Toeplitz Operators on Uniformly Rectifiable Domains

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Abstract. These notes discuss Hardy spaces of solutions to certain first-order elliptic systems of PDE on uniformly rectifiable (UR) domains, singular integral operators that yield projections onto the space of their boundary values, and Toeplitz operators associated with these projections. We produce results on the index of such Toeplitz operators, when they are Fredholm. This is a survey of work [MMT] with I. Mitrea and M. Mitrea, following work [HMT] with S. Hofmann and M. Mitrea.

Contents

1. Introduction

Uniformly rectifiable sets, introduced in [D], form a maximal class of sets for which one has a viable theory of the sort of singular integral operators associated with layer potentials. We will discuss the use of this in the study of some elliptic PDE, concentrating on the following situation.

Let M be a compact Riemannian manifold, D a first order elliptic differential operator (between sections of vector bundles $\mathcal{E}_j$) on M, and $\Omega \subset M$ a uniformly rectifiable domain. Then (under a few technical hypotheses) there is associated a projection $\mathcal{P}$ of $L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$ onto the space of boundary values of "Hardy spaces"

$$\mathcal{H}^p(\Omega, D) = \{u \in C(\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell) : Du = 0,\ \mathcal{N}(u) \in L^p(\partial\Omega), \quad \exists \text{ nontangential a.e. limit } u_b\}, \tag{1.1}$$

where $\mathcal{N}(u)$ is the nontangential maximal function (and $p \in (1,\infty)$). Given also $\Phi \in C(\partial\Omega, M(\ell,\mathbb{C}))$, then we can define the Toeplitz operator

$$T_\Phi f = \mathcal{P}\Phi\mathcal{P}f + (I-\mathcal{P})f. \tag{1.2}$$

We show that such operators are Fredholm if Φ takes values in $G\ell(\ell,\mathbb{C})$, and discuss properties of the index.

We also consider the more general case

$$\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, G\ell(\ell,\mathbb{C})), \tag{1.3}$$

obtaining simultaneously extensions to higher dimensions and to domains with rough boundary of index results of Brezis-Nirenberg [BN].

We briefly define a uniformly rectifiable domain as follows. (See [HMT], [MMT] for more details.) Let Ω be a relatively compact open subset of an n-dimensional Riemannian manifold M. We assume Ω has finite perimeter, i.e.,

$$d\chi_\Omega = \mu \tag{1.4}$$

is a finite, vector-valued measure. To avoid pathologies, we assume $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial^*\Omega) = 0$, where $\partial^*\Omega$ is the measure-theoretic boundary. Then (thanks to fundamental results of DeGiorgi and Federer) σ, the total variation measure associated to μ, is equal to $(n-1)$-dimensional Hausdorff measure, restricted to $\partial\Omega$. In this situation,

$$\mu = -\nu\,\sigma, \tag{1.5}$$

where ν is the outward-pointing unit normal to $\partial\Omega$, defined σ-a.e. on $\partial\Omega$. We assume $\partial\Omega$ is Ahlfors regular, i.e., there exist $C_0, C_1 \in (0,\infty)$ such that

$$C_0 r^{n-1} \le \sigma(B_r(p)) \le C_1 r^{n-1}, \tag{1.6}$$

for all $p \in \partial\Omega,\ 0 < r \le$ diam Ω. Then we call Ω an Ahlfors regular domain.

Such a domain is a UR domain provided $\partial\Omega$ is a uniformly rectifiable set, so it contains big pieces of Lipscihtz surfaces, at all length scales, satisfying uniform Lipschitz bounds. In more detail, there exist $\varepsilon, L \in (0,\infty)$ such that, for each $x \in \partial\Omega,\ R \in (0,1]$, there is a Lipschitz map $\varphi : B_R^{n-1} \to M$ (where B_R^{n-1} is a ball of radius R in $\mathbb{R}^{n-1}$) with Lipschitz constant $\le L$, such that

$$\mathcal{H}^{n-1}(\partial\Omega \cap B_R(x) \cap \varphi(B_R^{n-1})) \ge \varepsilon R^{n-1}. \tag{1.7}$$

The setting of UR domains allows for the following analytical results. Assume $E \in OPS^{-1}(M)$ is a pseudodifferential operator of order -1, with *odd* principal symbol, and integral kernel $E(x,y)$, so

$$Eu(x) = \int_M E(x,y)u(y)\,dV(y), \quad u \in C_0^\infty(M). \tag{1.8}$$

Consider the "principal value" singular integral

$$
\begin{aligned}
(1.9) \qquad Bf(x) &= \mathrm{PV} \int\limits_{\partial\Omega} E(x,y) f(y)\, d\sigma(y) \\
&= \lim_{\varepsilon\to 0} \int\limits_{\partial\Omega\setminus B_\varepsilon(x)} E(x,y) f(y)\, d\sigma(y).
\end{aligned}
$$

Then

$$
(1.10) \qquad B : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \quad \forall\, p \in (1,\infty).
$$

This was demonstrated in [D] when $M = \mathbb{R}^n$ and E is a convolution operator. Also [D] established associated L^p-estimates on the maximal function

$$
(1.11) \qquad \sup_{0<\varepsilon\le 1} \Bigl| \int\limits_{\partial\Omega\setminus B_\varepsilon(x)} E(x,y) f(y)\, d\sigma(y) \Bigr|,
$$

in the convolution setting. In [HMT] this was extended to the variable coefficient setting, and to manifolds. Also [HMT] studied the "double layer" potential

$$
(1.12) \qquad \mathcal{B}f(x) = \int\limits_{\partial\Omega} E(x,y) f(y)\, d\sigma(y), \quad x \in \Omega,
$$

complemented estimates on (1.11) with the nontangential maximal function estimate

$$
(1.13) \qquad \|\mathcal{N}\mathcal{B}f\|_{L^p(\partial\Omega)} \le C_p \|f\|_{L^p(\partial\Omega)}, \quad 1<p<\infty,
$$

and established the nontangential a.e. convergence

$$
(1.14) \qquad \mathcal{B}f\big|_{\partial\Omega}(x) = \frac{1}{2i}\sigma_E(x,\nu(x)) f(x) + Bf(x), \quad \text{a.e. } x \in \partial\Omega,
$$

where $\sigma_E(x,\xi)$ is the principal symbol of E and B is as in (1.9)–(1.10).

With these results in hand, one is in a position to study boundary problems for various elliptic PDE via layer potentials. Such a study carries on earlier works on analysis on Lipschitz domains, initiated by results of [CMM] and [Ver], among others, which in turn followed classical results treating domains with somewhat smoother boundaries.

Here we apply certain layer potentials to the study of spaces (1.1) of solutions to $Du = 0$ on Ω, when D is a first-order elliptic differential operator on M, acting between sections of vector bundles $\mathcal{E}_j$. If $D : H^{s+1}(M,\mathcal{E}_0) \to H^s(M,\mathcal{E}_1)$ is invertible, we can take $E = D^{-1}$ in (1.8). However, in many interesting cases, D will not be invertible. It often has nonzero index. One the other hand, under mild conditions, such as that D and D^* have UCP, one can construct $E \in OPS^{-1}(M)$ such that, for some neighborhood $\mathcal{O}$ of $\overline{\Omega}$,

$$\operatorname{supp} u \subset \mathcal{O} \Longrightarrow EDu = u. \tag{1.15}$$

Indeed, one can take $a \in C_0^\infty(M \setminus \overline{\Omega})$, ≥ 0 everywhere and > 0 on an open set, such that

$$\mathcal{D} = \begin{pmatrix} ia & D^* \\ D & ia \end{pmatrix} \tag{1.16}$$

is invertible, with inverse $\mathcal{D}^{-1} \in OPS^{-1}(M)$, and then

$$\mathcal{D}^{-1} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \Longrightarrow E_{12}D = I - iE_{11}a, \tag{1.17}$$

giving (1.15) with $E = E_{12}$. See [MMT] for details.

The structure of the rest of this paper is as follows. In §2 we pass from $\mathcal{B}$ and B, as in (1.12)–(1.14), to operators $\mathcal{C}$ and C, obtained by applying $\mathcal{B}$ and B to $\sigma_D(y, \nu(y))f(y)$; see (2.8) and (2.11). We approach these operators from two different perspectives, first via a reproducing formula, and then via the goal to obtain from (1.14) an operator such that the first term on the right side of (1.14) gets repaced by $(1/2)f(x)$. Comparison of these two approaches yields the basic result that

$$\mathcal{P} = \frac{1}{2}I + C \Longrightarrow \mathcal{P}^2 = \mathcal{P}. \tag{1.18}$$

This is the projection appearing in (1.2). In §3 we analyze (1.2) for $\Phi \in C(\partial\Omega, M(\ell, \mathbb{C}))$, obtain compactness of $T_{\Phi\Psi} - T_\Phi T_\Psi$, and deduce that (1.2) is Fredholm when $\Phi \in C(\partial\Omega, G\ell(\ell, \mathbb{C}))$. We note that the index of T_Φ on $L^p(\partial\Omega)$ is independent of p and that $\iota(\Phi) = \operatorname{Index} T_\Phi$ produces a group homomorphism

$$\iota : [\partial\Omega; G\ell(\ell, \mathbb{C})] \longrightarrow \mathbb{Z}, \tag{1.19}$$

where $[\partial\Omega; G\ell(\ell, \mathbb{C})]$ is the group of homotopy classes of continuous maps from $\partial\Omega$ to $G\ell(\ell, \mathbb{C})$.

In §4 we extend the scope from continuous Φ to

$$\Phi \in L^\infty \cap \operatorname{vmo}(\partial\Omega, M(n, \mathbb{C})). \tag{1.20}$$

If also Φ^{-1} satisfies (1.20), then T_Φ is Fredholm on $L^p(\partial\Omega)$ for $1 < p < \infty$. The appropriate homotopy invariance in this setting is more subtle than that in §3. We produce a result that extends the scope of some work of [BN], both to higher dimensions and to rough boundaries.

In §5 we extend the scope in another direction, allowing Φ to be a section of $\operatorname{End}\mathcal{C}$, when $\mathcal{C} \rightarrow M$ is a vector bundle, yielding "twisted" Toeplitz operators.

In §6 we introduce cobordism invariance as a useful tool to apply to the problem of computing the index of a Toeplitz operator. A key result is that if $\mathcal{C} \rightarrow M$ is a vector bundle, then

$$\Phi \in C(\overline{\Omega}, G\ell(\mathcal{C})) \Longrightarrow \operatorname{Index} T_\Phi = 0. \tag{1.21}$$

This cobordism invariance is applied in §7, in conjunction with some topological results of Bott and index results of [Ven], [B], and [BDT], to compute the index for a certain interesting class of Toeplitz operators. For example, if $\Omega \subset \mathbb{C}^\mu$ is a bounded UR domain whose boundary $\partial\Omega$ is homeomorphic to the sphere $S^{2\mu-1}$, and

$$D = \overline{\partial} + \overline{\partial}^* : \Lambda^{0,\mathrm{even}}(\mathbb{C}^\mu) \longrightarrow \Lambda^{0,\mathrm{odd}}(\mathbb{C}^\mu), \tag{1.22}$$

then

$$\operatorname{Index} T_\Phi = \pm\vartheta([\Phi]), \quad \text{for} \ \ \Phi \in C(\partial\Omega, U(\ell)), \tag{1.23}$$

provided $\ell \geq \mu$, where ϑ is the Bott isomorphism

$$\vartheta : [\partial\Omega; U(\ell)] \xrightarrow{\approx} \mathbb{Z} \tag{1.24}$$

(well defined up to sign). If $\ell < \mu$, then $\operatorname{Index} T_\Phi = 0$. We emphasize that, in this setting, Ω need not be pseudoconvex, and $\partial\Omega$ can be quite rough.

2. Reproducing formulas, Cauchy integrals, and Calderón projectors

We start with a sequence of reproducing formulas, valid for progressively less smooth functions u and for progressively less rough domains Ω. To begin, we assume

$$u \in C(M, \mathcal{E}_0), \quad Du \in L^1(M, \mathcal{E}_1). \tag{2.1}$$

We let $f \in \operatorname{Lip}(M)$ be scalar and note the Leibniz type formula

$$D(fu) = f Du + (D_0 f)u, \quad D_0 f(x) = \frac{1}{i}\sigma_D(x) df(x), \tag{2.2}$$

where the principal symbol of D is $\sigma_D(x,\xi) = \sigma_D(x)\xi$, linear in $\xi \in T_x^* M$. Assume $\operatorname{supp} f \subset \mathcal{O}$, with $\mathcal{O}$ as in (1.15). Then

$$fu(x) = E\big((D_0 f)u\big) + E(f\, Du). \tag{2.3}$$

Now, assume $\Omega \subset M$ is a finite perimeter domain, and replace f by a sequence $f_\nu \in \mathrm{Lip}(M)$, supported in $\mathcal{O}$, and satisfying

$$\begin{aligned} f_\nu &\longrightarrow \chi_\Omega, \quad \text{boundedly and a.e.,} \\ df_\nu &\longrightarrow d\chi_\Omega = \mu, \quad \text{weak}^* \text{ in measure on } M. \end{aligned} \tag{2.4}$$

Passing to the limit gives

$$\begin{aligned} u(x) = i \int\limits_{\partial\Omega} & E(x,y)\, \sigma_D(y,\nu(y)) u(y)\, d\sigma(y) \\ & + \int\limits_{\Omega} E(x,y) Du(y)\, dV(y), \quad x \in \Omega. \end{aligned} \tag{2.5}$$

This is our basic reproducing formula. Note that the second integral vanishes if $Du = 0$ on Ω. At this point we have (2.5) for Ω with finite perimeter, provided u satisfies (2.1). We will need this formula for much rougher functions u.

The following is established in §2.2 of [MMT], extending a Green formula given in §2.3 of [HMT]. To state it, we bring in the spaces

$$\begin{aligned} \mathfrak{L}^p(\Omega) = \{u \in C(\Omega, \mathcal{E}_0) : \mathcal{N}u \in L^p(\partial\Omega), \text{ and} \\ \exists \text{ nontangential limit } u_b, \sigma\text{-a.e.}\}. \end{aligned} \tag{2.6}$$

Proposition 2.1. *Assume Ω is Ahlfors regular and that, for some $p > 1$,*

$$u \in \mathfrak{L}^p(\Omega), \quad \textit{and} \quad Du \in L^1(\Omega, \mathcal{E}_1). \tag{2.7}$$

Then (2.5) holds.

We now specialize to the case that $\Omega \subset M$ is a UR domain. As stated in §1, the layer potential operator $\mathcal{B}$ defined by (1.12) satisfies (1.13)–(1.14), with B as in (1.9)–(1.10). Given the nontangential limit result (1.14), it follows that if

$$\mathcal{C}f(x) = \int\limits_{\partial\Omega} E(x,y)\sigma_D(y,\nu(y)) f(y)\, d\sigma(y), \quad x \in \Omega, \tag{2.8}$$

then

$$\|\mathcal{N}\mathcal{C}f\|_{L^p(\partial\Omega)} \le C_p \|f\|_{L^p(\partial\Omega)}, \quad 1 < p < \infty, \tag{2.9}$$

and, since $\sigma_E(x,\xi)\sigma_D(x,\xi) = I$, we have nontangential a.e. convergence

$$\mathcal{C}f\big|_{\partial\Omega}(x) = \frac{1}{2} f(x) + Cf(x), \quad x \in \partial\Omega, \tag{2.10}$$

where

$$Cf(x) = i \, \mathrm{PV} \int_{\partial\Omega} E(x,y)\sigma_D(y,\nu(y))f(y)\, d\sigma(y). \tag{2.11}$$

It follows that if $1 < p < \infty$ and $f \in L^p(\partial\Omega, \mathcal{E}_0)$, then

$$u = \mathcal{C}f \Longrightarrow u \in \mathcal{H}^p(\Omega, D), \tag{2.12}$$

defined in (1.1) (for now, with $\ell = 1$). Also (2.5) applies, with $Du = 0$ on Ω, hence

$$u = \mathcal{C}f \Longrightarrow u = \mathcal{C}\big(u\big|_{\partial\Omega}\big). \tag{2.13}$$

Comparing (2.10), we deduce that

$$\mathcal{P} = \frac{1}{2}I + C \Longrightarrow \mathcal{P}^2 = \mathcal{P}. \tag{2.14}$$

By (1.10), we have

$$\mathcal{P} : L^p(\partial\Omega, \mathcal{E}_0) \longrightarrow L^p(\partial\Omega, \mathcal{E}_0), \quad 1 < p < \infty. \tag{2.15}$$

The integral (2.8) is a multi-dimensional generalization of the familiar Cauchy integral, obtained when $M = \mathbb{C}$ and $D = \overline{\partial}$.

When $\partial\Omega$ is smooth, $\mathcal{P}$ is a classical Calderón-type projector. By the definition of $\mathcal{H}^p(\Omega, D)$ in (1.1), there is a bounded trace map

$$\tau : \mathcal{H}^p(\Omega, D) \longrightarrow L^p(\partial\Omega, \mathcal{E}_0), \tag{2.16}$$

and Proposition 2.1 together with (2.8)–(2.13) imply that, when Ω is a UR domain,

$$\tau : \mathcal{H}^p(\Omega, D) \longrightarrow \mathcal{H}^p(\partial\Omega, D), \tag{2.17}$$

with

$$\mathcal{H}^p(\partial\Omega, D) = \mathcal{P}L^p(\partial\Omega, \mathcal{E}_0). \tag{2.18}$$

In §3.1 of [MMT], it is shown that τ in (2.17) is an isomorphism.

In [MMT] the authors also treat a "Calderón-Szegö projector" S, defined initially on $L^2(\partial\Omega, \mathcal{E}_0)$ as the orthogonal projection onto $\mathcal{H}^2(\partial\Omega, D)$. Extensions of S to $L^p(\partial\Omega, \mathcal{E}_0)$, for a range of p, and relations with $\mathcal{P}$, are discussed there. Space considerations motivate us to pass over this topic here, so we point the reader to §3.2 of [MMT].

Remark. It is natural to consider the following variant of (1.1):

$$\widetilde{\mathcal{H}}^p(\Omega, D) = \{u \in C(\Omega, \mathcal{E}_0) : Du = 0,\ \mathcal{N}u \in L^p(\partial\Omega)\}, \tag{2.19}$$

dropping the hypothesis that the trace u_b exists. The assertion that

$$\widetilde{\mathcal{H}}^p(\Omega, D) = \mathcal{H}^p(\Omega, D) \tag{2.20}$$

is known as a Fatou theorem. Such a result is classical when Ω is smoothly bounded. In [MMT2] it is shown that (2.20) holds when Ω is a Lipschitz domain, and also when Ω is a regular SKT domain (a class of domains defined in §4.1 of [HMT]).

3. Toeplitz operators – Fredholmness

Here, Ω will be a UR domain.

The maps $\mathcal{C}, C$, and $\mathcal{P}$, defined in (2.8), (2.11), and (2.14), extend natually from acting on sections of $\mathcal{E}_0$ to acting on sections of $\mathcal{E}_0 \otimes \mathbb{C}^\ell$, giving rise to projections

$$\mathcal{P} : L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell) \longrightarrow L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell), \quad 1 < p < \infty, \tag{3.1}$$

and we have (2.17)–(2.18), with $\mathcal{H}^p(\Omega, D)$ as in (1.1) for general $\ell \geq 1$.

For notational simplicity, we will henceforth typically denote $L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathbb{C}^\ell)$ by $L^p(\partial\Omega)$, unless we need to explicitly specify the relevant vector bundle.

If $\Phi \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$, then multiplication by Φ also naturally acts on sections of $\mathcal{E}_0 \otimes \mathbb{C}^\ell$, and we have the following Toeplitz operator:

$$T_\Phi f = \mathcal{P}\Phi\mathcal{P}f + (I - \mathcal{P})f. \tag{3.2}$$

If also $\Psi \in L^\infty(\partial\Omega, M(\ell, \mathbb{C}))$, then

$$T_\Phi T_\Psi - T_{\Phi\Psi} = \mathcal{P}\Phi[\mathcal{P}, \Psi]\mathcal{P}, \tag{3.3}$$

which is then compact on $L^p(\partial\Omega)$ as long as $[\mathcal{P}, \Psi]$ is. Note that

$$[\mathcal{P}, \Psi]f(x) = [C, \Psi]f(x) = i\,\mathrm{PV}\int_{\partial\Omega} E(x, y)\{\Psi(y) - \Psi(x)\}g(y)\,d\sigma(y), \tag{3.4}$$

where $g(y) = \sigma_D(y, \nu(y))f(y)$. If Ψ is Hölder continuous,

$$\Psi \in C^\alpha(\partial\Omega, M(\ell, \mathbb{C})), \quad \alpha > 0, \tag{3.5}$$

then the integral in (3.4) is weakly singular, and compactness on $L^p(\partial\Omega)$ for $1 < p < \infty$ is elementary. If

$$\Psi \in C(\partial\Omega, M(\ell, \mathbb{C})), \tag{3.6}$$

then we can take $\Psi_\nu \in C^\alpha(\partial\Omega, M(\ell, \mathbb{C}))$, $\Psi_\nu \to \Psi$ uniformly, and deduce that $[\mathcal{P}, \Psi]$ is compact, hence

$$\begin{aligned} &\Psi \in L^\infty(\partial\Omega, M(\ell, \mathbb{C})),\ \Phi \in C(\partial\Omega, M(\ell, \mathbb{C})) \\ &\Longrightarrow T_\Phi T_\Psi - T_{\Phi\Psi} \ \text{ compact on } \ L^p(\partial\Omega), \ \ 1 < p < \infty. \end{aligned} \tag{3.7}$$

From here we readily get

Proposition 3.1. *Let Ω be a UR domain, and assume*

$$\Phi : \partial\Omega \longrightarrow G\ell(\ell, \mathbb{C}) \tag{3.8}$$

is continuous. Then $T_{\Phi^{-1}} T_\Phi - I$ and $T_\Phi T_{\Phi^{-1}} - I$ are compact on $L^p(\partial\Omega)$ for all $p \in (1, \infty)$, so

$$T_\Phi : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \ \textit{is Fredholm}, \quad \forall\, p \in (1, \infty). \tag{3.9}$$

In §4.1 of [MMT] it is shown that

$$\iota(\Phi) = \iota(\Phi; D) := \operatorname{Index} T_\Phi \ \text{ on } \ L^p(\partial\Omega) \tag{3.10}$$

is independent of p. In fact, $\operatorname{Ker} T_\Phi$ on $L^p(\partial\Omega)$ and $\operatorname{Ker} T_\Phi^*$ on $L^{p'}(\partial\Omega)$ are both independent of p. (This can be interpreted as a regularity result.) From (3.7), we have that, if also $\Psi : \partial\Omega \to G\ell(\ell, \mathbb{C})$ is continuous, then

$$\iota(\Phi\Psi) = \iota(\Phi) + \iota(\Psi). \tag{3.11}$$

Note that is Φ_t is a continuous family of elements of $C(\partial\Omega, G\ell(\ell, \mathbb{C}))$, then T_{Φ_t} is a norm continuous family of Fredholm operators, so has a constant index. That is, Index T_Φ depends only on the homotopy class of Φ in $[\partial\Omega; G\ell(\ell, \mathbb{C})]$, the group of homotopy classes of continuous maps $\partial\Omega \to G\ell(\ell, \mathbb{C})$. By (3.11), we obtain a group homomorphism

$$\iota : [\partial\Omega; G\ell(\ell, \mathbb{C})] \longrightarrow \mathbb{Z}, \tag{3.12}$$

We return to this in §7.

In §4.3 of [MMT] the authors show that, if Ω is a UR domain satisfying the two-sided John condition (defined in §4.1 of [HMT]), and if

$$\Phi \in L^q_1(\partial\Omega, G\ell(\ell, \mathbb{C})), \quad q \geq p > 1, \ q \in (n-1, \infty), \tag{3.13}$$

then

$$T_\Phi : L^p_1(\partial\Omega) \longrightarrow L^p_1(\partial\Omega) \ \text{ is Fredholm}, \tag{3.14}$$

where $L^p_1(\partial\Omega)$ is the L^p-Sobolev space (developed for $\partial\Omega$ Ahlfors regular in §A.2 of [MMT]). In particular, (3.13)–(3.14) hold for $\Phi \in \operatorname{Lip}(\partial\Omega, G\ell(\ell, \mathbb{C}))$. It is also shown that the index of T_Φ in (3.14) is equal to that in (3.9)–(3.10). There is the corresponding regularity result that $\operatorname{Ker} T_\Phi$ on $L^p_1(\partial\Omega)$ is equal to that on $L^p(\partial\Omega)$.

4. Toeplitz operators with coefficients in $L^\infty \cap \mathrm{vmo}(\partial\Omega)$

We begin by defining some relevant function spaces. We take Ω to be a relatively compact UR domain, with boundary $\partial\Omega$, and define $\mathrm{bmo}(\partial\Omega)$ and $\mathrm{vmo}(\partial\Omega)$. These definitions extend to a broader class of measured metric spaces; cf. [HMT], §2.4. We have the BMO-seminorm

$$\|\Phi\|_{\mathrm{BMO}} = \sup_B \frac{1}{\sigma(B)} \|\Phi - \Phi_B\|_{L^1(B)}, \tag{4.1}$$

where B runs over all balls in $\partial\Omega$ and

$$\Phi_B = \frac{1}{\sigma(B)} \int\limits_B \Phi(y)\, d\sigma(y). \tag{4.2}$$

This is only a seminorm since Φ const $\Rightarrow \|\Phi\|_{\mathrm{BMO}} = 0$. We use the norm

$$\|\Phi\|_{\mathrm{bmo}} = \|\Phi\|_{\mathrm{BMO}} + \|\Phi\|_{L^1(\partial\Omega)}. \tag{4.3}$$

The space $\mathrm{vmo}(\partial\Omega)$ is the closure in bmo-norm of $C(\partial\Omega)$.

Here we study Toeplitz operators T_Φ with

$$\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega). \tag{4.4}$$

The following is proved in [T3], p. 81, for scalar functions. It extends readily to functions with values in $\mathrm{End}(\mathbb{C}^\ell)$.

Lemma 4.1. *$L^\infty \cap \mathrm{vmo}(\partial\Omega)$ is a closed linear subspace of $L^\infty(\partial\Omega)$, closed under products, hence a closed $*$-subalgebra of the C^*-algebra $L^\infty(\partial\Omega)$.*

Generally, if $\mathcal{A}$ is a C^*-algebra with unit 1 and $\mathcal{B}$ a C^*-subalgebra containing 1, then an element $\varphi \in \mathcal{B}$ is invertible in $\mathcal{B}$ if and only if it is invertible in $\mathcal{A}$. This has the following consequence:

$$\begin{aligned} &\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell), \quad \Phi^{-1} \in L^\infty(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell) \\ &\quad \Longrightarrow \Phi^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell). \end{aligned} \tag{4.5}$$

When Φ satisfies (4.5), we say

$$\Phi \in L^\infty(\partial\Omega, G\ell(\ell, \mathbb{C})). \tag{4.6}$$

The following extends the compactness result on $[\mathcal{P}, \Psi]$ in §3.

Lemma 4.2. *If $\Psi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell)$, then*

$$[\mathcal{P}, \Psi] : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \quad \textit{is compact}, \quad \forall\, p \in (1, \infty). \tag{4.7}$$

Proof. The assertion is that (3.4) is compact on $L^p(\partial\Omega)$ for such Ψ. This is established in §4.2 of [HMT], building on a fundamental commutator estimate of [CRW].

This leads to the following extension of Proposition 3.1.

Proposition 4.3. *If Ω is a UR domain and Φ satisfies (4.6), then $T_{\Phi^{-1}}T_\Phi - I$ and $T_\Phi T_{\Phi^{-1}} - I$ are compact on $L^p(\partial\Omega)$ for all $p \in (1,\infty)$, so*

$$T_\Phi : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \quad \textit{is Fredholm}, \quad \forall\, p \in (1,\infty). \tag{4.8}$$

Again the index $\iota(\Phi) = \iota(\Phi; D) =$ Index T_Φ is independent of $p \in (1,\infty)$. Also, we have

$$\iota(\Phi\Psi) = \iota(\Phi) + \iota(\Psi), \tag{4.9}$$

when $\Phi, \Psi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, G\ell(\ell,\mathbb{C}))$.

The appropriate homotopy invariance is a bit more subtle in this setting than in §3. As a first step, given Φ as in (4.6), one can set

$$\begin{aligned} &\Phi = AU, \quad A = (\Phi\Phi^*)^{1/2}, \\ &U = A^{-1}\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, U(\ell)). \end{aligned} \tag{4.10}$$

Then

$$\iota(\Phi) = \iota(U) + \iota(A). \tag{4.11}$$

Now $(1-t)A + tI \in L^\infty \cap \mathrm{vmo}(\partial\Omega, G\ell(\ell,\mathbb{C}))$ for $0 \le t \le 1$, and the identity $T_{(1-t)A+tI} = (1-t)T_A + tT_I$ yields

$$\iota(A) = 0, \quad \text{hence} \quad \iota(\Phi) = \iota(U). \tag{4.12}$$

Hence to examine the index of T_Φ, it suffices to consider $\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, U(\ell))$. The following two propositions are established in §4.2 of [MMT].

Proposition 4.4. *Assume $\Phi_t \in L^\infty \cap \mathrm{vmo}(\partial\Omega, U(\ell))$ for each $t \in [0,1]$ and*

$$t \mapsto \Phi_t \quad \textit{is continuous from } [0,1] \textit{ to } \mathrm{bmo}(\partial\Omega, \mathrm{End}\,\mathbb{C}^\ell). \tag{4.13}$$

Then $\iota(\Phi_t)$ is independent of $t \in [0,1]$.

The following result reduces index computations for T_Φ to the continuous case.

Proposition 4.5. *Given $\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, U(\ell))$, there exists an explicit approximation procedure, producing*

$$\Phi_t \in C(\partial\Omega, U(\ell)), \quad t > 0, \tag{4.14}$$

such that

$$\|\Phi_t - \Phi\|_{\mathrm{bmo}} \longrightarrow 0, \quad \textit{as} \;\; t \to 0. \tag{4.15}$$

There exists $\varepsilon_1 > 0$ *such that*

$$\iota(\Phi) = \iota(\Phi_t), \quad \forall\, t \in (0, \varepsilon_1]. \tag{4.16}$$

In the special case where $\Omega \subset \mathbb{C}$ is the unit disk (hence has smooth boundary) and $D = \overline{\partial}$ (and $\ell = 1$), such results are obtained in [BN], making use of the homotopy theory of BMO maps $X \to Y$ obtained in [BN] when X and Y are smooth compact manifolds. The analysis in [MMT] requires extending the homotopy theory to allow X to be a compact, Ahlfors regular set. Among other things, a somewhat more complicated approximation procedure is required to produce Φ_t in (4.14)–(4.16). The arguments needed to prove Propositions 4.4–4.5 are fairly elaborate, so we refer to §4.2 of [MMT] for details.

5. Twisted Toeplitz operators

We extend the setting of Toeplitz operators from (3.1) to

$$T_\Phi : L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathcal{C}) \longrightarrow L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathcal{C}), \quad 1 < p < \infty, \tag{5.1}$$

where $\mathcal{C} \to M$ is a smooth vector bundle and

$$\Phi \in C(\partial\Omega, \operatorname{End} \mathcal{C}) \tag{5.2}$$

is a continuous section of $\operatorname{End}\mathcal{C}$ over $\partial\Omega$. The case treated in §3 amounts to taking $\mathcal{C}$ to be the trivial bundle of rank ℓ. In that setting, $\mathcal{P}$ was extended to act on sections of $\mathcal{E}_0 \otimes \mathbb{C}^\ell = \mathcal{E}_0 \oplus \cdots \oplus \mathcal{E}_0$ componentwise. The current setting requires a more elaborate construction.

To begin, we move from D to

$$D_{\mathcal{C}} : H^{s+1}(M, \mathcal{E}_0 \otimes \mathcal{C}) \longrightarrow H^s(M, \mathcal{E}_1 \otimes \mathcal{C}), \tag{5.3}$$

such that

$$\sigma_{D_{\mathcal{C}}}(x, \xi) = \sigma_D(x, \xi) \otimes I_{\mathcal{C}}. \tag{5.4}$$

To do this, we provide $\mathcal{C}$ with a smooth connection ∇. Then, to define (5.3), we take a cue from (2.2) and seek to set

$$D_{\mathcal{C}}(u \otimes v) = Du \otimes v + (D_0 v)u, \tag{5.5}$$

where u is a section of $\mathcal{E}_0$ and v a section of $\mathcal{C}$. We need to define $(D_0 v)u$, as a section of $\mathcal{E}_1 \otimes \mathcal{C}$, again taking a cue from (2.2). Now $\sigma_D(x, \xi) = \sigma_D(x)\xi$ is linear in ξ, and we have

$$\sigma_D(x) : T^*_x \longrightarrow \operatorname{Hom}(\mathcal{E}_{0x}, \mathcal{E}_{1x}), \tag{5.6}$$

or equivalently

$$\sigma_D(x) : \mathcal{E}_{0x} \otimes T_x^* \longrightarrow \mathcal{E}_{1x}. \tag{5.7}$$

Tensoring with $I_{\mathcal{C}}$ gives

$$\sigma_D(x) : \mathcal{E}_{0x} \otimes T_x^* \otimes \mathcal{C}_x \longrightarrow \mathcal{E}_{1x} \otimes \mathcal{C}_x, \tag{5.8}$$

and it is natural to set

$$(D_0 v)u(x) = \frac{1}{i}\sigma_D(x)\big(u(x) \otimes \nabla v(x)\big). \tag{5.9}$$

The symbol identity (5.4) is readily verified, and the analysis of §§2–3 is applicable to $D_{\mathcal{C}}$, yielding the projection

$$\mathcal{P}_{\mathcal{C}} : L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathcal{C}) \longrightarrow L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathcal{C}). \tag{5.10}$$

Actually, in light of (5.9), this operator depends on the choice of connection ∇ on $\mathcal{C}$, but we will not burden the notation with this. Instead, we lighten the notation and simply use $\mathcal{P}$ to denote (5.10), and again (usually) denote the L^p-spaces in (5.10) simply by $L^p(\partial\Omega)$. Thus we set

$$T_\Phi u = \mathcal{P}\Phi\mathcal{P}u + (I - \mathcal{P})u, \tag{5.11}$$

when u is a section of $\mathcal{E}_0 \otimes \mathcal{C}$, and (5.1) holds, for Φ of the form (5.2), and more generally for

$$\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathcal{C}). \tag{5.12}$$

Parallel to Lemma 4.2, we have:

Lemma 5.1. *If* $\Psi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathcal{C})$, *then*

$$[\mathcal{P}, \Psi] : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \quad \textit{is compact}, \quad \forall\, p \in (1, \infty). \tag{5.13}$$

Proof. This time, the identity (3.4) does not quite hold, but, via an argument involving (5.4), the difference between the left and the right sides of (3.4) is given by a weakly singular integral, whose compactness is elementary. See §4.5 of [MMT] for details.

This leads to the following extension of Proposition 4.3.

Proposition 5.2. *Assume Ω is a UR domain and*

$$\Phi,\ \Phi^{-1} \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathcal{C}), \tag{5.14}$$

which we also write as

$$\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, G\ell(\mathcal{C})). \tag{5.15}$$

Then $T_{\Phi^{-1}}T_\Phi - I$ and $T_\Phi T_{\Phi^{-1}} - I$ are compact on $L^p(\partial\Omega)$ for $p \in (1,\infty)$, so

$$T_\Phi : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \ \ \textit{is Fredholm}, \quad \forall\, p \in (1,\infty). \tag{5.16}$$

Thus we can set

$$\iota(\Phi) = \mathrm{Index}\, T_\Phi \ \text{ on } \ L^p(\partial\Omega, \mathcal{E}_0 \otimes \mathcal{C}), \quad p \in (1,\infty). \tag{5.17}$$

As before, this index is independent of $p \in (1,\infty)$. If Ψ also satisfies (5.15), then

$$\iota(\Phi\Psi) = \iota(\Phi) + \iota(\Psi). \tag{5.18}$$

It is useful to have the following.

Proposition 5.3. *Given Φ satisfying (5.15), the index of T_Φ is independent of the choice of connection on $\mathcal{C}$.*

Proof. Two connections on $\mathcal{C}$ give two elliptic operators $D_{\mathcal{C}}$ that differ by an operator of order zero. Hence the integral kernels of $E(x,y)$ differ by a weakly singular term, and so the two versions of T_Φ differ by a compact operator.

6. Localization and cobordism invariance

Tools developed in [MMT] to analyze the index of T_Φ include localization and cobordism invariance. We describe these here. To begin, suppose

$$\partial\Omega = \bigcup_{j=1}^{J} \Gamma_j, \quad \text{disjoint, closed subsets}. \tag{6.1}$$

Define $C_j : L^p(\Gamma_j) \to L^p(\Gamma_j)$ by restricting the integral (2.11) to Γ_j, and set $\mathcal{P}_j = (1/2)I + C_j$, $\mathcal{P}_j : L^p(\Gamma_j) \to L^p(\Gamma_j)$. We have

$$\mathcal{P} - \bigoplus_{j=1}^{J} \mathcal{P}_j \ \text{ compact on } \ L^p(\partial\Omega), \quad \mathcal{P}_j^2 - \mathcal{P}_j \ \text{ compact on } \ L^p(\Gamma_j). \tag{6.2}$$

Thus, with

$$T_{\Gamma_j,\Omega,\Phi} f = \mathcal{P}_j \Phi \mathcal{P}_j f + (I - \mathcal{P}_j) f, \quad f \in L^p(\Gamma_j), \tag{6.3}$$

we have

$$T_\Phi - \bigoplus_{j=1}^{J} T_{\Gamma_j,\Omega,\Phi} \ \text{ compact on } \ L^p(\partial\Omega), \tag{6.4}$$

for $p \in (1,\infty)$, if $\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathcal{C})$. Clearly $T_{\Gamma_j,\Omega,\Phi}$ depends only on $\Phi|_{\Gamma_j}$. If

$$\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, G\ell(\mathcal{C})), \tag{6.5}$$

then each operator $T_{\Gamma_j,\Omega,\Phi}$ is Fredholm on $L^p(\Gamma_j, \mathcal{E}_0 \otimes \mathcal{C})$, and

$$\mathrm{Index}\, T_\Phi = \sum_{j=1}^{J} \mathrm{Index}\, T_{\Gamma_j,\Omega,\Phi}. \tag{6.6}$$

Here is a related localization. Given the UR domain $\Omega \subset M$, assume there is another Riemannian manifold $\widetilde{M}$, a neighborhood $\mathcal{O}$ of $\overline{\Omega}$ in M, and an open $\widetilde{\mathcal{O}} \subset \widetilde{M}$, isometric to $\mathcal{O}$. (From here on, we identify $\mathcal{O}$ and $\widetilde{\mathcal{O}}$.) Assume there exists a first order elliptic differential operator $\widetilde{D}$ on $\widetilde{M}$ acting on sections of $\widetilde{\mathcal{E}}_0 \otimes \widetilde{\mathcal{C}} \to \widetilde{M}$, these bundles agreeing with $\mathcal{E}_0 \otimes \mathcal{C}$ on $\widetilde{\mathcal{O}} = \mathcal{O}$, such that $\widetilde{D} = D$ on $\mathcal{O}$. Then we have the Toeplitz operator

$$T_{\widetilde{M},\Phi} : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \tag{6.7}$$

and

$$\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, \mathrm{End}\,\mathcal{C}) \Longrightarrow T_\Phi - T_{\widetilde{M},\Phi} \ \text{ compact on } \ L^p(\partial\Omega), \tag{6.8}$$

for $p \in (1,\infty)$, so

$$\Phi \in L^\infty \cap \mathrm{vmo}(\partial\Omega, G\ell(\mathcal{C})) \Longrightarrow \mathrm{Index}\, T_\Phi = \ \mathrm{Index}\, T_{\widetilde{M},\Phi}. \tag{6.9}$$

The following cobordism result is established in §4.7 of [MMT].

Proposition 6.1. *If $\Phi \in C(\overline{\Omega}, G\ell(\mathcal{C}))$, then*

$$\mathrm{Index}\, T_\Phi = 0. \tag{6.10}$$

This proposition applies in the following setting. Take an open set $\mathcal{O} \subset \Omega$ such that

$$\mathcal{O} \ \text{ is a UR domain, and } \ \partial\mathcal{O} = \partial\Omega \cup \Gamma, \ \text{ disjoint closed sets.} \tag{6.11}$$

Let

$$\Phi \in C(\overline{\mathcal{O}}, G\ell(\mathcal{C})). \tag{6.12}$$

Then we have $T_\Phi : L^p(\partial\Omega) \to L^p(\partial\Omega)$. Also, we have an analogue, which we denote $T_{\mathcal{O},\Phi}$, defined by replacing Ω by $\mathcal{O}$. Proposition 6.1, with $\mathcal{O}$ in place of Ω, implies

$$\operatorname{Index} T_{\mathcal{O},\Phi} = 0. \tag{6.13}$$

Furthermore,

$$\widetilde{\Omega} = \Omega \setminus \overline{\mathcal{O}} \Longrightarrow \partial\widetilde{\Omega} = \Gamma, \tag{6.14}$$

and via (6.13) and a localization argument, one gets

$$\operatorname{Index} T_\Phi = \operatorname{Index} T_{\widetilde{\Omega},\Phi}. \tag{6.15}$$

See §4.7 of [MMT] for details.

The result (6.15) sometimes applies in cases where $\partial\Omega$ is rough but $\partial\widetilde{\Omega}$ is smooth. There are tools available for calculating the right side of (6.15), including the Atiyah-Singer index formula, when $\partial\widetilde{\Omega}$ is smooth, so the identity (6.15) provides a path for the calculation of the index of T_Φ, in many cases where $\partial\Omega$ is rough.

7. Further results on index computations

As usual, Ω is a relatively compact UR domain. For simplicity, we assume here that $\Phi \in C(\partial\Omega, G\ell(\ell, \mathbb{C}))$. In fact, going further, as in (4.10)–(4.12), we may as well take

$$\Phi \in C(\partial\Omega, U(\ell)). \tag{7.1}$$

As in (3.12), $\iota(\Phi) = \operatorname{Index} T_\Phi$ defines a group homomorphism

$$\iota : [\partial\Omega; U(\ell)] \longrightarrow \mathbb{Z}. \tag{7.2}$$

When (7.1) holds, we can write

$$\Phi(x) = \Phi_0(x)\Phi_1(x), \tag{7.3}$$

with

$$\Phi_0(x) = \begin{pmatrix} \varphi & \\ & I \end{pmatrix}, \quad \varphi(x) = \det \Phi(x), \quad \Phi_1 \in C(\partial\Omega, SU(\ell)), \tag{7.4}$$

and

$$\iota(\Phi) = \iota(\Phi_0) + \iota(\Phi_1) = \iota(\varphi) + \iota(\Phi_1), \tag{7.5}$$

with $\varphi \in C(\partial\Omega, S^1)$, $S^1 \subset \mathbb{C}$. Then

$$[\partial\Omega; S^1] = 0 \Longrightarrow \iota(\Phi) = \iota(\Phi_1), \tag{7.6}$$

and

$$[\partial\Omega; SU(\ell)] = 0 \Longrightarrow \iota(\Phi) = \iota(\varphi). \tag{7.7}$$

Note that (7.6) holds when $\partial\Omega$ is simply connected and (7.7) holds if $\ell = 2$ and $\dim\Omega \le 3$.

We now specialize to the case where $\partial\Omega$ is homeomorphic to a sphere:

$$\partial\Omega \approx S^m, \quad m = n-1 \quad (n = \dim\Omega). \tag{7.8}$$

In such a case, $[\partial\Omega; U(\ell)] \approx \pi_m(U(\ell))$. Thus we are in the setting of $\pi_m(Y)$, the group of homotopy classes of maps from the sphere S^m to a space Y (with $Y = U(\ell)$). Classical results of Bott (cf. [Mil]) imply

$$m = 2\mu - 1 \Longrightarrow \pi_m(U(\ell)) \approx \mathbb{Z}, \text{ if } \ell \ge \mu. \tag{7.9}$$

By contrast,

$$m \notin \{1, 3, \dots, 2\ell - 1\} \Longrightarrow \pi_m(U(\ell)) \text{ is finite.} \tag{7.10}$$

When (7.9) holds, let

$$\vartheta : [\partial\Omega; U(\ell)] \xrightarrow{\approx} \mathbb{Z} \tag{7.11}$$

denote the induced isomorphism (uniquely defined up to sign). We have the following.

Proposition 7.1. *Assume Ω is a UR domain and (7.8) holds. If $m = 2\mu - 1$ and $\ell \ge \mu$, there exists $\alpha = \alpha(\Omega, D) \in \mathbb{Z}$ such that*

$$\iota(\Phi; D) = \alpha\vartheta([\Phi]), \quad \forall\Phi \in C(\partial\Omega, U(\ell)). \tag{7.12}$$

If $m \notin \{1, 3, \dots, 2\ell - 1\}$, then

$$\iota(\Phi; D) = 0, \quad \forall\,\Phi \in C(\partial\Omega, U(\ell)). \tag{7.13}$$

An extra argument is required to show that α in (7.12) is independent of ℓ (up to sign, when ℓ satisfies $\ell \ge \mu$, $m = 2\mu - 1$). See §4.8 of [MMT] for details. This argument also yields the following.

Corollary 7.2. *In the setting of Proposition 7.1, if $m = 2\mu - 1$ and $\ell_1 \geq \mu$, and if there exists $\Phi_1 \in C(\partial\Omega, U(\ell_1))$ such that*

$$\operatorname{Index} T_{\Phi_1} = 1, \tag{7.14}$$

then (7.12) holds with $\alpha = \pm 1$, for all $\ell \geq \mu$.

In fact, we see that α must be a nonzero integer of magnitude ≤ 1.

We aim to produce some cases where Corollary 7.2 applies. We begin with an apparent digression. Let $B \subset \mathbb{C}^\mu$ be the unit ball. Assume $\mu \geq 2$. Let $S_h : L^2(\partial B) \to L^2(\partial B)$ be the Szegö projector onto the space of boundary values of functions holomorphic on B. Since holomorphic functions satisfy an overdetermined elliptic system, this is a different sort of projector from what we have been considering. For example,

$$S_h \in OPS^0_{1/2,1/2}(\partial B). \tag{7.15}$$

This is sufficient to imply that operators $\tau_\Phi = S_h \Phi S_h + (I - S_h)$ are Fredholm if $\Phi \in C(\partial B, U(\ell))$, and one has an analogue of (7.12):

$$\operatorname{Index} \tau_\Phi = \alpha_h \vartheta([\Phi]). \tag{7.16}$$

In [Ven], it is shown that (7.16) holds with $\alpha_h = \pm 1$. An alternative treatment of such an index formula, in a more general setting, was done by Boutet de Monvel in [B]. His formula, valid when $B \subset \mathbb{C}^\mu$ is a smoothly bounded, strongly pseudoconvex domain, can be described as follows. Consider

$$D = \overline{\partial} + \overline{\partial}^* : \Lambda^{0,\mathrm{even}}(\mathbb{C}^\mu) \longrightarrow \Lambda^{0,\mathrm{odd}}(\mathbb{C}^\mu). \tag{7.17}$$

This is an operator of Dirac type. Then

$$\operatorname{Index} \tau_\Phi = \iota(\Phi; D). \tag{7.18}$$

See also [BDT] for a proof of (7.18) using K-homology. We have the following consequence.

Proposition 7.3. *When $\Omega = B$ is the unit ball in $\mathbb{C}^\mu$ and D is given by (7.17), then (7.12) holds with $\alpha = \pm 1$, provided $\ell \geq \mu$.*

From here, we obtain the following.

Proposition 7.4. *Let $\Omega \subset \mathbb{C}^\mu$ be a bounded UR domain and let D be given by (7.17). Let $\ell \geq \mu$. Then*

$$\textit{there exists } \Phi_1 \in C(\partial\Omega, U(\ell)) \textit{ such that } \operatorname{Index} T_{\Phi_1} = 1. \tag{7.19}$$

Proof. We can assume $0 \in B \subset \overline{B} \subset \Omega$. Take $\Phi_1 \in C(\partial B, U(\ell))$ such that T_{B,Φ_1} has index 1, using Proposition 7.3. Then extend Φ_1 to an element of $C(\mathbb{C}^\mu \setminus 0, U(\ell))$, homogeneous of degree 0, and restrict to $\partial\Omega$. The cobordism argument of §6 implies

$$\operatorname{Index} T_{\Omega,\Phi_1} = \operatorname{Index} T_{B,\Phi_1}, \tag{7.20}$$

so we have (7.19).

Corollary 7.5. *Let $\Omega \subset \mathbb{C}^\mu$ be a bounded UR domain and let D be given by (7.17). If $\partial\Omega$ is homeomorphic to $S^{2\mu-1}$, then (7.12) holds, with $\alpha = \pm 1$.*

A. Examples of UR domains

We describe some classes of relatively compact domains Ω that are UR domains. Clearly each Lipschitz domain is a UR domain. Here is a nontrivial generalization of this, from §3.1 of [HMT].

Proposition A.1. *If $\partial\Omega$ is locally the graph of a function with gradient in* bmo*, then Ω is a UR domain.*

First, it is shown in §2.5 of [HMT] that such a domain is Ahlfors regular. The rest of the demonstration relies on the following result, from [DJ] and [Se].

Proposition A.2. *Assume $\Omega \subset M$ is Ahlfors regular and the following holds. There exists $C_0 \in (0, \infty)$ such that for each $x \in \partial\Omega$, $r \in (0, 1]$, there are two balls, with centers of distance $\le r$ from x, of radius r/C_0, one in Ω and one in $M \setminus \overline{\Omega}$. (We say Ω has the two balls property.) More generally, assume there are two $(n-1)$-dimensional disks, with centers of distance $\le r$ from x, radius r/C_0, one in Ω and one in $M \setminus \overline{\Omega}$. (We say Ω has the two disks property.) Then Ω is a UR domain.*

As for the applicability of Proposition A.2, there is the following, from [JK].

Proposition A.3. *Let $\Omega \subset M$ have the property that $\partial\Omega$ is locally the graph of a function in the Zygmund space $C^1_* = B^1_{\infty,\infty}$. Then the two balls property described in Proposition A.2 holds.*

The applicability of Proposition A.3 follows from the implication $\nabla f \in \mathrm{bmo} \Rightarrow f \in C^1_*$.

We now describe examples of UR domains in $\mathbb{R}^n$ of infinite topological type. We begin with an Ahlfors regular surface $\overline{\mathcal{O}}$ that is a bounded subset of $\mathbb{R}^{n-1} \subset \mathbb{R}^n$. For example, we might have

$$\mathcal{O} = D_1(0) \setminus \bigcup_{k \ge 1} \overline{D_{2^{-k-2}}(2^{-k} v_k)}, \tag{A.1}$$

where

$$D_\rho(p) = \{x' \in \mathbb{R}^{n-1} : |x' - p| < \rho\},$$

and v_k are unit vectors in $\mathbb{R}^{n-1}$.

Lemma A.4. *If $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is Lipschitz, then the set*

$$\Sigma = \{(x', f(x')) : x' \in \overline{\mathcal{O}}\} \tag{A.2}$$

is an Ahlfors regular surface.

Proof. Given $p = (q, f(q)) \in \Sigma$, $r \in (0, 1]$, the desired upper bound on $\mathcal{H}^{n-1}(B_r(p) \cap \Sigma)$ is straightforward. It remains to establish a lower bound. For this, assume the Lipschitz constant of f is $\le L$, and set $\beta = (1 + L^2)^{-1/2}$. Then

$$x' \in D_{\beta r}(q) \cap \overline{\mathcal{O}} \Longrightarrow (x', f(x')) \in B_r(p) \cap \Sigma,$$

so

$$\mathcal{H}^{n-1}(B_r(p) \cap \Sigma) \ge \mathcal{H}^{n-1}(D_{\beta r}(q) \cap \overline{\mathcal{O}}),$$

yielding the desired lower bound.

We then have the following UR domains.

Proposition A.5. *If $f, g : \mathbb{R}^{n-1} \to \mathbb{R}$ are Lipschitz,*

$$f = g \quad \text{on} \quad \partial\mathcal{O}, \quad \text{and} \quad f > g \quad \text{on} \quad \mathcal{O}, \tag{A.3}$$

then

$$\Omega = \{(x', x_n) : x' \in \mathcal{O},\ g(x') < x_n < f(x')\} \tag{A.4}$$

is a UR domain.

Proof. That Ω is an Ahlfors regular domain follows from Lemma A.4. The UR property then follows directly from the definition.

Remark. For $\mathcal{O}$ as in (A.1), one could take $f(x') = \operatorname{dist}(x', \mathbb{R}^{n-1} \setminus \mathcal{O})$, and $g \equiv 0$, or perhaps $g = -f$.

B. Compactness of weakly singular integral operators

In various places we have indicated that certain integral operators were weakly singular, and hence compact on L^p. Here we provide an explicit statement of such a result. A proof can be found in §2.4 of [HMT].

Proposition B.1. *Let X be a compact, $(n-1)$-dimensional, Ahlfors regular surface, with surface measure σ. Let $k(x,y)$ be a measurable function on $X \times X$ satisfying*

$$|k(x,y)| \le C\psi(d(x,y))\, d(x,y)^{-(n-1)}, \tag{B.1}$$

with

$$\int_0^1 \frac{\psi(t)}{t}\, dt < \infty. \tag{B.2}$$

Consider

$$Kf(x) = \int\limits_X k(x,y)f(y)\, d\sigma(y). \tag{B.3}$$

Then

$$K : L^p(X,\sigma) \longrightarrow L^p(X,\sigma) \quad \textit{is compact}, \quad \forall\, p \in (1,\infty). \tag{B.4}$$

This result typically applies with $\psi(r) = r^a$, with $a > 0$. In fact, for use in these notes, typically $a = 1$.

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