$\Phi(x) = \sum_j \psi_j(x) C_{y_j}. \tag{32}$$

Then $\Phi \in C(\partial\Omega, M(\ell, \mathbb{C}))$ and

$$\operatorname{Re} \Phi(x) f(x) \geq bI > 0, \quad \forall\, x \in \partial\Omega. \tag{33}$$

Then, with $G(x) = \Phi(x) f(x)$, we have the **factorization**

$$f(x) = F(x) G(x), \quad \forall\, x \in \partial\Omega, \tag{34}$$

where $G \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$, and

$$F = \Phi^{-1} \in C(\partial\Omega, G\ell(\ell, \mathbb{C})), \quad \operatorname{Re} G(x) \geq b > 0, \;\; \forall\, x \in \partial\Omega. \tag{35}$$

Proof of Theorem 5

Let f be locally sectorial. Factorization (34) implies

$$\mathcal{T}_f = \mathcal{T}_F \mathcal{T}_G, \quad \text{mod compacts.}$$

Also, $\mathcal{T}_F$ is Fredholm. Next,

$$\begin{aligned}\operatorname{Re}(\mathcal{T}_G u, u)_{L^2} &= \operatorname{Re}(GSu, Su)_{L^2} + \|(I - S)u\|_{L^2}^2 \\ &\geq C\|u\|_{L^2}^2,\end{aligned}$$

for $u \in L^2(\partial\Omega)$. By Lax-Milgram, $\mathcal{T}_G$ is invertible on $L^2(\partial\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$, hence also on L^p, for p close to 2. This implies Theorem 5, and yields

$$\operatorname{Index} \mathcal{T}_f = \operatorname{Index} \mathcal{T}_F. \tag{36}$$

Example. Take $\Omega = D$, open disk in $\mathbb{C}$. Define $\varphi \in L^\infty(\partial D)$ by

$$\varphi(\theta) = e^{i\theta/2}, \quad 0 \le \theta < 2\pi.$$

So range of φ is Γ, upper boundary of $D^+ = \{z \in D : \Im z > 0\}$. $\partial D^+ = \Gamma \cup (-1,1)$. Set

$$\varphi_\lambda = \varphi - \lambda, \quad \lambda \in \mathbb{C}.$$

Theorem 5 plus (36) yield:

$$\lambda \in D^+ \Rightarrow \mathcal{T}_{\varphi_\lambda} \text{ Fredholm, and Index}\, \mathcal{T}_{\varphi_\lambda} = 1,$$
$$\lambda \in \mathbb{C} \setminus \overline{D}^+ \Rightarrow \mathcal{T}_{\varphi_\lambda} \text{ Fredholm, and Index}\, \mathcal{T}_{\varphi_\lambda} = 0.$$

Corollary:

$$\lambda \in (-1,1) \Rightarrow \mathcal{T}_{\varphi_\lambda} \text{ not Fredholm.}$$

Note that

$$\lambda \in (-1,1) \Rightarrow \varphi_\lambda \text{ and } \varphi_\lambda^{-1} \in L^\infty(\partial D).$$

References

[BDT] P. Baum, R. Douglas, and M. Taylor, Cycles and relative cycles in analytic K-homology, J. Diff. Geom. 39 (1989), 761–804.

[BdM] L. Boutet de Monvel, On the index of Toeplitz operators in several complex variables, Inventiones Math. 50 (1979), 249–272.

[BrN] H. Brezis and L. Nirenberg, Degree theory and BMO I, Selecta Math. 1 (1995), 197–263; II, Selecta Math. 2 (1996), 309–368

[Dav] G. David, Operateurs d'integrale singulieres sur les surfaces regulieres, Ann. Scient. Ecole Norm. Sup. 21 (1988), 225–258.

[DW] R. Douglas and H. Widom, Toeplitz operators with locally sectorial symbols, Indiana Univ. Math. J. 20 (1970), 385–388.

[HMT] S. Hofmann, M. Mitrea, and M. Taylor, Singular integrals and elliptic boundary problems on Semmes-Kenig-Toro domains, IMRN (2010), 2567–2865.

[MMT] I. Mitrea, M. Mitrea, and M. Taylor, Cauchy integrals, Calderon projectors, and Toeplitz operators on uniformly rectifiable domains, Adv. in Math. 268 (2015), 666–757.

[T] M. Taylor, Multidimensional Toeplitz operators with locally sectorial symbols, Comm. PDE 22 (2017), 1322–1336.