Microlocal Concentration of Eigenfunctions of Subelliptic Operators

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Let M be a compact, n-dimensional Riemannian manifold, and let $L \in OPS^2(M)$ be a positive, self-adjoint operator. We assume L is not elliptic, but that it is subelliptic, in the sense that there exists $\sigma > 0$ (necessarily $\sigma < 2$) such that

$$(L+1)^{-1} : H^s(M) \longrightarrow H^{s+\sigma}(M), \quad \forall\, s \in \mathbb{R}. \tag{1}$$

Let $\{\varphi_k\}$ be an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of L:

$$L\varphi_k = \lambda_k \varphi_k, \quad 0 \le \lambda_1 \le \lambda_2 \le \cdots \nearrow +\infty. \tag{2}$$

We aim to prove the following.

Theorem 1. *Take L as above, and denote its principal symbol by L_2. Assume*

$$\int_{S^*M} L_2(x,\omega)^{-n/2}\, dS(x,\omega) = \infty. \tag{3}$$

Then, except perhaps for a "sparse" subsequence, the sequence $\{\varphi_k\}$ concentrates microlocally on the characteristic set $\Sigma \subset S^*M$, given by

$$\Sigma = \{(x,\omega) \in S^*M : L_2(x,\omega) = 0\}. \tag{4}$$

The proof will involve a study of the semigroup $\{e^{-tL} : t \ge 0\}$, and of products Ae^{-tL}, with $A \in OPS^0(M)$. The hypothesis (1) implies

$$e^{-tL} : \mathcal{D}'(M) \longrightarrow C^\infty(M), \tag{5}$$

for each $t > 0$. In particular, $\operatorname{Tr} e^{-tL} < \infty$ for each $t > 0$. We will show that, under the hypotheses of Theorem 1,

$$t^{n/2} \operatorname{Tr} e^{-tL} \longrightarrow +\infty, \quad \text{as} \ \ t \searrow 0. \tag{6}$$

Furthermore, if the principal symbol A_0 of A satisfies

$$A_0 = 0 \ \text{ on a neighborhood of } \Sigma \text{ in } S^*M, \tag{7}$$

we obtain

$$\operatorname{Tr} A e^{-tL} \sim C(A_0) t^{-n/2}, \quad \text{as} \ \ t \searrow 0, \tag{8}$$

with $C(A_0) \in \mathbb{C}$. From (8) we obtain

$$\sum_{k\geq 0} e^{-t\lambda_k}(A\varphi_k,\varphi_k) \sim C(A_0)t^{-n/2}, \quad t \searrow 0, \tag{9}$$

when (7) holds. Applying this observation to A^*A yields

$$\lim_{t\to 0} t^{n/2} \sum_{k\geq 0} e^{-t\lambda_k} \|A\varphi_k\|_{L^2}^2 = C(|A_0|^2). \tag{10}$$

Meanwhile, (6) implies

$$\lim_{t\to 0} t^{n/2} \sum_{k\geq 0} e^{-t\lambda_k} = +\infty. \tag{11}$$

In preparation for proving (6), we will find it useful to recall some properties of e^{-tM} when $M \in OPS^2(M)$ is an *elliptic*, positive, self-adjoint operator, with principal symbol M_2. In such a case, parametrix constructions yield

$$e^{-tM}u(x) = \int_M H(t,x,y)u(y)\,dV(y), \tag{12}$$

with

$$H(t,x,y) = C_n \int_{T_x^*M} e^{-tM_2(x,\xi)} e^{i(x-y)\cdot\xi}\,d\xi + \cdots . \tag{12A}$$

In particular,

$$H(t,x,x) = C_n \int_{T_x^*M} e^{-tM_2(x,\xi)}\,d\xi + o(t^{-n/2}), \tag{13}$$

as $t \searrow 0$. Now

$$\int_{T_x^*M} e^{-tM_2(x,\xi)}\,d\xi = C_n' t^{-n/2} \int_{S_x^*M} M_2(x,\omega)^{-n/2}\,dS_x(\omega), \tag{14}$$

hence

$$\operatorname{Tr} e^{-tM} = (4\pi t)^{-n/2} \frac{1}{A_{n-1}} \int_{S^*M} M_2(x,\omega)^{-n/2}\,dS(x,\omega) + o(t^{-n/2}), \tag{15}$$

where A_{n-1} is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$, so

$$\frac{1}{A_{n-1}} \int\limits_{S^*M} dS(x,\omega) = \operatorname{Vol} M. \tag{16}$$

In particular, if $M = -\Delta$, where Δ is the Laplace operator on M, we have

$$\operatorname{Tr} e^{t\Delta} = (4\pi t)^{-n/2} \operatorname{Vol} M + o(t^{-n/2}). \tag{17}$$

Behind (12)–(13) is a parametrix construction of e^{-tM} as a family of pseudodifferential operators. Then pseudodifferential operator calculus yields, for $A \in OPS^0(M)$, with principal symbol A_0,

$$\operatorname{Tr} Ae^{-tM} = (4\pi t)^{-n/2} \frac{1}{A_{n-1}} \int\limits_{S^*M} A_0(x,\omega) M_2(x,\omega)^{-n/2}\, dS(x,\omega) + o(t^{-n/2}). \tag{18}$$

To establish (6), we argue as follows. Take $\varepsilon > 0$ and set $M = L - \varepsilon\Delta$. We apply (15) to such M. The relevance of such an application arises as follows. Say $\{\psi_k\}$ is an orthonormal basis of $L^2(M)$ consisting of eigenfunctions of M:

$$L\psi_k = \mu_k \psi_k, \quad 0 \le \mu_1 \le \mu_2 \le \cdots \nearrow +\infty. \tag{19}$$

Lemma 2. *Let L, M be positive, self-adjoint operators with compact resolvents. Assume*

$$\mathcal{D}(M) \subset \mathcal{D}(L), \quad L \le M. \tag{20}$$

Let the eigenvalues be $\{\lambda_k\}$, $\{\mu_k\}$, as in (2) and (19). Then, for each k,

$$\lambda_k \le \mu_k. \tag{21}$$

Proof. Pick $\mu \in (0,\infty)$, and let $V_\nu \subset L^2(M)$ be the span of $\{\psi_k : \mu_k < \mu\}$, so $((M - \mu I)v, v) < 0$ for all nonzero $v \in V_\mu$, but not for all v in a linear space of larger dimension. The hypotheses above yield $((L - \mu I)v, v) < 0$, for all nonzero $v \in V_\mu$, so

$$\#\{\lambda_j : \lambda_j < \mu\} \ge \#\{\mu_j : \mu_j < \mu\}.$$

From the lemma, we deduce that

$$\operatorname{Tr} e^{-tL} \ge \operatorname{Tr} e^{-t(L-\varepsilon\Delta)}, \tag{22}$$

for each $\varepsilon > 0,\ t > 0$. Applying (15) to $M = L - \varepsilon\Delta$, we have

$$\lim_{t\to 0} (4\pi t)^{n/2} \operatorname{Tr} e^{-(L-\varepsilon\Delta)} = \frac{1}{A_{n-1}} \int\limits_{S^*M} (L_2(x,\omega) + \varepsilon)^{-n/2}\, dS(x,\omega). \tag{23}$$

Hence

$$\liminf_{t\to 0}\,(4\pi t)^{n/2}\operatorname{Tr} e^{-tL} \geq \frac{1}{A_{n-1}} \int\limits_{S^*M} (L_2(x,\omega)+\varepsilon)^{-n/2}\,dS(x,\omega), \tag{24}$$

for each $\varepsilon > 0$. Hence

$$\liminf_{t\to 0}\,(4\pi t)^{n/2}\operatorname{Tr} e^{-tL} \geq \frac{1}{A_{n-1}} \int\limits_{S^*M} L_2(x,\omega)^{-n/2}\,dS(x,\omega). \tag{25}$$

Thus, given the hypothesis (3), we have (6).

Next, we bring in the fact that, if $A \in OPS^0(M)$ satisfies (7), then the construction of a parametrix for $e^{-tL}A$ is microlocal, and yields, parallel to (18),

$$\operatorname{Tr} e^{-tL}A = (4\pi t)^{-n/2}\frac{1}{A_{n-1}} \int\limits_{S^*M} A_0(x,\omega)L_2(x,\omega)^{-n/2}\,dS(x,\omega) + o(t^{-n/2}), \tag{26}$$

and, of course,

$$\operatorname{Tr} Ae^{-tL} = \operatorname{Tr} e^{-tL}A, \tag{27}$$

so we have (8).

Examples. Let $M = S^2 \subset \mathbb{R}^3$ be the unit sphere, and let X_j be vector fields generating 2π-periodic rotation about the x_j-axis, for $1 \leq j \leq 3$. Then $\Delta = X_1^2 + X_2^2 + X_3^2$. Now

$$L = -(X_1^2 + X_2^2)$$

satisfies (1), with $\sigma = 1$, and we also have (3). On the other hand,

$$L = -(X_1^2 + X_2^2 + X_3 M_{x_1}^2 X_3)$$

also satisfies (1), with $\sigma = 1$, but (3) does not hold. In this case, the integral $\int_{S^*M} L_2(x,\omega)^{-1}\,dS$ is finite.