Microlocal Analysis and Nonlinear PDE

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The process of localization, i.e., a map of the form $u \mapsto \varphi(x)u$ where $\varphi \in C_0^\infty(\mathbb{R}^n)$, is of frequent use in both linear and nonlinear PDE. In addition to localizing in space, one often localizes in frequency, i.e., one uses $u \mapsto \varphi(D)u$, where $\varphi(D)$ has the effect of multiplying the Fourier transform $\hat{u}(\xi)$ of u by $\varphi(\xi)$. One can combine these two types of operations, to produce 'microlocal analysis.' Thus, we consider operators of the form

$$\varphi(x, D)u = \int \varphi(x, \xi)\hat{u}(\xi)e^{ix\cdot\xi}\, d\xi. \tag{1}$$

If $\varphi(x, \xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$, then $\varphi(x, D)$ is just the linear differential operator $\sum a_\alpha(x)D^\alpha$. There are various 'symbol classes,' such as $S^m_{\rho,\delta}$, introduced in [H1], where, with $\langle\xi\rangle = (1 + |\xi|)^{\frac{1}{2}}$,

$$\varphi(x, \xi) \in S^m_{\rho,\delta} \Longleftrightarrow |D_x^\beta D_\xi^\alpha \varphi(x, \xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}. \tag{2}$$

We say $\varphi(x, D) \in OPS^m_{\rho,\delta}$. Typically, we require $0 \le \delta \le \rho \le 1$. For example, the differential operator of order m mentioned above belongs to $OPS^m_{1,0}$.

If $\delta < \rho$, there is a useful symbol calculus, arising from

$$p_1(x, D)p_2(x, D) = a(x, D), \tag{3}$$

where

$$p_j(x, \xi) \in S^{m_j}_{\rho,\delta} \Longrightarrow a(x, \xi) = p_1(x, \xi)p_2(x, \xi) \bmod S^{m_1+m_2-(\rho-\delta)}_{\rho,\delta}. \tag{4}$$

On the other hand, if $\rho = 1$ and also $\delta < 1$, one has the following boundedness on Sobolev spaces. If $p(x, \xi) \in S^m_{\rho,\delta}$,

$$p(x, D) : H^{s,p}(\mathbb{R}^n) \longrightarrow H^{s-m,p}(\mathbb{R}^n), \quad 1 < p < \infty, \tag{5}$$

for all $s \in \mathbb{R}$. There are also Hölder estimates:

$$p(x, D) : C^s(\mathbb{R}^n) \longrightarrow C^{s-m}(\mathbb{R}^n), \tag{6}$$

if $s, s-m \in (0,\infty) \setminus \mathbb{Z}^+$. Proofs of these results can be found in a number of places, such as [H2], [S2], and [T1].

These results have a well-known role in linear PDE. For the simplest application, suppose $p(x,D)$ is an elliptic differential operator (with smooth coefficients) of order m, i.e., $|p(x,\xi)| \geq C|\xi|^m$ for $|\xi| \geq B$. Then

$$q(x,\xi) = \psi(\xi)p(x,\xi)^{-1} \in S^{-m}_{1,0}, \tag{7}$$

if $\psi \in C^\infty(\mathbb{R}^n),\ \psi(\xi) = 0$ for $|\xi| \leq B,\ \psi(\xi) = 1$ for $|\xi| \geq 2B$. It follows from (4) that

$$q(x,D)p(x,D)u = u + r(x,D)u, \quad r(x,\xi) \in S^{-1}_{1,0}. \tag{8}$$

Standard results on elliptic regularity follow easily from this.

There is a plethora of other applications to linear PDE, many of which are given in [H2], [T1], and [Tr]. Our aim here is to discuss some applications to nonlinear PDE, which have played a significant role since the foundational work of J.-M.Bony [B1] and Y.Meyer [M1], [M2] in about 1980.

We begin with an analysis of $F(u)$, for smooth F, given in [M1]. Take $\Psi_0 \in C_0^\infty(\mathbb{R}^n),\ \Psi_0(\xi) = 1$ for $|\xi| \leq 1$, and set $\Psi_k(\xi) = \Psi_0(2^{-k}\xi),\ u_k = \Psi_k(D)u$. Then

$$F(u) = M(x,D)u + F(u_0) \tag{9}$$

where the formula

$$M(x,D)u = \sum_{k\geq 0} \bigl\{F(u_{k+1}) - F(u_k)\bigr\} \tag{10}$$

yields

$$M(x,\xi) = \sum_k m_k(x)\psi_{k+1}(\xi), \quad m_k(x) = \int_0^1 F'\bigl(\Psi_k(\tau;D)u\bigr)\, d\tau, \tag{11}$$

with

$$\psi_{k+1}(\xi) = \Psi_{k+1}(\xi) - \Psi_k(\xi), \quad \Psi_k(\tau;D) = \Psi_k(D) + \tau\psi_{k+1}(D). \tag{12}$$

To estimate $M(x,\xi)$, given $u \in L^\infty(\mathbb{R}^n)$, we have, by the chain rule,

$$\|D_x^\ell m_k\|_{L^\infty} \leq C_\ell \sum_{1\leq\nu\leq\ell} \|D^{\ell_1}u_{k+1}\|_{L^\infty} \cdots \|D^{\ell_\nu}u_{k+1}\|_{L^\infty} \cdot \|F''\|_{C^{\nu-1}}. \tag{13}$$

Also,

$$\|D^{\ell_j}u_{k+1}\|_{L^\infty} \leq C_{\ell_j} 2^{k\ell_j}\|u\|_{L^\infty}. \tag{14}$$

Since $2^{k\ell} \sim \langle\xi\rangle^\ell$ on the support of $\psi_{k+1}(\xi)$, we see that

$$u \in L^\infty(\mathbb{R}^n) \Longrightarrow M(x,\xi) \in S^0_{1,1}. \tag{15}$$

Now $\delta = \rho = 1$ is the 'bad' case we avoided when writing down (4)-(6). However, as proved in [S1] (see also [Bour]), for $p(x,\xi) \in S^m_{1,1}$, (5) holds, provided $s - m > 0$, and so does (6).

One immediate consequence of this is that, for $s > 0,\ p \in (1,\infty)$,

$$\begin{aligned} \|F(u)\|_{H^{s,p}} &\le \|M(x,D)u\|_{H^{s,p}} + \|F(u_0)\|_{H^{s,p}} \\ &\le C\big(\|u\|_{L^\infty}\big)\big\{\|u\|_{H^{s,p}} + 1\big\}, \end{aligned} \tag{16}$$

which is a 'Moser estimate,' established in [Mos] and of frequent use in nonlinear PDE. The fact that this rather subtle and powerful estimate follows so readily is a good preliminary indication of the power of (9)-(15) as a tool in nonlinear analysis.

In order to have a symbol calculus available, one splits such a symbol as $M(x,\xi)$ into two pieces:

$$M(x,\xi) = M^\#(x,\xi) + M^b(x,\xi), \tag{17}$$

where

$$M^\#(x,\xi) = \sum_k J_k m_k(x)\ \psi_{k+1}(\xi), \tag{18}$$

and J_k are smoothing operators (in the x variable), forming an approximate identity. Possible choices of J_k are

$$J_k = \Psi_0(2^{-k\delta}D), \quad \text{or} \quad J_k = \Psi_{k-5}(D), \tag{19}$$

where $\delta \in (0,1)$. Given $r > 0$, we have

$$u \in C^r \Longrightarrow M^\#(x,\xi) \in S^0_{1,\delta}, \quad M^b(x,\xi) \in S^{-r\delta}_{1,1}. \tag{20}$$

If we take $\delta < 1$, then the symbol calculus (4) applies. If instead we take $J_k = \Psi_{k-5}(D)$, then there is a replacement operator calculus, given by [M1], [B1]. We have $M^\#(x,\xi)$ in the symbol class $\mathcal{B}S^0_{1,1}$, where

$$p(x,\xi) \in \mathcal{B}S^m_{1,1} \Longleftrightarrow p(x,\xi) \in S^m_{1,1}, \text{ and } \operatorname{supp} \hat{p}(\eta,\xi) \subset \{|\eta| \le \rho|\xi|\}, \tag{21}$$

for some $\rho \in (0,1)$. Operators in $OP\mathcal{B}S^m_{1,1}$ satisfy (5), for all $s \in \mathbb{R}$, not just $s > m$. A more general operator calculus has been developed in [Bour] and [H3].

For a nonlinear differential operator, there is a similar construction. We can write

$$F(x, D^m u) = M(x, D)u + R(u), \tag{22}$$

where $R(u) = F\big(x, D^m \Psi_0(D)u\big)$ and

$$u \in C^m \Longrightarrow M(x, \xi) \in S^m_{1,1}, \tag{23}$$

and we can write $M(x, \xi) = M^{\#}(x, \xi) + M^b(x, \xi)$ where, given $r > 0$,

$$u \in C^{m+r} \Longrightarrow M(x, \xi) \in S^m_{1,\delta}, \quad M^b(x, \xi) \in S^{m-r\delta}_{1,1}. \tag{24}$$

As an application of the results stated above, we can establish a Schauder type elliptic regularity result for a solution to a completely nonlinear elliptic PDE. Suppose

$$F(x, D^m u) = g(x). \tag{25}$$

That this is elliptic implies that $M(x, \xi) \in S^m_{1,1}$ is elliptic, and that $M^{\#}(x, \xi) \in S^m_{1,\delta}$ is elliptic. Pick $\delta < 1$. Then the symbol calculus (4) yields a parametrix $E \in OPS^{-m}_{1,\delta}$ of $M^{\#}(x, D)$, such that $EM^{\#}(x, D) = I$ modulo a smoothing operator. Writing (25) as $M^{\#}(x, D)u = g - M^b(x, D)u$ and applying E, we have

$$u = Eg - EM^b(x, D)u, \quad \text{mod} \quad C^\infty. \tag{26}$$

Suppose we assume initially that

$$u \in C^{m+\varepsilon}, \quad \varepsilon > 0. \tag{27}$$

Also assume that $g \in H^{s,p}(\mathbb{R}^n)$, $1 < p < \infty$, $s > 0$. The hypothesis (27) implies $M^b(x, D) \in OPS^{m-\varepsilon\delta}_{1,1}$. Hence the right side of (26) belongs to $H^{s+m,p}(\mathbb{R}^n) + C^{m+\varepsilon+\varepsilon\delta}(\mathbb{R}^n)$. This is contained in $H^{m+\gamma,p}(\mathbb{R}^n)$, for any $\gamma < \min(s, \varepsilon + \varepsilon\delta)$. Given this, we have $Eg - EM^b(x, D)u \in H^{s+m,p}(\mathbb{R}^n) + H^{m+\gamma+\varepsilon\delta,p}(\mathbb{R}^n)$. Iterating this argument, we obtain

$$g \in H^{s,p}(\mathbb{R}^n) \Longrightarrow u \in H^{s+m,p}(\mathbb{R}^n), \tag{28}$$

for a solution to (25) when this PDE is elliptic, given $1 < p < \infty$, $s > 0$, and (27). A similar argument yields

$$g \in C^s(\mathbb{R}^n) \Longrightarrow u \in C^{s+m}(\mathbb{R}^n), \tag{29}$$

for a solution u to (25), in the elliptic case, given $s \in \mathbb{R}^+ \setminus \mathbb{Z}^+$ and (27).

Note that the seminorms of $E(x,\xi)$ in $S^{-m}_{1,\delta}$ depend only on $\|u\|_{C^m}$ while those of $M^b(x,\xi)$ in $S^{m-\varepsilon\delta}_{1,1}$ depend on $\|u\|_{C^{m+\varepsilon}}$. Using this, we can establish the following Moser type estimate for a solution u to (25), when it is elliptic:

$$\|u\|_{C^{m+s}} \le A_s\big(\|u\|_{C^m}\big)\|g\|_{C^s} + B_s\big(\|u\|_{C^{m+\varepsilon}}\big), \tag{30}$$

the significant part of this estimate being the linear dependence on $\|g\|_{C^s}$. Note that the $m = 0$ case of (30), which is an estimate on a solution to $F(u) = g$ when F is invertible, is just a little weaker than the estimate (16) (with F replaced by F^{-1} and $H^{s,p}$ replaced by C^s).

Another important Moser estimate is the commutator estimate

$$\|P(fu) - fPu\|_{H^{s,p}} \le C\|f\|_{\mathrm{Lip}^1}\|u\|_{H^{m-1+s,p}} + C\|f\|_{H^{m+s,p}}\|u\|_{L^\infty}, \tag{31}$$

when $s \ge 0,\ p \in (1,\infty)$, and P is a differential operator of order $m \in \mathbb{Z}^+$. This was extended in [KP] to the case $P \in OPS^m_{1,0},\ m > 0$. In [T2] it was shown how this result can be derived from the paradifferential operator calculus.

In the approach of [T2], to establish (31), one starts with the following representation of a product:

$$fg = T_f g + T_g f + R(f,g), \tag{32}$$

where T_f is Bony's 'paraproduct,' defined by

$$T_f g = \sum_{k\ge 5} \Psi_{k-5}(D)f \cdot \psi_{k+1}(D)g.$$

This arises from the construction (9)-(19), applied to $F(f,g) = fg$, and with J_k given by the second formula in (19). Clearly $f \in L^\infty \Rightarrow T_f \in OPS^0_{1,1}$. In fact, it belongs to $OP\mathcal{B}S^0_{1,1}$, and hence is bounded on all the Sobolev spaces $H^{s,p}$, for $s \in \mathbb{R}, p \in (1,\infty)$. There are the following important estimates:

$$\|T_f g\|_{L^p} \le C_p\|f\|_{L^p}\|g\|_{\mathrm{BMO}}, \quad \|R(f,g)\|_{L^p} \le C_p\|f\|_{\mathrm{BMO}}\|g\|_{L^p}, \tag{33}$$

for $p \in (1,\infty)$, which follow from work of [CM]; proofs are also given in [T2]. Another useful estimate, established in [T2], is

$$\|R(f,g)\|_{H^{\sigma,p}} \le C\|f\|_{\mathrm{Lip}^1}\|g\|_{H^{\sigma-1,p}}, \quad \sigma \in [0,\infty),\ p \in (1,\infty). \tag{34}$$

To apply the decomposition (32) to (31), we write

$$\begin{aligned} f(Pu) &= T_f Pu + T_{Pu} f + R(f,Pu), \\ P(fu) &= PT_f u + PT_u f + PR(f,u). \end{aligned} \tag{35}$$

The operator calculus readily yields

$$\|T_f Pu - PT_f u\|_{H^{s,p}} \le C\|f\|_{\mathrm{Lip}^1}\|u\|_{H^{m-1+s,p}}, \tag{36}$$

for all $m, s \in \mathbb{R},\ p \in (1,\infty)$. Meanwhile, (34) implies

$$\|R(f,Pu)\|_{H^{s,p}} + \|PR(f,u)\|_{H^{s,p}} \le C\|f\|_{\mathrm{Lip}^1}\|u\|_{H^{m-1+s,p}}, \tag{37}$$

provided $s \ge 0,\ m+s \ge 0$, and $p \in (1,\infty)$. To estimate $PT_u f$, we use the fact that $u \in L^\infty \Rightarrow T_u \in OP\mathcal{B}S^0_{1,1}$, so $T_u : H^{\sigma,p} \to H^{\sigma,p}$ for all $\sigma \in \mathbb{R}$, to get

$$\|PT_u f\|_{H^{s,p}} \le C\|u\|_{L^\infty}\|f\|_{H^{m+s,p}}. \tag{38}$$

Finally, if $u \in L^\infty$ and $m > 0,\ Pu$ belongs to the 'Zygmund space' C^{-m}_* (which we define below; see (55)), and one can show that T_{Pu} belongs to $OP\mathcal{B}S^m_{1,1}$ and thus maps $H^{\sigma,p} \to H^{\sigma-m,p}$, for all $\sigma \in \mathbb{R}$. Hence

$$\|T_{Pu} f\|_{H^{s,p}} \le C\|u\|_{L^\infty}\|f\|_{H^{m+s,p}}, \tag{39}$$

provided $m > 0,\ 1 < p < \infty$. Thus we have (31). For details of the arguments sketched above, see [T2].

We next sketch a method for establishing existence of solutions to first order quasilinear symmetric hyperbolic systems:

$$\frac{\partial u}{\partial t} = K(u,D)u, \quad u(0,x) = f(x), \tag{40}$$

where

$$K(u,D)u = \sum_{j=1}^n K_j(u)\partial_j u, \tag{41}$$

with $\partial_j u = \partial u/\partial x_j$. Here, f and u take values in $\mathbb{R}^\ell$, and each K_j is an $\ell \times \ell$ matrix valued function. The hypothesis that (40) is symmetric hyperbolic is that $K_j(u)^* = K_j(u)$. We could just as easily have $K_j = K_j(x,u)$.

We obtain a solution to (40) as a limit of solutions u_ε to

$$\frac{\partial u_\varepsilon}{\partial t} = J_\varepsilon K(J_\varepsilon u_\varepsilon, D)J_\varepsilon u_\varepsilon, \quad u_\varepsilon(0) = f, \tag{42}$$

where $J_\varepsilon = \Psi_0(\varepsilon D)$. For simplicity, write $K_\varepsilon = K(J_\varepsilon u_\varepsilon, D)$. For any $\varepsilon > 0$, (42) has a unique solution on some t-interval containing 0. We need to show that the size of the t-interval does not shrink to zero as $\varepsilon \to 0$, and to obtain appropriate bounds on u_ε. To do this, we estimate the rate of change of

$\|\Lambda^m u_\varepsilon(t)\|_{L^2}^2$, where $\Lambda^m \in OPS^m_{1,0}$ is Fourier multiplication by $\langle\xi\rangle^m$. Here m is some positive real number. We have

$$\begin{aligned} \frac{d}{dt}\|\Lambda^m u_\varepsilon\|_{L^2}^2 &= 2(\Lambda^m J_\varepsilon K_\varepsilon J_\varepsilon u_\varepsilon, \Lambda^m u_\varepsilon)_{L^2} \\ &= 2(\Lambda^m K_\varepsilon J_\varepsilon u_\varepsilon, \Lambda^m J_\varepsilon u_\varepsilon)_{L^2} \\ &= A_1 + A_2, \end{aligned} \tag{43}$$

where

$$A_1 = 2(K_\varepsilon \Lambda^m J_\varepsilon u_\varepsilon, \Lambda^m J_\varepsilon u_\varepsilon)_{L^2}, \quad A_2 = 2\big([\Lambda^m, K_\varepsilon] J_\varepsilon u_\varepsilon, \Lambda^m J_\varepsilon u_\varepsilon\big)_{L^2}. \tag{44}$$

To estimate A_1, we have

$$\begin{aligned} (K_\varepsilon v, v)_{L^2} &= \sum \big(K_j(J_\varepsilon u_\varepsilon)\partial_j v, v\big)_{L^2} \\ &= -\sum \big(\partial_j[K_j(J_\varepsilon u_\varepsilon) v], v\big)_{L^2}, \end{aligned} \tag{45}$$

upon integrating by parts and using $K_j = K_j^*$. Taking the sum of the two expressions on the right, we obtain

$$(K_\varepsilon v, v)_{L^2} = \frac{1}{2}\sum \big([\partial_j K_j(J_\varepsilon u_\varepsilon)]v, v\big)_{L^2}. \tag{46}$$

Hence

$$\begin{aligned} |A_1| &\le \sum \|\partial_j K_j(J_\varepsilon u_\varepsilon)\|_{L^\infty} \|\Lambda^m J_\varepsilon u_\varepsilon\|_{L^2}^2 \\ &\le C\big(\|J_\varepsilon u_\varepsilon\|_{C^1}\big)\|\Lambda^m J_\varepsilon u_\varepsilon\|_{L^2}^2. \end{aligned} \tag{47}$$

To estimate A_2, we have

$$[\Lambda^m, K_\varepsilon]v = \sum_j [\Lambda^m, K_j(J_\varepsilon u_\varepsilon)]\partial_j v, \tag{48}$$

and applying the estimate (31), with $s = 0,\ p = 2$, gives

$$\begin{gathered} \|[\Lambda^m, K_\varepsilon]v\|_{L^2} \le \\ C\sum_j \big\{\|K_j(J_\varepsilon u_\varepsilon)\|_{H^m}\|\partial_j v\|_{L^\infty} + \|K_j(J_\varepsilon u_\varepsilon)\|_{\mathrm{Lip}^1}\|\partial_j v\|_{H^{m-1}}\big\}. \end{gathered} \tag{49}$$

Hence, using (16) to estimate $\|K_j(J_\varepsilon u_\varepsilon)\|_{H^m}$, we have

$$|A_2| \le C\big(\|J_\varepsilon u_\varepsilon\|_{C^1}\big)\|\Lambda^m J_\varepsilon u_\varepsilon\|_{L^2}^2. \tag{50}$$

Thus we have

$$\frac{d}{dt}\|\Lambda^m u_\varepsilon\|_{L^2}^2 \le C\big(\|J_\varepsilon u_\varepsilon\|_{C^1}\big)\|\Lambda^m u_\varepsilon\|_{L^2}^2. \tag{51}$$

If $f \in H^m(\mathbb{R}^n)$ and $m > \frac{n}{2} + 1$, so $\|v\|_{C^1} \le C\|v\|_{H^m}$, we have

$$\frac{d}{dt}\|u_\varepsilon\|_{H^m}^2 \le B\big(\|u_\varepsilon\|_{H^m}\big), \tag{52}$$

where the form of $B(\lambda)$ is independent of ε. Gronwall's inequality then yields an estimate

$$\big\|u_\varepsilon(t)\big\|_{H^m} \le C\big(\|f\|_{H^m}\big), \quad \text{for} \quad t \in (-T, T), \tag{53}$$

with T independent of ε. The PDE (42) then yields a bound on $\|\partial_t u_\varepsilon\|_{H^{m-1}}$, and a standard argument yields a limit $u_{\varepsilon_\nu} \to u$, solving (40). A variant of (43)-(51) demonstrates that the solution $u(t)$ persists as long as there is a bound on $\|u(t)\|_{C^1}$.

A more general class of first order quasilinear hyperbolic system (40) is the class of *symmetrizable* hyperbolic systems. If we do not assume that $K_j(u)^* = K_j(u)$, but that there exists $R(u)$, positive definite, such that $R(u)K_j(u)$ is symmetric for $1 \le j \le n$, then $R(u)$ is called a symmetrizer for (40). Basic examples arise in the equations of compressible fluid flow. There is a more general notion of symmetrizer, introduced by P.Lax. Namely, we consider $R(u, x, \xi)$, smooth on $\mathbb{R}^\ell \times \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$, homogeneous of degree 0 in ξ, which is positive definite and satisfies the property that $R(u, x, \xi) \sum K_j(u)\xi_j$ is symmetric. One can modify the analysis in (42)-(53) to treat such symmetrizable hyperbolic systems; see [T2].

Higher order quasilinear hyperbolic systems can be reduced to systems of a more complicated form than (40):

$$\frac{\partial u}{\partial t} = K(Au, x, D)u + F(x, Au), \quad u(0, x) = f(x), \tag{54}$$

where $A \in OPS^0_{1,0}$, and $K(v, x, \xi) \in S^1_{1,0}$. Again a treatment parallel to (42)-(53) is effective, though there are some significant differences. For example, dependence on $\|u\|_{C^1}$ is replaced by dependence on $\|Bu\|_{C^1}$, for some $B \in OPS^0_{1,0}$, which might not be bounded on C^1. This appears to lead to a weaker sort of persistence result than the one mentioned above. In fact, in all these cases such a persistence result can be strengthened, using a technique from [BKM]. Namely, for a solution to (54) to persist, in the hyperbolic case, it suffices to have a bound on $\|u(t)\|_{C^1_*}$, where we use the *Zygmund norm*:

$$\|u\|_{C^r_*} = \sup_{k \ge 0} \langle k \rangle^r \|\psi_k(D)u\|_{L^\infty(\mathbb{R}^n)}, \tag{55}$$

with $r = 1$. This result is proved in [T2].

Solutions to hyperbolic equations involve the propagation of waves, and at this point let us mention that the first major application of paradifferential operator calculus, in [B1], was to the study of propagation of singularities of solutions to nonlinear PDE. At the same time, an alternative approach to propagation of singularities, in the semilinear case, was given in [RR]. Variants of Bony's proof are also given in [H4] and [T2], and in [Be] there is an extensive discussion of the semilinear case. Further important developments include [Del], [MSZ]; see also [BMR] for a number of survey papers. A related subject is the propagation of highly oscillatory solutions to nonlinear wave equations; papers on this include [HMR], [JMR].

We next discuss another commutator estimate, first established in [CRW], and then used in [CLMS]: given $P \in OPS^0_{1,0},\ 1 < p < \infty$,

$$\|fPu - P(fu)\|_{L^p} \le C_p \|f\|_{\mathrm{BMO}} \|u\|_{L^p}. \tag{56}$$

In [AT], there is a proof of this result (with the BMO-norm replaced by the slightly stronger bmo-norm) using paradifferential operator calculus, in particular the decomposition (35). In place of (37)-(39), we have

$$\begin{aligned} \|T_{Pu}f\|_{L^p} + \|R(f,Pu)\|_{L^p} + \|PT_u f\|_{L^p} + \|PR(f,u)\|_{L^p} \\ \le C_p \|f\|_{\mathrm{BMO}} \|u\|_{L^p}, \end{aligned} \tag{57}$$

as a consequence of (33). One gets (56) (with bmo) from this and the following result of [AT]:

$$\|[P, T_f]u\|_{L^p} \le C \|f\|_{C^0_*} \|u\|_{L^p}, \tag{58}$$

where C^0_* is a Zygmund norm, as in (55). Several proofs of this are given in [AT]. One approach used there is to give a refinement of the analysis of products of operators in [M1] and then show that

$$f \in C^0_*(\mathbb{R}^n) \Longrightarrow [P, T_f] \in OP\mathcal{B}S^0_{1,1}, \tag{59}$$

modulo a smoothing operator, given $P \in OPS^0_{1,0}$, or more generally, (essentially) $P \in OP\mathcal{B}S^0_{1,1}$.

One primary corollary of the commutator estimate (56) is a 'div-curl lemma,' as discussed in [CLMS]. We give here an abstract version of such a result, using a formulation of P.Auscher and the author. Consider a bilinear form

$$\mathcal{P}u \cdot \mathcal{Q}v = \sum_{j=1}^{N} (P_j u)(Q_j v), \tag{60}$$

where $\mathcal{P},\ \mathcal{Q} \in OPS^0_{1,0}$ (more generally, we can take $\mathcal{P}, \mathcal{Q} \in OP\mathcal{B}S^0_{1,1}$). Here, u and v can take values in $\mathbb{R}^k$ and $\mathbb{R}^\ell$, respectively, so $\mathcal{P}$ is a $k \times N$ matrix of operators and $\mathcal{Q}$ is an $\ell \times N$ matrix of operators. Take

$$f \in \mathrm{bmo}, \quad u \in L^p(\mathbb{R}^n), \quad v \in L^{p'}(\mathbb{R}^n). \tag{61}$$

We have

$$\Big(f, \sum_j (P_j u)(Q_j v)\Big) = \int f(\mathcal{P}u)\cdot(\mathcal{Q}v)\,dx = \int v\ \mathcal{Q}^t(f\mathcal{P}u)\,dx \tag{62}$$
$$= \big(v, [\mathcal{Q}^t, M_f]\mathcal{P}u\big) + (fv, \mathcal{Q}^t\mathcal{P}u).$$

Now, we make the hypothesis that

$$h = \mathcal{Q}^t\mathcal{P}u = \sum Q_j^t P_j u \in L^r(\mathbb{R}^n), \quad r > p. \tag{63}$$

Then, if f has support in a compact set $K \subset \mathbb{R}^n$, we have

$$|(fv, h)| \le C_K \|f\|_{\mathrm{bmo}} \|v\|_{L^{p'}} \|h\|_{L^r}.$$

Since (56) implies that

$$\|[\mathcal{Q}^t, M_f]w\|_{L^p} \le C_p \|f\|_{\mathrm{bmo}} \|w\|_{L^p}, \tag{64}$$

for $1 < p < \infty$, we have (when supp $f \subset K$)

$$\Big|\Big(f, \sum (P_j u)(Q_j v)\Big)\Big| \le C_{pK}\|f\|_{\mathrm{bmo}}\Big(\|u\|_{L^p} + \|h\|_{L^r}\Big)\|v\|_{L^{p'}}. \tag{65}$$

In the standard div-curl lemma, $u = (u_1, \dots, u_n)$, $P_j u = u_j$, and $Q_j v = \partial_j \Lambda^{-1} v$, $1 \le j \le n$. Then (63) is the hypothesis that div $u \in H^{-1,r}(\mathbb{R}^n)$. One particularly successful application of this result is given in [Hel], on the regularity of harmonic maps of 2-dimensional Riemannian manifolds into spheres. See [Ev] for another proof of the div-curl lemma and application to results on partial regularity. We also mention the proof of [DM], which makes use of 'product renormalization.'

A number of variants of the div-curl lemma, discussed in [CLMS] and elsewhere, involve estimation of wedge products. We take the space here to mention an approach to such estimates, using the following estimate on a 'super-commutator.' Let M be a compact, oriented, n-dimensional Riemannian manifold. Let f be an ℓ-form, on M, set $W_f u = f \wedge u$, and define

$$[[\Lambda^{-1}d, W_f]] = [\Lambda^{-1}d, W_f] \ \text{ if } \ \ell \ \text{ is even,} \tag{66}$$
$$\{\Lambda^{-1}d, W_f\} \ \text{ if } \ \ell \ \text{ is odd,}$$

where $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$. Here, d is the exterior derivative and $\Lambda = (I - \Delta)^{\frac{1}{2}}$. We prove the following estimate.

Lemma. *For $1 < p < \infty$, we have*

$$\|\,[[\Lambda^{-1}d, W_f]]\beta\|_{L^p} \le C_p \|f\|_{\mathrm{bmo}} \|\beta\|_{L^p}. \tag{67}$$

Proof. Write $W_f = \sum M_{f_i} W_{e_i}$ where e_i are smooth ℓ-forms and $\sum \|f_i\|_{\mathrm{bmo}} \sim \|f\|_{\mathrm{bmo}}$. Then

$$[[\Lambda^{-1}d, W_f]]\beta = \sum_i [\Lambda^{-1}d, M_{f_i}] W_{e_i}\beta + \sum_i M_{f_i} [[\Lambda^{-1}d, W_{e_i}]]\beta. \tag{68}$$

Now the estimate (56) applies to the first sum on the right. Since the principal symbol of $\Lambda^{-1}d$ is wedge by $i|\xi|^{-1}\xi$, we have

$$[[\Lambda^{-1}d, W_{e_i}]] \in OPS^{-1}_{1,0}, \tag{69}$$

so the estimate on the second term on the right side of (68) is elementary.

We apply this lemma, first to an estimate of $du \wedge dv$. Let u be a j-form and v a k-form on M, $j+k \le n-2$. Let f be an ℓ-form, $\ell = n-j-k-2$. We set $u = \Lambda^{-1}\tilde{u}$, $v = \Lambda^{-1}\tilde{v}$, and desire to estimate

$$\int f \wedge du \wedge dv = (W_f d\Lambda^{-1}\tilde{u}, \delta\Lambda^{-1} * \tilde{v}). \tag{70}$$

Here, δ is the adjoint of d, and $*$ is the Hodge star operator. Since $W_f \Lambda^{-1} dd\Lambda^{-1} = 0$, the right side of (70) is equal to

$$(\Lambda^{-1} dW_f d\Lambda^{-1}\tilde{u}, *\tilde{v}) = \big([[\Lambda^{-1}d, W_f]] d\Lambda^{-1}\tilde{u}, *\tilde{v}\big). \tag{71}$$

Applying the Lemma, we deduce that

$$\Big| \int f \wedge du \wedge dv \Big| \le C_p \|f\|_{\mathrm{bmo}} \|u\|_{H^{1,p}} \|v\|_{H^{1,p'}}. \tag{72}$$

Next, we estimate k-fold wedge products. Assume u_j are ℓ_j-forms, $\sum_{j=1}^{k} (\ell_j + 1) = m \le n$. Let f be an $(n-m)$-form. Then we will show that

$$\Big| \int f \wedge du_1 \wedge \cdots \wedge du_k \Big| \le C_p \|f\|_{\mathrm{bmo}} \|u_1\|_{H^{1,p_1}} \cdots \|u_k\|_{H^{1,p_k}}, \tag{73}$$

provided $p_j \in (1, \infty]$ and

$$\frac{1}{p_1} + \cdots + \frac{1}{p_k} = 1, \quad p_k \in (1, \infty). \tag{74}$$

To prove this, note that, since $du_1 \wedge \cdots \wedge du_{k-1}$ is closed, we can use Hodge theory to write

$$du_1 \wedge \cdots \wedge du_{k-1} = du + h, \tag{75}$$

where h is a harmonic form and

$$\begin{gathered}\|u\|_{H^{1,p}} + \|h\|_{L^\infty} \le C\|u_1\|_{H^{1,p_1}} \cdots \|u_{k-1}\|_{H^{1,p_{k-1}}}, \\ \frac{1}{p_1} + \cdots + \frac{1}{p_{k-1}} = \frac{1}{p}, \quad p \in (1,\infty), \quad p_k = p'.\end{gathered} \tag{76}$$

Then, with $v = u_k$, we have

$$\int f \wedge du_1 \wedge \cdots \wedge du_k = \int f \wedge du \wedge dv + \int f \wedge h \wedge dv. \tag{77}$$

The last integral in (77) is easy to estimate, and the estimate (72) applies to the other integral on the right side of (77). This proves the desired estimate (73). The case $k = n$, $\ell_j = 0$ yields a Jacobian determinant estimate, which played a particularly significant role in [CLMS].

There are a number of other topics in nonlinear PDE in which microlocal analysis has been influential recently. We mention particularly the study of the Euler equation for incompressible fluid flow; see [Che]. Microlocal analysis in nonlinear PDE is still a young area, and one can expect a good deal of development to alter the landscape considerably over the next decade.

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Research partially supported by NSF grant DMS 9400905.